

# DECOMPOSING ALMOST COMPLETE GRAPHS BY RANDOM TREES

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ABSTRACT. An old conjecture of Ringel states that every tree with  $m$  edges decomposes the complete graph  $K_{2m+1}$ . The best lower bound for the order of a complete graph decomposed by a given tree with  $m$  edge is  $O(m^3)$ . We show that asymptotically almost surely a random tree with  $m$  edges and  $p = 2m + 1$  a prime decomposes  $K_{2m+1}(r)$  for every  $r \geq 2$ , the graph obtained from the complete graph  $K_{2m+1}$  by replacing each vertex by a coclique of order  $r$ . As a consequence of the main result we obtain approximations to Ringel's conjecture for random trees of almost complete graphs of linear order with the size of the tree.

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## 1. INTRODUCTION

Given two graphs  $H$  and  $G$  we say that  $H$  decomposes  $G$  if  $G$  is the edge-disjoint union of isomorphic copies of  $H$ . The following is a well-known conjecture of Ringel.

**Conjecture 1** (Ringel [13]). *Every tree with  $m$  edges decomposes the complete graph  $K_{2m+1}$ .*

The conjecture has been verified by a number of particular classes of trees, see the dynamic survey of Gallian [8]. By using the polynomial method, the conjecture was verified by Kézdy [10] for the more general class of so-called *stunted* trees. As mentioned by the author, this class is still small among the set of all trees.

Robinson and Schwenk [14] proved that the average number of leaves in an (unlabelled) random tree with  $m$  edges is asymptotically  $cm$  with  $c \approx 0.438$ . Drmota and Gittenberger [5] showed that the distribution of the number of leaves in a random tree with  $m$  edges is asymptotically normal with variance  $c_2m$  for some positive constant  $c_2$ . Thus, asymptotically almost surely a random tree with  $m$  edges has more than  $2m/5$  leaves. Drmota and the author [7] used further structural results on random trees to show that asymptotically almost surely a random tree with  $m$  edges decomposes the complete bipartite graph  $K_{2m,2m}$ , thus providing an approximate result to another decomposition conjecture by Graham and Haggkvist which asserts that in fact  $K_{m,m}$  can be decomposed by a given tree with  $m$  edges.

Let  $g(m)$  be the smallest integer  $n$  such that any tree with  $m$  edges decomposes the complete graph  $K_n$ . It was shown by Yuster [16] that  $g(m) = O(m^{10})$  and the upper bound was reduced by Kezdy and Snevily [11] to  $g(m) = O(m^3)$ . Since  $K_{2m,2m}$  decomposes the

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complete graph  $K_{8m^2+1}$  (see Snevily [15]), the above mentioned result on the decomposition of  $K_{2m,2m}$  shows that  $g(m) = O(m^2)$  asymptotically almost surely.

In this note we prove that one can decompose almost complete graphs by random trees, getting much closer to the original conjecture by Ringel.

For positive integers  $n, r$  we denote by  $K_n(r)$  the graph obtained from the complete graph  $K_n$  by replacing each vertex by a copy of the null graph  $N_r$  with  $r$  vertices, and joining every pair of vertices which do not belong to the same copy of  $N_r$ . Our main result is the following one.

**Theorem 1.** *For every  $m$  such that  $p = 2m + 1$  is a prime, and every  $r \geq 2$ , asymptotically almost surely a random tree with  $m$  edges decomposes  $K_{2m+1}(r)$ .*  $\square$

As an application of the former result we have the following corollaries, which are an approximate result of Ringel conjecture for random trees.

A direct consequence of Theorem 1 with  $r = 2$  is the following statement.

**Corollary 1.** *For every  $m$  such that  $p = 2m + 1$  is a prime, asymptotically almost surely, a random tree with  $m$  edges decomposes  $K_{4m+2} \setminus M$  where, where  $M$  is a complete matching.*

Next Corollary follows also from Theorem 1 with some additional work. Its proof is given at the end of the last Section.

**Corollary 2.** *For every  $m$  such that  $p = 2m + 1$  is a prime, asymptotically almost surely, a random tree with  $m + 1$  edges decomposes  $K_{6m+5} \setminus e$ , where  $e$  is an edge.*

By using similar techniques as the ones involved in the proofs of the above results one can show that, for each odd  $r \geq 3$ , almost all trees with  $m + (r - 1)/2$  edges decompose  $K_{r(2m+1)} \setminus K_{(r+1)/2}$ . This extension of Corollary 2 can be seen as an approximation to a more general conjecture by Ringel which states that every tree with  $m$  edges decomposes the complete graph  $K_{rm+1}$  whenever  $r$  and  $m$  are not both odd.

The paper is organised as follows. In section 2 we introduce the notion of rainbow embeddings in connection to graph decompositions and give some results which ensure that a tree can be rainbowly embedded in an appropriate Cayley graph. The embedding techniques use the polynomial method of Alon and bring the condition of primality in the statement of Theorem 1. Moreover the techniques work only with trees with sufficiently many leaves, which explain the use of random trees. In section 3 we describe the decomposition and provide the proof of Theorem 1 and Corollary 2.

## 2. RAINBOW EMBEDDINGS

The general approach to show that a tree  $T$  decomposes a complete graph or a complete bipartite graph consists in showing that  $T$  cyclically decompose the corresponding graphs. We next recall the basic principle behind this approach in slightly different terminology.

A rainbow embedding of a graph  $H$  into an oriented arc-colored graph  $X$  is an injective homomorphism  $f$  of some orientation  $\vec{H}$  of  $H$  in  $X$  such that no two arcs of  $f(\vec{H})$  have

the same color. According to its common use, even if a rainbow embedding is meant to be defined as a map  $f : V(H) \rightarrow V(X)$ , we still call  $f$  the induced map  $f : E(\vec{H}) \rightarrow E(X)$  on arcs defined as  $f(x, y) = (f(x), f(y))$ , and we think of  $f$  as a map  $f : \vec{H} \rightarrow X$ .

Let  $X = \text{Cay}(G, S)$  be a Cayley digraph of an abelian group  $G$  with respect to an antisymmetric subset  $S \subset G$  (that is,  $S \cap -S = \emptyset$ ). We consider  $X$  as an arc-colored oriented graph, by giving to each arc  $(x, x + s)$ ,  $x \in G, s \in S$ , the color  $s$ .

**Lemma 1.** *Suppose that the graph  $H$  admits a rainbow embedding  $f$  in  $X = \text{Cay}(G, S)$ , where  $S$  is an antisymmetric subset of  $G$ . Then  $T$  decomposes the underlying graph of  $X$ .*

**Proof.** For each  $a \in G$  the translation  $x \rightarrow x + a$ ,  $x \in G$ , is an automorphism of  $X$  which preserves the colors and has no fixed points. Therefore, each translation sends  $f(\vec{H})$  to an isomorphic copy which is edge disjoint from it. Thus the sets of translations for all  $a \in G$  give rise to  $n := |G|$  edge-disjoint copies of  $\vec{H}$  in  $X$ . By ignoring orientations and colors, we thus have  $n$  edge disjoint copies of  $H$  in the underlying graph of  $X$ .  $\square$

The proof of the main Theorem uses the above Lemma for a rainbow subgraph of an appropriate Cayley graph  $X$ . Instead of finding a rainbow embedding of the tree we will find a rainbow edge-injective homomorphism of  $T$  in  $X$  in two steps, first embedding the tree with some leaves removed and then embedding the remaining forest of stars to complete  $T$ . For the first step we use the so-called Combinatorial Nullstellensatz of Alon [1] that we next recall.

**Theorem 2** (Combinatorial Nullstellensatz). *Let  $F$  be a field and  $P \in F[x_1, \dots, x_k]$  a polynomial with  $k$  variables with coefficients in  $F$  with degree  $d$ .*

*If the coefficient of the monomial  $x_1^{d_1} \dots x_k^{d_k}$  is nonzero, where  $\sum_i d_i = d$ , then  $P$  takes a nonzero value in every grid  $A_1 \times \dots \times A_k$  with  $|A_1| > d_1, \dots, |A_k| > d_k$ .  $\square$*

A *peeling ordering* of a tree  $T$  is an ordering  $x_0, \dots, x_m$  of its vertices such that the subgraph  $T[x_0, \dots, x_t]$  induced by every initial segment is a subtree of  $T$ . We assume that  $T$  is a directed tree with all its edges oriented from the root  $x_0$  of a peeling ordering. In the following lemma we use Theorem 2 in a similar way as it was used by Kézdy [10].

**Lemma 2.** *Let  $p$  be a prime and let  $T$  be a tree with  $m < 3(p - 1)/10$  edges. There is an antisymmetric set  $S \subset \mathbb{Z}_p^*$  with  $|S| = m$  such that the tree  $T$  admits a rainbow embedding in  $\text{Cay}(\mathbb{Z}_p, S)$ .*

**Proof.** Let  $x_0, x_1, \dots, x_m$  be a peeling ordering of  $T$ . Let  $y_1, \dots, y_m$  be a labeling of the edges such that, for each  $i$ , the edge  $y_i$  joins  $x_i$  with  $T[x_0, x_1, \dots, x_{i-1}]$ . For each  $i$  we denote by  $T(0, i)$  the set of edges  $y_j$  which lie in the unique path from  $x_0$  to  $x_i$  in  $T$ . Consider the polynomial

$$P(y_1, \dots, y_m) = \prod_{1 \leq i < j \leq m} (y_i^2 - y_j^2) \prod_{1 \leq i < j \leq m} \left( \sum_{y_r \in T(0, i)} y_r - \sum_{y_r \in T(0, j)} y_r \right).$$

We note that, if  $P$  does not vanish in  $(a_1, \dots, a_m)$  with no entry zero then, since the first factor  $Q = \prod_{i < j} (y_i^2 - y_j^2)$  of  $P$  is nonzero at  $(a_1, \dots, a_m)$ , the set  $S = \{a_1, \dots, a_m\}$  consists of pairwise distinct elements and it is antisymmetric.

Moreover, since the second factor  $R = \prod_{i < j} (\sum_{y_r \in T(0,i)} y_r - \sum_{y_r \in T(0,j)} y_r)$  is nonzero, the map  $f : V(T) \rightarrow \text{Cay}(\mathbb{Z}_p, S)$  defined as  $f(x_i) = \sum_{a_r \in T(0,i)} a_r$  is injective and provides a rainbow embedding of  $T$  in  $\text{Cay}(\mathbb{Z}_p, S)$ .

Let us show that  $P$  is nonzero at some point of  $(\mathbb{Z}_p^*)^k$ . To this end we consider the monomial

$$y_m^{3(m-1)} y_{m-1}^{3(m-2)} \dots y_1^0.$$

This monomial can be obtained by collecting  $y_m$  in all the summands of  $P$  where it appears, giving  $y_m^{2(m-1)}$ , and also in all terms of  $Q$  where it appears, which, since  $y_m$  is a leave of  $T$ , gives  $y_m^{m-1}$ . This is the unique way to obtain  $y_m^{3(m-1)}$  in a monomial of  $P$ . Thus the coefficient of  $y_m^{3(m-1)}$  in  $f$  is

$$[y^{3(m-1)}]P = \pm P_1,$$

where

$$P_1(y_1, \dots, y_{m-1}) = \prod_{1 \leq i < j \leq m-1} (y_i^2 - y_j^2) \prod_{1 \leq i < j \leq m-1} \left( \sum_{y_r \in T(0,i)} y_r - \sum_{y_r \in T(0,j)} y_r \right).$$

By iterating the same argument we conclude that the coefficient of  $y_m^{3(m-1)} y_{m-1}^{3(m-2)} \dots y_1^0$  in  $P$  is  $\pm 1$  and, in particular, different from zero. Since  $3(m-1) < 9p/10 < p-1$  for  $p > 10$ , we conclude from Theorem 2 that  $P$  takes a nonzero value in  $(\mathbb{Z}_p^*)^k$ . This concludes the proof.  $\square$

In the second step we try to embed rainbowly the forest of stars which remains to complete our given tree. It may happen that a rainbow embedding of the forest of stars using the remaining colors of the host graph is not possible. However we still use Theorem 2 or rather the following consequence derived from it by Alon [2].

**Theorem 3.** *Let  $p$  be a prime. For every sequence  $a_1, \dots, a_k$  and every set  $\{b_1, \dots, b_k\}$  there is a permutation  $\sigma \in \text{Sym}(k)$  such that the sums  $a_1 + b_{\sigma(1)}, \dots, a_k + b_{\sigma(k)}$  are pairwise distinct.*  $\square$

One consequence of the above result is that every forest of stars with  $h$  edges almost admits a rainbow embedding in  $\text{Cay}(\mathbb{Z}_p, S)$  for every antisymmetric set  $S$  with  $h$  elements. Moreover, the centers of the stars in the forest can be placed at prescribed vertices. The rainbow map defined with the help of Theorem 3 may fail to be a rainbow embedding of the forest in the fact that some endvertices may be sent to some center of another star. The following is the precise statement.

**Lemma 3.** *Let  $p$  be a prime. Let  $F$  be a directed forest of  $k$  stars centered at  $x_1, \dots, x_k$  and  $m \leq (p-1)/2$  edges, each one directed from the center to its end vertex. Let  $S$  be an antisymmetric subset with  $h$  elements. Every injection  $f : \{x_1, \dots, x_k\} \rightarrow \mathbb{Z}_p$  can be*

extended to a rainbow edge-injective homomorphism of  $F$  in  $\text{Cay}(\mathbb{Z}_p, S)$  in such a way that the image of  $F$  by  $f$  is a directed graph with maximum indegree one.

**Proof.** Consider the sequence  $(f(x_1)^{h_1}, \dots, f(x_k)^{h_k})$ , where the multiplicity  $h_i$  of  $f(x_i)$  is the number of leaves of the star centered at  $x_i$ ,  $\sum_i h_i = h$ . By Theorem 3 there is a numbering  $\{s_1, \dots, s_h\}$  of the elements of  $S$  such that the sums

$$f(x_i) + s_j, \quad 1 \leq i \leq k, \quad h_1 + \dots + h_{i-1} < j \leq h_1 + \dots + h_{i-1} + h_i,$$

are pairwise distinct. If we label the leaves of  $x_i$  by  $y_j$ ,

$$h_1 + \dots + h_{i-1} < j \leq h_1 + \dots + h_{i-1} + h_i$$

we obtain the desired rainbow embedding by defining  $f(y_j) = f(x_i) + s_j$ : since all sums are distinct, no two endvertices of  $F$  are sent to the same vertex by  $f$  and each one has indegree one in  $f(F)$ ; by the same reason, every  $f(x_i)$  can coincide with at most one  $f(y_j)$  for some  $y_j$  not in the same star as  $x_i$ . Thus the image  $f(F)$  has indegree at most one.  $\square$

### 3. THE DECOMPOSITION

In this Section we give the proof of Theorem 1. Since a random tree with  $m$  edges has asymptotically almost surely at least  $2m/5$  leaves, the Theorem follows from Lemma 4 below.

For a directed graph  $G$  and a positive integer  $r$  we denote by  $G(r)$  the directed graph obtained from  $G$  by replacing each vertex with a coclique of order  $r$  and placing an arc from a vertex  $x$  to a vertex  $y$  if there was an arc from the vertex corresponding to the coclique containing  $x$  to the vertex of the coclique containing  $y$  in  $G$ .

**Lemma 4.** *Let  $p$  be a prime and  $r \geq 2$  an integer. Let  $T$  be a tree with  $m$  edges and at least  $2m/5$  leaves. Then  $T$  decomposes  $K_{2m+1}(r)$ .*

**Proof.** Let  $T_0$  be the subtree of  $T$  obtained by removing  $\lceil 2m/5 \rceil$  leaves from  $T$ . By Lemma 2, since  $T_0$  has at most  $3m/5 < 3(p-1)/10 < (p-1)/3$  edges, there is a rainbow embedding  $f_0 : T_0 \rightarrow \text{Cay}(\mathbb{Z}_p, S_0)$  for some antisymmetric set  $S_0 \subset \mathbb{Z}_p$  with  $|S_0| = |E(T_0)|$ . Let  $x_0, \dots, x_t$  be a peeling ordering of  $T_0$ . By exchanging elements of  $S$  by their opposite elements if necessary we may assume that  $f_0(T)$  has all its edges oriented from  $x_0$  to the leaves of  $T_0$ .

Let  $a_1, \dots, a_k$  be the image of the vertices which are adjacent to leaves of  $T$  removed to obtain  $T_0$ . Let  $S \supset S_0$  be an antisymmetric subset of  $\mathbb{Z}_p$  with  $|S| = (p-1)/2$  which contains  $S_0$ . By Lemma 3 there is an edge-injective rainbow homomorphism  $f_1 : F \rightarrow \text{Cay}(\mathbb{Z}_p, S \setminus S_0)$  of the forest  $F = T \setminus T_0$  into  $\text{Cay}(\mathbb{Z}_p, S \setminus S_0)$  in a way that the centers of the stars of  $F$  are mapped to  $a_1, \dots, a_k$ . Moreover  $\tilde{F} = f_1(F)$  is a directed graph with maximum indegree one.

Consider the map  $f : V(T) \rightarrow \text{Cay}(\mathbb{Z}_p, S)$  defined by  $f_0$  on  $V(T_0)$  and by  $f_1$  on  $V(T_1)$ . This map is well defined and  $f(T)$  is a rainbow subgraph, call it  $H$ , of  $\text{Cay}(\mathbb{Z}_p, S)$ . We consider  $H$  as a directed arc-colored subgraph. We note that  $f$  may fail to be a rainbow embedding of  $T$  in  $\text{Cay}(\mathbb{Z}_p, S)$  to the effect that some leaves of  $T$  may have been sent

through  $f_1$  to some points of  $f_0(V(T_0))$ . Thus  $H$  may be not isomorphic to  $T$  and contain some cycles (see Figure 3 for an illustration.) We observe however that if  $f_1(y) = f_0(x)$  for some endvertex  $y$  of  $T$  not in  $T_0$  and some vertex  $x$  of  $T_0$ , then  $y$  is not adjacent to  $x$  in  $T$  because  $f_1$  is an edge–injective homomorphism. In other words,  $f(T)$  has maximum indegree at most two. If there is a vertex with indegree two we call its incoming arcs to be conflicting. We also observe that has  $H$  a decomposition

$$H = T_0 \oplus \tilde{F}.$$

Our last step consists of splitting each vertex of  $X = \text{Cay}(\mathbb{Z}_p, S)$  into a coclique of order  $r$  and distribute the conflicting arcs as to have the resulting graph decomposed into copies of the original tree  $T$ . To this end let us consider the directed Cayley graph

$$Y = \text{Cay}(\mathbb{Z}_p \times \mathbb{Z}_r, S \times \mathbb{Z}_r) = \bigoplus_{i \in \mathbb{Z}_r} \text{Cay}(\mathbb{Z}_p \times \mathbb{Z}_r, S \times \{i\}) = \bigoplus_{i \in \mathbb{Z}_r} \bigoplus_{j \in \mathbb{Z}_r} X_{ij},$$

where each  $X_{i,j}$  is the subgraph of  $Y$  isomorphic to  $X$  formed by the edges colored  $(*, i)$  and which contains the vertex  $(0, j)$ . Denote by  $\pi_{i,j} : X \rightarrow X_{i,j}$  the natural isomorphism which send edges colored  $x \in \mathbb{Z}_p$  of  $X$  to edges colored  $(x, i)$  in  $X_{i,j}$  and let  $f^{(i,j)} : T \rightarrow X_{i,j}$  the natural extension of the rainbow map  $f : T \rightarrow X$  into  $X_{i,j}$ , namely,

$$(1) \quad f^{(i,j)} = f \circ \pi_{i,j}.$$

Denote by

$$H_{i,j} = T_{0,i,j} \oplus \tilde{F}_{i,j} = f^{(i,j)}(T),$$

the corresponding copy of  $H$  in  $Y$ , where  $T_{0,i,j}$  and  $\tilde{F}_{i,j}$  are the corresponding rainbow copies of  $T_0$  and of  $\tilde{F}$  respectively.

We observe that

$$H_0 = H_{0,0} \oplus H_{1,0} \oplus \cdots \oplus H_{r-1,0},$$

is a rainbow subgraph of  $Y$  which contains  $r(p-1)/2$  edges. Therefore the set of translates

$$\{H_0 + (x, j) : (x, j) \in \mathbb{Z}_p \times \mathbb{Z}_r\}$$

is an edge decomposition of  $Y$  by copies of  $H$ .

Let

$$H(r) = \bigoplus_{j \in \mathbb{Z}_r} H_0 + (0, j) = \bigoplus_{j \in \mathbb{Z}_r} \bigoplus_{i \in \mathbb{Z}_r} H_{i,j}.$$

The subgraph  $H(r)$  contains  $r^2$  copies of  $H$ . We will define a modified edge–decomposition

$$H(r) = \bigoplus_{j \in \mathbb{Z}_r} \bigoplus_{i \in \mathbb{Z}_r} H'_{i,j}$$

where now each  $H'_{i,j}$  is isomorphic to the original tree  $T$  (instead of being isomorphic to  $H$ .) Thus the set of translations

$$\{H(r) + (x, 0) : x \in \mathbb{Z}_p\}$$

will result in a decomposition of  $Y$  by copies of  $T$ . By ignoring the directions and the colors in  $Y$  we get an edge–decomposition of  $K_p(r)$  by copies of  $T$  concluding the proof.

Let us now describe the distribution of the arcs of  $H(r)$  within the  $H'_{i,j}$ 's. Recall that  $x_0, x_1, \dots, x_t$  is a peeling order of  $T_0$ . By abuse of notation we still denote by  $x_0, x_1, \dots, x_t$  their images in  $f_0(T_0)$ . Since  $t = |V(T_0)| > |V(F)|$  we may assume that  $x_0$  is not incident

to an endvertex of  $f_1(F)$ . We will construct the  $H'_{i,j}$  iteratively on the initial segments of the peeling order of  $T_0$ . Denote by  $H'_{i,j}[s]$  the directed subgraphs induced by the first  $s$  vertices in the peeling order (these vertices are of the form  $(x_0, *), \dots, (x_s, *) \in \mathbb{Z}_p \times \mathbb{Z}_r$  for some values in their second coordinates.)

Suppose that we have constructed  $H'_{i,j}[s']$  for each  $i, j \in \mathbb{Z}_r$  and some  $s' \geq 0$ . Let  $s > s'$  be the smallest subscript in the peeling order of  $T_0$  for which  $H$  has a cycle. Let  $(x_a, x_s)$  the arc of  $T_0$  which joints  $x_s$  with the preceding tree  $T_0[x_0, \dots, x_{s-1}]$  and let  $(x_b, x_s)$  be the leave of  $F$  which creates the cycle at  $x_s$  (see Figure 3 for an illustration.)

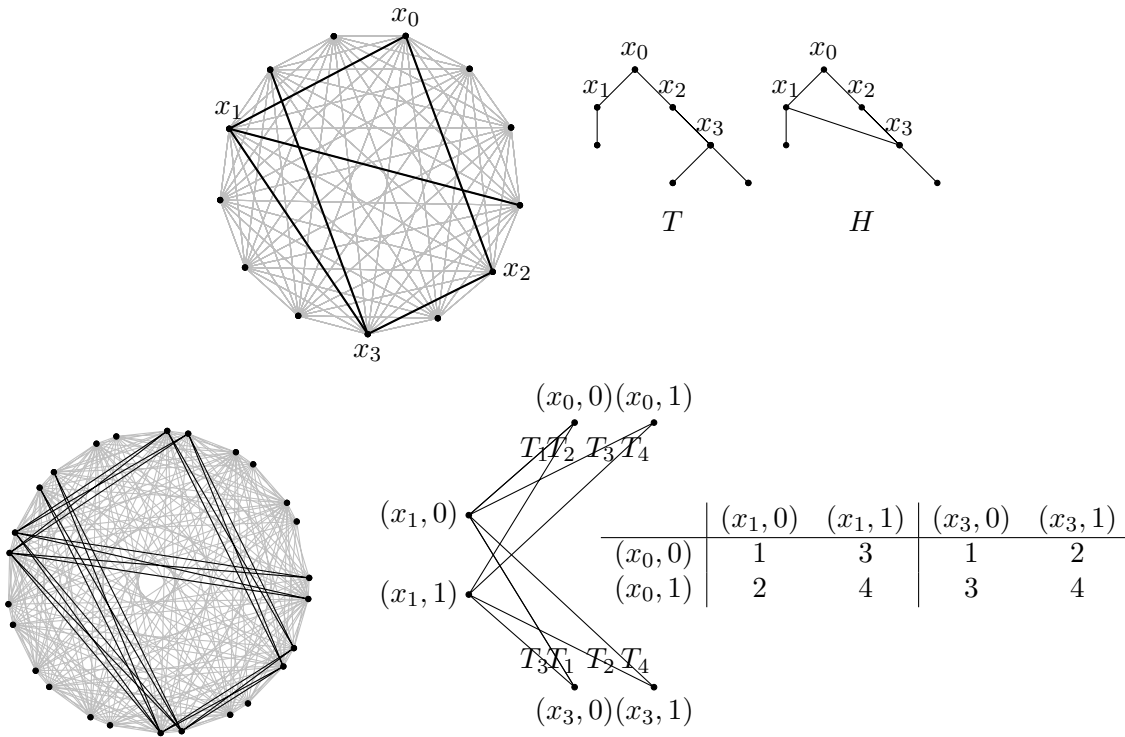


FIGURE 1. An illustration of the rainbow map of  $T$  and the conflicting arcs.

Each of the directed subgraphs of  $Y$  induced by the vertices in  $(x_a \times \mathbb{Z}_r) \cup (x_s \times \mathbb{Z}_r)$  and  $(x_b \times \mathbb{Z}_r) \cup (x_s \times \mathbb{Z}_r)$  is isomorphic to  $\vec{K}_2(r)$ , call them  $K_a$  and  $K_b$  respectively.

Each edge in  $K_a$  belongs to one of  $r^2$  trees isomorphic to  $T_0$  in the decomposition of  $Y$ , and likewise, each of the edges in  $K_b$  belongs to one of the  $r^2$  copies of  $\tilde{F}$ , label them with the numbers  $1, 2, \dots, r^2$  in such a way that the  $i$ -th copies of  $T_0$  and  $\tilde{F}$  form a copy of  $H$ . We construct the  $r \times r$  matrix  $M_a$  by placing at the entry  $i, j$  the number of the copy of  $T_0$  which contains the arc  $((x_a, i), (x_s, j))$ . Likewise the  $r \times r$  matrix  $M_b$  has the number of the copy of  $\tilde{F}$  which contains  $((x_b, i), (x_s, j))$  in the entry  $(i, j)$ . Without loss of generality

we may assume that

$$(M_a, M_b) = \left( \begin{array}{cccc|cccc} 1 & 2 & \cdots & r & \sigma_1 & \sigma_2 & \cdots & \sigma_r \\ r+1 & r+2 & \cdots & 2r & \sigma_{r+1} & \sigma_{r+2} & \cdots & \sigma_{2r} \\ \vdots & & & \vdots & \vdots & & & \vdots \\ r(r-1)+1 & r(r-1)+2 & \cdots & r^2 & \sigma_{r(r-1)+1} & \sigma_{r(r-1)+2} & \cdots & \sigma_{r^2} \end{array} \right),$$

for some permutation  $(\sigma_1, \dots, \sigma_{r^2})$  of  $\{1, \dots, r^2\}$ . If the  $i$ -th row of the matrix  $(M_a, M_b)$  has two repeated entries then the vertex  $(x_s, i)$  is incident to an arc of  $T_0[x_0, \dots, x_s]$  and of  $\tilde{F}$  creating a cycle of the copy of  $H$  corresponding to that entry. On the other hand, if all the rows of  $(M_a, M_b)$  have pairwise distinct entries, then no vertex in  $x_s \times \mathbb{Z}_r$  belongs to a cycle of the  $r^2$  copies of  $H$  contained in  $H(r)$ . Thus our goal is to distribute the edges of  $K_a$  and/or  $K_b$  among the  $r^2$  copies of  $H$  in such a way that the matrix  $(M_a, M_b)$  have no rows with repeated entries. We can freely assign the arcs coming out from each  $(x_a, i)$  to the different trees which go through that vertex, and similarly we can freely assign the arcs coming out from  $(x_b, i)$  to the different forests which go through that vertex. In other words, every permutation of the entries in one column of  $M_a$  and in one column of  $M_b$  does not affect the fact that we have an edge decomposition of  $H(r)[s]$ . Thus we can construct the desired  $H'_{i,j}[s]$ 's if we verify the following claim:

**Claim 1.** *There are permutations  $\tau_1, \dots, \tau_r$  of the entries in the columns of  $M_a$  and permutations  $\tau'_1, \dots, \tau'_r$  of the entries of the columns of  $M_b$  such that the resulting matrix  $(M'_a, M'_b)$  has no row with repeated entries.*

**Proof.** We proceed row by row as in the construction of a Latin rectangle. Recall that, by the definition of  $(M_a, M_b)$ , each column has  $r$  distinct entries. By identifying the columns by their sets of entries, for each pair  $I, J \subset \{1, 2, \dots, r\}$  we have

$$(2) \quad |(\cup_{i \in I} M_{a,i}) \cup (\cup_{j \in J} M_{b,j})| \geq r \max\{|I|, |J|\} \geq 2 \max\{|I|, |J|\} \geq |I| + |J|.$$

By Hall's theorem there is a transversal of  $M_{a,1}, \dots, M_{a,r}, M_{b,1}, \dots, M_{b,r}$  which can make the first row of  $(M'_a, M'_b)$  with no elements repeated. By deleting the element of the transversal in each column we get a family of  $(r-1)$ -sets from which the two last inequalities in (2) hold with  $r$  replaced by  $(r-1)$  as long as  $r-1 \geq 2$ . Hence there is a transversal of these new family of sets which can make the second row of  $(M'_a, M'_b)$ . We can proceed with the same argument up to the  $(r-1)$  row. Now if each of the first  $r-1$  rows of  $(M'_a, M'_b)$  have their entries pairwise distinct, the same holds for the remaining one.  $\square$

In order to construct  $H'_{i,0}[s]$  we assign the arcs to the numbered trees and forests according to the new matrices  $M'_a, M'_b$  which have no common entry in the same position. Therefore there are no conflicting arcs in  $H'_{i,0}[s]$ .

Once we have constructed  $H'_{i,0}[s]$  for each  $i$ , we label the trees going out from the vertices in  $x_s \times \mathbb{Z}_p$  in a coherent way and we can proceed with the construction of  $H'_{i,0}[s]$  for larger values of  $s$ . We observe that, since we follow a peeling order of the base tree  $T_0$  and no vertex in  $H$  has indegree larger than two, we perform the distribution in each arc at most once. Therefore the process can be completed until  $s = t$ . At this point we have obtained an edge decomposition of  $H(r)$  into the  $r^2$  digraphs  $H'_{i,j}$ , each one isomorphic to our given tree



$T$ . By repeating the argument in each translation of  $H(r)$  we end up with a decomposition of  $Y$  by copies of  $T$ . As we have pointed out before, this completes the proof.  $\square$

Since  $K_{2m+1}(2)$  is isomorphic to  $K_{4m+2} \setminus M$ , for  $M$  a matching of  $K_{4m+2}$ , Corollary 1 follows from Theorem 1 with  $r = 2$ . We next give the proof of the second corollary.

*Proof of Corollary 2 :* Let  $T$  be a random tree with  $m + 1$  edges. We know that (a.a.s.) the tree  $T$  has at least  $2m/5$  endvertices. Let  $T'$  be the tree obtained from  $T$  by deleting one leave  $xy$ , where  $y$  is an endvertex of  $T$ .

In what follows we use the notation from the proof of Lemma 4. It follows from that proof that the Cayley graph  $Y = \text{Cay}(\mathbb{Z}_{2m+1} \times \mathbb{Z}_3, S \times \mathbb{Z}_3)$  is decomposed into translations of the (oriented) graph  $H(3)$  and each  $H(3)$  admits the two decompositions

$$H(3) = \oplus_{1 \leq i, j \leq 3} H_{ij} = \oplus_{1 \leq i, j \leq 3} H'_{ij},$$

where each  $H'_{i,j}$  is isomorphic to  $T'$  with its arcs oriented from the root to the elaves of a peeling ordering. On the other hand  $H_{i,j}$  is isomorphic to the same  $T'$  but for the existence of some vertices with indegree two, whose incident arcs are said to be conflicting.

We next add two additional vertices  $\alpha, \beta$  to  $Y$  and make them adjacent from every vertex in  $Y$ . Moreover we add to  $Y$  the three arcs  $((z, j), (z, j + 1 \pmod{3}))$  joining the vertices in each stable set  $z \times \mathbb{Z}_3$  for each  $z \in \mathbb{Z}_{2m+1}$ . The resulting graph (omitting orientations and colors) is isomorphic to  $K_{6m+5} \setminus \{\alpha, \beta\}$ .

Denote by  $f'_{i,j} : T' \rightarrow Y$  the rainbow embedding obtained from  $f_{i,j}$  by making the necessary redistribution of arcs to transform  $H_{i,j} = f_{i,j}(T)$  into  $H'_{i,j} = f'_{i,j}(T)$ . We note that the images of  $x$  by the set  $\{f'_{i,j}, 1 \leq i, j, \leq 3\}$  are the vertices in the stable set  $x \times \mathbb{Z}_3$ .

Suppose first that  $x$  is not incident to a conflicting arc in  $H = f(T')$ . In this case each of  $(x, 0), (x, 1), (x, 2)$  has indegree three in  $H(3)$  and the incoming arcs belong to nine trees, say  $T'_1, \dots, T'_9$ , three in each vertex and each one isomorphic to  $T'$ . Moreover, each of  $(x, 0), (x, 1), (x, 2)$  is also incident to three arcs  $((x, j), \alpha), ((x, j), \beta), (x, j), (x, j + 1 \pmod{3}))$ . By assigning one of these three arcs to each of the three trees from the above list incident to each of the three vertices we obtain nine trees  $T_1, \dots, T_9$ , each one isomorphic to our original tree  $T$ . By repeating this procedure to each copy of  $H(3)$  in  $Y$  we eventually obtain a decomposition of  $K_{6m+5} \setminus e, e = \{\alpha, \beta\}$  into copies of  $T$  completing the proof.

Suppose now that  $x$  is incident to a conflicting arc in  $H = f(T')$  (as  $(x_3, j)$  in the illustration depicted in Figure 3). Then the above assignment of the leave  $xy$  to each of the trees  $T'_1, \dots, T'_9$  must be carefully defined in order to avoid new conflicts. In the remaining of the proof we show that, nevertheless, there is always an assignment which avoids creating new cycles.

In this case each vertex  $(x, j)$  has indegree six (as  $x$  has indegree two in  $H$ ) and the six incoming arcs belong to six different trees (this was already ensured by the Claim in the proof of Lemma 4.) We may assume that row  $j$  of the following matrix denotes the numbers

of the trees the arcs incident to each  $(x, j)$  belong to:

$$(M_a, M_b) = \left( \begin{array}{ccc|ccc} 1 & 2 & 3 & \sigma_1 & \sigma_2 & \sigma_3 \\ 4 & 5 & 6 & \sigma_4 & \sigma_5 & \sigma_6 \\ 7 & 8 & 9 & \sigma_7 & \sigma_8 & \sigma_9 \end{array} \right),$$

where  $(\sigma_1, \dots, \sigma_9)$  are pairwise distinct numbers in  $\{1, \dots, 9\}$  and each row has no repeated entries. Now the arc  $(x, j), (x, j + 1 \pmod{3})$  must be assigned to a tree numbered in row  $j$  of  $M_a$  but not contained in row  $j + 1$  of  $M_b$ . By doing so we ensure that the oriented graphs after the assignments of the leave  $xy$  to the copies of  $T'$  have indegree one, and thus each one of them is isomorphic to the given tree  $T$ . We call such an assignment *good* and the existence of a good assignment provides a decomposition of  $Y$  by copies of  $T$ .

We observe that a good assignment is always possible unless there is a row  $j$  in  $M_a$  with the same entries as the row  $j + 1 \pmod{3}$  of  $M_b$ . If this is the case, we invert the orientations of the arcs  $((x, j), (x, j + 1 \pmod{3}))$  to  $((x, j), (x, j - 1 \pmod{3}))$ . Since we can not have row  $j$  of  $M_a$  with the same entries as row  $j + 1 \pmod{3}$  of  $M_b$  and at the same time row  $j'$  of  $M_a$  with the same entries as row  $j' - 1 \pmod{3}$  of  $M_b$  (recall that each row has pairwise distinct entries), there is always a good assignment. This completes the proof.  $\square$

#### REFERENCES

- [1] N. Alon, Combinatorial Nullstellensatz, *Combinatorics, Probability and Computing* **8** (1999), 7–29.
- [2] N. Alon, Additive Latin Transversals, *Israel J. Math.* **117** (2000), 125–130.
- [3] M. Càmara, A. Lladó, J. Moragas. On a conjecture of Graham and Häggkvist with the polynomial method, *European J. Combin.* **30** (7) (2009) 1585–1592.
- [4] M. Drmota. Random Trees. Springer-Verlag, 2009.
- [5] M. Drmota, B. Gittenberger. The distribution of nodes of given degree in random trees, *Journal of Graph Theory* **31** (1999) 227–253.
- [6] M. Drmota, B. Gittenberger. The shape of unlabeled rooted random trees. *European J. Combin.* **31** (2010) 2028–2063
- [7] M. Drmota, A. Lladó. Almost every tree with  $m$  edges decomposes  $K_{2m, 2m}$ . *Combinatorics, Probability and Computing* **23** 1 (2014) 50–65.
- [8] J. A. Gallian, A Dynamic Survey of Graph Labeling, *The Electronic Journal of Combinatorics* **5** (2007) # DS6.
- [9] R. L. Häggkvist, Decompositions of Complete Bipartite Graphs, *Surveys in Combinatorics*, Johannes Siemons Ed., Cambridge University Press (1989) 115–146.
- [10] A.E. Kézdy.  $\rho$ -valuations for some stunted trees. *Discrete Math.* **306** (21) (2006) 2786–2789
- [11] A. E. Kézdy, H. S. Snevily. Distinct Sums Modulo  $n$  and Tree Embeddings. *Combinatorics, Probability and Computing*, **1**, Issue 1, (2002)
- [12] A. Lladó, S.C. López, J. Moragas. Every tree is a large subtree of a tree that decomposes  $K_n$  or  $K_{n,n}$ , *Discrete Math.* **310** (2010) 838–842
- [13] G. Ringel, Problem 25, *Theory of Graphs and its Applications*, Nakl. CSAV, Praha, (1964) pp. 162.
- [14] R.W. Robinson, A.J. Schwenk. The distribution of degrees in a large random tree. *Discrete Math.* **12**(4) (1975) 359–372.
- [15] H. Snevily. New families of graphs that have  $\alpha$ -labelings. *Discrete Math.* **170** (1997) 185–194.
- [16] R. Yuster. Packing and decomposition of graphs with trees. *J. Combin. Theory Ser. B* **78** (2000) 123–140.