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Continuation of the Exponentially Small Transversality for the Splitting of Separatrices to a Whiskered Torus with Silver Ratio

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Abstract—We study the exponentially small splitting of invariant manifolds of whiskered (hyperbolic) tori with two fast frequencies in nearly integrable Hamiltonian systems whose hyperbolic part is given by a pendulum. We consider a torus whose frequency ratio is the silver number $\Omega = \sqrt{2} - 1$. We show that the Poincaré−Melnikov method can be applied to establish the existence of 4 transverse homoclinic orbits to the whiskered torus, and provide asymptotic estimates for the transversality of the splitting whose dependence on the perturbation parameter ε satisfies a periodicity property. We also prove the continuation of the transversality of the homoclinic orbits for all the sufficiently small values of ε, generalizing the results previously known for the golden number.

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Dedicated to the 60th anniversary of Sergey Bolotin and the 50th anniversary of Dmitry Treschev

1. INTRODUCTION AND SETUP

1.1. Background and State of the Art

This paper is dedicated to the study of the transversality of the exponentially small splitting of separatrices in a perturbed 3-degree-of-freedom Hamiltonian system, associated to a 2-dimensional whiskered torus (invariant hyperbolic torus) whose frequency ratio is the silver number $\Omega = \sqrt{2} - 1$. This quadratic irrational number has nice arithmetic properties since it has a 1-periodic continued fraction.

We start with an integrable Hamiltonian H_0 having whiskered (hyperbolic) tori with a *separatrix*: coincident stable and unstable whiskers (invariant manifolds). We focus our attention on a torus, with a frequency vector of *fast frequencies*:

$$\omega_\varepsilon = \frac{\omega}{\sqrt{\varepsilon}}, \qquad \omega = (1, \Omega), \qquad \Omega = \sqrt{2} - 1. \tag{1.1}$$

This frequency ratio Ω is called the *silver number*. If we consider a perturbed Hamiltonian $H = H_0 + \mu H_1$, where μ is small, in general the stable and unstable whiskers do not coincide

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anymore, and this phenomenon has got the name *splitting of separatrices*. If we assume, for the two involved parameters, a relation of the form $\mu = \varepsilon^p$ for some $p > 0$, we have a problem of singular perturbation and in this case the splitting is *exponentially small* with respect to ε. Our aim is to detect homoclinic orbits associated to persistent whiskered tori, provide *asymptotic estimates* for both the splitting distance and its *transversality*, and use the arithmetic properties of the silver number Ω in order to show the *continuation* of the transversality of the homoclinic orbits for *all* sufficiently small ε. When transversality takes place, the perturbed system turns out to be nonintegrable and there is chaotic dynamics near the homoclinic orbits.

A very usual tool to measure the splitting is the *Poincaré–Melnikov method*, introduced by Poincaré in [29] and rediscovered much later by Melnikov and Arnold [1, 27]. By considering a transverse section to the stable and unstable perturbed whiskers, one can consider a function $\mathcal{M}(\theta)$, $\theta \in \mathbb{T}^2$, usually called the *splitting function*, giving the vector distance between the whiskers on this section. The method provides a first-order approximation to this function, with respect to the parameter μ, given by the *Melnikov function* $M(\theta)$, defined by an integral. We have

$$\mathcal{M}(\theta) = \mu M(\theta) + \mathcal{O}(\mu^2), \tag{1.2}$$

and hence for μ small enough the simple zeros θ_* of $M(\theta)$ give rise to transverse intersections between the perturbed whiskers. In this way, we can obtain asymptotic estimates for both the *maximal splitting distance* as the maximum of the function $|\mathcal{M}(\theta)|$ and for the *transversality* of the splitting, which can be measured by the minimal eigenvalue (in modulus) of the (2×2)-matrix $\mathrm{D}\mathcal{M}(\theta_*)$.

An important related fact is that both functions $\mathcal{M}(\theta)$ and $M(\theta)$ are gradients of scalar functions [3, 15]:

$$\mathcal{M}(\theta) = \nabla\mathcal{L}(\theta), \qquad M(\theta) = \nabla L(\theta).$$

Such scalar functions are called *splitting potential* and *Melnikov potential*, respectively, and the transverse homoclinic orbits correspond to the nondegenerate critical points of the splitting potential.

As said before, the case of fast frequencies ω_ε as in (1.1), with a perturbation of order $\mu = \varepsilon^p$, turns out to be a *singular problem*. The difficulty is that the Melnikov function $M(\theta)$ is exponentially small in ε, and the Poincaré–Melnikov method cannot be directly applied, unless we assume that μ is exponentially small with respect to ε. In order to validate the method in the case $\mu = \varepsilon^p$, with p as small as possible, the use of parameterizations of a complex strip of the whiskers (whose width is defined by the singularities of the unperturbed ones), together with flow-box coordinates, was introduced in [24] in order to ensure that the error term is also exponentially small, and that the Poincaré–Melnikov approximation dominates it. This tool was initially developed for the Chirikov standard map [24], for Hamiltonians with one and a half degrees of freedom (with 1 frequency) [13, 14, 18] and for area-preserving maps [12].

Later, those methods were extended to the case of whiskered tori with 2 frequencies. In this case, the arithmetic properties of the frequencies play an important role in the exponentially small asymptotic estimates of the splitting function, due to the presence of *small divisors*. This was first mentioned in [26] and later detected in [34], and then rigorously proved in [10] for the quasi-periodically forced pendulum, assuming a polynomial perturbation in the coordinates attached to the pendulum. Recently, a more general (meromorphic) perturbation has been considered in [21]. It is worth mentioning that, in some cases, the Poincaré–Melnikov method does not predict correctly the size of the splitting, as shown in [2].

As an alternative way to study the splitting, the parameterization of the whiskers as solutions of the Hamilton–Jacobi equation was used in [25, 32, 33], and exponentially small estimates were also obtained by this method, as well as the transversality of the splitting, provided some intervals of the perturbation parameter ε are excluded. Similar results were obtained in [5, 6]. Besides, in the case of golden ratio $\Omega = (\sqrt{5} - 1)/2$, the *continuation* of the transversality was shown in [6] for *all* sufficiently small values of ε, under a certain condition on the phases of the perturbation. Otherwise, homoclinic bifurcations can occur, which are studied, for instance, in [35] for Arnold's example. The generalization of this approach to some other quadratic frequency ratios was considered in [5], extending the asymptotic estimates for the splitting, but without a satisfactory result concerning the continuation of the transversality. Recently, we have carried out a parallel study [7] for the cases of 2 and 3 frequencies (in the case of 3 frequencies, with a frequency vector $\omega = (1, \Omega, \Omega^2)$, where Ω is a concrete cubic irrational number), obtaining also exponentially small estimates for the maximal splitting distance. We also mention the paper [30] for a different approach (based on a careful averaging out of the fast angular variables) which, in principle, should lead to analogous results, at least, concerning the sharp upper bounds of the splitting. We refer to [7, 11] for a more complete background and references concerning exponentially small splitting, and its relation to the arithmetic properties of the frequencies.

In this paper, we consider a 2-dimensional torus whose frequency ratio in (1.1) is given by the silver number. Our main objective is to develop a methodology, taking into account the arithmetic properties of the given frequencies, allowing us to obtain asymptotic estimates for both the maximal splitting distance and the transversality of the splitting, as well as its continuation for all values of $\varepsilon \to 0$. The results on transversality and continuation generalize the results obtained for the golden number in [6], and could be analogously extended to other quadratic frequency ratios by means of a specific study in each case.

1.2. Setup

Here we describe the nearly integrable Hamiltonian system under consideration. In particular, we study a *singular* or *weakly hyperbolic* (*a priori stable*) Hamiltonian with 3 degrees of freedom possessing a 2-dimensional whiskered tori with fast frequencies. In canonical coordinates $(x, y, \varphi, I) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^2 \times \mathbb{R}^2$, with the symplectic form $\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}\varphi \wedge \mathrm{d}I$, the Hamiltonian is defined by

$$H(x, y, \varphi, I) = H_0(x, y, I) + \mu H_1(x, \varphi), \tag{1.3}$$

$$H_0(x, y, I) = \langle \omega_\varepsilon, I \rangle + \frac{1}{2}\langle \Lambda I, I \rangle + \frac{y^2}{2} + \cos x - 1, \tag{1.4}$$

$$H_1(x, \varphi) = h(x) f(\varphi). \tag{1.5}$$

Our system has two parameters $\varepsilon > 0$ and μ, linked by a relation of the form $\mu = \varepsilon^p$, $p > 0$ (the smaller p, the better). Thus, if we consider ε as the unique parameter, we have a singular problem for $\varepsilon \to 0$. See [4] for a discussion about singular and regular problems.

Recall that we are assuming a vector of fast frequencies $\omega_\varepsilon = \omega/\sqrt{\varepsilon}$ as given in (1.1), with the *silver frequency vector* $\omega = (1, \Omega)$, where $\Omega = \sqrt{2} - 1$. It is well known that this vector satisfies a *Diophantine condition*

$$|\langle k, \omega \rangle| \geqslant \frac{\gamma}{|k|}, \quad \forall k \in \mathbb{Z}^2 \setminus \{0\} \tag{1.6}$$

with a concrete $\gamma > 0$. We also assume in (1.4) that Λ is a symmetric (2×2)-matrix, such that H_0 satisfies the condition of *isoenergetic nondegeneracy*,

$$\det \begin{pmatrix} \Lambda & \omega \\ \omega^\top & 0 \end{pmatrix} \neq 0. \tag{1.7}$$

For the perturbation H_1 in (1.5), we consider the following periodic even functions:

$$h(x) = \cos x, \tag{1.8}$$

$$f(\varphi) = \sum_{\substack{k \in \mathbb{Z}^2 \\ k_2 \geqslant 0}} \mathrm{e}^{-\rho|k|} \cos\langle k, \varphi \rangle, \tag{1.9}$$

where the restriction in the sum is introduced in order to avoid repetitions. The constant $\rho > 0$ gives the complex width of analyticity of the function $f(\varphi)$. With this perturbation, our Hamiltonian system given by (1.3)–(1.9) is *reversible* with respect to the involution

$$\mathcal{R} : (x, y, \varphi, I) \mapsto (-x, y, -\varphi, I) \tag{1.10}$$

(indeed, its associated Hamiltonian field satisfies the identity $X_H \circ \mathcal{R} = -\mathcal{R}\, X_H$). We point out that reversible perturbations have also been considered in some related papers [17, 20, 31]. The results can be presented in a somewhat simpler way under the assumption of reversibility. Nevertheless, this is not essential in our approach, and we show that our results are valid also in the nonreversible case if the even function $f(\varphi)$ in (1.9) is replaced by a much more general function (1.14), provided the phases in its Fourier expansion satisfy a suitable condition.

On the other hand, to justify the form of the perturbation H_1 chosen in (1.5) and (1.8)–(1.9), we stress that it makes easier the explicit computation of the Melnikov potential, which is necessary in order to compute explicitly the Melnikov approximation and show that it dominates the error term in (1.2), and therefore to establish the existence of splitting. Moreover, the fact that all harmonics in the Fourier expansion with respect to φ are nonzero, having an exponential decay, ensures that the study of the dominant harmonics of the Melnikov potential can be carried out directly from the arithmetic properties of the frequency vector ω (see Section 3). It is worth reminding that the Hamiltonian defined in (1.3)–(1.9) is paradigmatic, since it is a generalization of the famous Arnold example (introduced in [1] to illustrate the transition chain mechanism in Arnold's diffusion). It provides a model for the behavior of a near-integrable Hamiltonian system near a single resonance (see [4] for a motivation) and has often been considered in the literature (see, for instance, [11, 19, 25]). We also mention that a perturbation with an exponential decay as in (1.9) has also often been considered (see, for instance, [30]). Here, our aim is to emphasize the role of the arithmetic properties of the silver frequency vector ω in the study of the splitting.

Let us describe the invariant tori and whiskers, as well as the splitting and Melnikov functions. First, notice that the unperturbed system H_0 consists of the pendulum given by $P(x,y) = y^2/2 + \cos x - 1$, and 2 rotors with fast frequencies: $\dot\varphi = \omega_\varepsilon + \Lambda I$, $\dot I = 0$. The pendulum has a hyperbolic equilibrium at the origin, and the (upper) separatrix can be parameterized by $(x_0(s), y_0(s)) = (4 \arctan \mathrm{e}^s, 2/\cosh s)$, $s \in \mathbb{R}$. The rotors system (φ, I) has the solutions $\varphi = \varphi_0 + (\omega_\varepsilon + \Lambda I_0)\, t$, $I = I_0$. Consequently, H_0 has a 2-parameter family of 2-dimensional whiskered invariant tori which have a *homoclinic whisker*, i.e., coincident stable and unstable manifolds. Among the family of whiskered tori, we will focus our attention on the torus located at $I = 0$, whose frequency vector is ω_ε as in (1.1).

When adding the perturbation μH_1, the *hyperbolic KAM theorem* can be applied (see, for instance, [28]) thanks to the Diophantine condition (1.6) and the isoenergetic nondegeneracy (1.7). For μ small enough, the whiskered torus persists with some shift and deformation, as well as its local whiskers.

In general, for $\mu \neq 0$ the (global) whiskers do not coincide anymore, and one can introduce a *splitting function* giving the distance between the stable whisker $\mathcal{W}^{\mathrm{s}}$ and the unstable whisker $\mathcal{W}^{\mathrm{u}}$, in the directions of the action coordinates $I \in \mathbb{R}^2$: denoting by $\mathcal{J}^{\mathrm{s,u}}(\theta)$ the parameterizations of some concrete transverse section $x = \mathrm{const}$ of both whiskers, one can define the vector function $\mathcal{M}(\theta) := \mathcal{J}^{\mathrm{u}}(\theta) - \mathcal{J}^{\mathrm{s}}(\theta)$, $\theta \in \mathbb{T}^2$ (see [3, §5.2]). This function turns out to be the gradient of the (scalar) *splitting potential*: $\mathcal{M}(\theta) = \nabla \mathcal{L}(\theta)$ (see [3, 15]). Notice that the *nondegenerate critical points* of $\mathcal{L}$ correspond to simple zeros of $\mathcal{M}$ and give rise to *transverse homoclinic orbits* to the whiskered torus.

Due to the reversibility (1.10), the whiskers are related by the involution: $\mathcal{W}^{\mathrm{s}} = \mathcal{R}\, \mathcal{W}^{\mathrm{u}}$. Hence, their parameterizations can be chosen to satisfy the identity $\mathcal{J}^{\mathrm{s}}(\theta) = \mathcal{J}^{\mathrm{u}}(-\theta)$, provided the transverse section $x = \pi$ is considered in their definition. This implies that the splitting function is an odd function: $\mathcal{M}(-\theta) = -\mathcal{M}(\theta)$ (and the splitting potential $\mathcal{L}(\theta)$ is even). Taking into account its periodicity, we deduce that it has, at least, the following 4 zeros (which, in principle, might be nonsimple):

$$\theta_*^{(1)} = (0,0), \quad \theta_*^{(2)} = (\pi,0), \quad \theta_*^{(3)} = (0,\pi), \quad \theta_*^{(4)} = (\pi,\pi). \tag{1.11}$$

Applying the Poincaré–Melnikov method, the first-order approximation (1.2) is given by the (vector) *Melnikov function* $M(\theta)$, which is the gradient of the *Melnikov potential*: $M(\theta) = \nabla L(\theta)$.

The latter one can be defined by integrating the perturbation H_1 along a trajectory of the unperturbed homoclinic whisker, starting at the point of the section $s = 0$ with a given phase θ:

$$L(\theta) = -\int_{-\infty}^{\infty} [h(x_0(t)) - h(0)] f(\theta + \omega_\varepsilon t)\, \mathrm{d}t. \tag{1.12}$$

Our choice of the pendulum, whose separatrix has simple poles, makes it possible to use the method of residues in order to compute the coefficients of the Fourier expansion of $L(\theta)$ (see their expression in Section 3). We refer to [11] for estimates for the Melnikov potential and for the error term in our model (1.3)–(1.9). We stress that our approach can also be directly applied to other classical 1-degree-of-freedom Hamiltonians $P(x,y) = y^2/2 + V(x)$, with a potential $V(x)$ having a unique nondegenerate maximum, although the use of residues becomes more cumbersome when the separatrix has poles of higher orders (see some examples in [14]).

1.3. Main Result

We show in this paper that, for the Hamiltonian system (1.3)–(1.9) with the 2 parameters linked by $\mu = \varepsilon^p$, the Poincaré–Melnikov method can be applied to detect the splitting as long as we choose the exponent $p > p^*$, with some p^*. Namely, we provide asymptotic estimates for the *maximal distance* of splitting, in terms of the maximum size in modulus of the splitting function $\mathcal{M}(\theta)$, and for the *transversality* of the homoclinic orbits. The main goal of this paper is to show that $\mathcal{M}$ has 4 simple zeros (equivalently, that the splitting potential $\mathcal{L}$ has 4 nondegenerate critical points) for *all* sufficiently small ε and, hence, establish the existence of 4 transverse homoclinic orbits to the whiskered tori, generalizing the results on the *continuation* of the transversality, obtained in [6] for the golden number. We also obtain an asymptotic estimate for the minimal eigenvalue (in modulus) of the splitting matrix $\mathrm{D}\mathcal{M}$ at each zero. This estimate provides a measure of transversality of the homoclinic orbits.

Due to the form of $f(\varphi)$ in (1.9), the Melnikov potential $L(\theta)$ is readily represented in its Fourier series (see Section 3). We use this expansion of L in order to detect its *dominant harmonics* for every ε. The dominant harmonics of L correspond, for μ small enough, to the dominant harmonics of the splitting potential $\mathcal{L}$ and, as shown in [5], at least 2 dominant harmonics of $\mathcal{L}$ are necessary in order to prove the nondegeneracy of its critical points. Such dominant coefficients are closely related to the (quasi-)resonances of the silver frequency vector $\omega = (1, \Omega)$. For any quadratic frequency vector, a classification of the integer vectors k into *primary* and *secondary resonances* is established in [5]: the primary resonances are the ones which fit better the Diophantine condition (1.6). In the concrete case of the silver number $\Omega = \sqrt{2} - 1$, the primary resonances are related to the *Pell numbers* (see, for instance, [16, 22]), which play the same role as the Fibonacci numbers in the case of the golden number considered in [6]. With this in mind, we define the sequence of *Pell vectors* through the following recurrence:

$$s_0(0) = (0,1), \quad s_0(1) = (-1,2), \quad s_0(n+1) = 2s_0(n) + s_0(n-1), \quad n \geqslant 1. \tag{1.13}$$

We show that a change in the second dominant harmonic of the splitting potential $\mathcal{L}$ occurs when ε goes across some critical values $\widehat{\varepsilon}_n$ (called *transition values*). The nondegeneracy of the critical points of $\mathcal{L}$ can be proved in the case of 2 dominant harmonics for most values of ε, for some quadratic numbers including the silver number Ω (see [5]). But this excludes small neighborhoods of $\widehat{\varepsilon}_n$, where the second dominant harmonic coincides with some subsequent harmonics. In the present paper, we carry out the study near the transition values $\widehat{\varepsilon}_n$ assuming that the frequency ratio Ω in (1.1) is the silver number. In fact, for ε close to $\widehat{\varepsilon}_n$, we need to consider 4 dominant harmonics since the second, the third and the fourth dominant harmonics (two of them are associated to primary resonances and one is secondary) are of the same magnitude. We establish, for the concrete perturbation H_1 in (1.3)–(1.9), the nondegeneracy of the critical points of the splitting potential $\mathcal{L}$ for the values $\varepsilon \approx \widehat{\varepsilon}_n$ too, and this implies the continuation of the 4 homoclinic orbits for *all* $\varepsilon \to 0$, with no bifurcations.

We use the notation $f \sim g$ if we can bound $c_1|g| \leqslant |f| \leqslant c_2|g|$ with positive constants c_1, c_2 independent of ε, μ.

Theorem 1 (*main result*). *Assume for the Hamiltonian* (1.3)–(1.9) *that* ε *is small enough and that* $\mu = \varepsilon^p$, $p > 3$. *Then, for the splitting function* $\mathcal{M}(\theta)$ *we have:*

(a) $\max_{\theta\in\mathbb{T}^2} |\mathcal{M}(\theta)| \sim \frac{\mu}{\sqrt{\varepsilon}} \exp\left\{-\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/4}}\right\}$;

(b) *it has exactly 4 zeros* $\theta_*^{(j)}$ *as in* (1.11), *all simple, and the minimal eigenvalue of* $\mathrm{D}\mathcal{M}(\theta_*^{(j)})$ *at each zero satisfies*

$$m_*^{(j)} \sim \mu\varepsilon^{1/4} \exp\left\{-\frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/4}}\right\}.$$

The functions $h_1(\varepsilon)$ *and* $h_2(\varepsilon)$, *defined in* (3.6), *are* $4\ln(1+\sqrt{2})$*-periodic in* $\ln\varepsilon$, *with* $\min h_1(\varepsilon) = 1$, $\max h_1(\varepsilon) = \min h_2(\varepsilon) = \sqrt{(1+\sqrt{2})/2} \approx 1.0987$, $\max h_2(\varepsilon) = \sqrt{2} \approx 1.4142$. *On the other hand,* $C_0 = (\pi\rho)^{1/2}$.

In fact, we show in Section 4 that this result applies to a much more general perturbation in (1.9):

$$f(\varphi) = \sum_{\substack{k\in\mathbb{Z}^2 \\ k_2\geqslant 0}} e^{-\rho|k|} \cos(\langle k,\varphi\rangle - \sigma_k), \tag{1.14}$$

under a suitable condition on the phases $\sigma_k \in \mathbb{T}$ associated to primary vectors k, see (1.13). Such a condition, established in Lemma 5, will be clearly fulfilled in our concrete reversible case (1.9), given by $\sigma_k = 0$ for any k.

We stress that the result on continuation, given in Theorem 1, requires a careful study of the transitions in the second dominant harmonic, when the parameter ε goes through the values $\widehat{\varepsilon}_n$, where the results of [5] do not apply. A result on continuation was already obtained in [6], but for the golden number $\Omega = (\sqrt{5}-1)/2$, showing that, in this case, one only needs to take into account the primary resonances. We extend this result to the case of the silver number $\Omega = \sqrt{2}-1$ with the additional difficulty that at the transition values, we also have to take into account the harmonics associated to secondary resonances. We point out that the technique used in this paper could also be applied to any quadratic number by means of a specific study (in each case, assuming a suitable condition on the phases σ_k in (1.14)).

Remark. If the function $h(x)$ in (1.8) is replaced by $h(x) = \cos x - 1$, then the results of Theorem 1 are valid for $\mu = \varepsilon^p$ with $p > 2$ (instead of $p > 3$). The details of this improvement are not given here, since they work exactly as in [6].

2. THE SILVER FREQUENCY VECTOR

We review in this section the technique developed in [5] (see also [8]) for studying the resonances of quadratic frequency vectors ω, in (1.1), in the concrete case of the silver ratio. This ratio has the following 1-periodic continued fraction:

$$\Omega = \sqrt{2}-1 = [2,2,2,\ldots] = \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cdots}}}.$$

It is well known that $\omega = (1,\Omega)$, as well as any quadratic frequency vector, satisfies a *Diophantine condition* as in (1.6). With this in mind, we define the *"numerators"*

$$\gamma_k := |\langle k,\omega\rangle| \cdot |k|, \qquad k \in \mathbb{Z}^2 \setminus \{0\} \tag{2.1}$$

(for integer vectors, we use the norm $|\cdot| = |\cdot|_1$, i.e., the sum of absolute values of the components of the vector). Our goal is to provide a classification of the integer vectors k, according to the size of γ_k, in order to find the primary resonances (i.e., the integer vectors k for which γ_k is the smallest and, hence, fits best the Diophantine condition (1.6)), and study their separation with respect to the secondary resonances.

The key point in [5] is to use a unimodular matrix T having the vector ω as an eigenvector with eigenvalue $\lambda > 1$. This is a particular case of a result by Koch [23]. For quadratic numbers, the periodicity of the continued fraction can be used to construct T (see [8, 9]). Clearly, the iterations of the matrix T provide approximations to the direction of ω. Then, the associated quasi-resonances are given by the matrix $U := -(T^{-1})^\top$ according to the following important equality:

$$|\langle Uk, \omega\rangle| = \frac{1}{\lambda}\,|\langle k, \omega\rangle|\,.$$

For the silver number Ω, the matrices are

$$T = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}.$$

The eigenvalues of T are

$$\lambda := \Omega^{-1} = \sqrt{2} + 1 \tag{2.2}$$

and $-\lambda^{-1}$ with the eigenvectors $\omega = (1, \Omega)$ and $(1, -\Omega^{-1})$, respectively. The matrix U has the same eigenvectors with the eigenvalues $-\lambda^{-1}$ and λ, respectively. In fact, for a quadratic number *equivalent* to Ω, i.e., with a non-purely periodic continued fraction $\hat\Omega = [b_1, \ldots, b_l, 2, 2, \ldots] = [b_1, \ldots, b_l, \Omega]$, a linear change given by a unimodular matrix can be done between $\omega = (1, \Omega)$ and $\hat\omega = (1, \hat\Omega)$ in order to construct the corresponding matrices $\hat T$ and $\hat U$. This implies that the results of this paper can be extended to any other quadratic number equivalent to Ω.

We recall the results of [5], on the classification of quasi-resonances for any quadratic number Ω. The study can be restricted to integer vectors $k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{0\}$ with $|\langle k, \omega\rangle| < 1/2$, and we also assume that $k_2 \geqslant 1$. Such integer vectors have the form $k^0(j) = (-\operatorname{rint}(j\Omega), j)$, where $j \geqslant 1$ is an integer number, and $\operatorname{rint}(a)$ denotes the closest integer to a. An integer number j is said to be *primitive* if

$$\frac{1}{2\lambda} < |\langle k^0(j), \omega\rangle| < \frac{1}{2}\,.$$

Then, the integer vectors $k \in \mathbb{Z}^2$ with $|\langle k, \omega\rangle| < 1/2$ can be subdivided into *resonant sequences*:

$$s(j, n) := U^n k^0(j), \qquad n = 0, 1, 2, \ldots \tag{2.3}$$

generated by initial vectors $k^0(j)$ with a given primitive j. It was proved in [5, Theorem 2] (see also [7]) that, asymptotically, each resonant sequence $s(j, n)$ exhibits a geometric growth as $n \to \infty$, with ratio λ, and that the sequence of the numerators $\gamma_{s(j,n)}$ has a limit γ_j^*. More precisely,

$$|s(j, n)| = K_j \lambda^n + \mathcal{O}(\lambda^{-n}), \qquad \gamma_{s(j,n)} = \gamma_j^* + \mathcal{O}(\lambda^{-2n}), \tag{2.4}$$

where K_j and γ_j^* can be determined explicitly for each resonant sequence, from its primitive j (see explicit formulas in [5]). Since the lower bounds for γ_j^*, also provided in [5], are increasing in j, we can select the minimal of them, corresponding to some j_0. We denote

$$\gamma^* := \liminf_{|k|\to\infty} \gamma_k = \min_j \gamma_j^* = \gamma_{j_0}^* > 0. \tag{2.5}$$

The integer vectors of the sequence $s_0(n) := s(j_0, n)$ are called *the primary resonances*, and integer vectors belonging to any of the remaining resonant sequences $s(j, n)$, $j \neq j_0$, are called *secondary resonances*. One also introduces *normalized numerators* and their limits, after dividing by γ^*:

$$\tilde\gamma_k := \frac{\gamma_k}{\gamma^*}\,, \qquad \tilde\gamma_j^* := \frac{\gamma_j^*}{\gamma^*}\,.$$

For the concrete case of the silver number $\Omega=\sqrt{2}-1$, we have:

$$\gamma^*=\gamma_1^*=\frac{1}{2}\,,\qquad \tilde\gamma_k=2\gamma_k,\quad \tilde\gamma_j^*=2\gamma_j^*, \tag{2.6}$$

as well as the following data, which can be obtained from the results of [5]:

$$\begin{array}{llll}
j_0=1, & k^0(1)=(0,1), & \tilde\gamma_1^*=1, & K_1=\frac{1}{2}\Omega+1\approx 1.2071;\\[1ex]
j=3, & k^0(3)=(-1,3), & \tilde\gamma_3^*=2, & K_3=\frac{3}{2}\Omega+\frac{7}{2}\approx 4.1213;\\[1ex]
j=4, & k^0(4)=(-2,4), & \tilde\gamma_4^*=4, & K_4=2\Omega+5\approx 5.8284;\\[1ex]
j\geqslant 6 & & \tilde\gamma_j^*>6.5723 &
\end{array}$$

(notice that the integer vectors $k^0(j)$ for $j=2,5,\dots$ are not primitive, and belong to the sequence generated by some primitive). It is not hard to see from (2.3), applying induction with respect to n, that

$$s(j,n)=(-p(j,n-1),p(j,n)),$$

where $p(j,n)$ is a "generalized" Pell sequence: $p(j,n+1)=2p(j,n)+p(j,n-1)$, $n\geqslant 1$, starting from $p(j,0)=\mathrm{rint}(j\Omega)$ and $p(j,1)=j$. For $j=1$, since $\mathrm{rint}(\Omega)=0$, we get the (classical) Pell sequence: $P_{n+1}=2P_n+P_{n-1}$, with $P_0=0$ and $P_1=1$, and the primary resonances are $s_0(n)=s(1,n)=(-P_{n-1},P_n)$, as introduced in (1.13).

We denote by $s_1(n):=s(3,n)$ the sequence of secondary vectors generated by $k^0(3)=(-1,3)$. This sequence gives the second smallest limit $\tilde\gamma_3^*=2$ and, as shown in Section 4, it plays an essential role in the analysis of the transversality near the transition values. Because of this, the vectors in the sequence $s_1(n)$ will be called *the main secondary resonances*. Using induction, we can establish the following relation between the primary and the main secondary resonances:

$$s_1(n)=s_0(n)+s_0(n+1),\quad n\geqslant 0. \tag{2.7}$$

3. DOMINANT HARMONICS OF THE SPLITTING POTENTIAL

From now on, we consider the 3-degree-of-freedom Hamiltonian given as in (1.3)–(1.8) but, instead of (1.9), we consider a more general perturbation (1.14) with given phases σ_k. In fact, in order to guarantee the continuation of the transverse homoclinic orbits, a quite general condition on the phases σ_k will have to be fulfilled (see this condition in (4.17)).

We put our functions f and h defined in (1.14) and (1.8), respectively, into the integral (1.12) and, calculating it by residues, we get the Fourier expansion of the Melnikov potential:

$$L(\theta)=\sum_{\substack{k\in\mathbb{Z}^2\setminus\{0\}\\ k_2\geqslant 0}} L_k\cos(\langle k,\theta\rangle-\sigma_k),\qquad L_k=\frac{2\pi|\langle k,\omega_\varepsilon\rangle|\,\mathrm{e}^{-\rho|k|}}{\sinh|\frac{\pi}{2}\langle k,\omega_\varepsilon\rangle|}\,.$$

We point out that the phases σ_k are the same as in (1.14). Using (1.1) and (2.1), we can represent the coefficients in the form

$$L_k=\alpha_k\,\mathrm{e}^{-\beta_k},\qquad \alpha_k\approx\frac{4\pi\gamma_k}{|k|\sqrt{\varepsilon}}\,,\quad \beta_k=\rho|k|+\frac{\pi\gamma_k}{2|k|\sqrt{\varepsilon}}\,, \tag{3.1}$$

where an exponentially small term has been neglected in the denominator of α_k. For any given ε, the harmonics with the largest coefficients $L_k(\varepsilon)$ correspond essentially to the smallest exponents $\beta_k(\varepsilon)$. Thus, we have to study the dependence of such exponents on ε.

With this aim, we introduce for any X, Y the function

$$G(\varepsilon;X,Y):=\frac{Y^{1/2}}{2}\left[\left(\frac{\varepsilon}{X}\right)^{1/4}+\left(\frac{X}{\varepsilon}\right)^{1/4}\right],$$

having its minimum at $\varepsilon = X$, with the minimum value $G(X; X, Y) = Y^{1/2}$. Then, the exponents $\beta_k(\varepsilon)$ in (3.1) can be represented as

$$\beta_k(\varepsilon) = \frac{C_0}{\varepsilon^{1/4}}\, g_k(\varepsilon), \qquad g_k(\varepsilon) := G(\varepsilon; \varepsilon_k, \tilde{\gamma}_k), \tag{3.2}$$

where

$$\varepsilon_k := D_0\, \frac{\tilde{\gamma}_k^2}{|k|^4}\,, \qquad C_0 := (\pi\rho)^{1/2}, \qquad D_0 := \left(\frac{\pi}{4\rho}\right)^2 \tag{3.3}$$

(recall that the numerators $\tilde{\gamma}_k = 2\gamma_k$ were introduced in (2.5)–(2.6)). Consequently, for all k we have $\beta_k(\varepsilon) \geqslant \dfrac{C_0\tilde{\gamma}_k^{1/2}}{\varepsilon^{1/4}}$. This provides, according to (3.1), an asymptotic estimate for the exponent of the maximum value of the coefficient $L_k(\varepsilon)$ of each harmonic.

For any ε fixed we have to find the dominant terms L_k and the corresponding vectors k. Since the coefficients L_k are exponentially small in ε, it is more convenient to work with the functions g_k, whose smallest values correspond to the largest L_k. To this aim, it is useful to consider the graphs of the functions $g_k(\varepsilon)$, $k \in \mathbb{Z}^2 \setminus \{0\}$, in order to detect the minimum of them for a given value of ε.

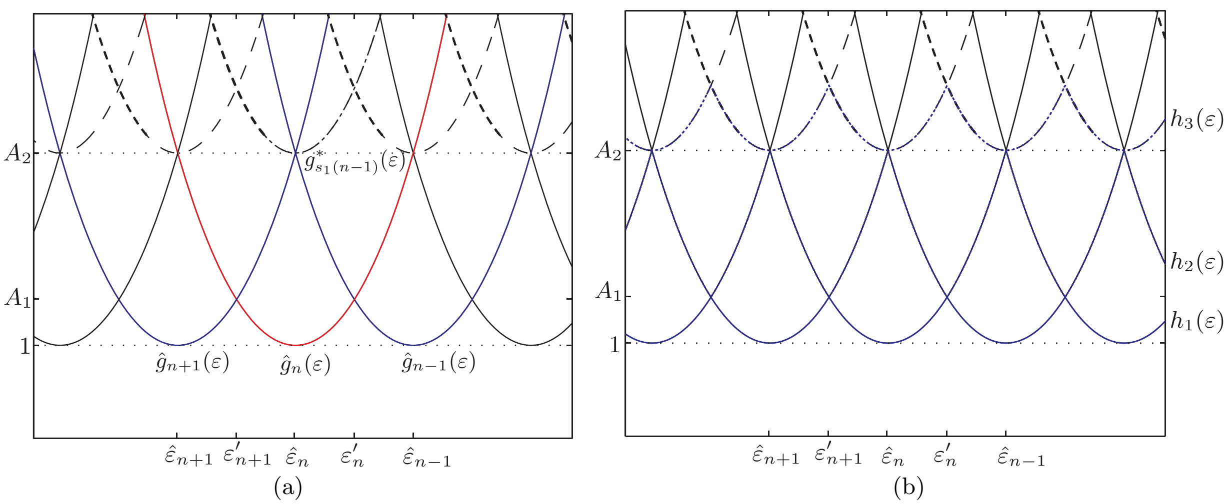

Fig. 1. (a) Graphs of the functions $g^*_{s(j,n)}(\varepsilon)$ using a logarithmic scale for ε; the ones with solid lines are the primary functions $\widehat{g}_n(\varepsilon)$. (b) Graphs of the minimizing functions $h_1(\varepsilon)$, $h_2(\varepsilon)$ and $h_3(\varepsilon)$. Here $A_1 = \sqrt{(1+\sqrt{2})/2} \approx 1.0987$ and $A_2 = \sqrt{2} \approx 1.4142$.

We know from (3.2) that the functions $g_k(\varepsilon)$ have their minimum at $\varepsilon = \varepsilon_k$ and the corresponding minimal values are $g_k(\varepsilon_k) = \tilde{\gamma}_k^{1/2}$. For the integer vectors $k = s(j,n)$ belonging to a resonant sequence (recall the definition in (2.3)), we use the approximations for $|s(j,n)|$ and $\gamma_{s(j,n)}$ as $n \to \infty$, given in (2.4). This provides the following approximations as $n \to \infty$,

$$g_{s(j,n)}(\varepsilon) \approx g^*_{s(j,n)}(\varepsilon) := G(\varepsilon; \varepsilon^*_{s(j,n)}, \tilde{\gamma}^*_j), \qquad \varepsilon_{s(j,n)} \approx \varepsilon^*_{s(j,n)} := \frac{D_0(\tilde{\gamma}^*_j)^2}{K_j^4 \lambda^{4n}}\,.$$

The graphs in Fig. 1a, where a logarithmic scale for ε is used, correspond to the approximations $g^*_k(\varepsilon)$, rather than the true functions $g_k(\varepsilon)$. Note that the functions $g^*_{s(j,n)}(\varepsilon)$ satisfy the following scaling property:

$$g^*_{s(j,n+1)}(\varepsilon) = g^*_{s(j,n)}(\lambda^4\varepsilon). \tag{3.4}$$

The case of the sequence of primary resonances plays an important role here, since it gives the smallest minimum values of the functions $g_k(\varepsilon)$. With this in mind, we denote

$$\widehat{g}_n(\varepsilon) := g^*_{s_0(n)} = G(\varepsilon; \widehat{\varepsilon}_n, 1), \qquad \widehat{\varepsilon}_n := \varepsilon^*_{s_0(n)} = \frac{16D_0}{\lambda^{4(n+1)}}, \tag{3.5}$$

where we have used the fact that $\tilde{\gamma}^*_1 = 1$ and $K_1 = \lambda/2$. On the other hand, for the main secondary resonances we can use that $\tilde{\gamma}^*_3 = 2$ and $K_3/K_1 = \sqrt{2}\,\lambda$, and obtain

$$g^*_{s_1(n-1)} = G(\varepsilon; \widehat{\varepsilon}_n, \sqrt{2}), \qquad \varepsilon^*_{s_1(n-1)} = \widehat{\varepsilon}_n.$$

Such facts are represented in Fig. 1a.

Now we define, for any given ε and for $i = 1, 2, 3, \ldots$, the function $h_i(\varepsilon)$ as the i-th minimum of the values $g^*_k(\varepsilon)$, $k \in \mathbb{Z}^2 \setminus \{0\}$, and denote $S_i = S_i(\varepsilon)$ the integer vectors where such minima are reached:

$$\begin{aligned} &h_1(\varepsilon) := \min_k g^*_k(\varepsilon) = g^*_{S_1}(\varepsilon), \qquad h_2(\varepsilon) := \min_{k \neq S_1} g^*_k(\varepsilon) = g^*_{S_2}(\varepsilon), \\ &h_3(\varepsilon) := \min_{k \neq S_1, S_2} g^*_k(\varepsilon) = g^*_{S_3}(\varepsilon), \quad \text{etc.} \end{aligned} \tag{3.6}$$

It is clear from the scaling property (3.4) that the functions $h_i(\varepsilon)$ are $4\ln\lambda$-periodic in $\ln\varepsilon$ and continuous. As we can see in Fig. 1b, the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ are given by primary vectors $s_0(n)$, and $h_3(\varepsilon)$ is given by secondary vectors $s_1(n)$. It is easy to check that the minimum and maximum values of h_1 and h_2 are the ones given in the statement of Theorem 1.

The functions $h_i(\varepsilon)$ provide estimates, for any ε, of the size of the corresponding dominant coefficients $L_{S_i(\varepsilon)}$ of the Melnikov potential. We say that a given value ε is a *transition value* if $h_2(\varepsilon) = h_3(\varepsilon)$, since a transition in the second dominant harmonic takes place at these values. In the case of the silver frequencies, these values correspond to the geometric sequence $\widehat{\varepsilon}_n$ defined in (3.5). In the next section, in order to prove the transversality in small neighborhoods of $\widehat{\varepsilon}_n$, we need to consider the 4 dominant harmonics of the splitting potential (one of which is a main secondary resonance $s_1(n)$). This is the main goal of this paper, since for the majority of values of ε (excluding such neighborhoods of $\widehat{\varepsilon}_n$) it is enough to consider the simpler case of 2 dominant harmonics in order to prove the transversality, and this is already considered in [5] for a wider class of quadratic frequency ratios. We also define the sequence of geometric means of the sequence $\widehat{\varepsilon}_n$,

$$\varepsilon'_n := \sqrt{\widehat{\varepsilon}_n \widehat{\varepsilon}_{n-1}} = \frac{16D_0}{\lambda^{4n+2}}, \tag{3.7}$$

at which the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$ coincide. For ε belonging to a given interval $(\varepsilon'_{n+1}, \varepsilon'_n)$, which contains the transition value $\widehat{\varepsilon}_n$, we have

$$S_1 = s_0(n), \quad S_3 = s_1(n+1), \tag{3.8}$$

and

$$\begin{aligned} &S_2 = s_0(n+1), \qquad S_4 = s_0(n-1) \qquad \text{for } \varepsilon < \widehat{\varepsilon}_n, \\ &S_2 = s_0(n-1), \qquad S_4 = s_0(n+1) \qquad \text{for } \varepsilon > \widehat{\varepsilon}_n \end{aligned} \tag{3.9}$$

(see also Fig. 1). We have the following important estimate: since we can choose $n = n(\varepsilon)$ such that $\varepsilon \in (\varepsilon'_{n+1}, \varepsilon'_n)$, from (2.4) and (3.5) we obtain

$$|S_i| \sim \lambda^n \sim \varepsilon^{-1/4}, \quad i = 1, 2, 3, 4 \tag{3.10}$$

(recall that the notation "$\sim$" was introduced just before Theorem 1).

We will use the next lemma of [5], which establishes that the 4 most dominant harmonics of the Melnikov potential are also dominant for the splitting potential,

$$\mathcal{L}(\theta) = \sum_{\substack{k\in\mathbb{Z}^2\setminus\{0\}\\ k_2\geqslant 0}} \mathcal{L}_k \cos(\langle k,\theta\rangle - \tau_k),$$

providing an estimate for such dominant harmonics $\mathcal{L}_{S_i}$ (and an upper bound for the difference of their phases), as well as an estimate for the sum of all other harmonics in terms of the first neglected harmonic $\mathcal{L}_{S_5}$. In fact, since we will be interested in some derivative of the splitting potential, we consider the sum of (positive) amounts of the type $|k|^l\mathcal{L}_k$. The constant C_0 in the exponentials is the one defined in (3.3).

For positive amounts, we use the notation $f \preceq g$ if we can bound $f \leqslant c\,g$ with some constant c not depending on ε and μ.

Lemma 1. *For ε small enough and $\mu = \varepsilon^p$ with $p > 3$, one has:*

(a) $\mathcal{L}_{S_i} \sim \mu\, L_{S_i} \sim \dfrac{\mu}{\varepsilon^{1/4}} \exp\left\{-\dfrac{C_0 h_i(\varepsilon)}{\varepsilon^{1/4}}\right\}, \quad |\tau_{S_i} - \sigma_{S_i}| \preceq \dfrac{\mu}{\varepsilon^3}, \quad i = 1,2,3,4;$

(b) $\displaystyle\sum_{k\neq S_1,\ldots,S_4} |k|^l\, \mathcal{L}_k \sim \frac{1}{\varepsilon^{l/4}}\, \mathcal{L}_{S_5}, \quad l \geqslant 0.$

4. BEHAVIOR NEAR THE TRANSITION VALUES

This section is devoted to the study of the transversality of the homoclinic orbits for values of the perturbation parameter ε near the transition values $\widehat{\varepsilon}_n$, defined in (3.5), where the second, the third and the fourth dominant harmonics are of the same magnitude. The difficulty is due to the fact that the third dominant harmonic is associated to a main secondary resonance: $S_3 = s_1(n-1)$.

We consider a concrete interval $\varepsilon \in (\varepsilon'_{n+1}, \varepsilon'_n)$ which contains $\widehat{\varepsilon}_n$ (the values ε'_n are defined in (3.7)). For $\varepsilon \approx \widehat{\varepsilon}_n$ we show that, under suitable conditions, the splitting potential $\mathcal{L}(\theta)$ has 4 nondegenerate critical points, which give rise to 4 transverse homoclinic orbits. First, we study the critical points of the approximation of $\mathcal{L}(\theta)$ given by the 4 dominant harmonics (3.8)–(3.9) in the considered interval,

$$\mathcal{L}^{(4)}(\theta) := \sum_{i=1,2,3,4} \mathcal{L}_{S_i} \cos(\langle S_i,\theta\rangle - \tau_{S_i})$$

and afterwards we prove the persistence of these critical points in the whole function $\mathcal{L}(\theta)$.

We perform the linear change

$$\psi_1 = \langle s_0(n-1),\theta\rangle - \tau_{s_0(n-1)}, \quad \psi_2 = \langle s_0(n),\theta\rangle - \tau_{s_0(n)}, \tag{4.1}$$

which can be written as

$$\psi = \mathcal{A}\theta - b, \quad \text{where } \mathcal{A} = \begin{pmatrix} s_0(n-1)^\top \\ s_0(n)^\top \end{pmatrix}, \quad b = \begin{pmatrix} \tau_{s_0(n-1)} \\ \tau_{s_0(n)} \end{pmatrix}.$$

Since $\det\mathcal{A} = (-1)^{n-1}$, as easily seen from (2.3), this change is one-to-one on $\mathbb{T}^2$. Taking into account (1.13) and (2.7), and recalling (3.8)–(3.9), we see that the function $\mathcal{L}^{(4)}(\theta)$ is transformed, by this change, into

$$\begin{aligned} K^{(4)}(\psi) &= B\cos\psi_2 + B\eta(1-Q)\cos\psi_1 + B\eta Q\cos(\psi_1 + 2\psi_2 - \triangle\tau) \\ &\quad + B\eta\widetilde{Q}\cos(\psi_1+\psi_2-\triangle\tau_1), \end{aligned} \tag{4.2}$$

where we define

$$B=B(\varepsilon):=\mathcal{L}_{s_0(n)},\qquad \eta=\eta(\varepsilon):=\frac{\mathcal{L}_{s_0(n-1)}+\mathcal{L}_{s_0(n+1)}}{\mathcal{L}_{s_0(n)}}. \tag{4.3}$$

$$Q=Q(\varepsilon):=\frac{\mathcal{L}_{s_0(n+1)}}{\mathcal{L}_{s_0(n-1)}+\mathcal{L}_{s_0(n+1)}},\quad \widetilde{Q}=\widetilde{Q}(\varepsilon):=\frac{\mathcal{L}_{s_1(n-1)}}{\mathcal{L}_{s_0(n-1)}+\mathcal{L}_{s_0(n+1)}}, \tag{4.4}$$

$$\begin{aligned}\triangle\tau &:= \tau_{s_0(n+1)}-2\tau_{s_0(n)}-\tau_{s_0(n-1)},\\ \triangle\tau_1 &:= \tau_{s_1(n-1)}-\tau_{s_0(n)}-\tau_{s_0(n-1)}.\end{aligned} \tag{4.5}$$

Let us describe the behavior of Q, $\widetilde{Q}$, η as ε varies in the interval $(\varepsilon'_{n+1},\varepsilon'_n)$, which contains the transition value $\widehat{\varepsilon}_n$ in which we are interested. On the one hand, we see from (3.8)–(3.9) and Lemma 1(a) that η is exponentially small in ε in the whole interval, and we will consider it as a perturbation parameter. On the other hand, Q goes from 1 to 0 and $\widetilde{Q}$ takes values between 0 and 1/2, as ε crosses $\widehat{\varepsilon}_n$. More precisely, as one can see in Figure 1, for $\varepsilon\simeq\varepsilon'_{n+1}$ we have $\widehat{g}_{n+1}<g_{s_1(n-1)}<\widehat{g}_{n-1}$ and hence, recalling (3.5), $\mathcal{L}_{s_0(n+1)}\gg\mathcal{L}_{s_1(n-1)}\gg\mathcal{L}_{s_0(n-1)}$ and $Q\simeq 1$, $\widetilde{Q}\simeq 0$. On the other hand, for $\varepsilon\simeq\varepsilon'_n$ we have $\widehat{g}_{n+1}>g_{s_1(n-1)}>\widehat{g}_{n-1}$ and hence $Q\simeq 0$, $\widetilde{Q}\simeq 0$. At $\varepsilon=\widehat{\varepsilon}_n$ we have $\widehat{g}_{n+1}=g_{s_1(n-1)}=\widehat{g}_{n-1}$, and therefore the harmonics coincide and we have $Q=\widetilde{Q}=1/2$. In the interval $(\varepsilon'_{n+1},\varepsilon'_n)$ considered, we see that Q is decreasing, and $\widetilde{Q}$ has a maximum at $\widehat{\varepsilon}_n$ and lies between Q and $1-Q$.

We are going to use the following lemma, whose proof is a simple application of the standard fixed point theorem.

Lemma 2. *If $F:\mathbb{T}\to\mathbb{R}$ is differentiable and satisfies $(F')^2+F^2<1$, then the equation $\sin x = F(x)$ has exactly two solutions $\overline{x}$ and $\overline{\overline{x}}$, which are simple. Furthermore, if $F(x)=\mathcal{O}(\eta)$ for any $x\in\mathbb{T}$ with η sufficiently small, then the solutions of the equation satisfy $\overline{x}=\mathcal{O}(\eta)$ and $\overline{\overline{x}}=\pi+\mathcal{O}(\eta)$.*

Now we introduce the following important quantity:

$$\begin{aligned}&E^*=E^*(\varepsilon):=\min(E^{(+)},E^{(-)}),\quad\text{where}\\ &E^{(\pm)}:=\sqrt{\left[1-Q+Q\cos\triangle\tau\pm\widetilde{Q}\cos\triangle\tau_1\right]^2+\left[Q\sin\triangle\tau\pm\widetilde{Q}\sin\triangle\tau_1\right]^2}.\end{aligned} \tag{4.6}$$

In the next lemma we prove the existence of 4 critical points of $K^{(4)}$ for η small enough, provided $E^*>0$.

Lemma 3. *Assume that in* (4.6)

$$E^*(\varepsilon)>0,\qquad \forall\varepsilon\in(\varepsilon'_{n+1},\varepsilon'_n). \tag{4.7}$$

If $\eta\preceq E^*$ in (4.3)*, the function $K^{(4)}(\psi)$ introduced in* (4.2) *has 4 nondegenerate critical points $\psi^{(j)}=\psi^{(j),0}+\mathcal{O}(\eta)$, $j=1,2,3,4$, where we define*

$$\begin{aligned}\psi^{(1),0}&=(\alpha^{(+)},0), &\qquad \psi^{(2),0}&=(\alpha^{(+)}+\pi,0),\\ \psi^{(3),0}&=(\alpha^{(-)},\pi), &\qquad \psi^{(4),0}&=(\alpha^{(-)}+\pi,\pi),\end{aligned} \tag{4.8}$$

with

$$\cos\alpha^{(\pm)}=\frac{1-Q+Q\cos\triangle\tau\pm\widetilde{Q}\cos\triangle\tau_1}{E^{(\pm)}},\quad \sin\alpha^{(\pm)}=\frac{Q\sin\triangle\tau\pm\widetilde{Q}\sin\triangle\tau_1}{E^{(\pm)}}. \tag{4.9}$$

At the critical points,

$$\begin{aligned}|\det \mathrm{D}^2K^{(4)}(\psi^{(1,2)})|&=B^2\eta(E^{(+)}+\mathcal{O}(\eta)),\\ |\det \mathrm{D}^2K^{(4)}(\psi^{(3,4)})|&=B^2\eta(E^{(-)}+\mathcal{O}(\eta)).\end{aligned}$$

Proof. The critical points of $K^{(4)}(\psi)$ are the solutions to the system of equations

$$\begin{aligned}&(1-Q)\sin\psi_1 + Q\sin(\psi_1+2\psi_2-\triangle\tau)+\widetilde{Q}\sin(\psi_1+\psi_2-\triangle\tau_1)=0,\\ &\sin\psi_2+2\eta Q\sin(\psi_1+2\psi_2-\triangle\tau)+\eta\widetilde{Q}\sin(\psi_1+\psi_2-\triangle\tau_1)=0.\end{aligned} \tag{4.10}$$

We can rewrite the second equation as follows:

$$\begin{aligned}&\sin\psi_2=\eta f(\psi_1,\psi_2),\\ &\text{where}\quad f(\psi_1,\psi_2):=-2Q\sin(\psi_1+2\psi_2-\triangle\tau)-\widetilde{Q}\sin(\psi_1+\psi_2-\triangle\tau_1).\end{aligned} \tag{4.11}$$

Since η is small enough and f is bounded with its derivatives, we can apply Lemma 2 with $F=\eta f$, and ψ_1 as a parameter, and we find that Eq. (4.11) has two solutions: $\overline{\psi}_2=\overline{\psi}_2(\psi_1)=\mathcal{O}(\eta)$ and $\overline{\overline{\psi}}_2=\overline{\overline{\psi}}_2(\psi_1)=\pi+\mathcal{O}(\eta)$.

Substituting $\overline{\psi}_2(\psi_1)$ into the first equation of (4.10), we get the equation $F_\eta^{(+)}(\psi_1)=0$ with the function

$$\begin{aligned}F_\eta^{(+)} &:= (1-Q)\sin\psi_1+Q\sin(\psi_1-\triangle\tau)+\widetilde{Q}\sin(\psi_1-\triangle\tau_1)\\ &\quad -\eta f^{(+)}(\psi_1,\overline{\psi}_2;\eta)\\ &= \left[1-Q+Q\cos\triangle\tau+\widetilde{Q}\cos\triangle\tau_1\right]\sin\psi_1\\ &\quad -\left[Q\sin\triangle\tau+\widetilde{Q}\sin\triangle\tau_1\right]\cos\psi_1-\eta f^{(+)}(\psi_1,\overline{\psi}_2;\eta)\\ &= E^{(+)}\sin(\psi_1-\alpha^{(+)})-\eta f^{(+)}(\psi_1,\overline{\psi}_2;\eta),\end{aligned}$$

where $E^{(+)}$ and $\alpha^{(+)}$ are the constants defined in (4.6) and (4.9), respectively, and a function $f^{(+)}$, which is bounded jointly with its derivatives. Thus, provided $E^{(+)}>0$, the equation $F_\eta^{(+)}=0$ is equivalent to

$$\sin(\psi_1-\alpha^{(+)})=\frac{\eta}{E^{(+)}}\,f^{(+)}(\psi_1,\overline{\psi}_2;\eta)$$

and, by Lemma 2, it has 2 solutions $\psi_1^{(1)}=\alpha^{(+)}+\mathcal{O}(\eta)$ and $\psi_1^{(2)}=\alpha^{(+)}+\pi+\mathcal{O}(\eta)$, since $\eta\preceq E^*\leqslant E^{(+)}$. In this way, we have 2 critical points as solutions of the system (4.10): $\psi^{(j)}=(\psi_1^{(j)},\overline{\psi}_2(\psi_1^{(j)}))$, $j=1,2$.

We proceed analogously for $\overline{\overline{\psi}}_2$ and rewrite the first equation of (4.10) as

$$F_\eta^{(-)}:=E^{(-)}\sin(\psi_1-\alpha^{(-)})-\eta f^{(-)}(\psi_1,\overline{\overline{\psi}}_2;\eta)=0.$$

Assuming that $E^{(-)}>0$, we get other two solutions $\psi_1^{(3)}=\alpha^{(-)}+\mathcal{O}(\eta)$ and $\psi_1^{(4)}=\alpha^{(-)}+\pi+\mathcal{O}(\eta)$, since $\eta\preceq E^*\leqslant E^{(-)}$. Such solutions give rise to the other 2 critical points $\psi^{(j)}$, $j=3,4$.

To compute the determinant at the critical points, we use the fact that

$$\begin{aligned}&\det \mathrm{D}^2K^{(4)}(\psi)\\ &\quad = B^2(\eta[(1-Q)\cos\psi_1+Q\cos(\psi_1+2\psi_2-\triangle\tau)\\ &\quad\quad +\widetilde{Q}\cos(\psi_1+\psi_2-\triangle\tau_1)]\cdot\cos\psi_2+\mathcal{O}(\eta^2))\end{aligned}$$

for any $\psi\in\mathbb{T}^2$. At $\psi^{(1)}$, for example, we have

$$\begin{aligned}\det \mathrm{D}^2K^{(4)}(\psi^{(1)}) &= B^2\left(\eta\,\frac{\partial F_\eta^{(+)}}{\partial\psi_1}\bigg|_{\psi^{(1)}}\cdot\cos\psi_2^{(1)}+\mathcal{O}(\eta^2)\right)\\ &= B^2(\eta E^{(+)}+\mathcal{O}(\eta^2)),\end{aligned}$$

and similarly with the other 3 critical points. □

Remark. In our case of a reversible perturbation, as introduced in (1.9), we obtain in (4.9) the values $\alpha^{(\pm)} = 0$. By the linear change (4.1) and using the fact that the phases are $\sigma_k = 0$, we get the 4 critical points of $\mathcal{L}^{(4)}$, as deduced in (1.11) from the reversibility property.

To ensure the existence of nondegenerate critical points of $K^{(4)}$, in Lemma 3 we have assumed condition (4.7). In the next lemma we see when this assumption fails.

Lemma 4. *Let $0 < Q < 1$, $0 < \widetilde{Q} \leqslant 1/2$ and $\triangle\tau, \triangle\tau_1 \in \mathbb{T}$ be given, and consider E^* defined as in (4.6). Then one has $E^* = 0$ if and only if the following three conditions are satisfied:*

$$|1-2Q| \leqslant \widetilde{Q}, \quad \cos\triangle\tau = -\frac{1-2Q+2Q^2-\widetilde{Q}^2}{2(1-Q)Q}, \quad \cos\triangle\tau_1 = \pm\frac{\widetilde{Q}^2+1-2Q}{2(1-Q)\widetilde{Q}}. \tag{4.12}$$

Proof. We prove this lemma geometrically. It is clear from (4.6) that $E^* = 0$ if and only if $E^{(+)} = 0$ or $E^{(-)} = 0$, i.e., one of the following two assertions holds:

$$1 - Q + Q\cos\triangle\tau = -\widetilde{Q}\cos\triangle\tau_1 \quad \text{and} \quad Q\sin\triangle\tau = -\widetilde{Q}\sin\triangle\tau_1,$$

$$1 - Q + Q\cos\triangle\tau = \widetilde{Q}\cos\triangle\tau_1 \quad \text{and} \quad Q\sin\triangle\tau = \widetilde{Q}\sin\triangle\tau_1.$$

Now we consider the points

$$P_1 = (1 - Q + Q\cos\triangle\tau, Q\sin\triangle\tau),$$

$$P_2 = (\widetilde{Q}\cos\triangle\tau_1, \widetilde{Q}\sin\triangle\tau_1), \quad P_3 = (-\widetilde{Q}\cos\triangle\tau_1, -\widetilde{Q}\sin\triangle\tau_1),$$

which lie on the circles represented in Fig. 2a, and, hence, $E^{(+)}$ is the distance P_1P_3, while $E^{(-)}$ is P_1P_2.

Varying Q and $\widetilde{Q}$ and changing the corresponding circles in Fig. 2a in order to see when P_1 coincides with P_2 and, thus, $E^{(-)} = 0$, one can get that if $|1-2Q| < \widetilde{Q}$, the circles intersect (at the point $P_1 \equiv P_2$) and there is a triangle with sides Q, $1-Q$, $\widetilde{Q}$ and angles satisfying (4.12). For $1-2Q = \pm\widetilde{Q}$, the circles are tangent, having $\triangle\tau = \pi$, and $\triangle\tau_1 = 0$ (if $1-2Q = \widetilde{Q}$) or $\triangle\tau_1 = \pi$ (if $1-2Q = -\widetilde{Q}$). In the case $|1-2Q| > \widetilde{Q}$, the circles do not intersect. The case $E^{(+)} = 0$ (which corresponds to $P_1 \equiv P_3$) can be studied in a similar way. □

In this way, we can ensure the existence and continuation of the 4 critical points of $K^{(4)}$ given by Lemma 3 if the three conditions (4.12) do no hold simultaneously for any $\varepsilon \in (\varepsilon'_{n+1}, \varepsilon'_n)$. Now we provide a simple *sufficient condition* allowing us to avoid the occurrence of (4.12) and, hence, to ensure (4.7).

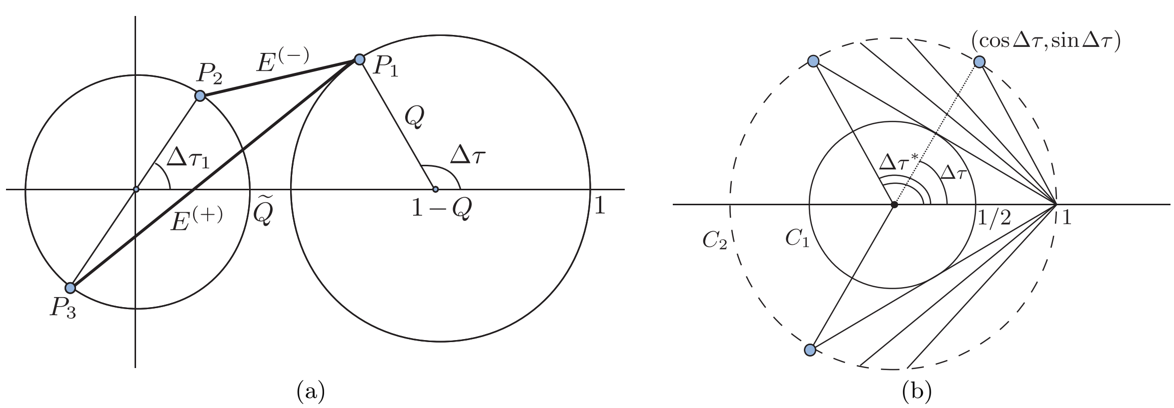

Fig. 2. (a) Geometrical representation of $E^{(+)}$ and $E^{(-)}$. (b) $E^{(\pm)} > 0$ (the straight lines do not intersect the circle C_1) if $|\triangle\tau| < \triangle\tau^* = 2\pi/3$.

Lemma 5. *If*

$$|\triangle\tau| < \frac{2\pi}{3}, \tag{4.13}$$

then the condition (4.7) *is fulfilled independently of* Q, $\widetilde{Q}$, $\triangle\tau_1$.

Proof. This is a corollary of Lemma 4. Indeed, inequality (4.13) implies that we have $\cos\triangle\tau > -1/2$. Then, if the second equality in (4.12) is satisfied, we have $1-\widetilde{Q}^2 < 3Q(1-Q)$, which contradicts the facts that $0 < Q < 1$ and $0 < \widetilde{Q} \leqslant 1/2$. □

Remark. We can provide a geometric interpretation for this lemma. In Fig. 2b we consider two circles centered at the origin: C_1 with radius $1/2$ (the maximum value for $\widetilde{Q}$) and the unit circle C_2. For any given $\triangle\tau$, the map $Q \mapsto (1-Q+Q\cos\triangle\tau, Q\sin\triangle\tau)$, for $0 \leqslant Q \leqslant 1$, gives us a family of straight lines (with $\triangle\tau$ as a parameter) connecting the points $(1,0)$ and $(\cos\triangle\tau, \sin\triangle\tau)$, both belonging to C_2. The straight lines corresponding to $\triangle\tau$ satisfying (4.13) do not intersect the circle C_1, which implies that $E^* > 0$ (see the proof of Lemma 4). Notice that, for $\widetilde{Q} = 1/2$, the critical value $\triangle\tau^* = 2\pi/3$ is sharp, but for $\widetilde{Q} < 1/2$ the critical value would be greater: $\triangle\tau^* > 2\pi/3$.

In the next lemma, we prove the persistence of the 4 critical points $\psi^{(j)}$ of the approximation $K^{(4)}(\psi)$, when the nondominant terms are also considered. With this aim, we denote by $K(\psi)$ the function obtained when the linear change (4.1) is applied to the *whole* splitting potential $\mathcal{L}(\theta)$. Recalling the definitions (4.3)–(4.4), we can write:

$$K(\psi) = K^{(4)}(\psi) + B\eta\eta' G(\psi),$$

where the term $B\eta\eta' G(\psi)$ corresponds to the sum of all nondominant harmonics, and $\mathcal{L}_{S_5} = B\eta\eta'$ is the largest among them with

$$\eta' := \frac{\mathcal{L}_{S_5}}{\mathcal{L}_{s_0(n-1)} + \mathcal{L}_{s_0(n+1)}} \ll Q, \widetilde{Q}. \tag{4.14}$$

Note that the function G is obtained via the linear change (4.1) applied to the nondominant harmonics of $\mathcal{L}(\theta)$. Thus, using Lemma 1(b), we get bounds for $G(\psi)$ and its partial derivatives:

$$|G| \preceq 1, \quad |\partial_{\psi_i} G| \preceq \varepsilon^{-1/2}, \quad |\partial^2_{\psi_i\psi_j} G| \preceq \varepsilon^{-1}, \quad i,j = 1,2, \tag{4.15}$$

where we have taken into account that, by (3.10), the entries of the matrix of the linear change (4.1) are $\sim \varepsilon^{-1/4}$.

Lemma 6. *Assuming condition* (4.7), *if* $\bar\eta := \max(\eta, \eta\eta'\varepsilon^{-1}) \preceq E^*$, *then the function* $K(\psi)$ *has 4 critical points, all nondegenerate:* $\psi_*^{(j)} = \psi^{(j),0} + \mathcal{O}(\bar\eta)$, $j = 1,2,3,4$, *with* $\psi^{(j),0}$ *defined in* (4.8). *At the critical points,*

$$|\det \mathrm{D}^2 K(\psi_*^{(1,2)})| = B^2\eta(E^{(+)} + \mathcal{O}(\bar\eta)),$$
$$|\det \mathrm{D}^2 K(\psi_*^{(3,4)})| = B^2\eta(E^{(-)} + \mathcal{O}(\bar\eta)).$$

Proof. The critical points of $K(\psi)$ are the solution of the following equations, which are perturbations of (4.10):

$$(1-Q)\sin\psi_1 + Q\sin(\psi_1 + 2\psi_2 - \triangle\tau) + \widetilde{Q}\sin(\psi_1 + \psi_2 - \triangle\tau_1) - \eta\eta'\partial_{\psi_1} G = 0,$$
$$\sin\psi_2 + 2\eta Q\sin(\psi_1 + 2\psi_2 - \triangle\tau) + \eta\widetilde{Q}\sin(\psi_1 + \psi_2 - \triangle\tau_1) - \eta\eta'\partial_{\psi_2} G = 0.$$

Now we can proceed as in the proof of Lemma 3. Indeed, applying Lemma 2 twice we can solve the second equation for ψ_2 with ψ_1 as a parameter, and we replace the solution in the first equation and solve it for ψ_1. The only difference with respect to Lemma 3 is that now we have additional perturbative terms $\eta\eta'\partial_{\psi_i}G$, which we have bounded in (4.15), and for this reason we consider $\bar\eta$ as the size of the perturbation. The determinant at the critical points can be computed as in Lemma 3. □

Remark. The smallness condition on $\bar\eta$ in Lemma 6 is clearly fulfilled in our case, since in (4.14) we have that η' is exponentially small in ε and, hence, can be bounded by any power of ε.

Applying the inverse (one-to-one) of the linear change (4.1), the 4 critical points $\psi_*^{(j)}$ of $K(\psi)$ give rise to 4 critical points of $\mathcal{L}(\theta)$, all nondegenerate:

$$\theta_*^{(j)} = \mathcal{A}^{-1}(\psi_*^{(j)} + b), \quad j = 1, 2, 3, 4. \tag{4.16}$$

Lemma 7. *Assuming condition* (4.7), *if* $\bar\eta := \max(\eta, \eta\eta'\varepsilon^{-1}) \preceq E^*$, *then the splitting potential* $\mathcal{L}(\theta)$ *has exactly 4 critical points* $\theta_*^{(j)}$, *given by* (4.16), *all nondegenerate, and the minimal eigenvalue* (*in modulus*) $m_*^{(j)}$ *of* $\mathrm{D}^2\mathcal{L}(\theta_*^{(j)})$ *satisfies*

$$E^*\sqrt{\varepsilon}\,\mathcal{L}_{S_2} \preceq m_*^{(j)} \preceq \sqrt{\varepsilon}\,\mathcal{L}_{S_2}, \quad j = 1, 2, 3, 4.$$

Proof. The proof is similar to the one of [5, Lemma 5] and, thus, we give here only a sketch of the proof. First, denoting $D = \det \mathrm{D}^2\mathcal{L}(\theta_*^{(j)})$ and $T = \operatorname{tr} \mathrm{D}^2\mathcal{L}(\theta_*^{(j)})$, it is not hard to see that if $|D| \ll T^2$, then $m_*^{(j)} \sim |D|/|T|$. Thus, we need to provide asymptotic estimates for $|D|$ and $|T|$.

Since $|\det \mathcal{A}| = 1$, the matrices $\mathrm{D}^2K(\psi_*^{(j)})$ and $\mathrm{D}^2\mathcal{L}(\theta_*^{(j)}) = \mathcal{A}^\top \mathrm{D}^2K(\psi_*^{(j)})\mathcal{A}$ have equal determinants, and, hence, by Lemma 6,

$$|D| = B^2\eta(E^{(\pm)} + \mathcal{O}(\bar\eta)) \sim E^{(\pm)}\mathcal{L}_{s_0(n)}(\mathcal{L}_{s_0(n-1)} + \mathcal{L}_{s_0(n+1)}) \sin E^{(\pm)}\mathcal{L}_{S_1}\mathcal{L}_{S_2},$$

where we have taken into account the definitions (4.3) and the relations (3.8)–(3.9) between the dominant harmonics and the primary resonances. Using the fact that $E^* \leqslant E^{(\pm)} \preceq 1$, we get a lower and an upper bound for $|D|$.

On the other hand, for the components of $\mathrm{D}^2K(\psi_*^{(j)}) = \begin{pmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{pmatrix}$, given in first approximation by derivatives of (4.2), we have $|k_{22}| \sim B(1 + \mathcal{O}(\bar\eta))$ as the main entry, and $|k_{11}|, |k_{12}| \preceq B\bar\eta$. By the linear change (4.1) the trace of $\mathrm{D}^2\mathcal{L}(\theta_*^{(j)})$ is given by

$$T = k_{11}\langle s_0(n-1), s_0(n-1)\rangle + 2k_{12}\langle s_0(n-1), s_0(n)\rangle + k_{22}\langle s_0(n), s_0(n)\rangle.$$

Then, applying (3.10) and the estimates of Lemma 1(a), we obtain

$$|T| \sim \frac{1}{\sqrt{\varepsilon}}\,\mathcal{L}_{S_1}.$$

Now we have an estimate for the quotient $|D|\,/\,|T|$, which gives us the desired estimate for the minimal eigenvalue. □

Proof of Theorem 1. Finally, we can complete the proof of our main result. As explained in Section 3, to establish the transversality for all sufficiently small ε, it is enough to consider a neighborhood of the transition values $\widehat{\varepsilon}_n$, since for other values of ε it is enough to consider 2 dominant harmonics, and the results of [5] apply.

For ε close to a transition value $\widehat{\varepsilon}_n$, recalling that $\mathcal{M}(\theta) = \nabla\mathcal{L}(\theta)$, it follows from Lemma 7 that, under (4.7), the splitting function $\mathcal{M}(\theta)$ has 4 simple zeros θ_*, given in (4.16). Likewise, by Lemma 5 the condition (4.7) is fulfilled if

$$|\sigma_{s_0(n+1)} - 2\sigma_{s_0(n)} - \sigma_{s_0(n-1)}| \approx |\triangle\tau| < \frac{2\pi}{3}, \qquad \forall n \geqslant 1 \tag{4.17}$$

(we have taken into account the bound on the difference of phases σ_k and τ_k given in Lemma 1(a)). The particular case of a reversible perturbation (1.9) corresponds to (1.14) with $\sigma_k = 0$ for every k, and hence condition (4.17) on the phases is clearly fulfilled. Moreover, we have $E^{(\pm)} = 1 \pm \widetilde{Q} \sim 1$ in (4.6), and hence $E^* = 1 - \widetilde{Q} \geqslant 1/2$, which implies that $1/2 \leqslant E^* \leqslant 1$. By Lemma 7, for the minimal eigenvalue of the splitting matrix $\mathrm{D}\mathcal{M}(\theta_*)$ at each zero we can write $m_* \sim \sqrt{\varepsilon}\,\mathcal{L}_{S_2}$. This estimate, together with the estimate on $\mathcal{L}_{S_2}$ given by Lemma 1, implies part (b).

As for part (a), the maximal splitting distance is given by the most dominant harmonic

$$\max_{\theta\in\mathbb{T}^2} |\mathcal{M}(\theta)| \sim |\mathcal{M}_{S_1}| \sim \mu\,|S_1|\,L_{S_1}$$

(see, for instance, [7]), and the corresponding estimate of Lemma 1 implies the desired estimate. □

Remark. For the sake of simplicity, we have restricted the statement of Theorem 1 to the case of a reversible perturbation given by (1.9) with the phases $\sigma_k = 0$. Nevertheless, our results apply to a much more general perturbation (1.14), provided the phases $\sigma_{s_0(n)}$, associated to the primary resonances, satisfy inequality (4.17).

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