Diffeomorphisms with Given Jacobians involving the Heat Equation

A. Avinyó, J. Solà-Morales, M. València

Universitat Politècnica de Catalunya,
Barcelona, Spain

1. Problem

- $\Omega \subset \mathbb{R}^n$ smooth bounded domain, $f : \overline{\Omega} \to \mathbb{R}^+$, $\inf_{x\in\overline{\Omega}} f > 0$, $\int_\Omega f =$ vol (Ω).
- We want to obtain $\Phi : \overline{\Omega} \to \overline{\Omega}$ such that

$$\boxed{\begin{cases} \det D\Phi(x) = f(x), & x \in \Omega, \\ \Phi(\partial\Omega) = \partial\Omega, \end{cases}} \tag{1}$$

by means of a definite algorithm free from arbitrary choices, easy to construct and with optimal regularity results.

- Non-uniqueness. For instance, if $\Omega = B_1((0,0))$, then $\Phi_k(r,\alpha) = \big(r\cos(\alpha + 2k\pi r^2), r\sin(\alpha + 2k\pi r^2)\big)$ is a solution of (1) with $f = 1$.
- Applications: Ergodic maps (Anosov, Katok, 71), equilibrium of gases (Dacorogna, 81), random distributions of particles (Russo, 90), Monge-Kantorovich problem (Brenier, 91), geographical representations (Dorling, 96), Brunn-Minkowski inequality (Alesker et alt., 99).

2. Algorithm of Moser

Step 1. $\rho : \Omega \times [0,T] \to \mathbb{R}$, $F : \Omega \times [0,T] \to \Omega$ solution of:

$$\begin{cases} \dfrac{\rho(t,x)}{\partial t} + \operatorname{div}\left(\rho(x,t)F(x,t)\right) = 0, & (x,t) \in \Omega \times (0,T), \\ \rho(x,0) = f(x), & x \in \Omega, \\ \rho(x,T) = 1, & x \in \Omega, \\ F(x,t) \cdot \nu(x) = 0, & (x,t) \in \partial\Omega \times [0,T]. \end{cases}$$

Step 2. $\Phi_t : \overline{\Omega} \to \overline{\Omega}$ solution of:

$$\begin{cases} \frac{d\Phi_t(x)}{dt} = F\left(\Phi_t(x)\right), & t > 0, \\ \Phi_0(x) = x. \end{cases}$$

Then,

$$\boxed{\det D\Phi_T(x) = f(x), \quad x \in \Omega}$$

- Dacorogna, Moser: $\rho = t + (1-t)f$, $F = \dfrac{\nabla u}{t + (1-t)f}$
- Evans, Gangbo: $\rho = t + (1-t)f$, $F = \dfrac{|\nabla u|^{p-2}\nabla u}{t + (1-t)f}$
- Avinyó, Solà, València: $\rho = u$, $F = \dfrac{-\nabla u}{u}$

3. Auxiliary result. A singular initial value problem

- $F : (0, \infty) \to \mathcal{C}^{1,0}(\overline{\Omega}, \mathbb{R}^n)$ continuous function such that

$$\|F(t, \cdot)\|_{\mathcal{C}^{1,0}} \in L^1(0, \infty), \ \ F(t, x) \cdot \nu(x) = 0, \ \text{ if } \ x \in \partial\Omega.$$

Then,

a) For all $x_0 \in \overline{\Omega}$, the initial value problem

$$\begin{cases} x'(t) &= F\left(t, x(t)\right), \quad t > 0, \\ \lim\limits_{t \to 0^+} x(t) &= x_0, \end{cases}$$

has a unique solution $x_{in}(t; x_0) \in \mathcal{C}^1((0, \infty), \overline{\Omega})$.
Moreover, $\lim_{t\to\infty} x_{in}(t; x_0)$ exists and belongs to $\overline{\Omega}$.

b) $\Phi(x_0) := \lim\limits_{t\to\infty} x_{in}(t; x_0)$ satisfies $\Phi \in \mathcal{C}^{1,0}\left(\overline{\Omega}\right)$ and

$$\det\left(D\Phi\right)(x_0) = \exp\left(\int_0^\infty \operatorname{div} F\left(t, x_{in}(t; x_0)\right) dt\right),$$

c) Moreover, if

$$\int_0^\infty \left|D_x F(s, x(s)) - D_x F(s, y(s))\right| ds \leq C \left\|x - y\right\|_0^\alpha,$$

then $\Phi \in \mathcal{C}^{1,\alpha}\left(\overline{\Omega}\right)$.

4. Theorem 1

- (without restrictions on the boundary)

  Let f be a function of class $\mathcal{C}^{0,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$ with $\inf_{\overline{\Omega}} f > 0$ and $\int_\Omega f = |\Omega|$. Let $u(t,x)$ be the solution of the heat equation with homogeneous Neumann boundary conditions

$$\begin{cases} u_t(t,x) & = \Delta u(t,x), \quad x \in \Omega, \quad t > 0, \\ u(0,x) & = f(x), \qquad x \in \Omega, \\ u_\nu(t,x) & = 0, \qquad x \in \partial\Omega, \quad t > 0. \end{cases}$$

  Then,

  a) For all $x_0 \in \overline{\Omega}$, the initial value problem

$$\begin{cases} x'(t) & = \dfrac{-\nabla u\,(t, x(t))}{u(t,x(t))}, \quad t > 0, \\ \lim\limits_{t \to 0+} x(t) & = x_0, \end{cases}$$

  has a unique solution $x(t) \in \mathcal{C}^1((0,\infty), \overline{\Omega})$, and $\lim_{t\to\infty} x(t)$ exists and belongs to $\overline{\Omega}$.

  b) $\Phi : \overline{\Omega} \to \overline{\Omega}$ defined by $\Phi(x_0) := \lim_{t\to\infty} x(t)$ is a diffeomorphism of $\overline{\Omega}$ such that $\Phi, \Phi^{-1} \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$, and it satisfies

$$\boxed{\det\left(D\Phi\right)(x) = f(x), \quad x \in \overline{\Omega}.}$$

5. Proof of Theorem 1

- We have to see that the hypotheses of the singular initial value problem (auxiliary result) are satisfied with $F = \nabla u/u$ where u is the solution of the heat equation with Neumann boundary conditions. For this, we need the following two results:

- **Lemma 1.1**

  If $u \in \mathcal{C}^{2,\beta}\left(\overline{\Omega}\right)$, then

$$\left\| \frac{\nabla u}{u} \right\|_{\mathcal{C}^{1,\alpha}} \leq C \left\| u - 1 \right\|_{\mathcal{C}^{2,\alpha}}$$

- **Lemma 1.2**

  Let $u(t,x)$ be the solution of the heat equation with $u(0,x) = f(x) \in \mathcal{C}^{0,\alpha}\left(\overline{\Omega}\right)$. Then:

  a) $t \to u(t,\cdot)$ is continuous from $(0,\infty)$ to $\mathcal{C}^{2,\alpha}\left(\overline{\Omega}\right)$

  b) If $0 < t < 1$ and $0 \leq \beta < \alpha < 1$,

$$\left\| u(t,\cdot) - 1 \right\|_{\mathcal{C}^{2,\beta}} \leq \frac{C}{t^{1-(\alpha-\beta)/2}} \left\| f - 1 \right\|_{\mathcal{C}^{0,\alpha}}$$

  c) If $t > 1$,

$$\left\| u(t,\cdot) - 1 \right\|_{\mathcal{C}^{2,\beta}} \leq C\, e^{-\mu_2 t}$$

  where μ_2 is the second eigenvalue of the Laplacian with Neumann boundary conditions.

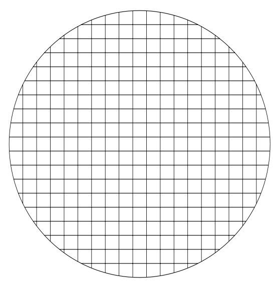
6. Numerical example

- Ω is the unit disc and the jacobian function f is defined by

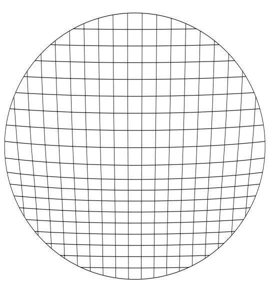
$$f(x,y) = \begin{cases} 1+\varepsilon, & \text{if } y > 0, \\ 1-\varepsilon, & \text{if } y < 0. \end{cases}$$

for $\varepsilon = 0.25$, 0.5 and 0.75.

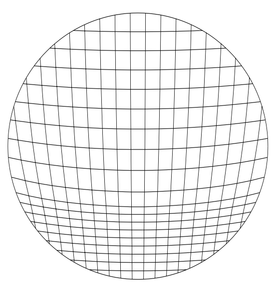
original grid



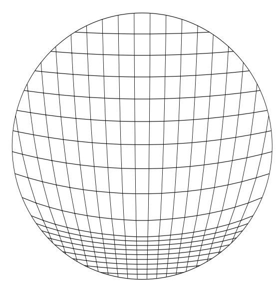
$\varepsilon = 0.25$



$\varepsilon = 0.50$



$\varepsilon = 0.75$



- Even though this function f does not satisfy the regularity hypothesis of Theorem 1, we used it because of its clear intuitive meaning.

- This example suggests us that our algorithm seems to be robust enough to work also in this discontinuous case.

7. Theorem 2

- (keeping fixed the boundary)

  Let $V(t,x)$ the solution, in a weak sense, of the Stokes problem

$$\begin{cases} \Delta V(x,t) &= \nabla p(x,t), \quad x \in \Omega, \quad t > 0, \\ \operatorname{div} V(x,t) &= 0, \quad x \in \Omega, \quad t > 0, \\ V(x,t) &= V^*(x,t), \quad x \in \partial\Omega, \quad t > 0. \end{cases}$$

  with $V^*(x,t) = \nabla u(x,t)$ and $u(t,x)$ is the solution of the heat equation of Theorem 1. Then,

  a) For all $x_0 \in \overline{\Omega}$, the initial value problem

$$\begin{cases} x'(t) &= -\dfrac{\nabla u\,(t, x(t)) - V(t, x(t))}{u(t, x(t))}, \\ \lim\limits_{t \to 0+} x(t) &= x_0, \end{cases}$$

  has a unique solution $x(t) \in \mathcal{C}^1((0,\infty), \overline{\Omega})$, and $\lim_{t\to\infty} x(t)$ exists and belongs to $\overline{\Omega}$.

  b) $\Phi : \overline{\Omega} \to \overline{\Omega}$ defined by $\Phi(x_0) := \lim_{t\to\infty} x(t)$ is a diffeomorphism of $\overline{\Omega}$ such that $\Phi, \Phi^{-1} \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$, and it satisfies

$$\boxed{\begin{aligned} \det\,(D\Phi)\,(x) &= f(x), \quad & x \in \overline{\Omega}, \\ \Phi(x) &= x, \quad & x \in \partial\Omega. \end{aligned}}$$

8. Proof of Theorem 2

- We have to see that the hypotheses of the singular initial value problem (auxiliary result) are satisfied with $F = (\nabla u + V)/u$ where u is the solution of the heat equation with Neumann boundary conditions and V is the solution of the Stokes system. For this, besides Lemmas 1.1 and 1.2, we need the two following results:

- **Lemma 2.1**

  If $u \in \mathcal{C}^{2,\alpha}\left(\overline{\Omega}\right)$, then

$$\left\| \frac{1}{u} \right\|_{\mathcal{C}^{1,\alpha}} \leq C \left(1 + \|u - 1\|_{\mathcal{C}^{2,\alpha}}^{\frac{1+\alpha}{2+\alpha}} \right)$$

- **Lemma 2.2**

  For all $V^* \in \mathcal{C}^{1,\alpha}(\partial\Omega)$ with $\int_{\partial\Omega} V^* \cdot \nu = 0$, there exists a unique weak solution V of the Stokes problem such that $V \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$ and satisfies:

$$\begin{aligned} \|V\|_{\mathcal{C}^{0,0}(\Omega)} &\leq C \, \|V^*\|_{\mathcal{C}^{0,0}(\partial\Omega)}, \\ \|V\|_{\mathcal{C}^{1,\alpha}(\Omega)} &\leq C \, \|V^*\|_{\mathcal{C}^{1,\alpha}(\partial\Omega)} \end{aligned}$$

9. Numerical example

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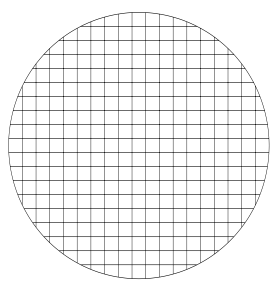

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