UNIFIED FORMALISM FOR THE GENERALIZED kth-ORDER HAMILTON-JACOBI PROBLEM

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Abstract

The geometric formulation of the Hamilton-Jacobi theory enables us to generalize it to systems of higher-order ordinary differential equations. In this work we introduce the unified Lagrangian-Hamiltonian formalism for the geometric Hamilton-Jacobi theory on higher-order autonomous dynamical systems described by regular Lagrangian functions.

Key words: Hamilton-Jacobi equation, Higher-order systems, Skinner-Rusk formalism.

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Contents

1 Introduction

The geometric formulation of the Hamilton-Jacobi theory given in [2] and [4] enables us to generalize it to systems of higher-order ordinary differential equations. This generalization has been done recently for the Lagrangian and Hamiltonian formalism of higher-order autonomous mechanical systems described by regular Lagrangian functions [3]. The aim of this work is to give a unified Lagrangian-Hamiltonian version of this theory for these kinds of systems, using the unified framework introduced by Skinner and Rusk [8]. The advantage of this formulation is that it compresses the Lagrangian and Hamiltonian Hamilton–Jacobi problems into a single formalism which allows to recover both of them in a simple way, and it is specially interesting when dealing with singular systems.

All the manifolds are real, second countable and C^∞. The maps and the structures are assumed to be C^∞. Sum over repeated indices is understood.

2 Higher-order tangent bundles

Let Q be a n-dimensional manifold, and $k \in \mathbb{Z}^+$. The *kth-order tangent bundle of Q* is the $(k+1)n$-dimensional manifold T^kQ made of the k-jets of the bundle $\pi\colon \mathbb{R}\times Q \to \mathbb{R}$ with fixed source point $t = 0 \in \mathbb{R}$; that is, $\mathrm{T}^kQ = J^k_0\pi$.

We have the following natural projections (for $r \leqslant k$):

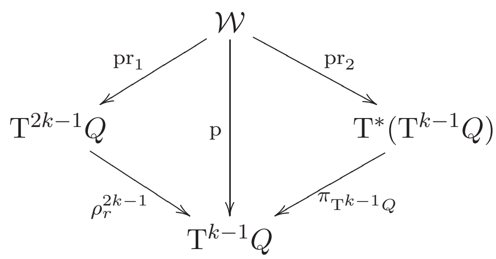
$$\begin{array}{rcl} \rho^k_r\colon \mathrm{T}^kQ & \longrightarrow & \mathrm{T}^rQ \\ j^k_0\phi & \longmapsto & j^r_0\phi \end{array} \quad ; \quad \begin{array}{rcl} \beta^k\colon \mathrm{T}^kQ & \longrightarrow & Q \\ j^k_0\phi & \longmapsto & \phi(0) \end{array}$$

where $j^k_0\phi$ denotes a point in T^kQ; that is, the equivalence class of a curve $\phi\colon I \subset \mathbb{R} \to Q$ by the k-jet equivalence relation. Notice that $\rho^k_0 = \beta^k$, where T^0Q is canonically identified with Q, and $\rho^k_k = \mathrm{Id}_{\mathrm{T}^kQ}$. Observe also that $\rho^l_s \circ \rho^r_l = \rho^r_s$, for $0 \leqslant s \leqslant l \leqslant r \leqslant k$.

If $\phi\colon \mathbb{R} \to Q$ is a curve in Q, the canonical lifting of ϕ to T^kQ is the curve $j^k\phi\colon \mathbb{R} \to \mathrm{T}^kQ$ defined as the k-jet lifting of ϕ restricted to $\mathrm{T}^kQ \hookrightarrow J^k\pi$ (see [5]).

3 The Hamilton-Jacobi problem in the Skinner-Rusk formalism

Let Q be a n-dimensional smooth manifold modeling the configuration space of a kth-order autonomous dynamical system with n degrees of freedom, and let $\mathcal{L} \in \mathrm{C}^\infty(\mathrm{T}^k Q)$ be a Lagrangian function for this system, which is assumed to be regular. In the Lagrangian-Hamiltonian formalism, we consider the bundle $\mathcal{W} = \mathrm{T}^{2k-1}Q \times_{\mathrm{T}^{k-1}Q} \mathrm{T}^*(\mathrm{T}^{k-1}Q)$ with canonical projections $\mathrm{pr}_1 \colon \mathcal{W} \to \mathrm{T}^{2k-1}Q$ and $\mathrm{pr}_2 \colon \mathcal{W} \to \mathrm{T}^*(\mathrm{T}^{k-1}Q)$. It is clear from the definition that the bundle $\mathcal{W}$ fibers over $\mathrm{T}^{k-1}Q$. Let $\mathrm{p} \colon \mathcal{W} \to \mathrm{T}^{k-1}Q$ be the canonical projection. Obviously, we have $\mathrm{p} = \rho^{2k-1}_{k-1} \circ \mathrm{pr}_1 = \pi_{\mathrm{T}^{k-1}Q} \circ \mathrm{pr}_2$. Hence, we have the following commutative diagram



We consider in $\mathcal{W}$ the presymplectic form $\Omega = \mathrm{pr}_2^*\, \omega_{k-1} \in \Omega^2(\mathcal{W})$, where $\omega_{k-1} \in \Omega^2(\mathrm{T}^*(\mathrm{T}^{k-1}Q))$ is the canonical symplectic form. In addition, from the Lagrangian function $\mathcal{L}$, and using the canonical coupling function $\mathcal{C} \in \mathrm{C}^\infty(\mathcal{W})$, we construct a Hamiltonian function $H \in \mathrm{C}^\infty(\mathcal{W})$ as $H = \mathcal{C} - \mathcal{L}$. Thus, the dynamical equation for the system is

$$i(X_{LH})\Omega = \mathrm{d}H \ , \quad X_{LH} \in \mathfrak{X}(\mathcal{W}) \, . \tag{1}$$

Following the constraint algorithm in [5], a solution to the equation (1) exists on the points of a submanifold $j_o \colon \mathcal{W}_o \hookrightarrow \mathcal{W}$ which can be identified with the graph of the Legendre-Ostrogradsky map $\mathcal{FL} \colon \mathrm{T}^{2k-1}Q \to \mathrm{T}^*(\mathrm{T}^{k-1}Q)$ associated to $\mathcal{L}$. If the Lagrangian function is regular, then there exists a unique vector field X_{LH} solution to (1) and tangent to $\mathcal{W}_o$ (see [8]).

3.1 The generalized Hamilton-Jacobi problem

We first state the generalized version of the Hamilton-Jacobi problem. Following the same patterns as in [2], [3] and [4] (see also an approach to the problem for higher-order field theories in [9]), the natural definition for the generalized Hamilton-Jacobi problem in the Skinner-Rusk setting [7], [8] is the following.

Definition 1 *The* generalized kth-order Lagrangian-Hamiltonian Hamilton-Jacobi problem *(or* generalized kth-order unified Hamilton-Jacobi problem*) consists in finding a section $s \in \Gamma(\mathrm{p})$ and a vector field $X \in \mathfrak{X}(\mathrm{T}^{k-1}Q)$ such that the following conditions are satisfied:*

1. *The submanifold* $\mathrm{Im}(s) \hookrightarrow \mathcal{W}$ *is contained in* $\mathcal{W}_o$.
2. *If $\gamma \colon \mathbb{R} \to \mathrm{T}^{k-1}Q$ is an integral curve of X, then $s \circ \gamma \colon \mathbb{R} \to \mathcal{W}$ is an integral curve of X_{LH}, that is,*
$$X \circ \gamma = \dot{\gamma} \Longrightarrow X_{LH} \circ (s \circ \gamma) = \dot{\overline{s \circ \gamma}} \, . \tag{2}$$

It is clear that the vector field $X \in \mathfrak{X}(\mathrm{T}^{k-1}Q)$ cannot be chosen independently from the section $s \in \Gamma(\mathrm{p})$. Indeed, following the same pattern as in [2] we can prove:

Proposition 1 *The pair $(s, X) \in \Gamma(\mathrm{p}) \times \mathfrak{X}(\mathrm{T}^{k-1}Q)$ satisfies the two conditions in Definition 1 if, and only if, X_{LH} and X are s-related.*

Corollary 1 *If $s \in \Gamma(\mathrm{p})$ and $X \in \mathfrak{X}(\mathrm{T}^{k-1}Q)$ satisfy the two conditions in Definition 1, then $X = \mathrm{Tp} \circ X_{LH} \circ s$.*

That is, the vector field $X \in \mathfrak{X}(\mathrm{T}^{k-1}Q)$ is completely determined by the section $s \in \Gamma(\mathrm{p})$, and it is called the *vector field associated to s*. Therefore, the search of a pair $(s, X) \in \Gamma(\mathrm{p}) \times \mathfrak{X}(\mathrm{T}^{k-1}Q)$ satisfying the two conditions in Definition 1 is equivalent to the search of a section $s \in \Gamma(\mathrm{p})$ such that the pair $(s, \mathrm{Tp} \circ X_{LH} \circ s)$ satisfies the same condition. Thus, we can restate the problem as follows.

Proposition 2 *The generalized kth-order unified Hamilton-Jacobi problem for X_{LH} is equivalent to finding a section $s \in \Gamma(\mathrm{p})$ satisfying the following conditions:*

1. *The submanifold $\mathrm{Im}(s) \hookrightarrow \mathcal{W}$ is contained in $\mathcal{W}_o$.*
2. *If $\gamma \colon \mathbb{R} \to \mathrm{T}^{k-1}Q$ is an integral curve of $\mathrm{Tp} \circ X_{LH} \circ s \in \mathfrak{X}(\mathrm{T}^{k-1}Q)$, then $s \circ \gamma \colon \mathbb{R} \to \mathcal{W}$ is an integral curve of X_{LH}, that is*

$$\mathrm{Tp} \circ X_{LH} \circ s \circ \gamma = \dot{\gamma} \Longrightarrow X_{LH} \circ (s \circ \gamma) = \dot{\overline{s \circ \gamma}}\,.$$

Proposition 3 *The following assertions on a section $s \in \Gamma(\mathrm{p})$ are equivalent.*

1. *s is a solution to the generalized kth-order unified Hamilton-Jacobi problem.*
2. *The submanifold $\mathrm{Im}(s) \hookrightarrow \mathcal{W}$ is invariant under the flow of the vector field X_{LH} solution to equation (1) (that is, X_{LH} is tangent to the submanifold $\mathrm{Im}(s)$).*
3. *The section s satisfies the dynamical equation $i(X)(s^*\Omega) = \mathrm{d}(s^*H)$, where $X = \mathrm{Tp} \circ X_{LH} \circ s$ is the vector field associated to s.*

(*Proof*) The proof is analogous to that of Proposition 6 and Theorem 2 in [2]. ■

Coordinate expression. Let (q_0^A) be a set of local coordinates in Q, with $1 \leqslant A \leqslant n$, and $(q_0^A, \ldots, q_{2k-1}^A, p_A^0, \ldots, p_A^{k-1})$ the induced local coordinates in $\mathcal{W}$ (see [7] for details). Then, local coordinates in $\mathcal{W}$ adapted to the p-bundle structure are (q_i^A, q_j^A, p_A^i), where $0 \leqslant i \leqslant k-1$, $k \leqslant j \leqslant 2k-1$. Hence, a section $s \in \Gamma(\mathrm{p})$ is given locally by $s(q_i^A) = (q_i^A, s_j^A, \alpha_A^i)$, where s_j^A, α_A^i are local functions in $\mathrm{T}^{k-1}Q$.

From Proposition 3, an equivalent condition for a section $s \in \Gamma(\mathrm{p})$ to be a solution of the generalized kth-order unified Hamilton-Jacobi problem is that the dynamical vector field X_{LH} is tangent to the submanifold $\mathrm{Im}(s) \hookrightarrow \mathcal{W}$, which is defined locally by the constraints $q_j^A - s_j^A = 0$ and $p_A^i - \alpha_A^i = 0$. From [7], the vector field X_{LH} solution to equation (1) is given locally by

$$X_{LH} = \sum_{l=0}^{2k-2} q_{l+1}^A \frac{\partial}{\partial q_l^A} + F^A \frac{\partial}{\partial q_{2k-1}^A} + \frac{\partial \mathcal{L}}{\partial q_0^A} \frac{\partial}{\partial p_A^0} + \left(\frac{\partial \mathcal{L}}{\partial q_i^A} - p_A^{i-1} \right) \frac{\partial}{\partial p_A^i}\,,$$

where F^A are the functions solution to the following system of n equations

$$(-1)^k(F^B - d_T(q^B_{2k-1}))\frac{\partial^2\mathcal{L}}{\partial q^B_k \partial q^A_k} + \sum_{i=0}^{k}(-1)^i d^i_T\left(\frac{\partial \mathcal{L}}{\partial q^A_i}\right) = 0\,.$$

Hence, requiring $X_{LH}(q^A_j - s^A_j) = 0$ and $X_{LH}(p^i_A - \alpha^i_A) = 0$ we obtain the following system of $2kn$ partial differential equations on $\mathrm{Im}(s)$

$$\begin{gathered}
s^A_{j+1} - q^B_{i+1}\frac{\partial s^A_j}{\partial q^B_i} - s^B_k \frac{\partial s^A_j}{\partial q^B_{k-1}} = 0 \ ; \ F^A - q^B_{i+1}\frac{\partial s^A_{2k-1}}{\partial q^B_i} - s^B_k \frac{\partial s^A_{2k-1}}{\partial q^B_{k-1}} = 0 \\
\frac{\partial \mathcal{L}}{\partial q^0_A} - q^B_{i+1}\frac{\partial \alpha^0_A}{\partial q^B_i} - s^B_k \frac{\partial \alpha^0_A}{\partial q^B_{k-1}} = 0 \ ; \ \frac{\partial \mathcal{L}}{\partial q^A_l} - \alpha^{l-1}_A - q^B_{i+1}\frac{\partial \alpha^l_A}{\partial q^B_i} - s^B_k \frac{\partial \alpha^l_A}{\partial q^B_{k-1}} = 0\,.
\end{gathered} \tag{3}$$

This is a system of $2kn$ partial differential equations with $2kn$ unknown function s^A_j, α^i_A. Hence, a section $s \in \Gamma(\mathrm{p})$ is a solution to the generalized kth-order Lagrangian-Hamiltonian Hamilton-Jacobi problem if, and only if, its component functions satisfy the local equations (3).

3.2 The Hamilton-Jacobi problem

In general, to solve the generalized kth-order Hamilton-Jacobi problem is a difficult task since we must find kn-dimensional submanifolds of $\mathcal{W}$ contained in the submanifold $\mathcal{W}_o$ and invariant by the dynamical vector field X_{LH}. Hence, it is convenient to consider a less general problem and require some additional conditions to the section $s \in \Gamma(\mathrm{p})$ [1, 2].

Definition 2 *The* kth-order Lagrangian-Hamiltonian Hamilton-Jacobi problem *consists in finding sections $s \in \Gamma(\mathrm{p})$ solution to the generalized kth-order unified Hamilton-Jacobi problem such that $s^*\Omega = 0$. Such a section is called a* solution to the kth-order Lagrangian-Hamiltonian Hamilton-Jacobi problem.

From the definition of $\Omega \in \mathit{\Omega}^2(\mathcal{W})$ we have

$$s^*\Omega = s^*(\mathrm{pr}_2^*\,\omega_{k-1}) = (\mathrm{pr}_2 \circ s)^*\omega_{k-1}\ .$$

Hence, $s^*\Omega = 0$ if, and only if, $(\mathrm{pr}_2 \circ s)^*\omega_{k-1} = 0$. As $\Gamma(\pi_{\mathrm{T}^{k-1}Q}) = \mathit{\Omega}^1(\mathrm{T}^{k-1}Q)$, the section $\mathrm{pr}_2 \circ s \in \Gamma(\pi_{\mathrm{T}^{k-1}Q})$ is a 1-form in $\mathrm{T}^{k-1}Q$, and from the properties of the tautological form θ_{k-1} of the cotangent bundle $\mathrm{T}^*(\mathrm{T}^{k-1}Q)$ we have

$$(\mathrm{pr}_2 \circ s)^*\omega_{k-1} = (\mathrm{pr}_2 \circ s)^*(-\mathrm{d}\theta_{k-1}) = -\mathrm{d}((\mathrm{pr}_2 \circ s)^*\theta_{k-1}) = -\mathrm{d}(\mathrm{pr}_2 \circ s)\,.$$

Hence, the condition $s^*\Omega = 0$ is equivalent to $\mathrm{pr}_2 \circ s \in \mathit{\Omega}^1(\mathrm{T}^{k-1}Q)$ being a closed 1-form. Therefore, the Hamilton-Jacobi problem can be reformulated as follows.

Proposition 4 *The kth-order unified Hamilton-Jacobi problem is equivalent to finding sections $s \in \Gamma(\mathrm{p})$ solution to the generalized kth-order unified Hamilton-Jacobi problem such that $\mathrm{pr}_2 \circ s$ is a closed 1-form in $\mathrm{T}^{k-1}Q$.*

Taking into account the new assumption $s^*\Omega = 0$ in Definition 2, a consequence of Proposition 3 is the following result.

Proposition 5 *The following assertions on a section $s \in \Gamma(\mathrm{p})$ satisfying $s^*\Omega = 0$ are equivalent:*

1. *s is a solution to the kth-order unified Hamilton-Jacobi problem.*
2. $\mathrm{d}(s^*H) = 0$.
3. $\mathrm{Im}(s)$ *is an isotropic submanifold of $\mathcal{W}$ invariant by X_{LH}.*
4. *The integral curves of X_{LH} with initial conditions in $\mathrm{Im}(s)$ project onto the integral curves of $X = \mathrm{Tp} \circ X_{LH} \circ s$.*

Coordinate expression. From [7], the Hamiltonian function in $\mathcal{W}$ has coordinate expression $H = q_{i+1}^A p_A^i - \mathcal{L}(q_0^A, \ldots, q_k^A)$. Thus, its differential is given locally by

$$\mathrm{d}H = -\frac{\partial \mathcal{L}}{\partial q_0^A}\mathrm{d}q_0^A + \left(p_A^i - \frac{\partial \mathcal{L}}{\partial q_{i+1}^A}\right)\mathrm{d}q_{i+1}^A + q_{i+1}^A \mathrm{d}p_A^i \, .$$

Hence, the condition $\mathrm{d}(s^*H) = 0$ in Proposition 5 holds if, and only if, the following kn partial differential equations are satisfied

$$\begin{aligned} & q_{i+1}^B \frac{\partial \alpha_B^i}{\partial q_0^A} + s_k^B \frac{\partial \alpha_B^{k-1}}{\partial q_0^A} + \alpha_B^{k-1} \frac{\partial s_k^B}{\partial q_0^A} - \left(\frac{\partial \mathcal{L}}{\partial q_0^A} + \frac{\partial \mathcal{L}}{\partial q_k^B}\frac{\partial s_k^B}{\partial q_0^A}\right) = 0 \, , \\ & q_{i+1}^B \frac{\partial \alpha_B^i}{\partial q_l^A} + s_k^B \frac{\partial \alpha_B^{k-1}}{\partial q_l^A} + \alpha_A^{l-1} + \alpha_B^{k-1} \frac{\partial s_k^B}{\partial q_l^A} - \left(\frac{\partial \mathcal{L}}{\partial q_l^A} + \frac{\partial \mathcal{L}}{\partial q_k^B}\frac{\partial s_k^B}{\partial q_l^A}\right) = 0 \, , \end{aligned} \tag{4}$$

where $1 \leqslant l \leqslant k-1$.

Equivalently, we can require the 1-form $\mathrm{pr} \circ s \in \Omega^1(\mathrm{T}^{k-1}Q)$ to be closed, that is, $\mathrm{d}(\mathrm{pr} \circ s) = 0$. Locally, this condition reads

$$\frac{\partial \alpha_A^i}{\partial q_j^B} - \frac{\partial \alpha_B^j}{\partial q_i^A} = 0 \text{ , with } A \neq B \text{ or } i \neq j \, . \tag{5}$$

Therefore, a section $s \in \Gamma(\mathrm{p})$ is a solution to the kth-order Lagrangian-Hamiltonian Hamilton-Jacobi problem if, and only if, the local functions s_j^A, α_A^i satisfy the system of partial differential equations given by (3) and (4), or, equivalently (3) and (5). Observe that the system of partial differential equations may not be $\mathrm{C}^\infty(U)$-linearly independent.

3.3 Relation with the Lagrangian and Hamiltonian formalisms

Finally, we state the relation between the solutions of the Hamilton-Jacobi problem in the unified formalism and the solutions of the problem in the Lagrangian and Hamiltonian settings given in [3].

Theorem 1 *Let $\mathcal{L} \in \mathrm{C}^\infty(\mathrm{T}^kQ)$ be a hyperregular Lagrangian function.*

1. *If $s \in \Gamma(\mathrm{p})$ is a solution to the (generalized) kth-order Lagrangian-Hamiltonian Hamilton-Jacobi problem, then the sections $s_{\mathcal{L}} = \mathrm{pr}_1 \circ s \in \Gamma(\rho_{k-1}^{2k-1})$ and $\alpha = \mathrm{pr}_2 \circ s \in \Omega^1(\mathrm{T}^{k-1}Q)$ are solutions to the (generalized) kth-order Lagrangian and Hamiltonian Hamilton-Jacobi problems, respectively.*

2. *If $s_{\mathcal{L}} \in \Gamma(\rho^{2k-1}_{k-1})$ is a solution to the (generalized) kth-order Lagrangian Hamilton-Jacobi problem, then $s = j_o \circ \overline{\mathrm{pr}}_1^{-1} \circ s_{\mathcal{L}} \in \Gamma(\mathrm{p})$ is a solution to the (generalized) kth-order Lagrangian-Hamiltonian Hamilton-Jacobi problem.*

   If $\alpha \in \Omega^1(\mathrm{T}^{k-1}Q)$ is a solution to the (generalized) kth-order Hamiltonian Hamilton-Jacobi problem, then $s = j_o \circ \overline{\mathrm{pr}}_2^{-1} \circ \alpha \in \Gamma(\mathrm{p})$ is a solution to the (generalized) kth-order Lagrangian-Hamiltonian Hamilton-Jacobi problem.

(*Proof*) The proof of the first item follows the same patterns that the proof of Theorem 1 in [3]. For the second item, the key point is to take into account that the maps $\overline{\mathrm{pr}}_1 \colon \mathcal{W} \to \mathrm{T}^{2k-1}Q$ and $\overline{\mathrm{pr}}_2 \colon \mathcal{W} \to \mathrm{T}^*(\mathrm{T}^{k-1}Q)$ are diffeomorphisms, and that the dynamical vector field $X_{LH} \in \mathfrak{X}(\mathcal{W})$ solution to equation (1) is tangent to $\mathcal{W}_o$, and therefore is j_o-related to a vector field $X_o \in \mathfrak{X}(\mathcal{W}_o)$ for which it is possible to state an equivalent Hamilton-Jacobi problem. ■

3.4 An example: A (homogeneous) deformed elastic cylindrical beam with fixed ends

Consider a deformed elastic cylindrical beam with both ends fixed (see [7] and references therein). The problem is to determinate its shape; that is, the width of every section transversal to the axis. This gives rise to a 1-dimensional second-order dynamical system, which is autonomous if we require the beam to be homogeneous. Let Q be the 1-dimensional smooth manifold modeling the configuration space of the system with local coordinate (q_0). Then, in the natural coordinates of T^2Q, the Lagrangian function for this system is

$$\mathcal{L}(q_0, q_1, q_2) = \frac{1}{2}\mu q_2^2 + \rho q_0 \,,$$

where $\mu, \rho \in \mathbb{R}$ are constants, and $\mu \neq 0$. This a regular Lagrangian function because the Hessian matrix

$$\left(\frac{\partial^2 \mathcal{L}}{\partial q_2 \partial q_2} \right) = \mu \,,$$

has maximum rank equal to 1 when $\mu \neq 0$.

In the induced natural coordinates $(q_0, q_1, q_2, q_3, p^0, p^1)$ of $\mathcal{W}$, the coordinate expressions of the presymplectic form $\Omega = \mathrm{pr}_2^*\,\omega_1 \in \Omega^2(\mathcal{W})$ and the Hamiltonian function $H = \mathcal{C} - \mathcal{L} \in \mathrm{C}^\infty(\mathcal{W})$ are

$$\Omega = \mathrm{d}q_0 \wedge \mathrm{d}p^0 + \mathrm{d}q_1 \wedge \mathrm{d}p^1 \quad ; \quad H = q_1 p^0 + q_2 p^1 - \frac{1}{2}\mu q_2^2 - \rho q_0 \,.$$

Thus, the semispray of type 1 $X_{LH} \in \mathfrak{X}(\mathcal{W})$ solution to the dynamical equation (1) and tangent to the submanifold $\mathcal{W}_o = \mathrm{graph}(\mathcal{FL}) \hookrightarrow \mathcal{W}$ has the following coordinate expression

$$X_{LH} = q_1 \frac{\partial}{\partial q_0} + q_2 \frac{\partial}{\partial q_1} + q_3 \frac{\partial}{\partial q_2} - \frac{\rho}{\mu}\frac{\partial}{\partial q_3} + \rho \frac{\partial}{\partial p^0} - p^0 \frac{\partial}{\partial p^1} \,.$$

In the following we state the equations for the (generalized) Lagrangian-Hamilonian Hamilton-Jacobi problem for this dynamical system.

In the generalized Lagrangian-Hamiltonian Hamilton-Jacobi problem we look for sections $s \in \Gamma(\mathrm{p})$, given locally by $s(q_0, q_1) = (q_0, q_1, s_2, s_3, \alpha^0, \alpha^1)$, such that the submanifold $\mathrm{Im}(s) \hookrightarrow \mathcal{W}$ is invariant under the flow of $X_{LH} \in \mathfrak{X}(\mathcal{W})$. Since the constraints defining locally $\mathrm{Im}(s)$ are

$q_2 - s_2 = 0$, $q_3 - s_3 = 0$, $p^0 - \alpha^0 = 0$, $p^1 - \alpha^1 = 0$, then the equations for the section s are

$$s_3 - q_1\frac{\partial s_2}{\partial q_0} - s_2\frac{\partial s_2}{\partial q_1} = 0 \ ; \ -\frac{\rho}{\mu} - q_1\frac{\partial s_3}{\partial q_0} - s_2\frac{\partial s_3}{\partial q_1} = 0 \,,$$
$$\rho - q_1\frac{\partial \alpha^0}{\partial q_0} - s_2\frac{\partial \alpha^0}{\partial q_1} = 0 \ ; \ -\alpha^0 - q_1\frac{\partial \alpha^1}{\partial q_0} - s_2\frac{\partial \alpha^1}{\partial q_1} = 0 \,.$$

For the Lagrangian-Hamiltonian Hamilton-Jacobi problem, we require in addition the section $s \in \Gamma(\rho_1^{\mathcal{W}})$ to satisfy $s^*\Omega = 0$ or, equivalently, the form $\mathrm{pr}_2 \circ s \in \Omega^1(\mathrm{T}Q)$ to be closed. In coordinates, if $s = (q_0, q_1, s_2, s_3, \alpha^0, \alpha^1)$, then the 1-form $\mathrm{pr}_2 \circ s$ is given by $\mathrm{pr}_2 \circ s = \alpha^0 \mathrm{d}q_0 + \alpha^1 \mathrm{d}q_1$. Hence, a section $s \in \Gamma(\mathrm{p})$ solution to the unified Hamilton-Jacobi problem for this system must satisfy the following system of 5 partial differential equations

$$s_3 - q_1\frac{\partial s_2}{\partial q_0} - s_2\frac{\partial s_2}{\partial q_1} = 0 \ ; \ -\frac{\rho}{\mu} - q_1\frac{\partial s_3}{\partial q_0} - s_2\frac{\partial s_3}{\partial q_1} = 0 \ ; \ \frac{\partial \alpha^1}{\partial q_0} - \frac{\partial \alpha^0}{\partial q_1} = 0 \,,$$
$$\rho - q_1\frac{\partial \alpha^0}{\partial q_0} - s_2\frac{\partial \alpha^0}{\partial q_1} = 0 \ ; \ -\alpha^0 - q_1\frac{\partial \alpha^1}{\partial q_0} - s_2\frac{\partial \alpha^1}{\partial q_1} = 0 \,.$$

Observe that, when the condition $\mathrm{Im}(s) \subseteq \mathcal{W}_o = \mathrm{graph}\mathcal{FL}$ is required, these equations project to the Lagrangian or Hamiltonian equations for the Hamilton-Jacobi problem [3].

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