COALITIONAL MULTINOMIAL PROBABILISTIC VALUES*

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Abstract

We introduce a new family of coalitional values designed to take into account players' attitudes with regard to cooperation. This new family of values applies to cooperative games with a coalition structure by combining the Shapley value and the multinomial probabilistic values, thus generalizing the symmetric coalitional binomial semivalues. Besides an axiomatic characterization, a computational procedure is provided in terms of the multilinear extension of the game and an application to the Catalonia Parliament, Legislature 2003–2007, is shown.

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1 Introduction

Value theory started in 1953 with Shapley [28], who introduced the axiomatic method in cooperative game theory to define a solution concept called now the *Shapley value*. The Shapley value is characterized as the unique value to satisfy efficiency, the null player property, symmetry, and additivity.

In 1988, Weber [30] obtained a wide generalization of the Shapley value by defining the family of *probabilistic values*, each one of which requires weighting coefficients p_S^i for each player i and each coalition $S \subseteq N\backslash\{i\}$. The payoff that a probabilistic value allocates to each player is a weighted sum of his marginal contributions in the game. We quote from Weber [30]:

> "Let player i view his participation in a game v as consisting merely of joining some coalition S and then receiving as a reward his marginal contribution to the coalition. If p_S^i is the probability that he joins coalition S, then $\phi_i[v]$ is his expected payoff from the game."

In 2000, Puente [27] (see also [19]) defined two special subfamilies of probabilistic values: (a) *binomial semivalues*, where the weighting coefficients depend on a unique parameter $q \in [0, 1]$; and (b) *multinomial probabilistic values*, where the weighting coefficients depend on n parameters, one per player and all in $[0, 1]$ too.[1] Of course, (a) is a subfamily of (b).

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[1]Incidentally, we point out that the "Banzhaf α–indices", introduced by Carreras [7] when generalizing the decisiveness notion studied in Carreras [8], are multinomial probabilistic values applied to simple games—i.e., used as power indices. Cf. Carreras and Puente [12] or [13] for a joint work on these values and [20] for their application to the study of the partnership formation, which generalizes [16] widely.

Introduced in 1974 by Aumann and Drèze [4], the notion of *game with a coalition structure* provided a new avenue for the development of value theory. Since then, a lot of work has been done in this new field, mainly addressed to define *coalitional* values that often represent extensions of classical values to this setup. The best known coalitional value is the *Owen value*, an extension of the Shapley value introduced in 1977 by Owen [23] (also in [25]). The Owen value is characterized uniquely by the following axioms: efficiency, the null player property, symmetry within unions, symmetry in the quotient game, and additivity.

In 2006, Carreras and Puente [10] (see also [11]) extended the binomial semivalues to games with a coalition structure: they used these values in the quotient game and the Shapley value within unions and obtained the *symmetric coalitional binomial semivalues*, a family depending on one parameter $q \in [0, 1]$ that includes (when $q = 1/2$) the *symmetric coalitional Banzhaf value* introduced in 2002 by Alonso and Fiestras [1]. The only axiomatic difference between these new coalitional values and the Owen value is that the former satisfy the total power property whereas the latter satisfies efficiency.

In the present paper we extend the multinomial probabilistic values to games with a coalition structure by introducing the *coalitional multinomial probabilistic values*, a family that includes all symmetric coalitional binomial semivalues.

A first main aspect of the coalitional multinomial probabilistic values is that each one of them depends on n parameters (n being the number of players), which are interpreted as the individual tendencies of the players to form coalitions. A second main aspect is that they apply a multinomial probabilistic value in the quotient game that arises once the coalition structure is actually formed, but share within each union the payoff so obtained by applying the Shapley value to a game that concerns only the players of that union.

Using the Shapley value looks highly interesting in voting contexts. Indeed, once an alliance is formed—and, especially, if it supports a coalition government—, cabinet ministries, parliamentary and institutional positions, budget management, and other political responsibilities should be distributed among the members of the alliance *efficiently*, so the Shapley value is useful here. We thus evaluate not only the parliamentary power of the alliance but also the way to share this power allocation among its members. This two–step procedure (first power, then cake) offers a balanced approach for dealing with coalitional bargaining.

We emphasize the role of these new values as a consistent alternative to classical coalitional values. The fact that they are based on tendency profiles provides new tools to encompass a wide variety of situations that derive from players' personalities when playing a game. In this sense, the coalitional multinomial probabilistic values constitute a significant generalization of the symmetric coalitional binomial semivalues, whose monoparametric condition implies a limited capability to analyze such situations. Of course, these situations cannot be analyzed, without modifying the game, by means of classical, nonparametric values, which can be concerned only with the (formal) structure of the game.

Finally, it is worth mentioning that the greater ability of the new values to deal with strategic features is achieved without losing standard properties satisfied by classical coalitional values, like e.g. linearity, positivity, the total power property, the dummy player property, symmetry within unions, or the quotient game property.

The organization of the paper is as follows. In Section 2, a minimum of preliminaries is provided. Section 3 is devoted to define and to study the family of coalitional multinomial probabilistic values, including an axiomatic characterization. In Section 4 we restrict such values to simple games. Section 5 presents a computation procedure for these values by means of multilinear extensions. Section 6 contains an application of the coalitional multinomial probabilistic values to the analysis of the Catalonia Parliament (Legislature 2003–2007). Section 7 includes concluding remarks. All proofs are collected in Appendix 1.

2 Preliminaries

We assume that the reader is generally familiar with the basic ideas of the cooperative game theory (including simple games, which will be briefly revised in Section 4).

2.1 Games and values

Let N be a finite set of *players*, usually denoted as $N = \{1, 2, \ldots, n\}$, and 2^N be the set of *coalitions* (subsets of N). A *(cooperative) game* in N is a function $v : 2^N \to \mathbb{R}$ that assigns a real number $v(S)$ to each coalition $S \subseteq N$, with $v(\emptyset) = 0$. A game v is *monotonic* if $v(S) \leq v(T)$ whenever $S \subset T \subseteq N$. Player $i \in N$ is a *dummy in* v if $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$, and *null in* v if, moreover, $v(\{i\}) = 0$. Two players $i, j \in N$ are *symmetric in* v if $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

Endowed with the natural operations for real–valued functions, i.e. $v + v'$ and λv for all $\lambda \in \mathbb{R}$, the set $\mathcal{G}_N$ of all games in N becomes a vector space. For every nonempty coalition $T \subseteq N$, the *unanimity game* u_T in N is defined by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise, and it is easily checked that the set of all unanimity games is a basis for $\mathcal{G}_N$.

By a *value* on $\mathcal{G}_N$ we will mean a map $f : \mathcal{G}_N \to \mathbb{R}^N$, that assigns to every game v a vector $f[v]$ with components $f_i[v]$ for all $i \in N$.

The *multilinear extension* [21] of a game $v \in \mathcal{G}_N$ is the real–valued function defined on $\mathbb{R}^N$ by

$$f(x_1, x_2, \ldots, x_n) = \sum_{S \subseteq N} \Big[\prod_{i \in S} x_i \prod_{j \in N \backslash S} (1 - x_j) \Big] v(S). \text{ [2]} \tag{1}$$

2.2 Multinomial probabilistic values

The multinomial probabilistic values form a subfamily of *probabilistic values* [30]. They were introduced in reliability by Puente [27] (see also [19]) as follows. Let $N = \{1, 2, \ldots, n\}$ and let $\mathbf{p} \in [0, 1]^n$, that is, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ with $0 \leq p_i \leq 1$ for $i = 1, 2, \ldots, n$, be given. Then the coefficients

$$p^i_S = \prod_{j \in S} p_j \prod_{\substack{k \in N \backslash S \\ k \neq i}} (1 - p_k) \quad \text{for all } i \in N \text{ and } S \subseteq N \backslash \{i\} \tag{2}$$

(where the empty product, arising if $S = \emptyset$ or $S = N \backslash \{i\}$, is taken to be 1) define a probabilistic value on $\mathcal{G}_N$ that is called the $\mathbf{p}$–*multinomial probabilistic value* and will be denoted here as $\lambda^{\mathbf{p}}$. Its action is then given by

$$\lambda^{\mathbf{p}}_i[v] = \sum_{S \subseteq N \backslash \{i\}} \Big[\prod_{j \in S} p_j \prod_{\substack{k \in N \backslash S \\ k \neq i}} (1 - p_k) \Big] [v(S \cup \{i\}) - v(S)] \quad \text{for all } i \in N \text{ and } v \in \mathcal{G}_N. \tag{3}$$

In particular, the action of $\lambda^{\mathbf{p}}$ on a unanimity game u_T is given by

$$\lambda^{\mathbf{p}}_i[u_T] = \prod_{\substack{j \in T \\ j \neq i}} p_j \quad \text{if } i \in T \qquad \text{and} \qquad \lambda^{\mathbf{p}}_i[u_T] = 0 \quad \text{otherwise.} \tag{4}$$

As was announced in Section 1, we will attach to each p_i the meaning of *tendency of player i to form coalitions*, and thus we will say that $\mathbf{p}$ is a *(tendency) profile* in N. The components of $\mathbf{p}$ will be assumed to be independent of each other. From Eq. (2) it follows that coefficient p^i_S, the probability of i to join S according to [30], is an increasing function of the tendency of each member of S to form coalitions and a decreasing function of the tendency in this sense of each outside player, i.e. each member of $N \backslash (S \cup \{i\})$.

[2] The term "multilinear" means that, for each $i \in N$, the function is linear in x_i, that is, of the form $f(x_1, x_2, \ldots, x_n) = g_i(x_1, x_2, \ldots, \widehat{x_i}, \ldots, x_n) x_i + h_i(x_1, x_2, \ldots, \widehat{x_i}, \ldots, x_n)$.

2.3 Games with a coalition structure

Given $N = \{1, 2, \ldots, n\}$, we will denote by $B(N)$ the set of all partitions of N. Each $B \in B(N)$ is called a *coalition structure* in N, and each member of B is called a *union*. The so–called *trivial coalition structures* are $B^n = \{\{1\}, \{2\}, \ldots, \{n\}\}$ (individual coalitions) and $B^N = \{N\}$ (grand coalition). A *(cooperative) game with a coalition structure* is a pair $[v; B]$, where $v \in \mathcal{G}_N$ and $B \in B(N)$ for a given N. Each partition B gives a pattern of cooperation among players. We denote by $\mathcal{G}_N^{cs} = \mathcal{G}_N \times B(N)$ the set of all games with a coalition structure and player set N.

If $[v; B] \in \mathcal{G}_N^{cs}$ and $B = \{B_1, B_2, \ldots, B_m\}$, the *quotient game* v^B is the game played by the unions or, rather, by the *quotient set* $M = \{1, 2, \ldots, m\}$ of their representatives, as follows:

$$v^B(R) = v\left(\bigcup_{r \in R} B_r\right) \quad \text{for all } R \subseteq M.$$

By a *coalitional value* on $\mathcal{G}_N^{cs}$ we will mean a map $g : \mathcal{G}_N^{cs} \to \mathbb{R}^N$, which assigns to every pair $[v; B]$ a vector $g[v; B]$ with components $g_i[v; B]$ for each $i \in N$.

If f is a value on $\mathcal{G}_N$ and g is a coalitional value on $\mathcal{G}_N^{cs}$, it is said that g is a *coalitional value of* f (or a *coalitional f–value*, for short) iff $g[v; B^n] = f[v]$ for all $v \in \mathcal{G}_N$.

3 Coalitional multinomial probabilistic values

We introduce here a new family of coalitional values, establish their main common properties and discuss some aspects.

3.1 Concept

When trying to extend multinomial probabilistic values to games with a coalition structure, a first question refers to the way by which, given a coalition structure $B = \{B_1, B_2, \ldots, B_m\}$ and a tendency profile $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ in N, a tendency profile $\overline{\mathbf{p}} = (\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_m)$ is defined in the quotient set M. Only two natural constraints will be imposed.

Definition 3.1 Given profile $\mathbf{p}$ and coalition structure B in N, profile $\overline{\mathbf{p}}$ is *a profile induced by* $\mathbf{p}$ in M iff: (i) each $\overline{p}_r$ depends only on those p_i such that $i \in B_r$; and (ii) if, for a given $B_r \in B$, there is some $q \in [0, 1]$ such that $p_i = q$ for all $i \in B_r$ then $\overline{p}_r = q$.[3]

Of course, the interpretation attached to $p_1, p_2, \ldots, p_n$ in Subsection 2.2 will be kept in passing to $\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_m$. Among the infinitely many possibilities to define an induced profile $\overline{\mathbf{p}} = (\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_m)$ in terms of $\mathbf{p}$, let us suggest a few ones only as a matter of example:

(α) $\overline{p}_r = \min_{i \in B_r}\{p_i\}$

(β) $\overline{p}_r = p_i$ for some $i \in B_r$ arbitrarily chosen

(γ) $\overline{p}_r = \frac{1}{b_r} \sum_{i \in B_r} p_i$, where $b_r = |B_r|$

(δ) $\overline{p}_r = \max_{i \in B_r}\{p_i\}$

[3] Condition (ii) guarantees consistency with symmetric coalitional binomial semivalues (cf. [10, 3, 11]).

We will not try to discuss here which is the best option (if it exists). It may happen that different situations require different ways to define $\overline{p}_1, \overline{p}_2, \dots, \overline{p}_m$. Even more, nothing prevents different unions to make different choices when defining their respective tendencies in the quotient game—this is the reason for having imposed conditions (i) and (ii) *for each union.* Fortunately, the great freedom in this choice will not affect the validity of the theoretical results: the theory developed in this paper will be of application provided that $\overline{\mathbf{p}}$ is a profile induced by $\mathbf{p}$, no matter by which mechanism.[4]

However, in order to avoid using a more cumbersome notation and any ambiguity, we will implicitly assume from now on in the theoretical development that, for any given N, a unique mechanism has been chosen to induce, given $\mathbf{p}$ and B in N, a profile $\overline{\mathbf{p}}$ in M.

Definition 3.2 Let $N = \{1, 2, \dots, n\}$ be a finite player set and $\mathbf{p} = (p_1, p_2, \dots, p_n)$ be a profile in N. The coalitional $\mathbf{p}$–*multinomial probabilistic value* is the coalitional value $\Lambda^{\mathbf{p}} : \mathcal{G}_N^{cs} \to \mathbb{R}^N$ defined as follows. If $[v; B] \in \mathcal{G}_N^{cs}$ and $i \in B_k \in B$,

$$\Lambda_i^{\mathbf{p}}[v; B] = \sum_{R \subseteq M \backslash \{k\}} \Big[\prod_{j \in R} \overline{p}_j \prod_{\substack{h \in M \backslash R \\ h \neq k}} (1 - \overline{p}_h) \Big] \sum_{T \subseteq B_k \backslash \{i\}} \frac{v(Q \cup T \cup \{i\}) - v(Q \cup T)}{b_k \binom{b_k - 1}{t}}, \qquad (5)$$

where $\overline{\mathbf{p}}$ is the profile induced by $\mathbf{p}$ in M, $Q = \bigcup_{r \in R} B_r$, $b_k = |B_k|$, and $t = |T|$.

Remark 3.3 The coalitional $\mathbf{p}$–multinomial probabilistic value $\Lambda^{\mathbf{p}}$ yields the result of a *two–step bargaining procedure* analogous to that used in [23, 24] and also in [1, 10, 3, 11]. Here we first apply the $\overline{\mathbf{p}}$–multinomial probabilistic value $\lambda^{\overline{\mathbf{p}}}$ in the quotient game to obtain a payoff for each union; next, we use within each union the Shapley value, denoted here by φ, to share the payoff efficiently by applying this value to a *reduced game* played in that union.[5] The proof is essentially the same as in [23, 24, 1, 10, 3, 11] and will be omitted. Moreover, the new value coincides with one of these two values when the coalition structure is trivial. Indeed, it is straightforward to check that, for all $v \in \mathcal{G}_N$ and any profile $\mathbf{p}$ in N,

(i) $\Lambda^{\mathbf{p}}[v; B^N] = \varphi[v]$;

(ii) $\Lambda^{\mathbf{p}}[v; B^n] = \lambda^{\mathbf{p}}[v]$, i.e. $\Lambda^{\mathbf{p}}$ is a *coalitional* $\lambda^{\mathbf{p}}$*–value.*

Example 3.4 To illustrate the two–step procedure, we consider a very simple instance: let $n = 4$ and game v be defined by $v(S) = 1$ if $|S| \geq 3$ and $v(S) = 0$ otherwise. Let $\mathbf{p} = (0.4, 0.8, 0.6, 0.2)$ and $B = \{\{1, 2\}, \{3\}, \{4\}\}$. Let $M = \{1, 3, 4\}$, where 1 denotes the representative of $\{1, 2\}$ and the remaining players represent themselves (a nonstandard but easier notation in particular cases like this). The quotient game is given by $v^B(R) = 1$ if $R = \{1, 3\}, \{1, 4\}, \{1, 3, 4\}$ and $v^B(R) = 0$ otherwise.

Let us choose e.g. option (γ), according to which each $\overline{p}_r$ is the arithmetic mean of the p_i's for all $i \in B_r$, and take therefore as induced profile $\overline{\mathbf{p}} = (0.6, 0.6, 0.2)$. By applying the

[4]Although the results obtained in practice will depend in general of this mechanism (cf. Example 3.4).

[5]The reduced game is as follows. First, if $S \subseteq B_k$, let $\overline{v}_S^B$ be the *pseudoquotient game* in M defined by

$$\overline{v}_S^B(R) = v\Big[(\bigcup_{r \in R} B_r) \backslash (B_k \backslash S)\Big] \quad \text{for each} \quad R \subseteq M.$$

This game is the modification of the standard quotient game v^B when S replaces union B_k, as if the players of $B_k \backslash S$ were temporarily inactive. The *reduced game* of v in B_k, denoted by w_k, is then given by

$$w_k(S) = \lambda_k^{\overline{\mathbf{p}}}[\overline{v}_S^B] \quad \text{for each} \quad S \subseteq B_k.$$

$\overline{\mathbf{p}}$–multinomial probabilistic value $\lambda^{\overline{\mathbf{p}}}$ to the quotient game, using Eq. (3) we have

$$\lambda^{\overline{\mathbf{p}}}[v^B] = (0.68, 0.48, 0.24).$$

Then, in the case of singletons $\{3\}$ and $\{4\}$, the payoff received by the union in the quotient game is allocated to its unique member. For union $\{1, 2\}$, the fact that its players are symmetric in v implies that they share the payoff equally. Thus (see Fig. 1),

$$\Lambda^{\mathbf{p}}[v; B] = (0.34, 0.34, 0.48, 0.24).^6$$

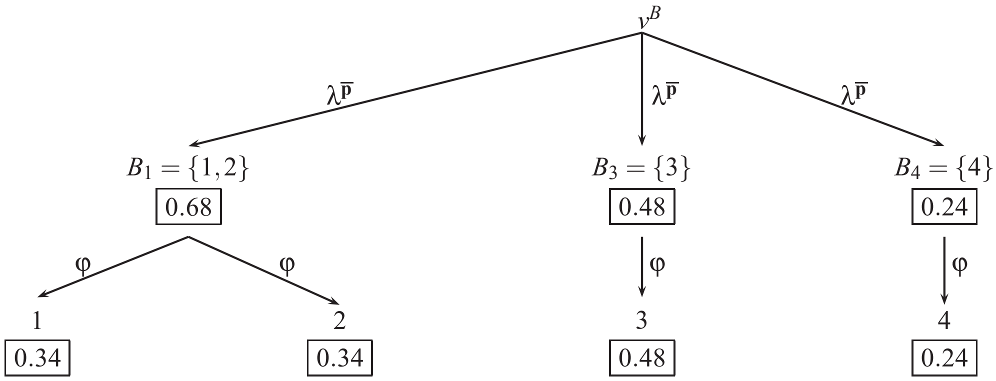

Figure 1: The two–step procedure: first $\lambda^{\overline{\mathbf{p}}}$, then φ

What has happened with the symmetry of all players in the original game? Players 3 and 4 are symmetric also in the coalition structure, and hence in the quotient game, but their final payoffs differ because of their different parameters. Instead, players 1 and 2 earn a common final payoff, in spite of having different parameters, because they are in the same union.

3.2 Properties

The rest of this section is devoted to the discussion of common properties for all coalitional multinomial probabilistic values. Our main aim is to reach an axiomatic characterization for each one of these values. We begin by considering standard properties for a generic coalitional value g on $\mathcal{G}_N^{cs}$:

- *linearity*: $g[\alpha v + \beta v'; B] = \alpha g[v; B] + \beta g[v'; B]$ for all $\alpha, \beta \in \mathbb{R}$, v, v' and B
- *positivity*: if v is monotonic, then $g[v; B] \geq 0$ for all B
- *dummy player property*: if i is a dummy in v, then $g_i[v; B] = v(\{i\})$ for all B
- *symmetry within unions:* if $i, j \in B_k$ are symmetric players in v then

$$g_i[v; B] = g_j[v; B]$$

- *symmetry in the quotient game*: if $r, s \in M$ are symmetric players in game v^B then

$$\sum_{i \in B_r} g_i[v; B] = \sum_{j \in B_s} g_j[v; B]$$

- *quotient game property*: for all $[v; B] \in \mathcal{G}_N^{cs}$ and all $k \in M$,

$$\sum_{i \in B_k} g_i[v; B] = g_k[v^B; B^m]. \tag{6}$$

[6] Using other mechanisms to define $\overline{\mathbf{p}}$ would give different final results: for example, $\Lambda^{\mathbf{p}}[v; B] = (0.34, 0.34, 0.32, 0.16)$ for option (α), $\Lambda^{\mathbf{p}}[v; B] = (0.34, 0.34, 0.64, 0.32)$ for option (δ), and one or another of these results for option (β), depending on which player's tendency (1 or 2) is chosen to define $\overline{p}_1$.

Remark 3.5 In principle, this last property makes sense only for coalitional values defined for all N; in such a case, one generally abuses the notation and uses a unique symbol g on both $\mathcal{G}_N^{CS}$ and $\mathcal{G}_M^{CS}$. However, the property also makes sense for a coalitional value g on a given $\mathcal{G}_N^{cs}$ provided, at least, that it induces a coalitional value $\overline{g}$ on $\mathcal{G}_M^{cs}$ for each $B \in B(N)$.

And this is precisely the case of the coalitional multinomial probabilistic values. Indeed, let $\mathbf{p}$ be a profile and B a coalition structure (both in N), M the quotient set, and $\overline{\mathbf{p}}$ the profile in M induced by $\mathbf{p}$. Profile $\overline{\mathbf{p}}$ defines a (multinomial probabilistic) value $\lambda^{\overline{\mathbf{p}}}$ on $\mathcal{G}_M$ and hence a coalitional (multinomial probabilistic) value $\Lambda^{\overline{\mathbf{p}}}$ on $\mathcal{G}_M^{cs}$ such that, by Remark 3.3(ii),

$$\Lambda^{\overline{\mathbf{p}}}[w; B^m] = \lambda^{\overline{\mathbf{p}}}[w] \quad \text{for all } w \in \mathcal{G}_M.$$

Now, two additional nonstandard properties will be considered.

Definition 3.6 Let $\mathbf{p}$ be a profile in N. A coalitional value g on $\mathcal{G}_N^{CS}$ satisfies the *coalitional* $\mathbf{p}$*–multinomial total power property* iff, for all $[v; B] \in \mathcal{G}_N^{cs}$,

$$\sum_{i \in N} g_i[v; B] = \sum_{k \in M} \sum_{R \subseteq M \setminus \{k\}} \Big[\prod_{j \in R} \overline{p}_j \prod_{\substack{h \in M \setminus R \\ h \neq k}} (1 - \overline{p}_h) \Big] [v(Q \cup B_k) - v(Q)],$$

where $\overline{\mathbf{p}}$ is the profile induced by $\mathbf{p}$ in M and $Q = \bigcup_{r \in R} B_r$.

Definition 3.7 Let $\mathbf{p}$ be a profile in N. A coalitional value g on $\mathcal{G}_N^{CS}$ satisfies the *property of* $\mathbf{p}$*–weighted payoffs for quotients of unanimity games* iff, for any $B \in B(N)$ and any nonempty $T \subseteq N$,

$$\overline{p}_k \sum_{i \in B_k} g_i[u_T; B] = \overline{p}_\ell \sum_{j \in B_\ell} g_j[u_T; B] \quad \text{for all } B_k, B_\ell \in B \text{ intersecting } T,$$

where $\overline{\mathbf{p}}$ is the profile induced by $\mathbf{p}$ in M.

Our axiomatic characterization theorem essentially holds for any coalitional $\mathbf{p}$–multibinary probabilistic value with a *positive* profile $\mathbf{p} = (p_1, p_2, \ldots, p_n)$.[7] "Positive" merely means that $p_i > 0$ for $i = 1, 2, \ldots, n$. To ease the proof of this theorem it is convenient to state a preliminary result, which holds for any profile.

Proposition 3.8 *Let* $\mathbf{p}$ *be a profile in* N*. The coalitional* $\mathbf{p}$*–multinomial probabilistic value satisfies positivity and the quotient game property.*

Theorem 3.9 *(Axiomatic characterization of each coalitional multinomial probabilistic value with positive profile) Let* $\mathbf{p}$ *be a positive profile in a given player set* N*. Then there is a unique coalitional value on* $\mathcal{G}_N^{cs}$ *that satisfies linearity, the dummy player property, symmetry within unions, the coalitional* $\mathbf{p}$*–multinomial total power property, and the property of* $\mathbf{p}$*–weighted payoffs for quotients of unanimity games. It is the coalitional* $\mathbf{p}$*–multinomial probabilistic value.*

Remark 3.10 Theorem 3.9, where linearity could be replaced with additivity, generalizes Theorem 1 in [1] and Theorem 3.5 in [10] (i.e., Theorem 3.6 in [11]). We have checked the logical independence of the axiomatic system for Theorem 3.9 in [14].

[7] This is revisited in the proof of Theorem 3.9.

3.3 Discussion

We include here some comments on the properties of coalitional multinomial probabilistic values considered in the previous subsection.

Remark 3.11 The reader might be surprised because the possibility of having $p_i \neq p_j$ does not cause any trouble for *symmetry within unions*. However, this is not so striking since, as follows from Eq. (5), if $i \in B_k$ then $\Lambda_i^{\mathbf{p}}[v; B]$ does not depend on $\overline{p}_k$ and hence on either p_i or p_j for any other $j \in B_k$. This is a special feature of these coalitional values: the payoff that they allocate to a player $i \in B_k$ depends only on the position of this player in the original game and the coalition structure, and also on the tendencies of the other unions which, in turn, depend only on the tendencies of their respective members according to 3.1(i).

Remark 3.12 With our notation Eq. (6) becomes, for the coalitional $\mathbf{p}$–multinomial probabilistic value,

$$\sum_{i \in B_k} \Lambda_i^{\mathbf{p}}[v; B] = \Lambda_k^{\overline{\mathbf{p}}}[v^B; B^m] = \lambda_k^{\overline{\mathbf{p}}}[v^B],$$

so the *quotient game property* is in this case a sort of *local efficiency* (that is, for unions).

Remark 3.13 The *coalitional* $\mathbf{p}$*–multinomial total power property* is the natural extension of a *total power property* that was first stated for the Banzhaf value [22] (cf. also [18, 17]) and gave rise later, among others, to the coalitional q–binomial total power property [10, 3, 11] and the $\mathbf{p}$–multinomial total power property [12, 13]. It reduces to this latter if $B = B^n$ but it also extends *efficiency*, to which it reduces if $B = B^N$.

Moreover, as $\Lambda^{\overline{\mathbf{p}}}$ is a coalitional $\lambda^{\overline{\mathbf{p}}}$–value, that property is a consequence of the quotient game property—maybe more compelling at first glance—and can be simply written as

$$\sum_{i \in N} \Lambda_i^{\mathbf{p}}[v; B] = \sum_{k \in M} \lambda_k^{\overline{\mathbf{p}}}[v^B],$$

thus establishing that the total amount shared according to $\Lambda^{\mathbf{p}}$ in $[v; B]$ coincides with the amount shared according to $\lambda^{\overline{\mathbf{p}}}$ in the quotient game v^B.

Remark 3.14 The coalitional $\mathbf{p}$–multinomial probabilistic value *fails to satisfy* the standard property of *symmetry in the quotient game*. This property would state that if $k, \ell \in M$ are symmetric players in v^B then

$$\sum_{i \in B_k} \Lambda_i^{\mathbf{p}}[v; B] = \sum_{j \in B_\ell} \Lambda_j^{\mathbf{p}}[v; B].$$

However, it is easy to see (even for $n = 2$) that this is not true in general. A simple argument shows that a coalitional multinomial probabilistic value satisfies symmetry in the quotient game iff it reduces to a symmetric coalitional binomial semivalue, so that *symmetry in the quotient game characterizes the family of symmetric coalitional binomial semivalues within the class of coalitional multinomial probabilistic values.*

It should be clear that the failure is essentially due to the fact that, in general, neither $\lambda^{\overline{\mathbf{p}}}$ nor $\lambda^{\mathbf{p}}$ satisfy anonymity. But this is precisely the positive reason by which we are considering here multinomial probabilistic values instead of binomial semivalues: the possibility, offered by profiles $\mathbf{p}$ and $\overline{\mathbf{p}}$, to discriminate among players and unions, respectively.

Remark 3.15 The property of **p**–*weighted payoffs for quotients of unanimity games* does not hold for general quotients. The corresponding statement, that could be called "**p**–weighted symmetry in the quotient game" would look as follows: if $k, \ell \in M$ are symmetric players in v^B then

$$\bar{p}_k \sum_{i \in B_k} \Lambda_i^{\mathbf{p}}[v; B] = \bar{p}_\ell \sum_{j \in B_\ell} \Lambda_j^{\mathbf{p}}[v; B].$$

Again it is easy to see (even for $n = 2$) that this is not true in general. Also it can be easily shown that *the* **p**–*coalitional multinomial probabilistic value satisfies* **p**–*weighted symmetry in the quotient game iff it is a symmetric coalitional binomial semivalue*, but then the property becomes just symmetry in the quotient game (and $p_1 = p_2 = \cdots = p_n$).

4 Simple games

Simple games constitute an especially interesting class of cooperative games. Not only as a test bed for many cooperative concepts, but also for the variety of their interpretations, often far from game theory. In particular, they have been intensively applied to describe and analyze collective decision–making mechanisms and the notion of voting power in the sense of, e.g., [29] and [5].

A cooperative game v in N is *simple* iff it is monotonic, $v(S) \in \{0, 1\}$ for every $S \subseteq N$, and $v(N) = 1$. A coalition $S \subseteq N$ is *winning in* v if $v(S) = 1$ (otherwise it is called *losing*), and $W(v)$ denotes the set of winning coalitions in v. Due to monotonicity, the subset $W^m(v)$ of all *minimal* winning coalitions determines $W(v)$ and hence the game. A simple game v is a *weighted majority game* iff there are nonnegative *weights* $w_1, w_2, \ldots, w_n$ allocated to the players and a positive *quota* q such that

$$v(S) = 1 \quad \text{iff} \quad \sum_{i \in S} w_i \geq q.$$

We then write $v \equiv [q; w_1, w_2, \ldots, w_n]$, although this representation is never unique.

Let $\mathcal{SG}_N$ denote the set of all simple games in a given player set N. It becomes a lattice under the standard composition laws given in $\mathcal{G}_N$ by $(v \vee v')(S) = \max\{v(S), v'(S)\}$ and $(v \wedge v')(S) = \min\{v(S), v'(S)\}$. A *power index* on $\mathcal{SG}_N$ is a function $f : \mathcal{SG}_N \to \mathbb{R}^N$ (we refer to [6] for a good survey on power indices and to [15] for the use of semivalues as such). The restriction to $\mathcal{SG}_N$ of any value on $\mathcal{G}_N$ is, of course, a power index, which will be denoted by the same symbol. We will say that a power index f satisfies the *transfer property* if

$$f[v \vee v'] + f[v \wedge v'] = f[v] + f[v'] \quad \text{for all } v, v' \in \mathcal{SG}_N.$$

The multinomial probabilistic values represent a natural extension, to all cooperative games, of the *Banzhaf* α*–indices* considered in [7] on the class of simple games. *Assessments* $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ used there are the analogue of our tendency profiles $\mathbf{p} = (p_1, p_2, \ldots, p_n)$. In the case of simple games, each p_i can also be attached the meaning of *tendency of player i to support a given proposal* when this proposal is submitted to the approval of the collectivity.

Now let $\mathcal{SG}_N^{cs} = \mathcal{SG}_N \times B(N)$ be the set of all simple games with a coalition structure in N. A *coalitional power index* on $\mathcal{SG}_N^{cs}$ is a function $g : \mathcal{SG}_N^{cs} \to \mathbb{R}^N$. All properties stated for coalitional values in this paper (except linearity), as well as the natural extension of the transfer property,

$$g[v \vee v'; B] + g[v \wedge v'; B] = g[v; B] + g[v'; B] \quad \text{for all } v, v' \in \mathcal{SG}_N \text{ and } B \in B(N),$$

make sense for coalitional power indices, and the restriction to $\mathcal{SG}_N^{cs}$ of any coalitional value on $\mathcal{G}_N^{cs}$ is a coalitional power index that will be denoted by the same symbol.

Our next result states the analogue of Theorem 3.9 when we restrict the coalitional multinomial probabilistic values to simple games.[8]

Theorem 4.1 *(Axiomatic characterization of each coalitional multinomial probabilistic power index with positive profile) Let* $\mathbf{p}$ *be a positive profile in a given player set* N*. Then there is a unique coalitional power index on* $\mathcal{SG}_N^{cs}$ *that satisfies the transfer property, the dummy player property, symmetry within unions, the coalitional* $\mathbf{p}$*–multinomial total power property, and the property of* $\mathbf{p}$*–weighted payoffs for quotients of unanimity games. It is (the restriction of) the coalitional* $\mathbf{p}$*–multinomial probabilistic value. Moreover, this coalitional power index satisfies positivity and the quotient game property.*

5 A computational procedure

In this section we present a method to compute any coalitional multinomial probabilistic value by means of the multilinear extension of the game. For the sake of completeness, we recall from [27, 19] that, if $f(x_1, x_2, \ldots, x_n)$ is the multilinear extension of game v in N and $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ is a profile in N, then

$$\lambda_i^{\mathbf{p}}[v] = \frac{\partial f}{\partial x_i}(p_1, p_2, \ldots, p_n) \quad \text{for each } i \in N. \tag{7}$$

Theorem 5.1 *Let* $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ *be a profile in* N*,* $[v; B] \in \mathcal{G}_N^{cs}$ *a game with a coalition structure in* N*, and* $\overline{\mathbf{p}}$ *the profile induced by* $\mathbf{p}$ *in* M*. Then the following steps lead to the coalitional* $\mathbf{p}$*–multinomial probabilistic value of any player* $i \in B_k \in B$ *in* $[v; B]$ *for any* k*.*

1. *Obtain the multilinear extension* $f(x_1, x_2, \ldots, x_n)$ *of game* v*.*
2. *For every* $r \neq k$ *and all* $h \in B_r$*, replace the variable* x_h *with* y_r*. This yields a new function of* x_j *for* $j \in B_k$ *and* y_r *for* $r \in M \backslash \{k\}$*.*
3. *In this new function, reduce to 1 all higher exponents, i.e. replace with* y_r *each* y_r^q *such that* $q > 1$*. This gives a new multilinear function denoted as* $g_k((x_j)_{j \in B_k}, (y_r)_{r \in M \backslash \{k\}})$*.*
4. *In the function obtained in step 3, substitute each* y_r *by* $\overline{p}_r$*. This provides a new function defined by* $\alpha_k((x_j)_{j \in B_k}) = g_k((x_j)_{j \in B_k}, (\overline{p}_r)_{r \in M \backslash \{k\}})$*.*
5. *The coalitional* $\mathbf{p}$*–multinomial probabilistic value of player* $i \in B_k$ *in* $[v; B]$ *is given by*

$$\Lambda_i^{\mathbf{p}}[v; B] = \int_0^1 \frac{\partial \alpha_k}{\partial x_i}(z, z, \ldots, z) dz.$$

Example 5.2 Let us consider the 5–person weighted majority game $v \equiv [68; 46, 42, 23, 15, 9]$ and the coalition structure $B = \{\{1\}, \{2, 3, 5\}, \{4\}\}$. We will compute $\lambda^{\mathbf{p}}[v]$ and $\Lambda^{\mathbf{p}}[v; B]$ for any profile $\mathbf{p}$ and the profile $\overline{\mathbf{p}}$ induced by $\mathbf{p}$ in $M = \{1, 2, 3\}$.

The set of minimal winning coalitions of the game is

$$W^m(v) = \{\{1, 2\}, \{1, 3\}, \{1, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}\}$$

and the multilinear extension of v is

$$\begin{aligned} f(x_1, x_2, x_3, x_4, x_5) &= x_1x_2 + x_1x_3 - x_1x_2x_3 + x_1x_4x_5 + x_2x_3x_4 + x_2x_3x_5 - \\ &- x_1x_2x_3x_4 - x_1x_2x_3x_5 - x_1x_2x_4x_5 - x_1x_3x_4x_5 - x_2x_3x_4x_5 + 2x_1x_2x_3x_4x_5. \end{aligned}$$

[8] In some manner, for simple games the transfer property replaces linearity. We have also checked the logical independence of the axiomatic system for Theorem 4.1 in [14].

The calculation of $\lambda^{\mathbf{p}}[v]$ derives from Eq. (7):

$$\begin{aligned}
\lambda_1^{\mathbf{p}}[v] &= p_2 + p_3 - p_2p_3(1 + p_4 + p_5 - p_4p_5) + p_4p_5(1 - p_2 - p_3 + p_2p_3),\\
\lambda_2^{\mathbf{p}}[v] &= p_1 + p_3(p_4 + p_5) - p_1p_3(1 + p_4 + p_5 - p_4p_5) - p_4p_5(p_1 + p_3 - p_1p_3),\\
\lambda_3^{\mathbf{p}}[v] &= p_1 + p_2(p_4 + p_5) - p_1p_2(1 + p_4 + p_5 - p_4p_5) - p_4p_5(p_1 + p_2 - p_1p_2),\\
\lambda_4^{\mathbf{p}}[v] &= p_2p_3(1 - p_1 - p_5 + p_1p_5) + p_1p_5(1 - p_2 - p_3 + p_2p_3),\\
\lambda_5^{\mathbf{p}}[v] &= p_2p_3(1 - p_1 - p_4 + p_1p_4) + p_1p_4(1 - p_2 - p_3 + p_2p_3).
\end{aligned}$$

For $\Lambda^{\mathbf{p}}[v;B]$ Theorem 5.1 applies. Steps 1–3 give $g_1(x_1, y_2, y_3) = y_2$, $g_3(x_4, y_1, y_2) = y_2$, and

$$\begin{aligned}
&g_2(x_2, x_3, x_5, y_1, y_3) = (x_2 + x_3)y_1 + x_5y_1y_3 + x_2x_3(-y_1 + y_3 - y_1y_3) - x_2x_5y_1y_3\\
&- x_3x_5y_1y_3 + x_2x_3x_5(1 - y_1 - y_3 + 2y_1y_3).
\end{aligned}$$

Step 4 leads to $\alpha_1(x_1) = \overline{p}_2$, $\alpha_3(x_4) = \overline{p}_2$, and

$$\begin{aligned}
&\alpha_2(x_2, x_3, x_5) = (x_2 + x_3)\overline{p}_1 + x_5\overline{p}_1\overline{p}_3 + x_2x_3(-\overline{p}_1 + \overline{p}_3 - \overline{p}_1\overline{p}_3) - x_2x_5\overline{p}_1\overline{p}_3\\
&- x_3x_5\overline{p}_1\overline{p}_3 + x_2x_3x_5(1 - \overline{p}_1 - \overline{p}_3 + 2\overline{p}_1\overline{p}_3).
\end{aligned}$$

Step 5 yields $\Lambda_1^{\mathbf{p}}[v;B] = 0$, $\Lambda_4^{\mathbf{p}}[v;B] = 0$ and, having in mind that $\overline{p}_1 = p_1$ and $\overline{p}_3 = p_4$,

$$\Lambda_2^{\mathbf{p}}[v;B] = \int_0^1 \frac{\partial \alpha_2}{\partial x_2}(z,z,z)dz = \frac{1}{3} + \frac{p_1 + p_4 - 2p_1p_4}{6}$$

and, similarly,

$$\Lambda_3^{\mathbf{p}}[v;B] = \frac{1}{3} + \frac{p_1 + p_4 - 2p_1p_4}{6} \quad \text{and} \quad \Lambda_5^{\mathbf{p}}[v;B] = \frac{1}{3} - \frac{p_1 + p_4 - 2p_1p_4}{3}\ .$$

6 An application to the political analysis

We consider here the Catalonia Parliament in Legislature 2003–2007, prematurely finished.[9]

The Catalan political life is rich and fairly complicated. There are two basic ideological axes to locate parties: the classical left–to–right axis and a crossed axis going from Spanish centralism to Catalanism (see Fig. 2). Restrictions to the cooperation between parties arise from ideological affinities or incompatibilities, strategic behavior, and even from links with parties of national level and the influence of the national politics developed in Madrid.

Five parties elected members to the Catalonia Parliament (135 seats) in the elections held on November 16, 2003. A brief ideological description of the parties and the seat distribution are as follows.

1: CiU (Convergència i Unió), Catalan nationalist middle–of–the–road coalition of two federated parties, CDC and UDC: 46 seats.

2: PSC (Partit dels Socialistes de Catalunya), moderate left–wing socialist party, federated to the Partido Socialista Obrero Español: 42 seats.

3: ERC (Esquerra Republicana de Catalunya), radical Catalan nationalist left–wing party: 23 seats.

4: PPC (Partit Popular de Catalunya), conservative party, Catalan delegation of the Partido Popular: 15 seats.

5: ICV (Iniciativa per Catalunya–Verds), coalition of Catalan eurocommunist parties, federated to Izquierda Unida, and ecologist groups ("Verds"): 9 seats.

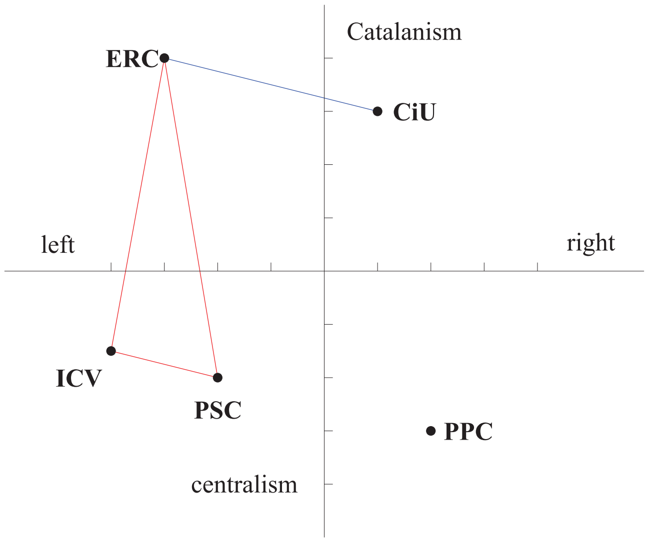

Figure 2: Position of parties in a two-dimensional ideological space

Under the standard absolute majority rule, and assuming voting discipline within parties, the structure of this parliamentary body can be represented by the weighted majority game

$$v \equiv [68; 46, 42, 23, 15, 9].$$

Therefore, the strategic situation is given by

$$W^m(v) = \{\{1,2\}, \{1,3\}, \{1,4,5\}, \{2,3,4\}, \{2,3,5\}\}.$$

Thus, in the fall of 2003, a main feature of the Catalonia Parliament issued from the elections was the absence of a party enjoying absolute majority, so a coalition government was expected to form. We will not try to give here a full description of the complexity of the Catalan politics, a task more suitable for a political science article. We wish only to state that the politically most likely coalitions to form, and the corresponding coalition structures to the analysis of which we will limit ourselves, were clearly the following:

- CiU + ERC, the Catalanist majority alliance: $B_C = \{\{1,3\}, \{2\}, \{4\}, \{5\}\}$.
- PSC + ERC + ICV, the left–wing majority alliance: $B_L = \{\{1\}, \{2,3,5\}, \{4\}\}$, which was finally formed.

We would like to analyze this situation. Of course, our main interest will center on the strategic possibilities of party 3 (ERC), whose position is crucial in the two–alternative scenario we are considering.

A classical approach to study the problem would consist in using either (a) the Shapley value and the Owen value, (b) the Banzhaf value and the Owen–Banzhaf modified value, or (c) the Banzhaf value and the symmetric coalitional Banzhaf value, in order to evaluate the strategic possibilities of each party under both hypotheses (Catalanist coalition vs. left–wing coalition). The results are given in Table 1, where (–) means no coalition formation, (C) means that CiU + ERC forms, and (LW) means that PSC + ERC + ICV forms.

[9] Deep disagreements among the parties participating in the coalition government led in 2006 to the dissolution of the Catalonia Parliament and early elections. However, our analysis remains valid for Legislature 2006–2010: in spite of the different seat distribution given by the elections held on November 1, 2006, the strategic possibilities were exactly the same.

	(a)			(b)			(c)		
	(–)	(C)	(LW)	(–)	(C)	(LW)	(–)	(C)	(LW)
CiU	0.4000	0.5000	0.0000	0.6250	0.5000	0.0000	0.6250	0.6250	0
PSC	0.2333	0.0000	0.3889	0.3750	0.0000	0.3750	0.3750	0	0.4167
ERC	0.2333	**0.5000**	0.3889	0.3750	0.**5000**	0.3750	0.3750	0.3750	**0.4167**
PPC	0.0667	0.0000	0.0000	0.1250	0.0000	0.0000	0.1250	0	0
ICV	0.0667	0.0000	0.2222	0.1250	0.0000	0.1250	0.1250	0	0.1667

Table 1: Classical measures of power in the Catalonia Parliament 2003–2007

According to (a), ERC would strictly prefer joining CiU instead of PSC and ICV. The same conclusion is obtained according to (b). Instead, according to (c), ERC would strictly prefer joining PSC and ICV instead of CiU.

Using in case (c) binomial semivalues—and symmetric coalitional binomial semivalues whenever a coalition structure exists—the conclusion of the analysis is that ERC was not necessarily forced to participate in the left–wing tripartite government but would have gained more political power in joining CiU depending on the (common) tendency parameter q of the parties to form coalitions. More precisely, ERC should have chosen PSC and ICV if, and only if, q was in the interval $(\frac{5-\sqrt{5}}{10}, \frac{5+\sqrt{5}}{10})$, and hence ERC should have preferred CiU for 44.72% of possibilities. The details can be found in [11].

Now we will apply multinomial probabilistic values. All allocations needed here have been computed using the multilinear extension technique, most of them in Example 5.2.

First, it is difficult in principle to say anything of importance about the expressions for $\lambda^{\mathbf{p}}[v]$ found in Example 5.2. Fortunately, a simplification can be reasonably achieved. Indeed, the almost isolated political position of party 4 (PPC) with regard to the remaining parties strongly suggests taking $p_4 \approx 0$, which will make things easier. In particular, this assumption will reduce the set of politically *feasible* minimal winning coalitions to

$$\{1,2\}, \{1,3\} \text{ and } \{2,3,5\}.$$

Then we find

$$\begin{aligned}
\lambda^{\mathbf{p}}_1[v] &\approx p_2 + p_3 - p_2p_3(1+p_5),\\
\lambda^{\mathbf{p}}_2[v] &\approx p_1(1-p_3) + p_3p_5(1-p_1),\\
\lambda^{\mathbf{p}}_3[v] &\approx p_1(1-p_2) + p_2p_5(1-p_1),\\
\lambda^{\mathbf{p}}_4[v] &= p_2p_3(1-p_1-p_5+p_1p_5) + p_1p_5(1-p_2-p_3+p_2p_3),\\
\lambda^{\mathbf{p}}_5[v] &\approx p_2p_3(1-p_1).
\end{aligned}$$

These allocations reflect the a priori power distribution. (a) The payoff to party 1 positively depends on the interest of parties 2 and 3 to form coalitions (e.g. $\{1,2\}$ or $\{1,3\}$), but not completely because it is partially damaged if party 5 is interested in entering a coalition ($\{2,3,5\}$). (b) The payoff to party 2 is improved by either the interest of party 1 combined with the apathy of parties 3 and 5 (that would favor $\{1,2\}$) or, conversely, by the interest of these parties combined with the unconcern of party 1, which would give more chances to $\{2,3,5\}$ instead of $\{1,2\}$ and $\{1,3\}$. (c) The same is true, *mutatis mutandis*, for party 3. (d) Finally, the possibilities of party 5 clearly rest upon the interest of parties 2 and 3 and the inertia of party 1.

Summing up, the evaluation of the game in terms of multinomial probabilistic values reflects the a priori tensions existing within each feasible (minimal winning) coalition, in terms of parameters concerning the will of each party in the coalition formation process. Tendency profiles allow us therefore to overcome the formal structure of the game and try to understand why a specific coalition forms.

Now we use the coalitional multinomial probabilistic value $\Lambda^{\mathbf{p}}$. We assume in the sequel $p_4 \approx 0$ and divide the study into two stages.

First stage. We apply $\Lambda^{\mathbf{p}}[v;B]$ in cases $B = B_C$ and $B = B_L$ for any profile $\mathbf{p}$. Then

- For the Catalanist majority alliance CiU + ERC:

$$\Lambda_1^{\mathbf{p}}[v;B_C] \approx \frac{1}{2} + \frac{p_2(1-p_5)}{2} \quad \text{and} \quad \Lambda_3^{\mathbf{p}}[v;B_C] \approx \frac{1}{2} - \frac{p_2(1-p_5)}{2} \ .$$

- For the left–wing majority alliance PSC + ERC + ICV:

$$\Lambda_2^{\mathbf{p}}[v;B_L] \approx \frac{1}{3} + \frac{p_1}{6} \ , \quad \Lambda_3^{\mathbf{p}}[v;B_L] \approx \frac{1}{3} + \frac{p_1}{6} \ , \quad \text{and} \quad \Lambda_5^{\mathbf{p}}[v;B_L] \approx \frac{1}{3} - \frac{p_1}{3} \ .$$

It is not true that all players get profit, with respect to the a priori power distribution, from entering a coalition. There always exist suitable values of the p_i's that produce a damage to a given player when joining.

Incidentally, let us notice that profile $\overline{\mathbf{p}}$ does not appear in these expressions for several reasons: (a) the payoffs to the members of a union depend only on the tendencies of the remaining unions; (b) if a union B_r reduces to a singleton $\{j\}$ then $\overline{p}_r = p_j$; (c) in both B_C and B_L, only one coalition with more than one member forms, so that the payoffs to its players can be expressed, according to (b), only in terms of profile $\mathbf{p}$ for any induced profile $\overline{\mathbf{p}}$; (d) since the coalition that forms is winning, the quotient game is a dictatorship, and hence the outside players become null and get 0. In both scenarios, the payoffs sum up to 1 by local efficiency.

Second stage. Finally, we will discuss the strategic possibilities of party 3 (ERC). The power of party 3 in B_C and B_L is, respectively,

$$\frac{1}{2} - \frac{p_2(1-p_5)}{2} \quad \text{and} \quad \frac{1}{3} + \frac{p_1}{6} \ .$$

The coincidence arises when

$$3p_2p_5 - 3p_2 - p_1 + 1 = 0 \tag{8}$$

and we distinguish two cases.

(a) We have $p_1 = 1$ iff $p_2 = 0$ or $p_5 = 1$. In this "degenerate" case, politically unlikely, the sharing in B_C gives 1/2 each to parties 1 and 3, and the sharing in B_L gives 1/2 to parties 2 and 3 and 0 to party 5. These are the best options for ERC in both possibilities.

(b) If $p_1 < 1$ and hence $p_2 > 0$ and $p_5 < 1$, from Eq. (8) it follows that

$$p_5 = 1 - \frac{1-p_1}{3p_2} \ .$$

Let us take $p_1 \in [0,1)$ as a parameter and consider p_5 as a function of $p_2 \in (0,1]$ solely. These functions (*ERC indifference curves*) are all increasing and concave, tend to $-\infty$ when p_2 tends to 0^+ and would tend to 1^- if p_2 were allowed to tend to $+\infty$ (see Fig. 3). When p_1 tends to 1, point $A = (\frac{1-p_1}{3}, 0)$ tends to $(0,0)$, while point $B = (1, 1-\frac{1-p_1}{3})$ tends to $(1,1)$, and the corresponding indifference curve moves towards the left and the top. The limit case,

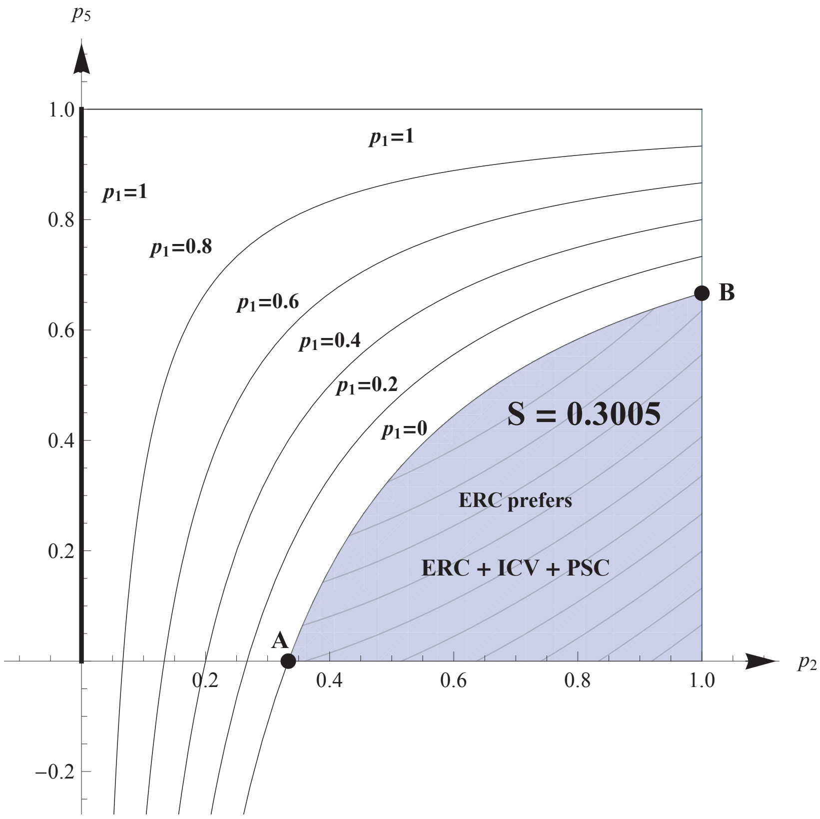

Figure 3: Indifference curves for ERC, parameterized by p_1

where the curve splits into sides $p_2 = 0$ and $p_5 = 1$ of the square $[0,1]^2$, corresponds to the degenerate case analyzed in (a).

The open region limited by lines $p_2 = 1$, $p_5 = 0$ and any indifference curve represents the cases where party 3 strictly prefers B_L for a given p_1. Its area, given by

$$S = \int_{\frac{1-p_1}{3}}^{1} \left(1 - \frac{1-p_1}{3x}\right) dx = \frac{1}{3}\left[3 - (1-p_1) + (1-p_1)\log\frac{1-p_1}{3}\right],$$

can be taken as a measure of the probability that ERC finally decides joining PSC and ICV instead of CiU. The minimum of S is 0.3005 and it is attained for $p_1 = 0$. As

$$\frac{dS}{dp_1} = -\frac{1}{3}\log\frac{1-p_1}{3} > 0 \quad \text{for all } p_1 \in [0,1),$$

this area increases with p_1 and tends to 1 (its maximum) in the degenerate case. Of course, $p_1 = 0$ gives the best payoff to party 3.

Hence there are “many” cases where party 3 should prefer party 1 instead of parties 2 and 5. In other words, party 1 could adapt its tendency (availability to form coalitions) to those of parties 2 and 5.

The conclusion of this analysis is that joining CiU would have not been a bad decision for ERC. But also that CiU should have been very careful in expressing its interest to form coalitions, a point where maybe its leaders were not strategically efficient enough. A scenario where the main party becomes excluded of the government must be considered exceptional.

7 Concluding remarks

We have introduced here *coalitional* multinomial probabilistic values. They apply to games with a coalition structure by combining the Shapley value and the corresponding multinomial probabilistic value. We first apply this latter value to the quotient game and obtain a payoff for each union; next, we apply within each union the Shapley value to a reduced game, played in the union, for sharing that payoff efficiently. These values form a n–parametric family since they depend on profiles $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ that supply information not necessarily included in the characteristic function of the game. We interpret each component p_i as the tendency of player i to form coalitions.

By using coalitional multinomial probabilistic values one can take into account the influence of players' different personalities in the study of the coalition formation process. The detailed analysis of the Catalonia Parliament in Section 6 illustrates this idea and also the good behavior of these values as power indices (i.e., acting on simple games).

In our opinion, by no means these values can qualify as "rare earths".[10] In effect: (a) they lie in the well recognized class of probabilistic values [30]; (b) they generalize the symmetric coalitional binomial semivalues [10] widely (and, in particular, the symmetric coalitional Banzhaf value [1]); and (c) they do not admit technical objections, enjoy good properties, and provide a promising framework for applications. Moreover, the model presented here is open to modifications.

Thus, future work should focus on several aspects concerning the use and interpretation of the tendency profiles and hence of the values. Below is a list of possibilities to act, some of which agree with suggestions made by the reviewers.

- To drop the independence assumption for the p_i's. This will open the possibility of having profiles depending on (i.e., defined in terms of) the game and/or the coalition structure.
- To search new ways to define the induced profile in the quotient game, that is, the $\overline{p}_r$'s. And even making them dependent on B and the quotient game or independent of the p_i's.
- To discuss who defines the tendencies, in both the game and the quotient game, and whether they can be assumed common knowledge.
- To discover new properties of the values. This task has been already initiated by members of our research group for multinomial probabilistic values and good results have been obtained that will be published soon. Some of these properties will probably extend to the coalitional case.
- To reach new individual axiomatic characterizations for each value.
- To establish an axiomatic characterization of the full family of coalitional multinomial probabilistic values (so independent of profiles $\mathbf{p}$ that define each member). A similar characterization has been already reached for multinomial probabilistic values [13].
- To extend the coalitional version to all probabilistic values introduced by Weber [30].

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[10] A term borrowed from chemistry.

Appendix: Proofs

Proof of Proposition 3.8. (a) Positivity. If v is monotonic then $v(Q\cup T\cup\{i\}) \geq v(Q\cup T)$ for all R and all T, so that $\Lambda_i^{\mathbf{p}}[v;B] \geq 0$ for all $i \in N$.

(b) Quotient game property. If $k \in M$ and $R \subseteq M\backslash\{k\}$ we define $v_R \in \mathcal{G}_{B_k}$ by

$$v_R(T) = v(Q\cup T) - v(Q) \quad \text{for all } T \subseteq B_k,$$

where $Q = \bigcup_{r\in R} B_r$. For any $i \in B_k$ the Shapley value gives

$$\varphi_i[v_R] = \sum_{T\subseteq B_k\backslash\{i\}} \frac{v_R(T\cup\{i\}) - v_R(T)}{b_k\binom{b_k-1}{t}} = \sum_{T\subseteq B_k\backslash\{i\}} \frac{v(Q\cup T\cup\{i\}) - v(Q\cup T)}{b_k\binom{b_k-1}{t}},$$

so that

$$\Lambda_i^{\mathbf{p}}[v;B] = \sum_{R\subseteq M\backslash\{k\}} \Big[\prod_{j\in R}\overline{p}_j \prod_{\substack{h\in M\backslash R\\ h\neq k}} (1-\overline{p}_h)\Big]\varphi_i[v_R].$$

As $\sum_{i\in B_k}\varphi_i[v_R] = v_R(B_k) = v(Q\cup B_k) - v(Q)$ we have

$$\sum_{i\in B_k}\Lambda_i^{\mathbf{p}}[v;B] = \sum_{R\subseteq M\backslash\{k\}} \Big[\prod_{j\in R}\overline{p}_j \prod_{\substack{h\in M\backslash R\\ h\neq k}} (1-\overline{p}_h)\Big][v(Q\cup B_k) - v(Q)].$$

Using Eqs. (2) and (3) and the definition of quotient game, we obtain

$$\sum_{i\in B_k}\Lambda_i^{\mathbf{p}}[v;B] = \sum_{R\subseteq M\backslash\{k\}} \overline{p}_R^k[v^B(R\cup\{k\}) - v^B(R)] = \lambda_k^{\overline{\mathbf{p}}}[v^B],$$

and the statement follows since, according to Remark 3.3(ii),

$$\lambda_k^{\overline{\mathbf{p}}}[v^B] = \Lambda_k^{\overline{\mathbf{p}}}[v^B;B^m]. \quad \square$$

Proof of Theorem 3.9. (Existence) 1. Linearity. It clearly follows from Eq. (5).

2. Dummy player property. Let i be a dummy player in v. Then $v(Q\cup T\cup\{i\}) - v(Q\cup T) = v(\{i\})$. Furthermore,

$$\sum_{R\subseteq M\backslash\{k\}} \prod_{j\in R}\overline{p}_j \prod_{\substack{h\in M\backslash R\\ h\neq k}} (1-\overline{p}_h) = \sum_{R\subseteq M\backslash\{k\}} \overline{p}_R^k = 1$$

by definition of probabilistic value and using that

$$\sum_{T\subseteq B_k\backslash\{i\}} \frac{1}{b_k\binom{b_k-1}{t}} = 1$$

because $1/b_k\binom{b_k-1}{t}$ is the coefficient of the Shapley value for a b_k–person game. Therefore $\Lambda_i^{\mathbf{p}}[v;B] = v(\{i\})$.

3. Symmetry within unions. From Eq. (5) we have

$$\Lambda_i^{\mathbf{p}}[v;B] - \Lambda_j^{\mathbf{p}}[v;B] = \sum_{R\subseteq M\backslash\{k\}} \Big[\prod_{\ell\in R}\overline{p}_\ell \prod_{\substack{h\in M\backslash R\\ h\neq k}} (1-\overline{p}_h)\Big]U(R,i,j),$$

where

$$U(R,i,j) = \sum_{T\subseteq B_k\setminus\{i\}} \frac{v(Q\cup T\cup\{i\}) - v(Q\cup T)}{b_k\binom{b_k-1}{t}} - \sum_{T\subseteq B_k\setminus\{j\}} \frac{v(Q\cup T\cup\{j\}) - v(Q\cup T)}{b_k\binom{b_k-1}{t}}$$

and $Q = \bigcup_{r\in R} B_r$. A little algebra yields

$$U(R,i,j) = \sum_{T\subseteq B_k\setminus\{i,j\}} \frac{1}{b_k}\left[\frac{1}{\binom{b_k-1}{t}} + \frac{1}{\binom{b_k-1}{t+1}}\right][v(Q\cup T\cup\{i\}) - v(Q\cup T\cup\{j\})]$$

and each term of this sum vanishes by the symmetry of i and j in v.

4. Coalitional **p**–multinomial total power property. During the proof of Proposition 3.8 we have established that if $k\in M$ and $R\subseteq M\setminus\{k\}$ then

$$\sum_{i\in B_k}\Lambda_i^{\mathbf{p}}[v;B] = \sum_{R\subseteq M\setminus\{k\}}\Big[\prod_{j\in R}\overline{p}_j \prod_{\substack{h\in M\setminus R\\ h\neq k}}(1-\overline{p}_h)\Big][v(Q\cup B_k) - v(Q)]$$

and the desired property readily follows from this, since

$$\sum_{i\in N}\Lambda_i^{\mathbf{p}}[v;B] = \sum_{k\in M}\sum_{i\in B_k}\Lambda_i^{\mathbf{p}}[v;B].$$

5. Property of **p**–weighted payoffs for quotients of unanimity games. Eq. (4) for $u_{\overline{T}}$ and $\lambda^{\overline{\mathbf{p}}}$ yields

$$\lambda_k^{\overline{\mathbf{p}}}[u_{\overline{T}}] = \prod_{\substack{h\in\overline{T}\\ h\neq k}}\overline{p}_h \quad \text{for each } k\in\overline{T}.$$

Using Proposition 3.8, Remark 3.3(ii) and the fact that $(u_T)^B = u_{\overline{T}}$,

$$\sum_{i\in B_k}\Lambda_i^{\mathbf{p}}[u_T;B] = \Lambda_k^{\overline{\mathbf{p}}}[u_{\overline{T}};B^m] = \lambda_k^{\overline{\mathbf{p}}}[u_{\overline{T}}] = \prod_{\substack{h\in\overline{T}\\ h\neq k}}\overline{p}_h.$$

Therefore

$$\overline{p}_k\sum_{i\in B_k}\Lambda_i^{\mathbf{p}}[u_T;B] = \prod_{h\in\overline{T}}\overline{p}_h = \overline{p}_\ell\sum_{j\in B_\ell}\Lambda_j^{\mathbf{p}}[u_T;B] \quad \text{for all } k,\ell\in\overline{T}$$

because the intermediate product is symmetrical in k and ℓ.

(Uniqueness) Let g be a coalitional value on $\mathcal{G}_N^{cs}$ that satisfies the stated properties. We will show that g is uniquely determined on all $[v;B]\in\mathcal{G}_N^{cs}$, so that it must coincide with $\Lambda^{\mathbf{p}}$.

From now on, assume $B\in B(N)$ is given. Using linearity, we need only to prove that g is uniquely determined on each unanimity game u_T. By the dummy player property, $g_i[u_T;B] = 0$ if $i\notin T$. This leaves us with the players of T. By the coalitional **p**–multinomial total power property,

$$\sum_{i\in N} g_i[u_T;B] = \sum_{i\in N}\Lambda_i^{\mathbf{p}}[u_T;B]. \tag{9}$$

Let $\overline{T} = \{k\in M : T\cap B_k\neq\emptyset\}$ and, if $k\in\overline{T}$, let $T_k = T\cap B_k$. Then Eq. (9) reduces to

$$\sum_{k\in\overline{T}}\sum_{i\in T_k} g_i[u_T;B] = \sum_{i\in N}\Lambda_i^{\mathbf{p}}[u_T;B]. \tag{10}$$

The property of **p**–weighted payoffs for quotients of unanimity games implies that

$$\overline{p}_k \sum_{i\in T_k} g_i[u_T;B] - \overline{p}_\ell \sum_{j\in T_\ell} g_j[u_T;B] = 0 \quad \text{for all } k,\ell \in \overline{T}. \tag{11}$$

Setting, for short, $\alpha_k = \sum_{i\in T_k} g_i[u_T;B]$ for $k \in \overline{T}$ and using Eqs. (10) and (11), we obtain a linear system of $\bar{t}$ equations and $\bar{t}$ unknowns $\alpha_k = \sum_{i\in T_k} g_i[u_T;B]$ for $k \in \overline{T}$ with determinant

$$D_{\bar{t}}(\overline{p}_1, \cdots, \overline{p}_{\bar{t}}) = (-1)^{\bar{t}-1} \sum_{i\in\overline{T}} \overline{p}_1 \cdots \hat{\overline{p}}_i \cdots \overline{p}_{\bar{t}} \neq 0.$$

Then, this system has a unique solution for $\{\alpha_k\}_{k\in\overline{T}}$ and this means that $\sum_{i\in T_k} g_i[u_T;B]$ is uniquely determined for every $k \in \overline{T}$. Finally, by symmetry within unions

$$g_i[u_T;B] = \frac{1}{|T_k|} \sum_{j\in T_k} g_j[u_T;B]$$

is also uniquely determined for all $i \in T_k$.[11] □

Proof of Theorem 4.1. (Existence) In view of Theorem 3.9, we only need to prove that the coalitional **p**–multinomial probabilistic value satisfies the transfer property, but this easily follows from linearity (or, simply, from additivity) and the fact that, for all $v, v' \in \mathcal{G}_N$ (and, in particular, for all simple games),

$$v \vee v' + v \wedge v' = v + v'.$$

(Uniqueness) Let g be a coalitional power index on $\mathcal{SG}_N^{cs}$ that satisfies the stated properties. We will show that g is uniquely determined on all $[v;B] \in \mathcal{SG}_N^{cs}$, so that it must coincide with (the restriction of) $\Lambda^{\mathbf{p}}$.

If $v \in \mathcal{SG}_N$ and $W^m(v) = \{T_1, T_2, \ldots, T_r\}$ then

$$v = u_{T_1} \vee u_{T_2} \vee \cdots \vee u_{T_r}.$$

By recurrence, from the transfer property we obtain, for each $B \in B(N)$,

$$g[v;B] = \sum_{j=1}^{r} (-1)^{j+1} \sum_{1\le i_1<\cdots<i_j\le r} g[u_{T_{i_1}} \wedge u_{T_{i_2}} \wedge \cdots \wedge u_{T_{i_j}}; B].$$

Moreover

$$u_{T_{i_1}} \wedge u_{T_{i_2}} \wedge \cdots \wedge u_{T_{i_j}} = u_{T_{i_1}\cup T_{i_2}\cup\cdots\cup T_{i_j}}$$

and therefore we need only to prove that g is uniquely determined on each unanimity game u_T. The remaining of the proof is exactly the same as in the case of Theorem 3.9. □

[11] The uniqueness proof for $g[u_T;B]$ works in some cases where **p** is not a positive profile. For example, if B and T are such that $\overline{p}_k > 0$ for all $k \in \overline{T}$. And even when $\overline{p}_k = 0$ for just one $k \in \overline{T}$, because in this case

$$D_{\bar{t}}(\overline{p}_1, \cdots, \overline{p}_{\bar{t}}) = (-1)^{\bar{t}-1} \overline{p}_1 \cdots \hat{\overline{p}}_k \cdots \overline{p}_{\bar{t}} \neq 0.$$

Thus, Theorem 3.9 remains valid if **p** is not positive but $p_i = 0$ for just one $i \in N$. And even when dropping the positivity condition on **p** but adding positivity, in the sense of Proposition 3.8, as a requirement for g, because the conditions lead therefore to $\alpha_k = 0$ for those $k \in \overline{T}$ such that $\overline{p}_k > 0$ and $\sum_h \alpha_h = 0$ for the remaining $h \in \overline{T}$, i.e. those such that $\overline{p}_h = 0$. Then the positivity of g implies that each such α_h vanishes and the proof can be finished successfully.

Proof of Theorem 5.1. Steps 1–3 have been already used in [26, 9, 27, 19, 2, 10, 11]. It will be of interest to recall the common argument here.

By the second and third steps, we obtain a multilinear function where all terms of (1) corresponding to coalitions S such that $S \cap B_r \neq \emptyset$ and $(N \backslash S) \cap B_r \neq \emptyset$ for some $r \in M\backslash\{k\}$ vanish. Indeed, in step 2, the terms corresponding to these coalitions are of the form $cy_r^{q_1}(1-y_r)^{q_2}$, with $q_1, q_2 \in \mathbb{N}$, and in step 3 they turn on $c(y_r - y_r)$ thus getting zero.

Hence, the only coalitions S for which the corresponding term of the multilinear extension in Eq. (1) may not vanish after steps 2 and 3 are those of the form $S = Q \cup T$, where $T \subseteq B_k$ and $Q = \bigcup_{r \in R} B_r$ for some $R \subseteq M\backslash\{k\}$. The function arising from step 3 is therefore

$$g_k((x_j)_{j\in B_k},\, (y_r)_{r \in M\backslash\{k\}}) =$$

$$\sum_{R \subseteq M\backslash\{k\}} \sum_{T \subseteq B_k} \Big[\prod_{j\in T} x_j \prod_{h \in B_k\backslash T} (1-x_h) \prod_{r\in R} y_r \prod_{\substack{s\in M\backslash R \\ s\neq k}} (1-y_s)\Big] v(Q\cup T).$$

Replacing each y_r with $\overline{p}_r$ (step 4) gives

$$\alpha_k((x_j)_{j\in B_k}) = \sum_{R \subseteq M\backslash\{k\}} \sum_{T \subseteq B_k} \Big[\prod_{r\in R} \overline{p}_r \prod_{\substack{s\in M\backslash R \\ s\neq k}} (1-\overline{p}_s) \prod_{j\in T} x_j \prod_{h \in B_k\backslash T} (1-x_h)\Big] v(Q\cup T).$$

By differentiating function $\alpha_k((x_j)_{j\in B_k})$ with respect to x_i we obtain

$$\frac{\partial \alpha_k}{\partial x_i}((x_j)_{j\in B_k}) =$$

$$\sum_{R \subseteq M\backslash\{k\}} \Big[\prod_{r\in R} \overline{p}_r \prod_{\substack{s\in M\backslash R \\ s\neq k}} (1-\overline{p}_s)\Big] \sum_{T \subseteq B_k\backslash\{i\}} \Big[\prod_{j\in T} x_j \prod_{\substack{h \in B_k\backslash T \\ h\neq i}} (1-x_h)\Big] [v(Q\cup T\cup\{i\}) - v(Q\cup T)].$$

Finally, by step 5,

$$\int_0^1 \frac{\partial \alpha_k}{\partial x_i}(z, z, \ldots, z)dz =$$

$$\sum_{R \subseteq M\backslash\{k\}} \Big[\prod_{r\in R} \overline{p}_r \prod_{\substack{s\in M\backslash R \\ s\neq k}} (1-\overline{p}_s)\Big] \sum_{T \subseteq B_k\backslash\{i\}} [v(Q\cup T\cup\{i\}) - v(Q\cup T)] \int_0^1 z^t(1-z)^{b_k-t-1}dz =$$

$$\sum_{R \subseteq M\backslash\{k\}} \Big[\prod_{r\in R} \overline{p}_r \prod_{\substack{s\in M\backslash R \\ s\neq k}} (1-\overline{p}_s)\Big] \sum_{T \subseteq B_k\backslash\{i\}} \frac{t!(b_k-t-1)!}{b_k!} [v(Q\cup T\cup\{i\}) - v(Q\cup T)],$$

which is equivalent to Eq. (5) for $\Lambda_i^{\mathbf{P}}[v; B]$. $\square$

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