

Geometric Classification of Monogenic Subspaces and Uniparametric Linear Control Systems ^{*}

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Abstract

We present a geometric approach to the classification of monogenic invariant subspaces, alternative to the classical algebraic one, which allows us to obtain several matricial canonical forms for each class. Some applications are derived: canonical coordinates of a vector with regard to an endomorphism, and a canonical form for uniparametric linear control systems, not necessarily controllable, with regard to linear changes of state variables. Moreover, the pointwise construction can be extended to differentiable families of changes of basis when differentiable families of equivalent monogenic subspaces are considered.

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1 Introduction

In this paper we present a geometric classification, alternative to the algebraic one in [11], of the invariant subspaces of an endomorphism in the particular case that they are “monogenic”, that is to say, spanned by one vector and its successive images. For the general case, [14] shows that this is a so-called “wild” problem when the degree of the minimal polynomial is greater than 6.

For monogenic subspaces, the classifying geometric parameters are the so-called marked and perturbation indices defined by means of the LR-sequence associated to each invariant subspace in [2] (Definition 3.4). The key tool in this paper is the geometrical approach there to the Carlson problem. We recall that it asks for conditions to ensure when three given Weyr characteristics can be realized as the ones corresponding to an endomorphism, its restriction to an invariant subspace and the one induced in the quotient space. The LR-sequences give an implicit answer to this problem ([12], [2]), but they do not characterize the equivalence class of a general invariant

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subspace, whereas we show that it is so for monogenic ones. Indeed, we find (Corollary 3.9) families of classifying elements, alternative to the Ulm sequence in [11], which can be easily listed (Corollary 6.8). The key point (Proposition 3.3) is the special shape of the LR-sequence, so that it is unique for each compatible triple of Weyr characteristics, and in turn determines uniquely the class of the monogenic subspace. Then the “marked” and “perturbation” indices (Definition 3.4) describe easily this LR-sequence. Moreover, they have an interesting interpretation in terms of Arnold’s deformations (Remark 4.5).

This geometrical approach allows us to construct a canonical representative of each equivalence class. We recall (see Definition 2.2) that, if A is the matrix in some basis of an endomorphism f and V is a f -invariant subspace, then the pairs equivalent to (V, A) are obtained by means of $(S^{-1}V, S^{-1}AS)$, where S is a matrix of change of basis. Thus, it seems natural to ask for suitable S in order to reduce (V, A) to a more simple representative pair. For example, by taking S a Jordan basis and J the Jordan form of A , we obtain reduced forms of the type $(S^{-1}V, J)$, where additional restrictions could be considered in $S^{-1}V$. Alternatively, if S is a basis adapted to V (i.e., a basis of E obtained by extending a basis of V), one obtains (\mathbb{R}^h, A_0) , where $h = \dim(V)$, $A_0 = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}$ and additional restrictions could be considered for A_1, A_2, A_3 . For example, we can take A_1 and A_2 as being Jordan matrices.

Here we construct three canonical forms when V is a monogenic subspace. Indeed in Section 4 we obtain (Theorem 4.1) a canonical PM-form (\mathbb{R}^h, A_{PM}) where A_{PM} is a so-called “perturbed marked” matrix. Next, in Section 5 we present a canonical C-form (\mathbb{R}^h, A_C) , where A_C is a Carlson matrix. Finally, also in Section 5, we construct (Proposition 5.4) the canonical J-form $(S^{-1}V, J)$, where the generator of $S^{-1}V$ has coordinates 0 or 1 and the number of non-zero coordinates is minimal. All of them are obtained by using in a suitable way the vectors determined in Lemma 3.6.

These three canonical forms contain essentially the same information, but their usefulness varies depending on further properties to be studied. For example, the PM-form seems more convenient if one asks for the perturbations of the initial pair (see Remark 4.5). The C-form makes clearer the relationship with the Carlson Problem. The J-form is reformulated in Section 6 as the canonical J-form of a vector, that is to say, we obtain canonical coordinates of a vector among those corresponding to Jordan bases. In particular it is applied to obtain a canonical form for linear control uniparametric systems (not necessarily controllable) with regard to changes of basis in the state space (Corollary 6.9). Finally, we improve the reduced form in [6] for bimodal piecewise dynamical systems with regard to changes of basis which preserve the separating hyperplane (Corollary 6.12).

Moreover, our geometric construction allows us an affirmative answer to the classical local-global question ([15]: “does pointwise solvability imply the existence of nicely parameterized solution?”). For example, a differentiable family of reducing changes of basis exists when one considers a differentiable family of equivalent monogenic subspaces (Proposition 4.6) or a differentiable family of equivalent vectors (Proposition 6.6).

The paper is organized as follows. In Section 2 we recall the basic definitions and results concerning invariant subspaces which will be used in the sequel. In particular, the Carlson problem, the LR-sequences and the techniques in [2] based in the double Jordan filtration which will be the key tool in our reasonings. In Section 3 we focus in the monogenic case. We characterize the LR-sequences, we introduce the marked and perturbed indices and relate them

to the other classifying elements. Then we determine the basic vectors associated to the LR-sequences. Section 4 is devoted to the PM-form and in Section 5 we construct the C-form and the J-form. The latter one is reformulated in Section 6 to obtain the J-form of a vector. Finally we apply it to uniparametric control systems and to bimodal dynamical systems.

In all the paper $M_{m \times n}(K)$ is the set of matrices with m rows and n columns with coefficients in the field K . If $A \in M_{m \times n}(K)$ then A^* means its transpose.

2 Invariant Subspaces

We recall some definitions and results concerning the classification of invariant subspaces.

Definition 2.1 *Let E be a n -dimensional vector space over \mathbb{C} , and f an endomorphism. A subspace $V \subset E$ is called invariant (or f -invariant) if $f(V) \subset V$. We write (V, f) for V in this situation.*

Equivalently, V is f -invariant if the matrix of f in any basis of E adapted to V (that is to say, a basis of E obtained by extending one of V) has the form

$$A = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}, \text{ where } A_1 \in M_h(\mathbb{C}), h = \dim(V).$$

Then, A_1 is the matrix of the restriction \hat{f} of f to V in the corresponding basis of V , and A_2 the one of the quotient endomorphism \hat{f} of E/V in the induced basis of E/V .

In particular, V is called marked if there is some Jordan basis for \hat{f} which can be extended to a Jordan basis for f . See [7] for a matrix characterization.

Definition 2.2 *Two invariant subspaces (V, f) and (V', f') are called equivalent if there exists $\varphi \in \text{Aut}(E)$ such that $\varphi(V) = V'$ and $\varphi \circ f = f' \circ \varphi$.*

If A and A' are the matrices of f and f' in some bases respectively, this is equivalent to the existence of a non-singular matrix S such that:

$$V' = S^{-1}V, \quad A' = S^{-1}AS.$$

In particular, if each basis is adapted to the corresponding subspace, then $S = \begin{pmatrix} S_1 & S_3 \\ 0 & S_2 \end{pmatrix}$, so that $A'_1 = S_1^{-1}A_1S_1, A'_2 = S_2^{-1}A_2S_2$.

Bearing in mind the decomposition $V = \bigoplus_{\lambda} (\text{Ker}(f - \lambda I)^{m_{\lambda}} \cap V)$, where λ runs over the eigenvalues of f , we can restrict ourselves to f being nilpotent.

Moreover, we focus in the *indecomposable* invariant subspaces, because the generalization is obvious:

Definition 2.3 *An invariant subspace (V, f) is called indecomposable if there do not exist two non-null invariant subspaces (U, f) and (U', f) such that $U \oplus U'$ is the total space and $V = (V \cap U) \oplus (V \cap U')$.*

It is obvious that the equivalence relation in Definition 2.2 preserves the Jordan type of the matrices A, A_1 and A_2 . We will confirm in a moment that this triple does not characterize the equivalence classes (different classes having the same triple can exist). Previously we have remarked that these three Jordan types are not independent. The Carlson problem asks for conditions characterizing the compatible triples, that is to say, those which occur for some invariant subspace. The next theorem gives an implicit answer in terms of the existence of a Littlewood-Richardson sequence (LR-sequence).

Definition 2.4 A partition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m, 0, \dots)$ will be any non-increasing finite sequence of nonnegative integers

$$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_m > \alpha_{m+1} = 0$$

where $\ell(\alpha) = m$ is its length and $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_m$ is its weight.

Its conjugate partition α^* is defined by $\alpha_j^* = \#\{1 \leq i \leq \ell(\alpha) : \alpha_i \geq j\}$.

If f is a nilpotent endomorphism, the Weyr characteristic is $\alpha = (\dim \text{Ker } f, \dim \text{Ker } f^2 - \dim \text{Ker } f, \dim \text{Ker } f^3 - \dim \text{Ker } f^2, \dots)$ and its conjugate partition is the Segre characteristic, formed by the sizes of Jordan blocks.

Theorem 2.5 [12], [3] Let α, γ, β be three partitions with $|\alpha| = n, |\gamma| = d, |\beta| = n - d, \ell(\alpha) = m$. The following conditions are equivalent:

- (I) There exist a nilpotent endomorphism $f \in \text{End}(E)$ having Weyr characteristic α and a f -invariant subspace V such that the restriction \hat{f} and the quotient \tilde{f} applications have Weyr characteristic γ and β respectively.
- (II) There exists a finite sequence of partitions $\gamma^0, \gamma^1, \dots, \gamma^m$ such that $\gamma^0 = \gamma, \gamma^m = \alpha$, and for all $i, j \geq 1$:

$$(a) |\gamma^j| - |\gamma^{j-1}| = \beta_j$$

$$(b) \gamma_i^j \geq \gamma_i^{j-1} \geq \gamma_{i+1}^j$$

$$(c) \sum_{\ell \leq i} (\gamma_\ell^{j+1} - \gamma_\ell^j) \leq \sum_{\ell \leq i-1} (\gamma_\ell^j - \gamma_\ell^{j-1})$$

taking $\gamma_0^{j-1} = 0, j \geq 1$.

The sequence $\gamma^0, \gamma^1, \dots, \gamma^m$ appearing in (II) is called a Littlewood-Richardson sequence.

For convenience, (see Example 3.7) we represent a LR-sequence as a table:

γ_m^0	\dots	γ_m^m
\dots	\dots	\dots
γ_1^0	\dots	γ_1^m

The geometric proof of Theorem 2.5 in [3] gives an explicit computation of the LR-sequence for an invariant subspace which we recall in Lemma 2.7. The construction for the converse (II) \Rightarrow (I) will be used in the next section.

Definition 2.6 Given a nilpotent endomorphism f and an invariant subspace V , we consider the double Jordan filtration defined by $V_i^j \doteq \text{Ker } f^i \cap f^{-j}(V)$, $1 \leq i \leq m$, $0 \leq j \leq m$.

$$\begin{array}{ccccccc}
V & = & V & \subset & f^{-1}(V) & \subset \cdots \subset & f^{-m}(V) & = & E \\
\parallel & & \parallel & & \parallel & & \parallel & & \parallel \\
\text{Ker } \widehat{f}^m & = & V_m^0 & \subset & V_m^1 & \subset \cdots \subset & V_m^m & = & \text{Ker } f^m \\
& & \cup & & \cup & & \cup & & \\
& & \vdots & & \vdots & & \vdots & & \\
& & \cup & & \cup & & \cup & & \\
\text{Ker } \widehat{f}^i & = & V_i^0 & \subset & V_i^1 & \subset \cdots \subset & V_i^m & = & \text{Ker } f^i \\
& & \cup & & \cup & & \cup & & \\
& & \vdots & & \vdots & & \vdots & & \\
& & \cup & & \cup & & \cup & & \\
\text{Ker } \widehat{f} & = & V_1^0 & \subset & V_1^1 & \subset \cdots \subset & V_1^m & = & \text{Ker } f
\end{array}$$

Notice that $V_i^{j-1} \cap V_{i-1}^j = V_{i-1}^{j-1}$ and $V_i^j = V_i^i$ if $i \leq j$.

Also, considering the quotient spaces $\tilde{V}_i^j \doteq \frac{V_i^j}{V_{i-1}^j + V_i^{j-1}}$, given $v \in V_i^j$, \tilde{v} will be its quotient class.

Lemma 2.7 [3] Let $f \in \text{End}(E)$ be nilpotent, $V \subset E$ an invariant subspace and (α, γ, β) as in Theorem 2.5. Notice that with the notation in Theorem 2.5:

- $\alpha_i = \dim(V_i^m) - \dim(V_{i-1}^m)$, $1 \leq i \leq m$
- $\gamma_i = \dim(V_i^0) - \dim(V_{i-1}^0)$, $1 \leq i \leq m$
- $\beta_j = \dim(V_m^j) - \dim(V_m^{j-1})$, $1 \leq j \leq m$.

Then a LR-sequence verifying Theorem 2.5(II) is $\gamma_i^j = \dim(V_i^j) - \dim(V_{i-1}^j)$, $1 \leq i \leq m$, $0 \leq j \leq m$.

That is to say, the partitions $\gamma^0 = \gamma, \gamma^1, \dots, \gamma^m = \alpha$ are the Weyr characteristics of the restriction of f to the invariant subspaces $V, f^{-1}(V), \dots, f^{-m}(V) = E$ respectively.

Also, $\dim(\tilde{V}_i^j) = \gamma_i^j - \gamma_i^{j-1}$ and $E \cong V \oplus_{1 \leq i \leq m, 1 \leq j \leq i} \tilde{V}_i^j$.

Hence, not only the triple (α, γ, β) but also the LR-sequence in Lemma 2.7 are preserved by the equivalence relation defined in Definition 2.2. However, the following examples show that they do not characterize the equivalence classes.

Example 2.8 A compatible triple (α, γ, β) as in Theorem 2.5(I) can be realized by different LR-sequences. For example, given $E = [e_1, e_2, \dots, e_6]$ and $V = [e_1, e_2, e_2]$, the compatible triple $\gamma = (2, 1)$, $\alpha = (3, 2, 1)$, $\beta = (2, 1)$, could be realized by the LR-sequences $\gamma^1 = (3, 2)$ or $\gamma^1 = (3, 1, 1)$, corresponding to the endomorphisms $e_1 \mapsto e_2 \mapsto 0, e_4 \mapsto e_5 \mapsto e_3 \mapsto 0, e_6 \mapsto 0$ and $e_4 \mapsto e_1 \mapsto e_2 \mapsto 0, e_3 \mapsto 0, e_5 \mapsto e_6 \mapsto 0$ respectively.

Example 2.9 A LR-sequence can be realized by non-equivalent invariant subspaces. For example, $\gamma = (2, 1), \gamma^1 = (3, 2), \alpha = (3, 2, 1)$ above corresponds also to the endomorphism $e_1 \mapsto e_2 \mapsto 0, e_4 \mapsto e_1 + e_6, e_5 \mapsto e_3 \mapsto 0, e_6 \mapsto 0$. V is marked with regard to the endomorphism in Example 2.8, but it is not for the one here.

In the next section we show that the triple (α, γ, β) as well as the LR-sequences do characterize the equivalence classes in the particular case of V being spanned by only one vector u and its images, that is to say, if $\gamma = (1, 1, \dots, 1)$. Moreover, the possible LR-sequences for a given α are easily characterized and computed, so that the β partitions compatible with α, γ are also easily obtained.

3 The monogenic case: families of classifying invariants

From now on, we restrict ourselves to f -invariant subspaces $V \subset E$ spanned by only one vector u and its images, that is to say:

$$V = [u, f(u), f^2(u), \dots, f^{n-1}(u)] \doteq \langle u \rangle$$

We will show that for this kind of subspace the equivalence classes are determined by the triple (α, γ, β) , as well as by its LR-sequence (see Examples 2.8, 2.9). Moreover we prove that the LR-sequences are easily described in terms of the so-called “marked” and “perturbation” indices.

Firstly, we consider f being nilpotent. Then, if $\dim(V) = h$, we have $V = [u, f(u), \dots, f^{h-1}(u)]$ or equivalently $\gamma = (1, \dots, 1, 0, \dots, 0), |\gamma| = h$.

From an algebraic point of view, the classification of monogenic invariant subspaces can be seen as a particular case of the classification of elements in a primary reduced module over a complete discrete valuation ring in [11], p. 57, by means of the so-called Ulm sequence:

Definition 3.1 Given $f \in \text{End}(E)$ nilpotent, for each $u \in E, u \neq 0$:

1. its depth, $d(u)$, is defined by $d(u) = \max\{j : u \in \text{Im} f^j\}$,
2. the (finite) sequence $d_i(u) = d(f^i(u)), i \geq 0$, is called its Ulm Sequence.

Theorem 3.2 ([11], Theorem 24) Given $f \in \text{End}(E)$ nilpotent, two monogenic subspaces $\langle u \rangle, \langle u' \rangle$ are equivalent if and only if u, u' have the same Ulm Sequence.

We are interested in a geometric approach by means of the LR-sequences, in order to obtain matrix canonical forms in the next sections. The key point is that, for V monogenic, the LR-sequence is specially simple: the lower-right triangle is row constant, having just the values of $\alpha_1, \alpha_2, \dots$; in the upper-left corner one has a row-echelon zero matrix; the intermediate positions (including the diagonal) have value 1 (see Example 3.7). Then, the length of the 0-rows will be called the “perturbation indices” and the number of the 0-rows will be called the “marked index”.

Proposition 3.3 *Let $f \in \text{End}(E)$ be nilpotent, and $V \subset E$ an f -invariant monogenic subspace, $\dim(V) = h$. Let α, γ, β be the Weyr characteristics of f , its restriction and quotient respectively, $l(\alpha) = m$.*

There exists only one possible LR-sequence that realizes the triple (α, γ, β) (see Lemma 2.7) and it is determined by:

- (1) $\gamma_i^i = \alpha_i$
- (2) *There is an integer $s, 0 \leq s \leq m - h$, such that $\gamma_i^{i-1} = 1$ for $1 < i \leq h + s$ and $\gamma_i^{i-1} = 0$ for $i > h + s$.*
- (3) *For $h \leq i \leq m$, there is an integer $j(i)$ such that $\gamma_i^j > 0$ if and only if $j \geq j(i)$.*

In addition, $j(h), j(h + 1), \dots, j(m)$ satisfy:

- (a) $j(i) < j(i + 1)$,
- (b) $j(i) < i$ if and only if $h \leq i \leq h + s$,
- (c) $\alpha_j = \beta_j$ if and only if $j = j(i)$ for some $i, h < i \leq m$; and $\alpha_j = \beta_j + 1$ in any other case.

Proof. Theorem 2.5 ensures the existence of a LR-sequence.

- (1) This is an immediate consequence of Definition 2.6 and Lemma 2.7.
- (2) Condition (II) (b) of Theorem 2.5 implies that $1 = \gamma_1 \geq \gamma_2^1 \geq \dots \geq \gamma_{m+1}^m = 0$ and $1 = \gamma_h \leq \gamma_h^{h-1}$. Let $h + s$ be the last natural i such that $\gamma_i^{i-1} = 1$.
- (3) We apply again (II) (b) in Theorem 2.5, for $h \leq i \leq m$, $\gamma_i \leq \gamma_i^1 \leq \dots \leq \gamma_i^m = \alpha_i \neq 0$, so $j(i)$ exists.
 - (a) In addition, $0 = \gamma_i^{j(i)-1} \geq \gamma_{i+1}^{j(i)} \geq 0$ and we conclude that $j(i) < j(i + 1)$.
 - (b) If $h \leq i \leq h + s$, $\gamma_i^{i-1} = 1$, so $j(i) < i$; if $h + s < i \leq m$, $\gamma_i^{i-1} = 0$, so $j(i) = i$ because $\gamma_i^i = \alpha_i \neq 0$.
 - (c) Finally, bearing in mind the double Jordan filtration (Definition 2.6), the relationship between α_i and β_i is clear.

■

Definition 3.4 *With the above notation, we call s the marked index of V and the sequence $j(i), h < i \leq h + s$, its perturbation indices (these names will be justified in remark 4.5).*

From Proposition 3.3, it is clear that:

Corollary 3.5 *In the conditions of Proposition 3.3, each of the following elements determines uniquely the other ones: the partition β , the LR-sequence; the marked and perturbation indices.*

We will see in a moment (Corollary 3.8) that the three elements above are in turn determined by the Ulm sequence and conversely. Indeed, from [11] we can derive that the Ulm sequence is the set of indices $\{i : \alpha_{i+1} > \beta_{i+1}\}$ in increasing order. For example, if $\alpha = (4, 4, 4, 3, 3, 3, 2, 2, 1, 1)$ and $\beta = (4, 3, 3, 3, 3, 2, 2, 1, 1)$, then the Ulm sequence is $(1, 2, 5, 7, 9)$.

Starting from the above simple structure of the LR-sequence, we construct two families of vectors, which will be used in the next sections to obtain reducing bases.

Lemma 3.6 *In the conditions of Proposition 3.3, let $T \doteq \{i : h < i \leq h + s, j(i) - j(i-1) > 1\}$. There exist:*

- $v_i \in V_i^{j(i)}$, $\tilde{v}_i \neq 0$ for $h \leq i \leq h + s$,
- $w_i \in V_{j(i)-1}^{j(i)-1}$, $\tilde{w}_i \neq 0$ for $i \in T$

such that $f(v_i) = v_{i-1} + w_i$ for $i \in T$ and $f(v_i) = v_{i-1}$ otherwise (for $i > h$).

Proof. For all $v_i \in V_i^{j(i)}$, $\tilde{v}_i \neq 0$, we have $f(v_i) \in V_{i-1}^{j(i)-1}$ but $f(v_i) \notin V_{i-2}^{j(i)-1}$ and $f(v_i) \notin V_{i-1}^{j(i)-2}$.

If $i \notin T$, $j(i-1) = j(i) - 1$ then, by Proposition 3.3, $\gamma_{i-1}^{j(i)-2} = 0$ and $V_{i-1}^{j(i)-2} = V_{i-2}^{j(i)-2}$. Therefore $V_{i-2}^{j(i)-1} + V_{i-1}^{j(i)-2} = V_{i-2}^{j(i)-1}$, $v_{i-1} \doteq f(v_i) \in V_{i-1}^{j(i)-1}$ and $\tilde{v}_{i-1} \neq 0$.

If $i \in T$, $j(i-1) \leq j(i) - 2$ then, by Proposition 3.3, $V_{i-1}^{j(i)-1} = V_{j(i)-1}^{j(i)-1} + V_{i-1}^{j(i)-1}$, and we can select $v_{i-1} \in V_{i-1}^{j(i)-1}$, $w_i \in V_{j(i)-1}^{j(i)-1}$ such that $f(v_i) = v_{i-1} + w_i$ with $\tilde{v}_{i-1} \neq 0$, $\tilde{w}_i \neq 0$ because $V_{i-1}^{j(i)-1} \subset V_{i-2}^{j(i)-1}$ and $V_{j(i)-1}^{j(i)-2} \subset V_{j(i)-1}^{j(i)-1}$.

We define v_i, w_i by recurrence for $i = h + s, h + s - 1, \dots, h$ following the scheme above. We start by choosing $v_{h+s} \in V_{h+s}^{j(h+s)}$, $\tilde{v}_{h+s} \neq 0$; this is possible because $\gamma_{h+s}^{j(h+s)} - \gamma_{h+s}^{j(h+s)-1} = 1$. Note that $v_h \in V_h$ is a generator of V . ■

Example 3.7 *Let $\gamma = (1, 1, 1, 1, 1)$, $\alpha = (4, 4, 4, 3, 3, 3, 2, 2, 1, 1)$, $\beta = (4, 3, 3, 3, 3, 2, 2, 1, 1)$. Then, $h = 5, s = 5$ and $j(6) = 1, j(7) = 4, j(8) = 5, j(9) = 7, j(10) = 9$ and the LR-sequence and the “places” of the vectors v_i, w_i above are summarized in the following tables. Then $f(v_{10}) = v_9 + w_{10}, f(v_9) = v_8 + w_9, f(v_8) = v_7, f(v_7) = v_6 + w_7, f(v_6) = v_5$.*

	γ									α	
10										1	1
9								1	1	1	1
8						1	1	1	2	2	2
7					1	1	1	2	2	2	2
6		1	1	1	1	1	3	3	3	3	3
5	1	1	1	1	1	3	3	3	3	3	3
4	1	1	1	1	3	3	3	3	3	3	3
3	1	1	1	4	4	4	4	4	4	4	4
2	1	1	4	4	4	4	4	4	4	4	4
1	1	4	4	4	4	4	4	4	4	4	4
i/j	0	1	2	3	4	5	6	7	8	9	10

										v_{10}
10										
9										v_9
8							v_8			w_{10}
7							v_7			
6		v_6						w_9		
5	v_5									
4										
3						w_7				
2										
1										
	0	1	2	3	4	5	6	7	8	9

A first consequence of this lemma is that the LR-sequence gives immediately the Ulm sequence (and conversely):

Corollary 3.8 *The lengths of the 1-diagonals in the table of the LR-sequence are $1 + d_i(u)$, where $d_i(u) = \max\{k : \gamma_{h-i+k}^k = 1\}$.*

In Example 3.7 the Ulm sequence is $(1, 2, 5, 7, 9)$. From Theorem 3.2 and Corollary 3.8, we conclude that the families in Corollary 3.5 are classifying sets. See remark 4.3 for an autonomous proof (without using [11]).

Corollary 3.9 *Given $f \in \text{End}(E)$ nilpotent and $V = \langle u \rangle$ an f -invariant monogenic subspace, its equivalence class is fully determined by any of the following sets:*

1. *The Weyr characteristic of the quotient.*
2. *The LR-sequence.*
3. *The marked and perturbation indices.*

4 The canonical PM-form

The geometric approach in section 3 leads to a natural basis of E adapted to V such that the initial pair (V, A) is reduced to (\mathbb{R}^h, A_{PM}) (Theorem 4.1), where A_{PM} is a particular case of the miniversal deformation of a marked matrix in [4]. Indeed, the central marked matrix is determined by the marked index, while the perturbation indices determine the non zero miniversal parameters (remark 4.5). This fact justifies the denomination of these indices, as well as that of the canonical form (the ‘‘perturbed marked’’ form). Notice that this canonical form classifies the monogenic subspaces, so that it gives a proof of Corollary 3.9 alternative to that in [11]. Moreover, it seems a natural starting point for further work about perturbations of monogenic subspaces.

From now on, N_p will be a $p \times p$ nilpotent square matrix with ones in the below-diagonal and zeros in the remaining entries.

Now, from Lemma 3.6 we obtain the canonical PM-form:

Theorem 4.1 *Let $f \in \text{End}(E)$ be nilpotent, $\dim(E) = n$, and $V \subset E, \dim(V) = h$, be a monogenic indecomposable f -invariant subspace having marked and perturbation indices $s, j(h+1), j(h+2), \dots, j(h+s)$. Then there exists a basis of E adapted to V such that the matrix of f is:*

$$\text{diag}(N_h, N_s, N_{j(i_1)-1}, \dots, N_{j(i_t)-1}) + (M_0, M_1, \dots, M_t)^*$$

where $i_1 > i_2 > \dots > i_t$ are the indices in T (see Lemma 3.6), $M_0 \in M_{(h+s) \times n}(\mathbb{C}), M_l \in M_{(j(i_l)-1) \times n}(\mathbb{C})$ are matrices whose the only non-zero entries are ones in the $(1, h+s), (1, 2h+s+1-i_l)$ positions for $1 \leq l \leq t$. In particular, V is generated by the first vector of this basis.

Proof. Taking $j_0 = h+s$ and $j_l = j_{l-1} + j(i_l) - 1$ for $1 \leq l \leq t$, such a basis is given by:

- For $1 \leq i \leq h$, $e_i \doteq f^{i-1}(v_h)$ (it is a Jordan basis of V),
- for $h < i \leq h+s$, $e_i \doteq v_{j_0+h+1-i}$,
- For $1 \leq l \leq t, 1 \leq i \leq j(i_l) - 1$, $e_{j_{l-1}+i} \doteq f^{i-1}(w_{i_l})$.

Note that $\dim(E) = j_t = h+s + \sum_{1 \leq l \leq t} j(i_l) - t$. ■

5 The canonical forms C and J

In section 4, given f nilpotent and V a monogenic h -dimensional subspace, we have obtained for the pair (V, f) a matrix representative of the form (\mathbb{R}^h, A_{PM}) , where A_{PM} is a PM-matrix. An alternative form (\mathbb{R}^h, A_C) seems natural, where A_C is a Carlson matrix

$$A_C = \begin{pmatrix} J_1 & A'_3 \\ 0 & J_2 \end{pmatrix}$$

J_1 is the Jordan form for the restriction (corresponding to γ), J_2 is the Jordan form for the quotient map (corresponding to β) and the only nonzero elements of A'_3 are placed in the rows corresponding to null rows of J_1 . We recall that the ‘‘Carlson condensation Lemma’’ (see for example [3]) ensures that any global characteristic α can be obtained by means of a suitable matrix A'_3 of this kind provided that α is compatible with γ, β . We recall also that given α, γ , then β classifies the monogenic subspaces.

Finally we obtain a third canonical form (V', J) , where J is the Jordan matrix of f and $V' = [v, Jv, \dots, J^{h-1}v]$. Moreover, some applications of these canonical forms are presented in section 6.

Proposition 5.1 *Let $f \in \text{End}(E)$ be nilpotent, $\dim(E) = n$, and let $V \subset E, \dim(V) = h$, be a monogenic indecomposable f -invariant subspace having marked and perturbation indices $s, j(h+1), j(h+2), \dots, j(h+s)$. Then there exists a basis of E adapted to V such that the matrix of f is:*

$$A_C = \text{diag}(N_h, N_{j(r_1)}, \dots, N_{j(r_{t+1})}) + (F_0, F_1, \dots, F_{t+1})$$

where $r_1 > r_2 > \dots > r_{t+1}$ are the indices in $P \doteq \{h+s\} \cup \{i : h < i, i+1 \in T\}$ (see Lemma 3.6 and note that $\text{card}(P) = \text{card}(T) + 1$), $F_0 = 0 \in M_{n \times h}(\mathbb{C})$ and $F_q \in M_{n \times j(r_q)}(\mathbb{C})$ are matrices whose only nonzero entries are ones in the $(1, r_q - h)$ position for $1 \leq q \leq t+1$.

Proof. Taking $k_0 = h, k_q = k_{q-1} + j(r_q)$ for $1 \leq q \leq t+1$, such a basis is given by:

1. For $1 \leq i \leq h$, $e_i \doteq f^{i-1}(v_h)$ (it is a Jordan basis of V),
2. For $1 \leq q \leq t+1$,
 - $e_{k_{q-1}+i} \doteq f^{i-1}(v_{r_q})$ if $1 \leq i \leq r_q - h$,
 - $e_{k_{q-1}+r_q-h+1} \doteq f^{r_q-h}(v_{r_q}) - v_h$ if $r_q - h < j(r_q)$
 - $e_{k_{q-1}+i} \doteq f^{i-r_q+h-1}(e_{k_{q-1}+r_q-h+1})$ if $r_q - h + 1 < i \leq j(r_q)$. ■

Definition 5.2 *The matrix A_C in the above proposition will be called the Carlson (C) canonical form of f relative to V .*

Example 5.3 *Following with the example in 3.7 and 4.4, in the basis:*

- $v_5, f(v_5), \dots, f^4(v_5),$

It is obvious that:

Lemma 6.2 *In the above conditions, the vectors u, u' are f -equivalent if and only if the f -invariant subspaces $V = \langle u \rangle$ and $V' = \langle u' \rangle$ are f -equivalent.*

Thus, from Proposition 5.4 we have:

Corollary 6.3 *Let $f \in \text{End}(E)$, J be its Jordan matrix and $u \in E$. Assume that the monogenic f -invariant subspace $V = \langle u \rangle$ is indecomposable and let $s, j(h+1), j(h+2), \dots, j(h+s)$ and $i_1 > i_2 > \dots > i_t$ be as in Proposition 5.4. Then the components $U_J = (E_0, E_1, \dots, E_t)$ are canonical representatives of u with regard to the f -equivalence. If V is not indecomposable, it is sufficient to add zero components for the complementary Jordan chains.*

Clearly, two vectors u, u' are f -equivalent if and only if they have the same f -representative as given above.

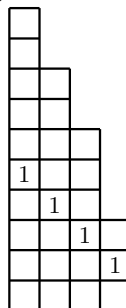
Definition 6.4 *In the conditions of the above corollary, the f -representative U_J of u will be called its canonical f -form (or J -form)*

Example 6.5 *In Example 5.5 the column matrix is the canonical J -form of the generator v_5 of V .*

Analogously to Proposition 4.6, we have:

Proposition 6.6 *J being a Jordan matrix, let W be a differentiable contractible manifold and $u(z), z \in W$, a differentiable family of J -equivalent vectors having U_J as J -form. Then, there exists a differentiable family of Jordan bases $S(z)$ such that $S(z)^{-1}u(z) = U_J$.*

Remark 6.7 *Often, the Jordan structure is represented by means of a Young diagram, each column corresponding to a Jordan block. Then, a Jordan basis fills the cells in the diagram with top-down chains. Thus, the J -form of a vector can be sketched as 1's placed in the corresponding cells (that is, the ones corresponding to non-zero coordinates). For example, the J -form of the vector v_5 in Example 5.5 is represented by:*



Empty columns can be added if the monogenic subspace is not indecomposable.

Note that the dimension of $V = \langle u \rangle$ is given by the highest position where a one appears. In addition, the heights of the cells having a one are decreasing, as well as their depths (where the depth of a cell means the number of cells above it in the corresponding column, that is to say, the depth of the Jordan vector corresponding to this cell).

These rules allow us to list all possible f -classes.

Corollary 6.8 *Let $f \in \text{End}(E)$ be nilpotent and $\alpha^* = (\alpha_1^*, \alpha_2^*, \dots, \alpha_n^*)$ be its Segre characteristic. Consider the sets $\{(\alpha_{p_i}^*, q_i) : 0 \leq i \leq t\}$, where:*

- (1) $\alpha_{p_i}^* - 1 > \alpha_{p_{i+1}}^* > 0, \quad 0 \leq i < t,$
- (2) $q_0 > q_1 > \dots > q_t > 0, \quad \alpha_{p_0}^* - q_0 > \alpha_{p_1}^* - q_1 > \dots > \alpha_{p_t}^* - q_t \geq 0.$

Then, each class of monogenic invariant subspaces is determined by one of these sets and conversely.

Proof. In Proposition 5.4 we have seen that $(\alpha_{p_0}^*, \alpha_{p_1}^*, \dots, \alpha_{p_t}^*) = (s + h, j(i_1) - 1, \dots, j(i_t) - 1)$ and $(q_0, q_1, \dots, q_t) = (s + 1, i_1 - h, \dots, i_t - h)$ which satisfy the required conditions. ■

Corollary 6.3 can be applied to control systems, giving a canonical form of uniparametric non-controllable linear control systems, with regard to changes of basis in the state variables. We recall that a linear control system is given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad A \in M_n(\mathbb{R}), B \in M_{n \times p}(\mathbb{R})$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^p$ are respectively the state and control variables. A change of basis $\bar{x} = S^{-1}x$ in the state space transforms the above equation into

$$\dot{\bar{x}}(t) = (S^{-1}AS)\bar{x}(t) + (S^{-1}B)u(t).$$

Canonical forms in this sense have been obtained (see, for example, [16]) for controllable systems, that is to say, when $\text{rank}(B, AB, A^2B, \dots, A^{n-1}B) = n$. In particular, for the so-called Jordan canonical form in [10], the matrix $S^{-1}AS$ is required to be in Jordan form. The same requirement is considered in [13] for general uncontrollable systems, but explicit descriptions are only obtained for the single input case (that is, $p = 1$). For these systems, from Corollary 6.3 an alternative form follows immediately, which we call their *J-canonical form*.

Corollary 6.9 *Let us consider the linear control system*

$$\dot{x}(t) = Ax(t) + bu(t), \quad A \in M_n(\mathbb{R}), b \in M_{n \times 1}(\mathbb{R}). \quad (1)$$

There exists a change of basis $\bar{x} = S^{-1}x$ in the state space such that the equation (1) is transformed into the following canonical form

$$\dot{\bar{x}}(t) = J\bar{x}(t) + b_J u(t),$$

where J is the Jordan form of A and b_J is the canonical J -form of b .

Example 6.10 *The matrices in Example 5.5 can be viewed as the J -canonical form of an uniparametric control linear system.*

Remark 6.11 *Note that, for any uniparametric linear control system as above (uncontrollable, in general), the reachable subspace $V = [b, Ab, \dots, A^{n-1}b]$ is a monogenic subspace generated by b . Changes of basis in the state space give equivalent reachable subspaces, and conversely.*

Finally, we apply Corollary 6.9 to improve the reduced forms in [6] for a bimodal continuous linear dynamical system. We recall that such systems are given by

$$\begin{aligned} \dot{x}(t) &= A_1 x(t) + b & \text{if } x_1 \leq 0 \\ \dot{x}(t) &= A_2 x(t) + b & \text{if } x_1 \geq 0 \end{aligned} \quad \text{where } A_1, A_2 \in M_n(\mathbb{R}) \text{ and } b = (b_1 \ b_2 \ \dots \ b_n)^* \in M_{n \times 1}(\mathbb{R}).$$

A change of basis $\bar{x} = S^{-1}x$ is called admissible if the hyperplanes $x_1 = k$ are preserved, that is to say, if

$$S = \begin{pmatrix} 1 & 0 \\ U & T \end{pmatrix}, \quad T \in Gl_{n-1}(\mathbb{R}).$$

In [6] one proves that there exists an admissible change of basis such that

$$S^{-1}A_1S = \begin{pmatrix} K_1 & 0 \\ \bar{A}_1 & J \end{pmatrix}, \quad S^{-1}A_2S = \begin{pmatrix} K_2 & 0 \\ \bar{A}_2 & J \end{pmatrix}, \quad (S^{-1}b)^* = (b_1 \ \bar{b}_2 \ \dots \ \bar{b}_n). \quad (2)$$

where

$$K_1 = \begin{pmatrix} a_1 & 1 & 0 & \dots & 0 & 0 \\ a_2 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & & \dots & \dots \\ a_{r-1} & 0 & 0 & \dots & 0 & 1 \\ a_r & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} \alpha_1 & 1 & 0 & \dots & 0 & 0 \\ \alpha_2 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & & \dots & \dots \\ \alpha_{r-1} & 0 & 0 & \dots & 0 & 1 \\ \alpha_r & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

$$\bar{A}_1 = \begin{pmatrix} \bar{a}_{r+1} & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & & \dots & \dots \\ \bar{a}_n & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad \bar{A}_2 = \begin{pmatrix} \bar{\alpha}_{r+1} & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & & \dots & \dots \\ \bar{\alpha}_n & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

and $J \in M_{n-r}(\mathbb{C})$ is a Jordan matrix.

Moreover $r, a_1, \dots, a_r, \alpha_1, \dots, \alpha_r, b_1, J$ are uniquely determined by (A_1, A_2, b) , whereas $\bar{a}_{r+1}, \dots, \bar{a}_n, \bar{\alpha}_{r+1}, \dots, \bar{\alpha}_n, \bar{b}_2, \dots, \bar{b}_n$ depend on the change S .

The above results allow us to improve this reduced form by means of an additional change of basis which simplifies $\bar{b}_{r+1}, \dots, \bar{b}_n$.

Corollary 6.12 *The expression of $(S^{-1}b)^*$ in (2) can be reduced by means of an admissible change of basis*

$$S' = \begin{pmatrix} I & 0 \\ 0 & Q \end{pmatrix}, \quad Q \in Gl_{n-r}(\mathbb{C}), \quad Q^{-1}JQ = J$$

such that

$$Q^{-1}(\bar{b}_{r+1} \ \dots \ \bar{b}_n)^*$$

is the J -canonical form of $(\bar{b}_{r+1} \ \dots \ \bar{b}_n)^*$.

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