Discrete Serrin's Problem☆

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Abstract

We consider here the discrete analogue of Serrin's problem: if the equilibrium measure of a network with boundary satisfies that its normal derivative is constant, what can be said about the structure of the network and the symmetry of the equilibrium measure? In the original Serrin's problem, the conclusion is that the domain is a ball and the solution is radial. To study the discrete Serrin's problem, we first introduce the notion of radial function and then prove a generalization of the minimum principle, which is one of the main tools in the continuous case. Moreover, we obtain similar results to those of the continuous case for some families of networks with a ball–like structure, which include spider networks with radial conductances, distance–regular graphs or, more generally, regular layered networks.

Keywords: Serrin's Problem, overdetermined boundary value problems, equilibrium measure, spider networks, minimum principle
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1. Introduction

In 1971 J. Serrin stated the following problem in the continuum field; see [16]: if Ω is a connected open bounded domain of $\mathbb{R}^n$ with smooth boundary $\delta(\Omega)$, and u is a smooth function on Ω such that $-\Delta(u) = 1$ on Ω, $u = 0$ on $\delta(\Omega)$, then the normal derivative of u, $\dfrac{\partial u}{\partial \mathsf{n}}$, is constant on $\delta(\Omega)$ if and only if Ω is a ball on $\mathbb{R}^n$. Furthermore, the solution is radial. The main tools used in [16] for solving the problem were the moving planes method and a refinement of the maximum principle. H. F. Weinberger gave in [15] an alternative proof by means of elementary arguments; mainly by describing the Laplacian in polar coordinates and applying the minimum principle and Green's identity. In the last decade, there have been generalizations of the problem; for instance to the case when the Laplacian is replaced by a quasilinear or nonlinear elliptic operator; to the case when the elliptic problem is stated on an exterior domain, or to the case when the

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overdetermined boundary condition is placed only in a part of the boundary, see [1, 5, 11, 10, 13] and references therein.

Our objective is to consider this very same problem in the discrete field. Specifically, if we consider a network with boundary $\Gamma = (F \cup \delta(F), E)$, in [3] it was proved that there exists a function $\nu^F \in \mathcal{C}^+(F)$ such that $\mathsf{supp}(\nu^F) = F$ and satisfying $\mathcal{L}(u) = 1$ on F. Then, the *Discrete Serrin's Problem* consists in characterizing those networks with boundary such that the normal derivative of ν^F is constant. We pose the question about the structure of the network and the properties of the solution of the problem. First of all, we prove a generalized minimum principle that determines how the level sets of a superharmonic function are distributed on F. As a consequence, we show that strictly superharmonic functions cannot have local minima on F.

Concerning Serrin's problem we prove that if a network satisfies Serrin's condition, then the value of the constant only depends on the ratio between the number of vertices in the interior and the number of vertices of the boundary, but not on the conductances. Notice that this property is the same as in the continuous case, where the constant is the ratio between the volume of Ω and the area of its boundary. Next, we consider two families of networks with ball–like structure, namely, spider networks with radial conductances and regular layered networks, and we show that they satisfy Serrin's condition. Finally, for a class of regular networks we provide a characterization of those satisfying Serrin's condition.

2. Preliminaries

Let $\Gamma = (V, E, c)$ be a finite network; that is, a finite connected graph without loops nor multiple edges, with vertex set V and edge set E, in which each edge $\{x, y\}$ has been assigned a *conductance* $c(x, y) > 0$. Moreover, $c(x, y) = c(y, x)$ and $c(x, y) = 0$ if $\{x, y\} \notin E$. We say that x is adjacent to y, $x \sim y$, if $\{x, y\} \in E$ and for all $x \in V$, the value $\kappa(x) = \sum_{y \in V} c(x, y)$ is called *total conductance at x* or *degree of x*. A *path of length $m \geq 1$* is a sequence $\{x_1, \dots, x_{m+1}\}$ of vertices such that $c(x_i, x_{i+1}) > 0$, or equivalently $x_i \sim x_{i+1}$, $i = 1, \dots, m$. That Γ is connected means that any two vertices of V can be joined by a path. More generally, a subset F of V is said to be *connected* if each pair of vertices of F is joined by a path entirely contained in F. If $x \neq y$, we denote by $d(x, y)$ the minimum length between the paths joining x and y.

The set of real functions on V, denoted by $\mathcal{C}(V)$, and the set of non–negative functions on V, $\mathcal{C}^+(V)$, are naturally identified with $\mathbb{R}^n$ and the positive cone of $\mathbb{R}^n$, respectively, where $n = |V|$. If $u \in \mathcal{C}(V)$, its *support* is given by $\mathsf{supp}(u) = \{x \in V : u(x) \neq 0\}$ and we denote by $\int_V u$ the value $\sum_{x \in V} u(x)$. Moreover, if F is a non empty subset of V, we consider the sets $\mathcal{C}(F) = \{u \in \mathcal{C}(V) : \mathsf{supp}(u) \subset F\}$ and $\mathcal{C}^+(F) = \mathcal{C}(F) \cap \mathcal{C}^+(V)$. For each $F \subset V$, the characteristic function of F will be denoted by χ_F. When $F = V$ we will omit the subscript.

Given $F \subset V$ a proper subset, for any $x \in V$ we consider $d(x, F) = \min_{y \in F}\{d(x, y)\}$. Therefore, $x \in F$ iff $d(x, F) = 0$ whereas $x \in F^c = V \setminus F$, the *complementary of F in V*, iff $d(x, F) \geq 1$. In addition, we consider

$$r(F) = \max_{x \in V}\{d(x, F)\} = \max_{x \in F^c}\{d(x, F)\} = \max_{x \in F^c} \min_{y \in F}\{d(x, y)\} \geq 1.$$

Given $F \subset V$, we call *interior, vertex boundary, closure* and *exterior of F* the subsets

$$\begin{aligned}
\overset{\circ}{F} &= \{x \in F : y \in F \text{ for all } y \sim x\}, \\
\delta(F) &= \{x \in V : d(x, F) = 1\}, \\
\bar{F} &= \{x \in V : d(x, F) \leq 1\}, \\
\mathsf{Ext}(F) &= \{x \in V : d(x, F) \geq 2\},
\end{aligned}$$

respectively. Observe that when F is connected then $\bar{F}$ is also connected, but $\overset{\circ}{F}$ is not necessarily connected. Moreover, when $F \neq \emptyset$, unlike the topological case, $\delta(F) \cap \delta(F^c) = \emptyset$ and either $\bar{F} = F$ or $\overset{\circ}{F} = F$ iff $F = V$. However, the following relations, which are similar to the topological ones, are satisfied.

Lemma 2.1. *If $F \subset V$, then $F \cap \delta(F) = \delta(F^c) \cap \overset{\circ}{F} = \emptyset$, $\delta(\overset{\circ}{F}) \subset \delta(F^c)$ and*

$$\bar{F} = F \cup \delta(F), \qquad F = \delta(F^c) \cup \overset{\circ}{F}, \qquad \mathsf{Ext}(F) = (\bar{F})^c.$$

Moreover, $(\overset{\circ}{F})^c = \overline{F^c}$ and $(\overset{\circ}{F^c}) = (\bar{F})^c$.

Proof. If $x \in \delta(\overset{\circ}{F})$, then $x \notin \overset{\circ}{F}$; that is, there exists $z \notin F$ such that $d(x, z) = 1$, but there exists $y \in \overset{\circ}{F}$ such that $d(x, y) = 1$. Therefore, $x \in F$ and in conclusion $x \in \delta(F^c)$. On the other hand, from equality $V = F^c \cup \underbrace{\overset{\circ}{F} \cup \delta(F^c)}_{F}$ we get that $(\overset{\circ}{F})^c = F^c \cup \delta(F^c) = \overline{F^c}$, whereas from equalities

$$V = F \cup \underbrace{(\overset{\circ}{F^c}) \cup \delta(F)}_{F^c} = (\overset{\circ}{F^c}) \cup \underbrace{F \cup \delta(F)}_{\bar{F}}$$

we obtain that $(\overset{\circ}{F^c}) = (\bar{F})^c$. □

We can generalize the above definitions and properties in the following way: Given $F \subset V$ a proper subset, for any $i = 0, \ldots, r(F)$ we consider the subsets

$$\begin{aligned}
B_i(F) &= \{x \in V : d(x, F) \leq i\}, \\
B^i(F) &= \{x \in V : d(x, F) \geq i\}, \\
S_i(F) &= \{x \in V : d(x, F) = i\}.
\end{aligned}$$

Therefore, $B_0(F) = S_0(F) = F$, $B^0(F) = V$, $B_1(F) = \bar{F}$ and $B^1(F) = F^c$, whereas $S_1(F) = \delta(F)$ and $B_{r(F)}(F) = V$. Moreover, $B^i(F) = B_{i-1}(F)^c$, for any $i = 1, \ldots, r(F)$.

Lemma 2.2. *Given $F \subset V$ a proper subset, for any $i = 0, \ldots, r(F) - 1$ the following properties hold:*

$$\delta(B_i(F)) = S_{i+1}(F), \qquad \bar{B}_i(F) = B_{i+1}(F) \qquad \textit{and} \qquad B_i(F) \subseteq \overset{\circ}{B}_{i+1}(F).$$

In particular, $\delta(B_{i+1}(F)^c) \subseteq S_{i+1}(F)$ for any $i = 0, \ldots, r(F) - 1$.

Proof. Clearly $B_{i+1}(F) = B_i(F) \cup S_{i+1}(F)$ for any $i = 0, \dots, r(F) - 1$ and hence $\bar{B}_i(F) = B_{i+1}(F)$ iff $\delta(B_i(F)) = S_{i+1}(F)$.

Given $x \in \delta(B_i(F))$, then $i < d(x, F)$ and there exists $y \in B_i(F)$ such that $d(x, y) = 1$. Therefore, $i+1 \le d(x, F) \le d(x, y)+d(y, F) \le i+1$, which implies that $d(x, F) = i+1$. Conversely, if $x \in S_{i+1}(F)$, then there exists $x_1, \dots, x_i \in V$ such that $d(x_j, x_{j+1}) = 1$ for $j = 1, \dots, i-1$, $x_1 \in F$ and $d(x_i, x) = 1$. Therefore, $x_i \in B_i(F)$ and hence $x \in \delta(B_i(F))$.

On the other hand, if $x \in B_i(F) \subset B_{i+1}(F)$ and we consider $y \in V$ such that $d(y, x) = 1$, then $d(y, F) \le d(x, y) + d(x, F) \le i + 1$ and hence $x \in \overset{\circ}{B}_{i+1}(F)$. The last claim is consequence of the following identities

$$\delta(B_{i+1}(F)^c) = B_{i+1}(F) \setminus \overset{\circ}{B}_{i+1}(F) \subseteq B_{i+1}(F) \setminus B_i(F) = S_{i+1}(F).$$

□

Lemma 2.3. *Given $F \subset V$ a proper subset, then for any $i = 1, \dots, r(F)$ the following properties hold:*

$$\delta(B^i(F)^c) = S_i(F), \qquad \overset{\circ}{B^i}(F) = B^{i+1}(F) \quad \text{and} \quad \bar{B}^{i+1}(F) \subseteq B^i(F).$$

In particular, $\delta(B^i(F)) \subseteq S_{i-1}(F)$ for any $i = 1, \dots, r(F)$.

3. Generalized minimum principle

The *combinatorial Laplacian* of Γ is the linear operator $\mathcal{L} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function defined as

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x, y)\,(u(x) - u(y)).$$

Observe that if $u \in \mathcal{C}^+(V)$ and $F = \mathsf{supp}(u)$, then $\mathcal{L}(u) < 0$ on $\delta(F)$ and $\mathcal{L}(u) = 0$ on $\mathsf{Ext}(F)$.

It is well–known that the Laplacian is a self–adjoint and positive semi–definite operator, in the sense that

$$\int_V v\,\mathcal{L}(u) = \int_V u\,\mathcal{L}(v) \qquad \text{for any} \quad u, v \in \mathcal{C}(V),$$

and $\displaystyle\int_V u\,\mathcal{L}(u) \ge 0$ and $\displaystyle\int_V u\,\mathcal{L}(u) = 0$ iff $u = \chi$. Moreover, given F a proper subset of V and $u \in \mathcal{C}(\bar{F})$, we define the *normal derivative* of u as the function in $\mathcal{C}(\delta(F))$ given by

$$\frac{\partial u}{\partial \mathsf{n}}(x) = \sum_{y \in F} c(x, y)\,(u(x) - u(y)), \qquad x \in \delta(F).$$

The discrete version of the Gauss Theorem, see for instance [4],

$$\int_F \mathcal{L}(u) = -\int_{\delta(F)} \frac{\partial u}{\partial \mathsf{n}},$$

will be useful.

A function $u \in \mathcal{C}(V)$ is called *harmonic*, *superharmonic* or *subharmonic* on F iff $\mathcal{L}(u) = 0$, $\mathcal{L}(u) \ge 0$ or $\mathcal{L}(u) \le 0$ on F, respectively. Moreover, $u \in \mathcal{C}(V)$ is called *strictly superharmonic* or *strictly subharmonic* on F iff $\mathcal{L}(u) > 0$ or $\mathcal{L}(u) < 0$ on F. The positive semi–definiteness of $\mathcal{L}$ implies that the harmonic functions on V are multiples of χ. In fact, if $u \in \mathcal{C}(V)$ is either superharmonic or subharmonic on V, then it is harmonic and hence constant.

The following results establish the minimum principle and the monotonicity of the Laplacian operator and were proved in [4] in a more general context, see also [9]. We include here these results because they are the basis for the new ones. In the sequel, we assume that F is a non empty connected proper subset of V.

Proposition 3.1. *If $u \in \mathcal{C}(V)$ is superharmonic on F, then*

$$\min_{x\in\delta(F)}\{u(x)\} \le \min_{x\in F}\{u(x)\}$$

and the equality holds iff u coincides on $\bar{F}$ with a multiple of $\chi_{\bar{F}}$.

Proposition 3.2. *If $u \in \mathcal{C}(V)$ is superharmonic on F and $u \geq 0$ on $\delta(F)$, then $u \in \mathcal{C}^+(\bar{F})$. In addition, either $u = 0$ on $\bar{F}$ or $u > 0$ on F.*

In the following result we show that in fact the values of superharmonic functions increase with the distance from $\delta(F)$.

Theorem 3.3 (General Minimum Principle)**.** *If $u \in \mathcal{C}(\bar{F})$ is superharmonic on F, then for any $i = 1,\dots,r(F^c)-1$*

$$\min_{x\in\delta(F)}\{u(x)\} \le \min_{x\in S_i(F^c)}\{u(x)\} \le \min_{x\in S_{i+1}(F^c)}\{u(x)\}\,.$$

Moreover, if for some i the left inequality is an equality, then u is constant on $\bar{F}$; whereas if the second inequality is an equality, then u is constant on $\bar{B}^{i+1}(F^c)$.

Proof. Notice that for any $i = 1,\dots,r(F^c)-1$ it is satisfied that

$$\min_{x\in\delta(F)}\{u(x)\} \le \min_{x\in F}\{u(x)\} \le \min_{x\in B^i(F^c)}\{u(x)\}$$

since $B^i(F^c) \subset B^1(F^c) = F$ and we have applied Proposition 3.1 to obtain the first inequality.

On the other hand, fixed $i = 1,\dots,r(F^c)-1$, from Lemma 2.3 we know that

$$B^i(F^c) = \overset{\circ}{B^i}(F^c) \cup \delta(B^i(F^c)^c) = B^{i+1}(F^c) \cup S_i(F^c)$$

and then it suffices to prove that

$$\min_{x\in S_i(F^c)}\{u(x)\} \le \min_{x\in B^{i+1}(F^c)}\{u(x)\}\,.$$

If we consider $H = B^{i+1}(F^c)$ and $v = u_{|_{\bar{H}}}$, then $v \in \mathcal{C}(\bar{H})$ and $\mathcal{L}(v) \geq 0$ on H. Keeping in mind that $\delta(H) \subseteq S_i(F^c)$, from Proposition 3.1, we obtain

$$\min_{x\in S_i(F^c)}\{u(x)\} \le \min_{x\in\delta(H)}\{u(x)\} \le \min_{x\in H}\{u(x)\} \le \min_{x\in S_{i+1}(F^c)}\{u(x)\}\,.$$

Therefore, if the $\min\limits_{x\in S_i(F^c)}\{u(x)\} = \min\limits_{x\in S_{i+1}(F^c)}\{u(x)\}$, then $\min\limits_{x\in\delta(H)}\{u(x)\} = \min\limits_{x\in H}\{u(x)\}$ and hence u is constant on $\bar{H}$. □

From the above results we can conclude that there exist strictly superharmonic functions on F that are null on $\delta(F)$ and strictly positive on F; see [4, Corollary 4.3]. The next result shows that strictly superharmonic functions cannot have local minima on F.

Lemma 3.4. *If $u \in C^+(F)$ is a strictly superharmonic function on F, then for any $x \in F$ there exists $y \in \bar{F}$ such that $c(x, y) > 0$ and $u(y) < u(x)$.*

Proof. Let $x \in F$ and suppose that for all $y \in \bar{F}$ such that $c(x, y) > 0$, $u(y) \geq u(x)$. Then,

$$0 < \mathcal{L}(u)(x) = \sum_{y \in V} c(x, y)(u(x) - u(y)) \leq 0,$$

which is a contradiction. □

Consider $u \in C^+(F)$ a strictly superharmonic function on F, we denote by $u_0 = 0$ and s the number of different values $u_1, \dots, u_s$ that u takes on F. We suppose that these values are ordered as $0 = u_0 < u_1 < \dots < u_s$ and we consider the level set of u, denoted by $U_i = \{ x \in \bar{F} \mid u(x) = u_i \}$ for $i = 0, \dots, s$. Observe that $U_0 = \delta(F)$ because u is strictly positive on F and that $U_i \cap U_j = \emptyset$ if $i \neq j$.

For simplicity of notation, we denote by $D_0 = \delta(F)$ and by $D_i = S_i(F^c)$ for $i = 1, \dots, r$, where $r = r(F^c)$. Observe that if $x \in D_i$, $i \geq 1$, its neighbours belong to $D_{i-1} \cup D_i \cup D_{i+1}$.

Proposition 3.5. *If $u \in C^+(F)$ is a strictly superharmonic function on F, then $U_0 = D_0$ and $U_i \subset \bigcup_{j=1}^{i} D_i$, for any $i = 1, \dots, s$.*

Proof. It suffices to prove that if $x \in U_i$, then $d(x, \delta(F)) \leq i$, for any $i = 0, \dots, s$. We prove the result by mathematical induction. The result is true for $i = 0$ from Proposition 3.1. Suppose that the result is true for $j = 0, \dots, i$ and consider $x \in U_{i+1}$. From Lemma 3.4 there exists $y \in \bar{F}$ such that $c(x, y) > 0$ and $y \in U_j$, $j \leq i$ and hence the result follows by the induction hypothesis. □

Observe that $U_1 \subset D_1$, but it is not true in general that $U_i \subset D_i$ as the example on Figure 1 shows. However, if the subsets U_i and D_i coincide until a certain layer, the above inclusion is true for the next layer.

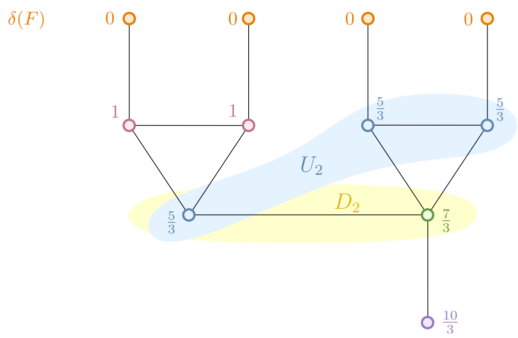

Figure 1: A graph Γ and u strictly superharmonic such that $U_2 \not\subset D_2$.

Corollary 3.6. *If $u \in C^+(F)$ is a strictly superharmonic function on F satisfying that $U_j = D_j$ for all $j = 0, \dots, i$, then $U_{i+1} \subset D_{i+1}$.*

The following definition is inspired by the above behavior. If $u \in C^+(F)$ is strictly superharmonic on F, it is called *radial* if $U_i = D_i$ for any $i = 0, \dots, s$. In this case, $s = r(F^c)$ and for any $x \in D_i$, $i = 1, \dots, s$, it is satisfied that

$$\mathcal{L}u(x) = k_{i+1}(x)(u_i - u_{i+1}) + k_{i-1}(x)(u_i - u_{i-1}) > 0, \tag{1}$$

where $k_{i+1}(x) = \sum_{y\in D_{i+1}} c(x,y)$ and $k_{i-1}(x) = \sum_{y\in D_{i-1}} c(x,y)$. Moreover, if $x \in D_0 = \delta(F)$, we get that

$$\frac{\partial u}{\partial \mathsf{n}}(x) = -k_1(x)u_1 < 0, \tag{2}$$

where $k_1(x) = \sum_{y\in D_1} c(x,y)$.

4. Discrete Serrin's Problem

In this section we study to what extent both discrete and continuous Serrin's problems are analogue as well as the differences between them and the difficulties that appear in the discrete setting. From now on we suppose fixed a non–empty and connected subset of vertices F with boundary $\delta(F)$. For the sake of simplicity we will assume that $V = \bar{F}$; that is, $\Gamma = (F \cup \delta(F), c)$, which is sometimes called *Network with boundary*; see [7].

The existence and uniqueness of solution of the discrete Dirichlet problem that appears in the Serrin's problem was proved for some of the authors.

Lemma 4.1. [3, Proposition 2.1] *For any non–empty connected proper subset F there exists a unique function $\nu^F \in C^+(F)$ such that $\mathsf{supp}(\nu^F) = F$ and satisfying $\mathcal{L}(u) = 1$ on F.*

The function $\nu^F \in C^+(F)$ is called *equilibrium measure of F*. Then, the *Discrete Serrin's Problem* consists in characterizing those networks with boundary such that $\dfrac{\partial \nu^F}{\partial \mathsf{n}}$ is constant on $\delta(F)$. The last boundary condition is known as *Serrin's condition*. From the Gauss Theorem, if the equilibrium measure satisfies Serrin's condition, then $\dfrac{\partial \nu^F}{\partial \mathsf{n}} = -\dfrac{|F|}{|\delta(F)|}$.

Before continuing with the study of Discrete Serrin's Problem, we need to observe that the equilibrium measure ν^F does not depend on the structure of the boundary edges but on the total conductance flowing from a vertex. However, its normal derivative on the boundary is indeed affected by them, and therefore Serrin's condition depends on the structure of the boundary edges. This fact creates a lack of precision on determining the structure of a network that fits discrete Serrin's Problem premises, as we can see with the example on Figure 2: both graphs

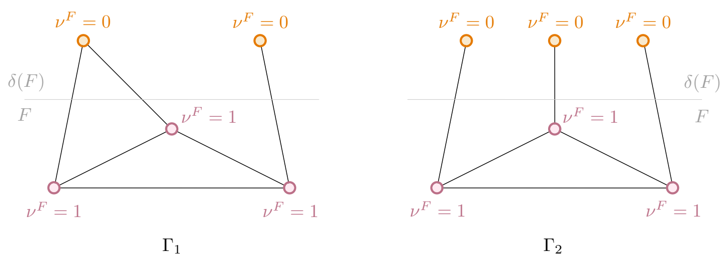

Figure 2: The equilibrium measure is not affected by the boundary edges.

Γ_1 and Γ_2 have the same equilibrium measure $\nu^F = 1$ on F. However, ν^F satisfies Serrin's condition on Γ_2 but not on Γ_1. In order to avoid this kind of ambiguities, we suppose that given a network with boundary Γ all its boundary vertices have a unique adjacent in D_1. This choice is in correspondence with the continuous concept of normal derivative and with the terminology of *separated boundary* introduced in [12].

So, throughout this section we will assume that the boundary is separated; that is, for any $x \in \delta(F)$ there exists a unique $\hat{x} \in D_1$ such that $c(x, \hat{x}) > 0$. Observe that given two different vertices $x, y \in \delta(F)$, it can happen that $\hat{x} = \hat{y}$. We also suppose that $|\delta(F)| \geq 2$.

Proposition 4.2. *Any two of the following conditions implies the third one:*

(i) ν^F *satisfies Serrin's condition.*

(ii) $U_1 = D_1$.

(iii) $c(x, \hat{x})$ *is constant for any* $x \in \delta(F)$.

Proof. From Proposition 3.5, we know that $U_1 \subset D_1$. Moreover, $\dfrac{\partial \nu^F}{\partial \mathsf{n}}(x) = -c(x, \hat{x})\nu^F(\hat{x})$, for any $x \in \delta(F)$.

If ν^F satisfies Serrin's condition, then $C = -c(x, \hat{x})\nu^F(\hat{x})$ for every $x \in \delta(F)$ and hence (ii) and (iii) are equivalent. On the other hand, (ii) and (iii) clearly imply (i). □

The above Proposition shows that if Serrin's condition is satisfied, the conductances on the boundary edges are constant iff the solution is radial on the first layer; that is, $D_1 = U_1$. The next example shows that in the discrete setting Serrin's condition is not enough to guarantee that the equilibrium measure is a radial function, see Figure 3. We remark that a similar situation happens in the continuous case when the considered operator is non–linear, see [6]. Let us consider a

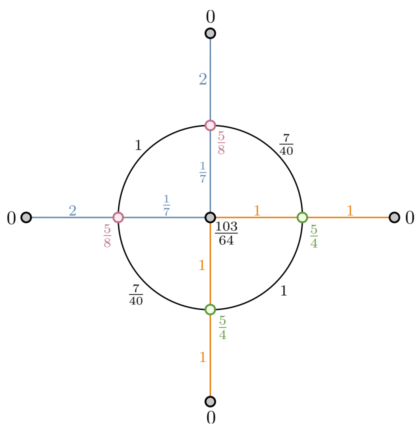

Figure 3: A network satisfying Serrin's condition with non radial equilibrium measure

family of networks with a similar structure to two–dimensional balls, the *spider networks with radial conductances*. We compute the equilibrium measure of these networks and we show that it is a radial function and that its normal derivative is constant; that is, they satisfy all the conclusion of Serrin's problem.

We use the same definitions and notations for spider networks as in [2] although they were first introduced by E. Curtis and J. Morrow, see [7]. A *spider network* (Γ, c) has n boundary nodes and the following structure: n *radii* and m *circles* distributed as in Figure 4, where the vertices lay on the intersections and the edges are given by these radii and circle lines. The vertex x_{ji} is defined as the intersection between the radius j and the circle i for all $i = 1, \ldots, m+1$, $j = 1, \ldots, n$, whereas the vertex x_{00} is the intersection of all the radii; that is, the vertex on the center. Note that $x_{j0} = x_{00}$ for all $j = 1, \ldots, n$. The boundary circle does not give any edge as it is not a proper circle of the network –it is an imaginary one such that the vertices on it are the n boundary nodes. For all $j = 1, \ldots, n$, we call $v_j = x_{jm+1}$ the vertices on the boundary circle. Let F be the set of

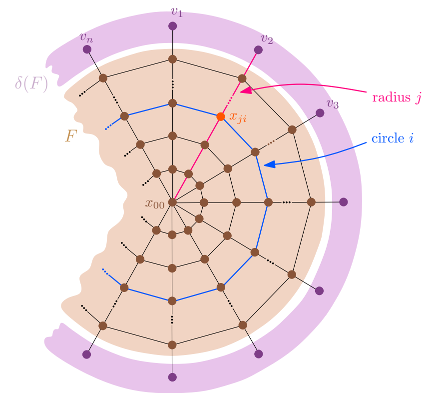

Figure 4: Structure of a spider network.

interior vertices of the spider network, where $F = \{x_{11}, \ldots, x_{n1}, \ldots, x_{1m}, \ldots, x_{nm}, x_{00}\}$, and let the set of boundary vertices be $\delta(F) = \{v_1, \ldots, v_n\}$.

Given real positive values $\{a_i\}_{i=0,\ldots,m}$, from now on we suppose that the conductances are given by $c(x_{ji}, x_{ji-1}) = a_{m-i+1}$ for $i = 1, \ldots, m+1$, $j = 1, \ldots, n$ on the radial edges and are free on the circle edges. We call these networks *spider networks with radial conductances*.

Proposition 4.3. *Let (Γ, c) be a spider network on n radii and m circles with radial conductances $\{a_i\}_{i=0,\ldots,m}$. Then, the equilibrium measure is radial and it is given by*

$$\nu^F(x_{js}) = \frac{1}{n}\sum_{i=0}^{m-s} \frac{n(m-i)+1}{a_i}$$

for all $j = 1, \ldots, n$ and $s = 0, \ldots, m+1$ and hence it satisfies Serrin's condition. In particular, when all the radial conductances are equal to a, we get

$$\nu^F(x_{js}) = \frac{(m-s+1)(nm+sn+2)}{2an},$$

for all $j = 1, \ldots, n$ and $s = 0, \ldots, m+1$.

Proof. From Lemma 4.1, the equilibrium measure of F exists. Suppose that it is a radial function, then $q(s) = \nu^F(x_{js})$, for any $j = 1 \ldots, n$ and $s = 0, \ldots, m+1$, where $q(m+1) = 0$.

By imposing that $\mathcal{L}(\nu^F) = 1$ on F, from (1) we get the following recurrence equation

$$1 = a_{m-s+1}(q(s)-q(s-1)) - a_{m-s}(q(s+1)-q(s)) \text{ for all } s = 1, \ldots, m \text{ and } 1 = -na_m(q(1)-q(0)).$$

If the above system has solution, then it will determine the equilibrium measure, since it is unique. If we define $\psi(s) = a_{m-s+1}(q(s) - q(s-1))$, for $s = 1, \ldots, m+1$, the recurrence relation becomes

$$1 = \psi(s) - \psi(s+1) \quad \text{for all} \quad s = 1, \ldots, m \quad \text{and} \quad \psi(1) = -\frac{1}{n}.$$

Hence,

$$q(s) - q(s-1) = -\frac{[n(s-1)+1]}{na_{m+1-s}}, \quad s = 1, \dots, m+1.$$

Therefore, keeping in mind that $q(m+1) = 0$, we get that

$$q(s) = \frac{1}{n}\sum_{k=s}^{m}\frac{[nk+1]}{a_{m-k}} = \frac{1}{n}\sum_{k=0}^{m-s}\frac{[n(m-k)+1]}{a_k}, \quad s = 0, \dots, m+1. \quad \square$$

If we consider a spider network that not fulfills the hypothesis of radial conductances, then Serrin's condition cannot be satisfied, as the following example shows.

Given four real values $a_1, a_2, b_1, b_2 > 0$, a spider network on n radius and $m = 1$ circle with conductances on the radius $c(x_{j1}, v_j) = a_1$, $c(x_{00}, x_{j1}) = a_2$ for $j = 1, \dots, n-1$ and $c(x_{n1}, v_n) = b_1$, $c(x_{00}, x_{n1}) = b_2$ (see Figure 5), satisfies Serrin's condition iff $b_i = a_i$, $i = 1, 2$; that is, the conductances are radial.

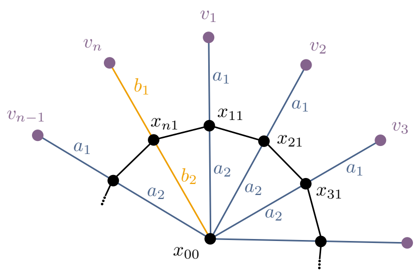

Figure 5: A spider network with $m = 1$ and *almost radial conductances*.

If Serrin's condition holds, then $\nu^F(x_{j1}) = -\dfrac{C}{a_1}$ for all $j = 1, \dots, n-1$ and $\nu^F(x_{n1}) = -\dfrac{C}{b_1}$. As $\mathcal{L}(\nu^F)(x_{j1}) = 1$ for all $j = 2, \dots, n-2$, we obtain $\nu^F(x_{00}) = -\dfrac{(a_1+a_2)C}{a_1 a_2} - \dfrac{1}{a_2}$ and then

$$\begin{aligned}1 = \mathcal{L}(\nu^F)(x_{11}) &= \big(c(x_{n1}, x_{11}) + a_1 + a_2\big)\nu^F(x_{11}) - c(x_{n1}, x_{11})\nu^F(x_{n1}) - a_2\nu^F(x_{00}) \\ &= -c(x_{n1}, x_{11})C\left(\frac{1}{a_1} - \frac{1}{b_1}\right) + 1.\end{aligned}$$

Therefore $a_1 = b_1$ because $c(x_{n1}, x_{11}) \neq 0$ and $C \neq 0$. Proceeding in the same way when using $\mathcal{L}(\nu^F)(x_{n1}) = 1$, we can also see that necessarily $a_2 = b_2$.

5. Discrete Serrin's Problem on regular layered networks

In this section we study a family of networks called *regular layered networks*. In 1973 P. Delsarte introduced the concept of *completely regular codes* as sets of vertices of a distance–regular graph, see [8, 14]. We readapt this concept considering the boundary of a network as a completely regular code of the network itself, as it is defined in the following. In this section, we do not assume that the network has separated boundary.

A *regular layered network* is a network with boundary $\Gamma = (F \cup \delta(F), c)$ such that there exist two sequences of positive numbers $\{b_i\}_{i=0,\dots,m-1}$ and $\{c_i\}_{i=1,\dots,m}$ and $c_0 = b_m = 0$, $m = r(\delta(F))$

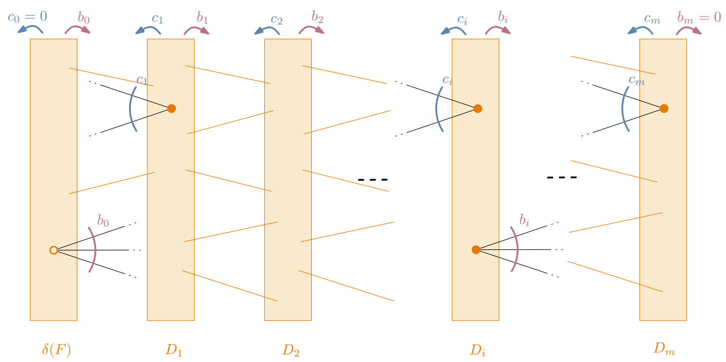

Figure 6: A regular layered graph.

satisfying that for any vertex $x \in D_i$, $k_{i-1}(x) = c_i$ and $k_{i+1}(x) = b_i$, $i = 1 \dots, m$. See Figure 6 for a better understanding in the case of graphs.

Observe that spider networks with radial conductances are regular layered networks. Futhermore, distance–regular graphs are also regular layered graphs when we consider $F = V \setminus \{x\}$ for a given $x \in V$.

Lemma 5.1. *Let* Γ *be a regular layered network. Then,*

$$\frac{|F|}{|\delta(F)|} = \sum_{i=1}^{m} \left(\prod_{\ell=0}^{i-1} \frac{b_\ell}{c_{\ell+1}} \right).$$

Proof. Clearly, $|F| = \sum_{i=1}^{m} |D_i|$ and moreover, for all $i = 0, \dots, m-1$,

$$b_i |D_i| = \sum_{x \in D_i} \sum_{y \in D_{i+1}} c(x,y) = \sum_{y \in D_{i+1}} \sum_{x \in D_i} c(x,y) = c_{i+1} |D_{i+1}|.$$

Therefore,

$$|D_i| = |D_0| \frac{b_0}{c_i} \prod_{\ell=1}^{i-1} \frac{b_\ell}{c_\ell} = |D_0| \prod_{\ell=0}^{i-1} \frac{b_\ell}{c_{\ell+1}}$$

for all $i = 1, \dots, m$ and the result follows. □

Proposition 5.2. *Given a regular layered network* Γ*, the equilibrium measure is*

$$\nu^F(x) = \sum_{j=1}^{s} \frac{1}{b_{j-1}} \sum_{k=j}^{m} \left(\prod_{\ell=j-1}^{k-1} \frac{b_\ell}{c_{\ell+1}} \right) \quad \textit{for all} \quad x \in D_s, \quad s = 0, \dots, m.$$

Therefore, it is radial and moreover ν^F *satisfies Serrin's condition.*

Proof. From Lemma 4.1, the equilibrium measure of a regular layered network exists. Suppose that it is a radial function, then $q(s) = \nu^F(x)$ for any $x \in D_s$ and $s = 0, \dots, m$, where $q(0) = q(m+1) = 0$.

By imposing that $\mathcal{L}(\nu^F) = 1$ on F, from Equation (1) we get

$$1 = c_s\big(q(s) - q(s-1)\big) - b_s\big(q(s+1) - q(s)\big) \quad \text{for all } s = 1, \dots, m.$$

If the above system has solution, then it will determine the equilibrium measure, since it is unique.

If we define for any $s = 1, \dots, m$, $\rho_s = \prod_{j=0}^{s-1} \dfrac{b_j}{c_{j+1}}$, then multiplying the above expressions by ρ_s we get

$$\rho_s = \rho_{s-1} b_{s-1}\big(q(s) - q(s-1)\big) - \rho_s b_s\big(q(s+1) - q(s)\big) \quad \text{for all } s = 1, \dots, m,$$

since $\rho_s c_s = \rho_{s-1} b_{s-1}$. By considering the function $\psi(s) = \rho_{s-1} b_{s-1}(q(s) - q(s-1))$, for any $s = 1, \dots, m+1$, then the recurrence relation becomes

$$\rho_s = \psi(s) - \psi(s+1) \quad \text{for all} \quad s = 1, \dots, m.$$

Observe that with this notation $\psi(m) = \rho_m$ since $b_m = 0$. Hence,

$$\psi(s) = \sum_{k=s}^{m} \rho_k, \quad \text{for all} \quad s = 1, \dots, m$$

and therefore,

$$q(s) - q(s-1) = \frac{1}{\rho_{s-1} b_{s-1}} \sum_{k=s}^{m} \rho_k, \quad s = 1, \dots, m.$$

Therefore, keeping in mind that $q(0) = 0$, we get that

$$q(s) = \sum_{j=1}^{s} \frac{1}{b_{j-1}\rho_{j-1}} \sum_{k=j}^{m} \rho_k = \sum_{j=1}^{s} \frac{1}{b_{j-1}} \sum_{k=j}^{m} \prod_{\ell=j-1}^{k-1} \frac{b_\ell}{c_{\ell+1}}, \quad s = 0, \dots, m. \quad \square$$

Theorem 5.3. *Let Γ be a network such that for all $i = 1, \dots, m-1$, $k_{i+1}(x) + k_{i-1}(x) = d_i$ for all $x \in D_i$, $U_i = D_i$ and $m = s$. Then, Γ satisfies the Serrin's condition iff Γ is a regular layered network.*

Proof. The necessary condition follows from Proposition 5.2. In order to prove the sufficient condition, we first observe that from Corollary 3.6, we get that $U_m \subset D_m$ and hence $U_m = D_m$ since $s = m$. On the other hand, if $x \in D_i$, $i = 1, \dots, m-1$, from Equation (1), we get that

$$1 = k_{i+1}(x)(u_i - u_{i+1}) + k_{i-1}(x)(u_i - u_{i-1}).$$

In particular, if $x, y \in D_i$ then $k_{i-1}(x) - k_{i-1}(y) = k_{i+1}(y) - k_{i+1}(x)$, and subtracting the above identities we get that

$$\begin{aligned} 0 &= \Big(k_{i+1}(y) - k_{i+1}(x)\Big)(u_{i+1} - u_i) + \Big(k_{i-1}(x) - k_{i-1}(y)\Big)(u_i - u_{i-1}) \\ &= \Big(k_{i+1}(y) - k_{i+1}(x)\Big)(u_{i+1} - u_{i-1}). \end{aligned}$$

As $u_{i+1} - u_{i-1} > 0$, then the last equality holds if a only if $k_{i+1}(y) = k_{i+1}(x)$ for all $x, y \in D_i$ and hence also $k_{i-1}(y) = k_{i-1}(x)$ for all $x, y \in D_i$. Let now $x \in D_m$, from Equation (1) we get that

$$1 = k_{m-1}(x)(u_m - u_{m-1}),$$

and therefore $k_{m-1}(x)$ does not depend on x. Finally, as Γ satisfies Serrin's condition, then $k_1(x) = -\dfrac{C}{u_1}$. Thus, Γ is a regular layered graph. $\square$

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