

ON STAR-FOREST ASCENDING SUBGRAPH DECOMPOSITION

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ABSTRACT. The Ascending Subgraph Decomposition (ASD) Conjecture asserts that every graph G with $\binom{n+1}{2}$ edges admits an edge decomposition $G = H_1 \oplus \cdots \oplus H_n$ such that H_i has i edges and it is isomorphic to a subgraph of H_{i+1} , $i = 1, \dots, n - 1$. We show that every bipartite graph G with $\binom{n+1}{2}$ edges such that the degree sequence d_1, \dots, d_k of one of the stable sets satisfies $d_i \geq n - i + 1$, $1 \leq i < k$, admits an ascending subgraph decomposition with star forests. We also give a necessary condition on the degree sequence which is not far from the above sufficient one.

1. INTRODUCTION

A graph G with $\binom{n+1}{2}$ edges has an Ascending Subgraph Decomposition (ASD) if it admits an edge-decomposition $G = G_1 \oplus \cdots \oplus G_n$ such that G_i has i edges and is isomorphic to a subgraph of G_{i+1} , $1 \leq i < n$. Throughout this paper we use the symbol \oplus to denote edge-disjoint union of graphs. It was conjectured by Alavi, Boals, Chartrand, Erdős and Oellerman [1] that every graph of size $\binom{n+1}{2}$ admits an ASD. The conjecture has been proved for a number of particular cases, including forests [5], regular graphs [9], complete multipartite graphs [8] or graphs with maximum degree $\Delta \leq n/2$ [6].

In the same paper Alavi et al. [1] conjectured that every star forest of size $\binom{n+1}{2}$ in which each connected component has size at least n admits an ASD in which every graph in the decomposition is a star. This conjecture was proved by Ma, Zhou and Zhou [13], and the condition was later on weakened to the effect that the second smaller component of the star forest has size at least n by Chen, Fu, Wang and Zhou [4].

The class of bipartite graphs which admit a star-forest-ASD is clearly larger than the one which admit star-ASD, see Figure 1 for a simple example. This motivates the study of star-forest-ASD for bipartite graphs in terms of the degree sequence of one of the stable sets, which is the purpose of this paper.

Faudree, Gyárfás and Schelp [5] proved that every forest of stars admits a star-forest-ASD. These authors mention, in the same paper, that “Surprisingly [this result], is the most difficult to prove. This could indicate that the conjecture (if true) is a difficult one to prove”. In the same paper the authors propose the following question: Let G be a graph

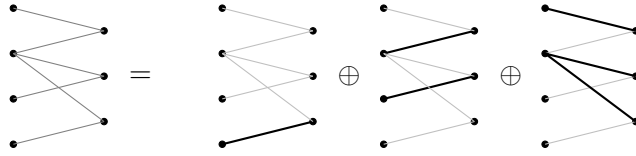


FIGURE 1. Bipartite graph with Star–Forest–ASD but with no Star–ASD.

with $\binom{n+1}{2}$ edges. Does G have an ASD such that each member is a star forest? In this paper we address this question for bipartite graphs when the centres of the stars in the star–forest–ASD belong to the same stable set. Our main result is the following one.

Theorem 1. *Let G be a bipartite graph with $\binom{n+1}{2}$ edges. Let $d_1 \leq d_2 \leq \dots \leq d_k$ be the degree sequence of one of the stable sets of G . If*

$$d_{k-i} \geq n - i \text{ for each } 0 \leq i \leq k - 1,$$

then, there is a star–forest–ASD of G .

The proof of Theorem 1 is made in two steps. First we prove the result for a class of bipartite graphs which we call reduced graphs. For this we use a representation of a star–forest decomposition by the so–called ascending matrices and certain multigraphs and reduce the problem to the existence of a particular edge–coloring of these multigraphs. The terminology and the proof for reduced graphs is contained in Section 2. In Section 3 we present an extension lemma, which uses a result of Häggkvist [10] on list edge–colorings, which allows one to extend the decomposition from reduced to all bipartite graphs with the same degree sequence on one of the stable sets, completing the proof of Theorem 1. The final section contains some concluding remarks.

The sufficient condition on the degrees given in Theorem 1 is not far from being necessary.

Lemma 1. *If every bipartite graph G with degree sequence $d_1 \leq d_2 \leq \dots \leq d_k$ of one stable sets X of G admits a star–forest decomposition (ascending or not) with the centers of the stars in X then*

$$\sum_{i=0}^t d_{k-i} \geq \sum_{i=0}^t (n - i), \quad t = 0, \dots, k - 1.$$

2. STAR-FOREST-ASD FOR REDUCED BIPARTITE GRAPHS

Let $G = G(X, Y)$ be a bipartite graph with $\binom{n+1}{2}$ edges. We denote by $d = (d_1 \leq \dots \leq d_k)$ the degree sequence of the vertices in the stable set X of G . With some abuse of notation,

we call d the *degree sequence* of G . We focus on star forest ASD with the stars of the decomposition centered at the vertices in X .

We first introduce some definitions.

We denote by G_R the bipartite graph with stable sets X and $Y' = \{y_1, \dots, y_{d_k}\}$ where $x_i \in X$ is adjacent to the vertices y_1, \dots, y_{d_i} , $i = 1, \dots, k$, and call G_R the *reduced graph* of G , see Figure 2 for an example.

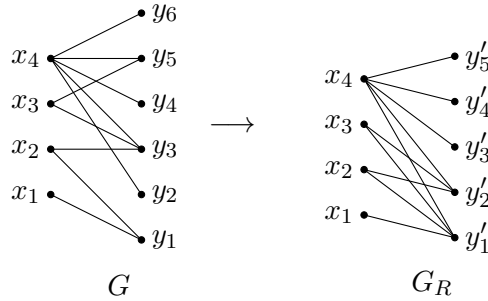


FIGURE 2. A bipartite graph and its reduced graph.

Given two k -dimensional vectors $c = (c_1, \dots, c_k)$ and $c' = (c'_1, \dots, c'_k)$, we say that $c \preceq c'$ if after reordering the components of each vector in nonincreasing order, the i -th component of c is not larger than the i -th component of c' . This definition is motivated by the following remark.

Remark 1. *Let F, F' be two forests of stars with centers x_1, \dots, x_k and x'_1, \dots, x'_k respectively. Then F is isomorphic to a subgraph of F' if and only if $(d_F(x_1), \dots, d_F(x_k)) \preceq (d_{F'}(x'_1), \dots, d_{F'}(x'_k))$.*

Given two sequences $d = (d_1 \leq \dots \leq d_k)$ and $b = (b_1 \leq \dots \leq b_n)$ of nonnegative integers with $\sum_i a_i = \sum_j b_j = b_j$, we say that a $k \times n$ -matrix A with nonnegative integer entries is a (d, b) -matrix if the row sums satisfy $\sum_j a_{ij} = d_i$, $i = 1, \dots, k$ and the column sums satisfy $\sum_i a_{ij} = b_j$, $j = 1, \dots, n$.

We say that A is (d, b) -*ascending* if in addition to the rowsum being the sequence d and the columnsum the sequence b , we have

$$A_1 \preceq A_2 \preceq \dots \preceq A_n,$$

where A_j denotes the j -th column of A .

We will use appropriate ascending matrices to define multigraphs which will lead to starforest-ASD as stated in Proposition 1 below. We recall that the bipartite adjacency matrix of

a bipartite multigraph H with color classes $X = \{x_1, \dots, x_k\}$ and $Z = \{z_1, \dots, z_n\}$ is the $(k \times n)$ matrix A where a_{ij} is the number of edges joining x_i with z_j .

A bipartite multigraph H with degree sequence $d_X = (d_1 \geq \dots \geq d_k)$ on the color class X has a *sequential coloring* for X if there is a proper edge coloring of H such that the edges incident with vertex x_i receive colors $\{1, \dots, d_i\}$ for each i .

Proposition 1. *A reduced bipartite graph $G = (X \cup Y, E)$ with degree-sequence $d_X = (d_1 \leq \dots \leq d_k)$ has a star-forest-ASD with centers of stars in X if and only if there is a (d_X, n^-) -ascending matrix A such that the bipartite multigraph with bipartite incidence matrix A admits a sequential coloring.*

Proof. Let $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_n\}$. Suppose that A is a (d_X, n^-) ascending matrix and that the multigraph $H = (X \cup Z, E)$ with bipartite adjacency matrix A has a sequential coloring.

Denote by f_1, \dots, f_n the vertices of the stable set Z of H , where vertex f_i has degree i (the sum of entries of column i of A), $1 \leq i \leq n$. Let $c : E \rightarrow [1, d_k]$ be a sequential coloring of H , so that the edges incident to x_i receive colors $\{1, \dots, d(x_i)\}$. Each f_j will be associated to the subgraph F_j of G defined as follows. For each edge $x_i f_j$ of color l we declare the edge $x_i y_l$ to be in F_j . This way we obtain a subgraph of G because $l \leq d(x_i)$ (the coloring is good) and the graph G is reduced. Moreover, since the coloring is proper, the degree of every vertex y_l in F_j is at most one. Hence F_j is a forest of stars and it has j edges. Moreover, for $j \neq j'$, the subgraphs F_j and $F_{j'}$ are edge-disjoint again from the fact that the coloring is proper. hence

$$G = F_1 \oplus \dots \oplus F_n$$

is a starforest-ASD for G .

Reciprocally, from a star-forest-ASD $G = F_1 \oplus \dots \oplus F_n$ we can define the multigraph $H = (X \cup Z, E)$ with $Z = \{f_1, \dots, f_n\}$ where we place $d_{F_j}(x_i)$ parallel edges joining x_i with f_j . The bipartite adjacency matrix A of H is a (d_X, n^-) -matrix and, since F_j is isomorphic to a subgraph of F_{j+1} , the matrix is ascending. Now consider the coloring of H obtained by giving to the $d_{F_j}(x_i)$ parallel edges joining x_i with f_j all the colors of the (subscripts of) vertices adjacent to x_i in F_j . Since we use a star-forest decomposition, the edge-coloring is proper. Moreover, since the graph G is reduced, every vertex x_i receives colors in $\{1, \dots, d(x_i)\}$ and we have a sequential coloring. \square

Figure 2 illustrates the statement of Proposition 1.

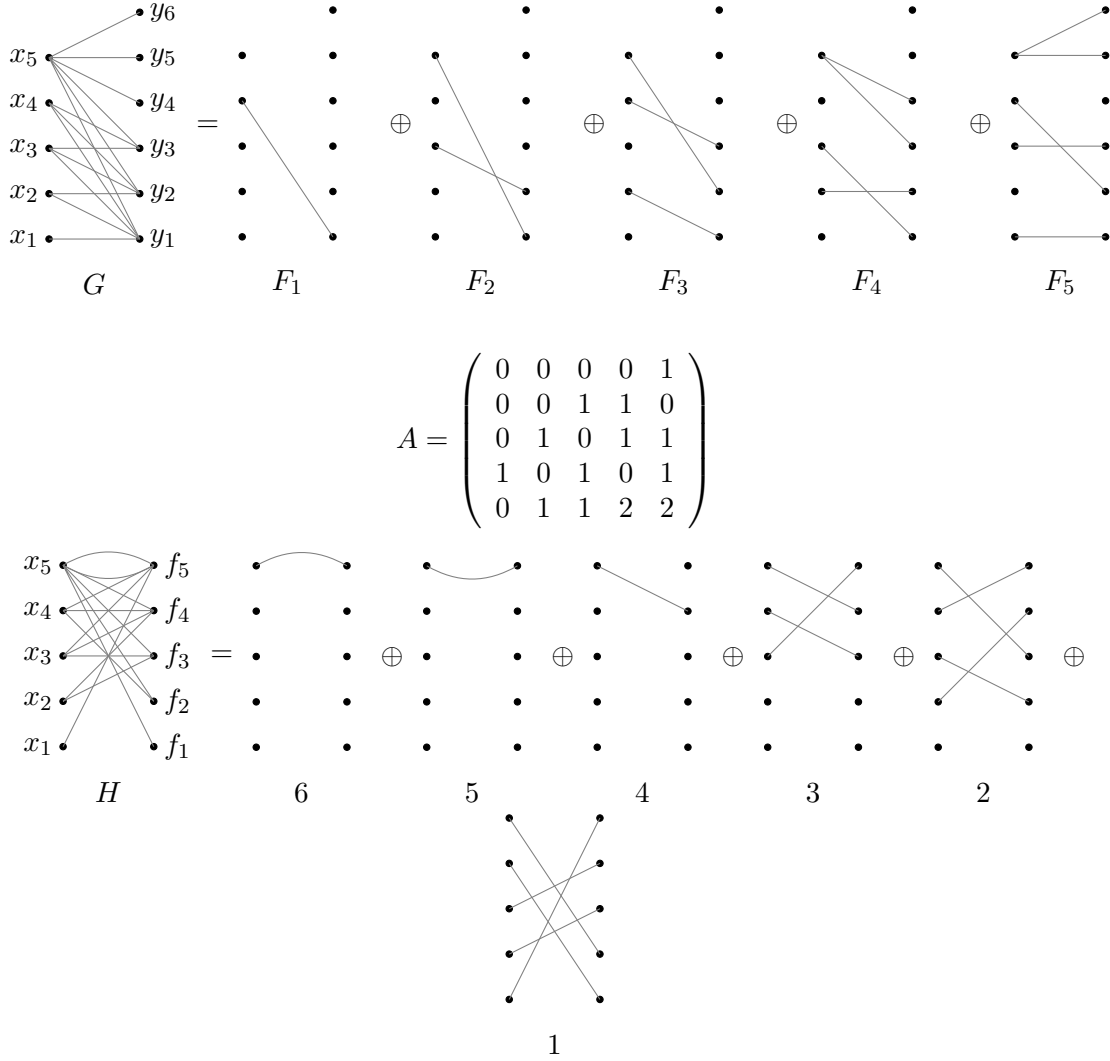


FIGURE 3. Illustration of Proposition 1 with the degree sequence $d_X = (6, 3, 3, 2, 1)$ and $n = 5$.

In order to prove the main result for reduced graphs we will show the existence of an appropriate ascending matrix and that the multigraph associated to it admits a sequential coloring.

We note that, if $b_1 \geq \dots \geq b_n$, then an (a, b) -matrix with $(0, 1)$ entries is ascending. The support of a matrix B is the set of positions with nonzero entries. We observe that if B is (a, b) -ascending, T is (a', b') -ascending and B and T have disjoint support then $B + T$ is

$(a + a', b + b')$ -ascending. The last sentence also holds if T has constant entries instead of having disjoint support from B .

For convenience we use the following notation for sequences. The constant sequence with r entries equal to x is denoted by x^r and (x^r, y^s) denotes the concatenation of x^r and y^s . Also, for an integer x we denote by x^- the ascending sequence $x^- = (1, 2, \dots, x - 1, x)$. Sums and differences of sequences of the same length are understood to be componentwise.

According to Remark 1, the above mentioned result [5] on the existence of a star-forest-ASD of every star forest with $\binom{n+1}{2}$ edges can be reformulated as the existence of a (d, n^-) -ascending matrix for every sequence d with $\sum_i d_i = \binom{n+1}{2}$:

Lemma 2. *For every sequence $d = (d_1 \leq \dots \leq d_k)$ there is a (d, n^-) -ascending matrix C .*

We next show that there exists a d -ascending matrix of a particular shape that will be useful to prove the existence of star-forest-ASD for reduced graphs with that degree sequence.

Lemma 3. *Let $d = (d_1 \leq \dots \leq d_k)$ be a sequence satisfying $\sum d_i = \binom{n+1}{2}$ and*

$$d_{k-i} \geq n - i, \quad i = 0, \dots, k - 1.$$

There is a (d, n^-) -ascending matrix A such that $a_{ij} \geq 1$ for each (i, j) with $i + j \geq k + 1$.

Proof. Consider the $(k \times n)$ matrix T with $t_{ij} = 1$ for $i + j \geq k + 1$ and $t_{ij} = 0$ otherwise. Let $d'_k = d_k - n, d'_{k-1} = d_{k-1} - (n - 1), \dots, d'_1 = d_1 - (n - k)$. Since $\sum_i d'_i = (n - k) + (n - k - 1) + \dots + 2 + 1$, by Lemma 3 there is a $(d', (n - k)^-)$ -ascending $(k \times (n - k))$ matrix for $d' = (d'_1, \dots, d'_k)$. Extend this matrix to a $k \times n$ matrix A' by adding zero columns to the right. Since the last $(n - k)$ columns of T are the all-ones vectors, the matrix $A = A' + T$ still has the ascending column property and, by construction, it is a (d, n^-) -ascending matrix with nonzero entries in the positions (i, j) with $i + j \geq k + 1$. \square

Next Lemma gives a sufficient condition for a degree sequence of a reduced graph to admit a star-forest-ASD.

Lemma 4. *Let $d = (d_1 \leq \dots \leq d_k)$ be a sequence of positive integers with $\sum_i d_i = \binom{n+1}{2}$. If*

$$d_{k-i} \geq n - i, \quad i = 0, 1, \dots, k - 1,$$

then the reduced graph with degree sequence d admits a star-forest ASD.

Proof. Let A be a (d, n^-) -ascending matrix such that $a_{ij} \geq 1$ for each (i, j) with $i + j \geq k + 1$, whose existence is ensured by Lemma ??.

Let H be the bipartite multigraph with stable sets $X = \{x_1, \dots, x_k\}$ and $Z = \{z_1, \dots, z_n\}$ whose bipartite adjacency matrix is A . We next show that H admits a sequential coloring. The result will follow by Proposition 1.

Let $\alpha_i = d_i - (n - k)$, $1 \leq i \leq k$. For each $i = 1, \dots, k$ denote by M'_j the matching in H formed by the k edges

$$M'_j = \{x_r z_s : r + s \equiv j \pmod{n}\}.$$

Such matchings exist in H by the condition $a_{ij} \geq 1$ for each pair (i, j) with $i + j \leq k + 1$, and they are pairwise edge-disjoint. For each $j = 0, 1, \dots, k - 1$, let $M_j \subset M'_j$ be obtained by selecting from M'_j the edges incident to x_i whenever $\alpha_i \geq j$. In this way each x_i is incident with the matchings $M_1, \dots, M_{t(i)}$ with $t(i) = \min\{k, \alpha_i\}$ and has degree at least $n - k$ in $M_1 \oplus \dots \oplus M_k$. On the other hand, by the condition on the degree sequence d_X , since $\alpha_i \geq k - i + 1$, the vertex z_{n-i} is incident to $k - i$ edges in $M_1 \oplus \dots \oplus M_k$.

Let H' denote the bipartite multigraph obtained from H by removing the edges in $M_1 \oplus \dots \oplus M_k$. Let $d'_X = (d'_1 \leq \dots \leq d'_k)$ be the degree sequence of X in H' , where $d'_i = d_i - t(i) \geq n - k$. Moreover, each vertex z_i , $1 \leq i \leq n$ has degree at most $n - k$ in H' .

Let $\Delta'(X)$ be the maximum degree in H' of the vertices in X . If $\Delta'(X) > n - k$ then there is a matching $M'_{\Delta'(X)}$ in H' from the vertices of maximum degree in X to Z . Color the edges of this matching with $\Delta'(X)$. By removing this matching from H' we obtain a bipartite multigraph in which the maximum degree of vertices in X is $\Delta'(X) - 1$. By iterating this process we eventually reach a bipartite multigraph H'' in which every vertex in X has degree $n - k$ while the maximum degree of the vertices in Z still satisfies $\Delta''(Z) \leq n - k$. By König's theorem, the edge-chromatic number of H'' is $n - k$. By combining an edge-coloring of H'' with $n - k$ colors with the ones obtained in the process of reducing the maximum degree of H_1 , the multigraph H' can be properly edge-colored in such a way that vertex x_i is incident in H' with colors $\{1, \dots, d'_i\}$, $1 \leq i \leq k$. We finally obtain a sequential coloring of H by adding to H' the matchings M_1, \dots, M_k , where M_j gets color $\Delta'(X) + (k - j)$: the largest color of an edge incident to x_i is $d_{H'}(x_i) + t(i) = d_i$. By Proposition 1, G has a starforest-ASD. \square

The next result is a reformulation of Lemma 1 which shows that, if a reduced bipartite graph can be decomposed into n star forests with sizes $1, 2, \dots, n$, regardless of the fact that it is ascending, then the degree sequence of the graph satisfies a condition which is necessary for the existence of a star-forest-ASD.

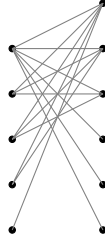


FIGURE 4

Lemma 5. *Let G_R be a reduced bipartite graph with degree sequence $(d_1 \geq \dots \geq d_k)$. If G_R admits an edge-decomposition into stars forests*

$$G = F_n \oplus F_{n-1} \oplus \dots \oplus F_1,$$

where F_i has i edges, then

$$\sum_{i=1}^t d_i \geq \sum_{i=1}^t (n - i + 1) \text{ for each } t = 1, \dots, k. \quad (1)$$

Proof. Let $A = \{a_1, \dots, a_k\}$, $B = \{b_1, \dots, b_{d_1}\}$ be the bipartition of G_R where a_i is adjacent to b_1, \dots, b_{d_i} for each i . Since the graph is reduced, the last $d_1 - d_2$ vertices of B have degree one, the preceding $d_2 - d_3$ have degree 2 and, in general, the consecutive $d_{j-1} - d_j$ vertices $b_{d_{j-1}}, \dots, b_{d_j-1}$ have degree $j - 1$.

Since F_n has n leaves in B we clearly have $|B| = d_1 \geq n$. Thus (1) is satisfied for $t = 1$.

Assume that (1) is satisfied for some $t = j - 1 < k$. Consider the subgraph G_j of G_R induced by $F_n \oplus F_{n-1} \oplus \dots \oplus F_{n-j-1}$. Since each every vertex in B has degree at most one in each forest, it has degree at most j in G_j . By combining this remark with the former upper bound on the degrees of vertices in B we have

$$\sum_{i=1}^j (n - i + 1) = \sum_i d_{G_j}(b_i) \leq jd_j + (j - 1)(d_{j-1} - d_j) + \dots + 2(d_2 - d_3) + (d_1 - d_2) = \sum_{i=1}^j d_i,$$

and (1) is satisfied for $t = j$. This concludes the proof. \square

3. AN EXTENSION LEMMA

In this section we prove an extension Lemma which shows that, if G_R admits a star-forest-ASD then so does G . This reduces the problem of giving sufficient conditions on the degree sequence of one stable set to ensure the existence of a star-forest-ASD to bipartite reduced

graphs. For the proof of our extension Lemma we use the following result by Häggkvist [10] on edge list-colorings of bipartite multigraphs.

Theorem 2. [10] *Let H be a bipartite multigraph with stable sets A and B . If H admits a sequential coloring, then H can be properly edge-colored for an arbitrary assignment of lists $\{L(a) : a \in A\}$ such that $|L(a)| = d(a)$ for each $a \in A$.*

Lemma 6 (Extension Lemma). *Let G be a bipartite graph with bipartition $A = \{a_1, \dots, a_k\}$ and B and degree sequence $d = (d_1 \geq \dots \geq d_k)$, $d = d(a_i)$, of the vertices in A . If the reduced graph G_R of G admits a decomposition*

$$G_R = F'_1 \oplus \dots \oplus F'_t,$$

where each F'_i is a star forest, then G has an edge decomposition

$$G = F_1 \oplus \dots \oplus F_t$$

where $F_i \cong F'_i$ for each $i = 1, \dots, t$.

Proof. Let C be the $(k \times t)$ matrix whose entry c_{ij} is the number of edges incident to a_i in the star forest F'_j of the edge decomposition of G_R .

Consider the bipartite multigraph H with stable sets A and $U = \{u_1, \dots, u_t\}$, where a_i is joined with u_j with c_{ij} parallel edges. Now, for each pair (i, j) , color the c_{ij} parallel edges of H with the neighbors of a_i in the forest F'_j bijectively. Note that in this way we get a proper edge-coloring of H : two edges incident with a vertex a_i receive different colors since the bipartite graph G_R has no multiple edges, and two edges incident to a vertex u_j receive different colors since F'_j is a star forest.

By the definition of the bipartite graph G_R , each vertex $a_i \in A$ is incident in the bipartite multigraph H with edges colored $1, 2, \dots, d_i$. Let $L(a_i)$ be the list of neighbours of a_i in the original bipartite graph G . By Theorem ??, there is a proper edge-coloring χ' of H in which the edges incident to vertex a_i in A receive the colors from the list $L(a_i)$ for each $i = 1, \dots, k$. Now construct F_s by letting the edge $a_i b_j$ be in F_s whenever the edge $a_i u_s$ is colored b_j in the latter edge-coloring of H . Thus F_s has the same number of edges than F'_s and the degree of a_i in F_s is c_{is} , the same as in F'_s . Moreover, since the coloring is proper, F'_s is a star forest. This concludes the proof. \square

We are now ready to prove our main result.

Proof of Theorem 1: Let G be a bipartite graph with degree sequence $d = (d_1 \geq \dots \geq d_k)$ satisfying $\sum_i d_i = \binom{n+1}{2}$ and $d_i \geq n - i + 1$, $i = 1, \dots, k$. By Lemma 3 there is a d -ascending

matrix C such that $c_{ij} \geq 1$ for each (i, j) with $i + j \leq k + 1$. By Lemma 4 there is a star-forest-ASD of the reduced graph G_R with degree sequence d . By Lemma 6 there is also a star-forest-ASD for G . \square

4. CONCLUDING REMARKS

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