Tensegrity frameworks: Static analysis review.

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Abstract

This paper hands in a review of the basic issues about the statics of tensegrity structures. Definitions and notation for the most important concepts, borrowed from the vast existing literature, are summarized. All of these concepts and definitions provide a complete mathematical framework to analyze the rigidity and stability properties of tensegrity structures from three different, but related, points of view: motions, forces and energy approaches. Several rigidity and stability definitions are presented in this paper and hierarchically ordered, from the strongest condition of infinitesimal rigidity to the more wide concept of simple rigidity, so extending some previous classifications already available.

Important theorems regarding the relationship between these definitions are also put together to complete the static overview of tensegrity structures. Examples of different tensegrity structures belonging to each of the rigidity and stability categories presented are described and analyzed. Concluding the static analysis of tensegrity structures, a review of existing form-finding methods is presented.

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1 What is a tensegrity?

Throughout history, most of the human construction engineering has been of a purely compression matter. Examples of this are the post and lintel construction technique (from Stonehenge to modern buildings) or the Roman arch. However, and in contrast to human engineering, the universe itself balances compression with tension, think about gravity balancing inertial forces, exhibiting a harmony of both. Although not commonly seen, tensegrity structures are already present in both natural (i.e. the structural framework of non-woody plants) and man-made (i.e. pneumatic structures) environments [7].

Buckminster Fuller, was capable of making an abstraction about the physical world into two complementary principles: compression and tension. He introduced structures based on such principles, later known as *tensegrities* [27, 29], as early as 1927 with his *Dymaxion House* [82], although he was unable to integrate them to what he called *Energetic Geometry.*

It was not until 1947, when he gave a lecture on Energetic-Synergetic Geometry at the Black Mountain College, that the nowadays accepted starting point for tensegrity structures appeared. A young artist, named Kenneth Snelson, built different models of the structures proposed by Fuller in his lecture. The term *Tensegrity*, was later coined by Fuller, in the early 60's, as the contraction for tensile integrity [27].

However, it seems to be some confusion on the origins of the first tensegrity system. The research performed by Russian constructivists (specially Ioganson) reported in Nagy [55] include references to an *equilibrium structure*, shown in an exhibition held in Moscow in 1921. Moreover, different patents have been applied for tensegrity systems almost simultaneously: Fuller [28] and Snelson [68] in EEUU, and Emmerich [23] in France. All these patents describe the same tensegrity structure, which is built from 3 compressive elements and 9 tensile elements, but from different points of view. A deeper historical background on tensegrity structures was given by Motro [51, 52].

A quite intuitive description of a tensegrity was given by Fuller [28] in his patent:

> *Islands of compression inside an ocean of tension.*

In his own patent Emmerich [23], gave almost the same definition but emphasizing the self-stress condition:

> *Self-stressing structures consist of bars and cables assembled in such a way that the bars remain isolated in a continuum of cables. All these elements*

must be spaced rigidly and at the same time interlocked by the prestressing resulting from the internal stressing of cables without the need for external bearings and anchorage. The whole is maintained firmly like a self-supporting structure, whence the term self-stressing.

Perhaps the most widely accepted definition of tensegrity is the one proposed by Pugh [60], which is the result of merging the definitions proposed by Fuller [28], Emmerich [23] and Snelson [68] in their respective patents:

A tensegrity system is established when a set of discontinuous compression components interacts with a set of continuous tensile components to define a stable volume in space.

Pugh's definition only takes into account two different kinds of elements: compressive and tensile, which can be regarded as struts and cables respectively. Roth and Whiteley [61] gave a more formal definition of tensegrity which expands the original definition by introducing a third kind of element, the bar, which can withstand both tension and compression. They used a geometrical point of view and defined the struts as elements which place a lower bound on the distance between their vertices, the cables as elements which place an upper bound on the distance between their vertices and the bars as elements that keep their length constant.

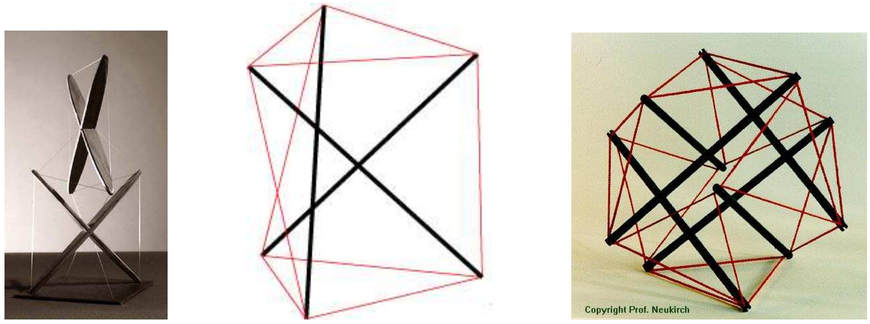
Several authors have also proposed their own definition of tensegrity which better suits their works. Of special interest is the one from Motro [51] which explicitly states the necessary condition of self stress to reach an equilibrium configuration:

Tensegrity systems are systems whose rigidity is the result of a state of self stressed equilibrium between cables under tension and compression elements.

Some examples of basic tensegrity modules both in $\mathbb{R}^2$ and $\mathbb{R}^3$ are shown in Fig. 1.

The idea of putting together several basic tensegrity modules to build more complex structures has been also studied, for example for double layered tensegrity frameworks [33, 34]. When assembling basic modules, care must be taken in how they are joined. Motro [51] analyzed three possible methods:

- node on node: This method joints a node from one module with a node from another module. Such a structure does not comply with the definition of tensegrity proposed by Pugh. Even though, this new structure leads to the concept of contiguous strut tensegrity grid proposed later by Wang [79, 80]. Some examples of this kind of structures are the Reciprocal prism (RP) (Fig. 2(a)) and the Crystal-cell pyramid (CP) (Fig. 2(b)). Skelton et al. [67] generalized this new kind of structures by defining a class k tensegrity



(a) The Snelson's X. It is the simplest tensegrity structure in $\mathbb{R}^2$.

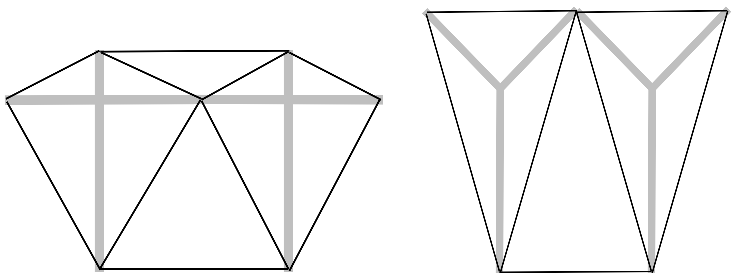
(b) The regular triangular prism. It is the simplest tensegrity structure in $\mathbb{R}^3$

(c) A tensegrity sphere made of 6 compressive elements and 24 tensile elements.

Fig. 1. Examples of basic tensegrity modules both in $\mathbb{R}^2$ and $\mathbb{R}^3$.

framework as a tensegrity with a maximum of k compressive elements in each node.

- node on cable: This method breaks the cable of one module to insert the node from another module, and therefore the structure obtained still complies with Pugh's definition. Examples of such structures are the needle tower, shown in Fig. 3 in exhibition at the Hirshhorn Museum, and the Sculpture Garden in Washington D.C.
- cable on cable: This method binds two cables from different modules to create a new node without any compressive element.



(a) An example of a Reciprocal Prism grid proposed by Wang

(b) An example of a Cell Pyramidal grid proposed by Wang

Fig. 2. Examples of contiguous strut tensegrity grids proposed by Wang. Solid lines are struts and dashed lines are cables.

Static and dynamic analysis of tensegrity frameworks have experienced a fast development over the last few decades due to its benefits over traditional approaches in several fields such as civil engineering, architecture [26], geometry, art and even biology. Skelton et al. [66] summarizes these benefits as:

- **Efficiency**: It has been shown [5] that structural material is only needed

Fig. 3. Needle tower built in 1968 with a height of about 18 m.

in the loads paths, so tensegrity structures, by carefully placing the compressive elements, are capable of increasing the resistance/weight ratio of traditional structures. Tensegrity are also energetically efficient since their members store energy in the form of tension or compression; the overall power needed to actuate such structures would be small since it is partially stored in the structure itself [43].

- **Deployability**: Stiff structures tend to have limited mobility, but, since compressive elements in tensegrity structures are disjoint, large displacements are allowed thus making it possible to create deployable structures than can be stored in small volumes. This is especially important in space applications such as deployable antennas and masts [30, 75].
- **Easily tunable**: The existence of pre-stress in the elements of the tensegrity allow the designer to modify its stiffness. Therefore, the way the structure behaves when external forces are applied as well as its natural oscillation frequency [53, 12], can be easily modified.
- **Easily modeled**: Due to the tensegrity design rules, whichever the external force applied to its elements, they only carry axial forces (either tension or compression). The model used to characterize its behavior is more reliable since it does not take into account bending phenomena.
- **Redundant**: Tensegrity can be seen as a special class of structures whose elements may simultaneously work as sensors, actuators and load-carrying elements. So, it is possible to have multiple elements capable of dealing with a given task, and, in the case one of them fails, other element can play its role and allow the whole structure to continue working. This is the principle of smart structures, and particularly, of smart sensors [72].
- **Scalability**: The main mathematical properties of tensegrity structures, not considering physical material limitations, are given by its geometry, so

they are applicable from small to large scale.

- **Biology inspired**: Ingber [36, 37] proposed that a tensegrity model could be used to explain how basic elements combine to form more complex structures (self-assembly). In the human body this model can be applied for both, the macro scale (≪*the* 206 *bones that constitute our skeleton are pulled up against the force of gravity and stabilized in a vertical form by the pull of tensile muscles, tendons and ligaments*≫), and the micro scale (≪*proteins and other key molecules in the body also stabilize themselves through the principles of tensegrity*≫). Vogel [78] also showed that the tensegrity model can be applied to the muscle-skeleton structures of some land animals.

2 Motivation

The high energetic efficiency, deployability, deformability and redundancy properties, as well as the biological inspiration, make this kind of structures interesting to design mobile robots, which is one of the main interests of the authors. On this subject, Aldrich [1] already built and controlled redundant manipulators based on tensegrity frameworks and, Paul et al. [57], built the first mobile tensegrity robot based on the regular triangular prism. Even though, little research has been done in this area.

In order to be able to use such kind of structures for robotic applications, different problems must be addressed: the first one, present in any application of tensegrity structures, is to find a stable configuration from a given topology or even design new topologies in order to achieve some desired results. This first goal has to do with the static behavior of tensegrity structures and the form-finding methods, both addressed in this paper.

The second main problem is to plan trajectories and motions taking into account the advantages that tensegrity structures offer. Traditional path planning algorithms together with the dynamic characterization of tensegrity structures and some kind of node trajectory planning can be used to plan complex tasks both with mobile robots and static manipulators.

In this survey the authors try to give a comprehensive, self-contained introduction to the rigidity and stability of tensegrity structures and also to the most important form-finding methods, thus covering the whole static analysis of tensegrity structures. Authors are presently working on issuing a sequel of this paper covering the dynamics and control of tensegrity in order to complete the study of these structures.

This paper is organized as follows. First, definitions on different mathematical concepts and notation used all throughout the paper is given in section 3. Then, the static issue is addressed in section 4. Rigidity, on its very different levels, as well as stability of tensegrity structures are explained from three different points of view: motions, forces and energy. In this section, several examples of tensegrity structures showing different rigidity and stability levels are also analyzed so to make the document more comprehensive. Next, in section 5, a review summarizing existing form-finding methods is presented. Finally, some conclusions are outlined in section 6.

3 Definitions

Notation for the most important concepts about tensegrity frameworks used all throughout the paper is introduced in this section. Several different notations and terminologies are used in the available literature depending on the field of expertise of the authors, from the most formal and rigorous formulation of mathematicians, passing through the more pragmatic and practical formulation of architects and engineers, to the intuitive point of view of artists and sculptors.

Roth and Whiteley [61] were the first to use graph theory to formulate tensegrity frameworks, and this approach have been used since then. An *abstract tensegrity framework* is a graph $G(V, E)$ in which the vertex set $V = \{1, 2, \ldots, n\}$ is the set of all the nodes of the structure, and, the edge set $E = C \cup B \cup S = \{1, 2, \ldots, e\}$, is the union of three independent sets: C, whose elements are cables which have an upper bound on their length, B whose elements are bars with a fixed length, and, S whose elements are struts, with a complementary behavior to that of the cables, i.e. imposing a lower bound on their length. An element of E is called indistinctly an *element* [83] or *member* [61, 14] of the tensegrity framework.

The term abstract means that the graph only carries topological information about the framework, i.e. which vertices are connected to other through edges. There is no particular realization in an Euclidean space ($\mathbb{R}^d$).

A *tensegrity framework* $G(\underline{p})$ is an abstract tensegrity framework together with an application $\underline{p} : x \in V \mapsto \mathbb{R}^d$ that maps each of the n vertices into a d-space. The set of vertex coordinates is arranged as $\underline{p} = (\underline{p}_1^T, \underline{p}_2^T, \ldots, \underline{p}_n^T) \in \mathbb{R}^{nd}$ and is called a *placement* [83], *embedding* [14], *realization* [61] or *configuration* [19].

Given a placement for an abstract tensegrity framework it is possible to define an application which assigns a length to each of the edges, which is known as the *edge function* [61], *rigidity map* [14] or *length map* [83]. Such a function is ($f : \mathbb{R}^{nd} \mapsto \mathbb{R}^e$):

$$f(\underline{p}_1^T, \ldots, \underline{p}_n^T) = (\ldots, |\underline{p}_i - \underline{p}_j|^2, \ldots) \tag{1}$$

Then, the *rigidity matrix* $\underline{\underline{R}}(\underline{p})$ [19], *compatibility matrix* (B) [62] or *geometrical matrix* (Π) [83], which are equivalent concepts, can be defined. Such a matrix has a row for each edge and a column for each vertex and dimension, so it is an e by nd matrix. Each row of this matrix is full of 0 except for the elements in the columns corresponding to the edge terminal vertices.

$$
\{i,j\} \begin{pmatrix} \vdots & \vdots & \vdots & \overset{i}{\vdots} & \vdots & \vdots & \vdots & \overset{j}{\vdots} & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & (\underline{p}_i^T - \underline{p}_j^T) & 0 & \dots & 0 & (\underline{p}_j^T - \underline{p}_i^T) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} . \tag{2}
$$

The rigidity matrix sets up the relationship between the nodal displacements and the edge's elongations (stretching). Similarly, its transpose, also known as the *equilibrium matrix* [62], sets up the relationship between the stresses present in the edges of the tensegrity framework and the nodal forces.

Formally, a *regular placement* [61, 83] is such that:

$$
rank_E(\underline{\underline{R}}(\underline{p})) = max(rank_E(\underline{\underline{R}}(\underline{q})|\underline{q} \in \mathbb{R}^{nd})) \tag{3}
$$

From a geometric point of view, a regular placement is such that the tensegrity framework as a whole is embedded in the maximum Euclidean space possible.

A *general placement* is such that:

$$
rank_A(\underline{\underline{R}}(\underline{p})) = max(rank_A(\underline{\underline{R}}(\underline{q})|\underline{q} \in \mathbb{R}^{nd})), \forall A \subset E \tag{4}
$$

where $rank_A(\underline{\underline{R}}(\underline{p}))$ is the rank of the rigidity matrix built from any nonempty subset of edges $A \subset E$. From a geometrical point of view, a general placement is such that any substructure derived from the whole tensegrity framework is also embedded in the maximum Euclidean space possible, meaning, that there are no two coincident vertices, there are no three collinear vertices, there are no four coplanar vertices, and so on. Note that the former definition is less restrictive since it allows sets of vertices to be embedded in Euclidean spaces of smaller dimension than the maximum dimension where a realization of the tensegrity framework is possible.

Given two different configurations $\underline{p}, \underline{q} \in \mathbb{R}^{nd}$ for the same abstract framework G, denoted $G(\underline{p})$ and $G(\underline{q})$, they are said to be *congruent* [61, 19] if one is the result of a rigid motion from the other. This means that both realizations have the same topology and distances between vertices, but their position in the d-space is different.

Congruences only account for rigid motions, if there exist multiple solutions for a given topology and node distances, the tensegrity framework is said to have multiple embeddings, each one with its own congruent placements. Even more, if the framework can be deformed complying with all the geometrical constraints, the possible embeddings form a continuum of solutions, otherwise, the different embeddings are isolated.

Such deformations are known as *mechanisms* and may be finite, if there is a perceptible change in the position of the framework nodes, or infinitesimal if there is a first or higher order change in the position of the framework nodes. In both cases, such change in position may or may not change the length of the framework edges.

Any rigid placement or *congruence* can be formulated as $h(\underline{p}) = \underline{\underline{S}}\underline{p}+\underline{p}_0$ where $\underline{\underline{S}}$ is a skew symmetric rotation matrix and $\underline{p}_0$ is the translation vector. The dimension of the subspace of rigid placements is $\frac{d(d+1)}{2}$ if no motion constraints are imposed; otherwise, the dimension is reduced by the number of constraints. Similarly, an *infinitesimal motion* is a velocity vector that can be stated as $\underline{p}' = \underline{\underline{R}}\underline{p}' + \underline{p}'_0$. The dimension of this subspace is also $\frac{d(d+1)}{2}$.

It is possible to define, for an abstract tensegrity framework G, a partial order of its placements. A tensegrity framework $G(\underline{p})$ dominates another $G(\underline{q})$ (noted as $G(\underline{p}) \geqslant G(\underline{q})$) if the following conditions are verified:

$$\begin{array}{l} \| \underline{p}_i - \underline{p}_j \| \geqslant \| \underline{q}_i - \underline{q}_j \| \ \{i,j\} \in C \\ \| \underline{p}_i - \underline{p}_j \| = \| \underline{q}_i - \underline{q}_j \| \ \{i,j\} \in B \\ \| \underline{p}_i - \underline{p}_j \| \leqslant \| \underline{q}_i - \underline{q}_j \| \ \{i,j\} \in S \end{array} , \tag{5}$$

that is, the configuration that dominates has longer cables and shorter struts than the other. Furthermore, two tensegrity frameworks are *equivalent* if all the distances between their vertices are the same.

Since tensegrity frameworks are a kind of prestressed structures, it is possible to assign a force value to each of the edges ω_{ij}. The collection of such scalars $\underline{\omega} = (\cdots, \omega_{ij}, \cdots)$ is called a *stress.* Motro [52] distinguishes between the stress due to the tensegrity topology itself, which is called *self-stress*, and the one due to the determination of the precise fixed location of some of the nodes in the space, which is called *pre-stress.* In any case, the stress must comply with the equilibrium conditions shown in eq. 6 which state that the resultant force at each node must be null:

$$\sum_j \omega_{ij}(\underline{p}_i - \underline{p}_j) = 0. \tag{6}$$

Using matrix notation, self-stress states are the solution of the linear equations $R(p)^T\omega = 0$.

A given stress is *proper* if for $\{ij\} \in C$, $\omega_{ij} \geqslant 0$ and $\{ij\} \in S$, $\omega_{ij} \leqslant 0$, that is, the scalar ω_{ij} associated to each cable is non-negative and the scalar associated to each strut is non-positive [14]. Other authors [61] make a more restrictive definition of a *proper stress* not allowing any scalar ω_{ij} to be 0. Crapo and Whiteley [22] refer to such stress as *strict.*

Given a stress vector it is possible to build the *reduced stress matrix* [14] $\underline{\underline{\overline{\Omega}}}$ which is a kind of adjacency matrix. In the literature it is possible to find both, a mathematical definition, [19], or a more intuitive physical derivation [32]. Its non-diagonal elements are the stress, with changed sign, between any two adjacent nodes, and the elements of the main diagonal are the sum of the corresponding row (column) as shown in eq. 7.

$$\underline{\underline{\overline{\Omega}}}_{ij} = \begin{cases} -\omega_{ij} & if\ i \neq j \\ \sum_k \omega_{ik} & if\ i = j \end{cases} . \tag{7}$$

It turns out that the matrix form of the equilibrium condition in eq. 6 can be reformulated in terms of the stress matrix, $p^T \Omega = 0$. Also, ω is a self-stress if it is a solution to this homogeneous system of equations.

An *equilibrium force* is the result of applying different forces on the structure nodes with a null total resultant force and torque onto the whole tensegrity framework [81, 61], that is, structure deformations are possible but there is not any resultant translation or rotation of the structure. On the other hand, a *resolvable force* is a force that the structure can compensate for by means of a proper stress vector $\underline{\omega}$ in the current configuration; note that in this case no structure deformations are allowed, so, in a general case, the set of resolvable forces ($\mathfrak{R}$) is a subset of the equilibrium forces ($\mathfrak{E}$) ($\mathfrak{R} \subset \mathfrak{E}$). The notation and terminology used throughout the paper is summarized in table 1.

As final comment, over the years some different notations have been proposed by several authors to identify specific tensegrity structures and the most common structures have even received names. Skelton et al. [67] used a simple notation which identifies the number of compressive (C) and tensile (T) elements, so the Snelson's X shown in 1(a) can be coded as $C2T4$. Motro [52] introduced an other notation which specifies the number of nodes n, the number of compressive elements (S), the number of tensile elements (C), if the structure is regular (R) or not (I) and if the structure is homeomorphic to an sphere (SS) or not. In this case, the Snelson's X is coded as $N4-S2-C4-R$.

Table 1
Used notation and terminology.

Symbol	Dimension	Name
n	-	Number of vertices
e	-	Number of edges
d	-	Dimension of the Euclidean space
V	n	Set of vertices
C	-	Set of cables
B	-	Set of bars
S	-	Set of struts
$\underline{\underline{R}}(\underline{p})$	$e \times nd$	Rigidity matrix
$\underline{\underline{R}}(\underline{p})^T$	$nd \times e$	Equilibrium matrix
$\underline{p}$	nd	Placement
$\underline{p}'$	nd	Motion
$\underline{\omega}$	e	Stress vector
$\overline{\underline{\underline{\Omega}}}$	$n \times n$	Reduced stress matrix
$\underline{\underline{\Omega}}$	$nd \times nd$	Stress matrix
$\mathfrak{R}$	-	Set of resolvable forces
$\mathfrak{E}$	-	Set of equilibrium forces

4 Static analysis

The static analysis of general frameworks has reached a certain level of maturity. Nevertheless, tensegrity frameworks introduced new questions to be solved such as: Why are they stable?, How can them be built?, or more recently, How can we control them?.

In the mid seventies, Grünbaum and Shephard [31] formulated a set of conjectures about tensegrity frameworks which renewed the interest in the study of such structures. Fuller used the idea behind some basic tensegrity structures, such as simple regular prisms (tetrahedrons, octahedrons, etc.), to build geodesic domes and introduced the tensegrity concept into his study of synergetics.

Pugh [60], in his book *An Introduction to tensegrity*, presents a topological classification for elementary cells of tensegrity frameworks:

- *Rhombic*: as quoted by Pugh, ≪*Each compressed member of a rhombic system constitutes the longest diagonal of a rhombus of cables, folded according to this axis*≫.
- *Circuit*: this class of tensegrity is characterized by the existence of compressed members circuits, which doesn't exactly comply with the standard definition of tensegrity.
- *Z-configurations*: as quoted by Pugh, ≪*A type Z 6 tensegrity system is such that between the two extremities of each compressive member there exist a totality of 3 non aligned cables*≫.

Initial works by Fuller [29], Emmerich [24] and Pugh [60] were mainly based on geometrical considerations and intuitive methods, but without any equilibrium criteria, which occasionally lead to non-stable tensegrity frameworks.

The first rigorous, mathematical, study on the tensegrity framework's statics does not appear until few years later. Roth and Whiteley [61] extended some well known concepts about bar frameworks, such as rigidity or infinitesimal and static rigidity, to tensegrity frameworks. They also set up some important implications between these concepts and between tensegrity frameworks and their equivalent bar frameworks. Their results were applied to tensegrities in both $\mathbb{R}^2$ and $\mathbb{R}^3$.

Connelly [14] studied some important properties of rigid tensegrity frameworks. In his research an energy approach was proposed which leads to a redefinition of the rigidity concept based on the positive definiteness of the stress matrix ($\underline{\underline{\Omega}}$) associated to the stress vector ($\underline{\omega}$).

Motro [49] studied the spheric tensegrity frameworks, that is, those which are homeomorphic to a sphere. He proved that the graph built from the prestressed elements is bipartite, and the one built from the tension elements is planar. He also proposed an algorithm to design such structures.

A tensegrity is in general, both, kinematically and statically indeterminate, regarding to the determination of the vertices position and tensions in the framework elements respectively. This means that there may exist multiple self-stress states and mechanisms for a given framework. Because the stiffness of tensegrity frameworks is conditioned by the stabilization of infinitesimal mechanisms with states of self-stress [52], which is known as the prestressability problem, it seems reasonable to study such mechanisms. Standard structural analysis techniques can not be applied in these structures, and a geometrical non-linear iterative scheme is normally used instead [48, 6].

Calladine [9] studied this subject and proved that such assemblies are capable of self-stress and that under some conditions, a state of self-stress may stiffen a mechanism. Further research in this area were carried out by S. Pellegrino [62, 63], Pellegrino [59] and Calladine [10], Calladine and Pellegrino [11], who developed an algebraic method to find out the number of mechanisms and equilibrium configurations as well as a base for the subspaces of mechanisms, self-stress states, resolvable and non-resolvable forces. They also presented a method to distinguish between first order infinitesimal mechanisms and higher order infinitesimal or finite mechanisms. Pellegrino [59] also studied the effect of external loads applied to a static and kinematically indeterminate framework.

Similar work was parallely developed by Kuznetsov [40, 42, 41]. His methods are based on the decomposition of the system in several subsystems and it also make extensive use of linear algebra techniques. Also Motro et al. [53], Hanaor [35], and Salerno [64], the last one working with energetical properties of the system, gave numerical solutions to this issue.

From a different point of view, Kenner [39] and Tarnai [73] provided an analytical approach to the prestressability problem. Later on, Tarnai [74], presented a geometrical method that can only be used in highly symmetric frameworks. Vassart et al. [77] further characterized first and higher order mechanisms, and proposed a method to find out the order of a given mechanism based on geometrical considerations. More recently, Sultan et al. [71] has provided analytical solutions for particular structures, and Guest [32] has presented a comparison between different existing formulations for the problem of analyzing prestressed structures.

In the late 80's, there was and attempt by several authors to write a book on the rigidity of frameworks [22] where some of the chapters were dedicated to tensegrity frameworks. Connelly wrote some chapters on the subject of basic rigidity concepts for general frameworks as well as its extension to tensegrities. Roth wrote the introductory chapter and another one on the application of the basic concepts to triangulated convex surfaces. Finally, Whiteley also contributed to the book with a chapter on global and second order rigidity for tensegrity frameworks.

Connelly and Whiteley [19] adapted two concepts for the rigidity and stability of tensegrity frameworks: Second order rigidity, which is an extension to tensegrity frameworks of the second-order rigidity concept for bar frameworks, previously introduced by Connelly [13], and pre-stress stability, which is based on the energy methodology introduced by Connelly [14]. For the first time a general hierarchy for some rigidity and stability definitions was presented by those authors, and the conditions for some of the direct and inverse implications were established. In Fig. 4 we depict a complete hierarchical relationship for all rigidity and stability definitions completing the one given by Connelly and Whiteley [19].

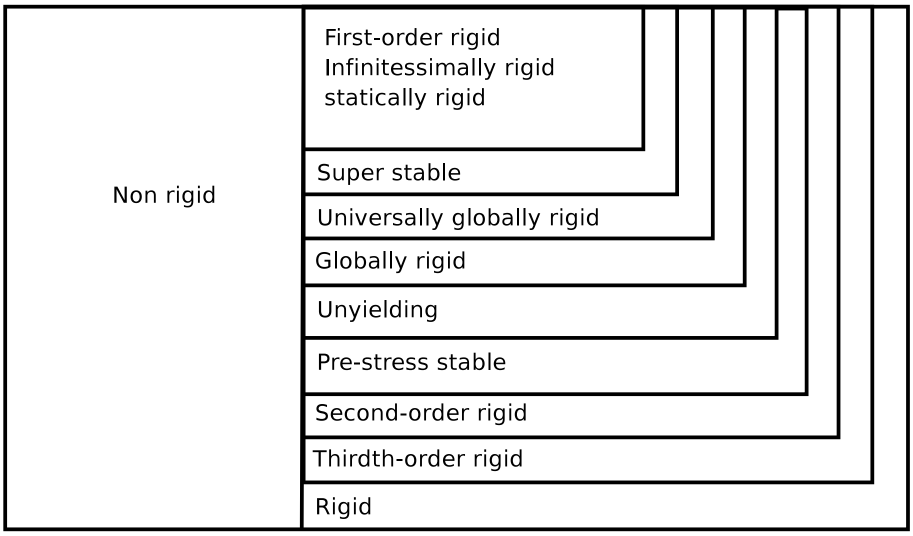


Fig. 4. Extended tensegrity framework rigidity and stability hierarchy.

Connelly and Whiteley [19] also introduced the positive definiteness study of the Hessian matrix (second derivative) of the energy function which links intrinsic parameters such as the physical stiffness of the material used and the framework self-stress which depends on design criteria and external loads.

In the late 90's, Connelly and Back [17], Connelly [16], presented a good overview of rigidity and stability theory and introduced a new concept of stability: super stability.

Most of the work carried out regarding the tensegrity framework statics, assumes that the actual deformation of the initial configuration is negligible when external forces are applied, but there are also some studies from Kebiche (see for instance Kebiche et al. [38]), that directly handle the non-linearities associated with large deformations of the structure. Using the virtual work approach, Correa [20], Correa et al. [21], presented a mathematical model derivation that addresses the static analysis problem and finds the position reached by the structure when external forces are applied. The study is performed for 3 and 4 bar tensegrities based on regular prisms.

More recently, Williams [83] summarizes the more fundamental results regarding the stability and rigidity analysis of tensegrity structures. He used a purely formal approach and substantially changed the notation used until then.

4.1 Rigidity and stability

Multiple in-equivalent definitions of both rigidity and stability for tensegrity frameworks exist in the available literature depending on the approach used by the authors: motions ($\underline{p}'$), forces ($\underline{\omega}$) or energies (E_p).

In this section we try to give a comprehensive presentation of these concepts as well as the proposed methods to check for them. For a deeper understanding of this section, the reading of Roth and Whiteley [61], Connelly [14], Connelly and Whiteley [19], Connelly [16] is recommended as those are the works we consider basic on these topics. For the sake of clarity no demonstrations are included in the following sections, instead, references to the corresponding original work are included when necessary.

For tensegrity based robots, the static analysis presented in this section give some important constraints that the structure must verify at any time in order to keep its shape and do not collapse. In whatever method used to control the tensegrity robot, the stability and rigidity conditions should constraint either the possible motions or the member stresses to reach a given configuration.

4.1.1 Motion approach

Intuitively, rigidity means the absence of relative motion between the members of a structure [22], which implies that the length of the members linking the vertices of the framework are kept constant:

$$\| \underline{p}_i - \underline{p}_j \|^2 = c_{ij} \tag{8}$$

where c_{ij} is the squared length of the $\{ij\}$ edge.

To mathematically formulate this concept, Roth and Whiteley [61] defined two sets of placements: $X(\underline{p})$ which are the placements that comply with the framework constraints given an initial configuration $\underline{p}$ and $M(\underline{p})$ which is the set of congruent placements to a given one. Note that $M(p) \subset X(p)$ denotes the set of all rigid placements for the structure in $\mathbb{R}^d$.

$$M(\underline{p}) = \{\underline{q} = (\underline{q}_1, \ldots, \underline{q}_n) \in \mathbb{R}^{nd} | \parallel \underline{p}_i - \underline{p}_j \parallel^2 = \parallel \underline{q}_i - \underline{q}_j \parallel^2, 1 \leqslant i, j \leqslant n\} \quad (9)$$

$$X(\underline{p}) = \left\{\underline{x} = (\underline{x}_1, \ldots, \underline{x}_n) \in \mathbb{R}^{nd} | \parallel \underline{x}_i - \underline{x}_j \parallel^2 \begin{cases} = \parallel \underline{p}_i - \underline{p}_j \parallel^2 & \{i, j\} \in B \\ \leqslant \parallel \underline{p}_i - \underline{p}_j \parallel^2 & \{i, j\} \in C \\ \geqslant \parallel \underline{p}_i - \underline{p}_j \parallel^2 & \{i, j\} \in S \end{cases}\right. \quad (10)$$

So, from a motion point of view, a tensegrity framework is said to be *rigid* in $\mathbb{R}^d$ if all the neighbour configurations $\underline{q}_i$ of a given configuration $\underline{p}$ are congruent [61, 16]. In other words, the two subsets 9 and 10 presented earlier are the same $(X(\underline{p}) \bigcap U = M(\underline{p}) \bigcap U)$. Otherwise, the tensegrity is said to be flexible, i.e. it has a continuous path that belong to the set of non-rigid motions (see Roth and Whiteley [61] for a proposition establishing the equivalence of non-rigidity and flexibility).

In the book by Crapo and Whiteley [22], an alternate but equivalent topological definition for rigidity is proposed: a given tensegrity framework is rigid in $\mathbb{R}^d$ if there exists an $\epsilon > 0$ such that, if $G(\underline{p}) \geqslant G(\underline{q})$ then $\parallel \underline{p} - \underline{q} \parallel < \epsilon$, meaning that both placements are congruent.

It is possible to define a motion, also known as a *flex* or *mechanism* [14, 61], as a continuous (analytic) path $\underline{p}(t)$, $0 \leqslant t \leqslant 1$ and $\underline{p}(0) = \underline{p}$, such that $p(t) \in X(p) - M(p)$ for which cables do not increase their length, struts do not decrease their length and bars keep their length constant.

If this motion is the same for each node of the framework ($\underline{p}(t) = \underline{\underline{S}}(t)\underline{p}(0) + \underline{T}(t)$ where $\underline{\underline{S}}$ is a skew symmetric rotation matrix and $\underline{T}$ is the translation vector), it is called a *trivial flex*. Therefore, a tensegrity framework is rigid if all the continuous (analytic) paths $\underline{p}(t)$, $0 < t \leqslant 1$, are trivial [14, 22].

The rigidity conditions presented so far have to deal with non-linearities due to the length function given in eq. 8. To overcome this problem, it is better to

Table 2
Relationship between the different sets defined by Roth and Whiteley [61].

Placement	$\Rightarrow$	$M(\underline{p})$	$\subseteq$	$X(\underline{p})$
		$\downarrow \frac{d}{dt}$		$\downarrow \frac{d}{dt}$
Velocity	$\Rightarrow$	$T(\underline{p})$	$\subseteq$	$I(\underline{p})$

work with the first derivative of the length function:

$$\left.\frac{d \parallel \underline{p}_i - \underline{p}_j \parallel^2}{dt}\right|_{t=0} = (\underline{p}_i - \underline{p}_j)^T(\underline{p}'_i(0) - \underline{p}'_j(0)) \begin{cases} = 0 \; \{ij\} \in B \\ \leqslant 0 \; \{ij\} \in C \\ \geqslant 0 \; \{ij\} \in S \end{cases}, \tag{11}$$

where $\underline{p}'_i(0)$ and $\underline{p}'_j(0)$ are velocities at $t = 0$. In order to be rigid, the above conditions for the first derivative and also for higher order derivatives must be verified with equality.

Roth and Whiteley [61] defined two sets of velocities analogous to those of placements: $T(\underline{p})$ which is the set of rigid velocities (only those which are the same for all the nodes) and $I(\underline{p})$ which is the set of all admissible velocities (those which verify all the constraints on the length of the members). Both $T(\underline{p})$ and $I(\underline{p})$ are tangent to the sets $M(\underline{p})$ and $X(\underline{p})$ respectively. Table 2 summarizes the relationship between these sets. This new formulation leads to the concept of *infinitesimal rigidity* [61] or *first-order rigid* [19] which only refers to the initial velocity.

$$T(\underline{p}) = M(\underline{p}') = \{p' \in \mathbb{R}^{nd}, \underline{p}'_i = \underline{\underline{S}}\underline{p}_i + \underline{T} \; \forall i \in V\}, \tag{12}$$

$$I(\underline{p}) = X(\underline{p}') = \{\underline{p}' \in \mathbb{R}^{nd}, (\underline{p}_i - \underline{p}_j)^T(\underline{p}'_i(0) - \underline{p}'_j(0)) \left\{ \begin{array}{l} \leqslant 0 \;\; \{i,j\} \in C \\ = 0 \;\; \{i,j\} \in B \\ \geqslant 0 \;\; \{i,j\} \in S \end{array} \right\}. \tag{13}$$

Every velocity vector $(\underline{p}')$ which belongs to $I(\underline{p})$ is called an *infinitesimal flex* [61, 22], or *first-order flex* [19]; it is called a *trivial infinitesimal flex* if it just belongs to $T(\underline{p})$ [22], i.e., velocities are exactly the same for all the vertices. Physically, a first-order flex is a velocity vector field associated with $G(p)$. A tensegrity framework is infinitesimally rigid in $\mathbb{R}^d$ if all admissible initial velocities are trivial, Roth and Whiteley [61], Crapo and Whiteley [22] ($T(\underline{p}) = I(\underline{p})$).

The condition for infinitesimal rigidity presented above is still hard to verify, but, compacting eq. 11 for all edges in matrix notation, it is easy to see that a trivial first order flex $\underline{p}'_h$ is a solution to the linear homogeneous equations $\underline{\underline{R}}(\underline{p})\underline{p}'_h = 0$. Note that besides the rigid velocities (trivial infinitesimal flexes), there are other possible solutions to the system of equations, that is, those velocity vectors $(\underline{p}'_i - \underline{p}'_j)$ which are orthogonal to the edge vectors $(\underline{p}_i - \underline{p}_j)$ (orthogonal non-trivial infinitesimal flexes) since they also verify eq. 11 with equality.

So, if k is the number of coordinate constraints of the nodes, the infinitesimal rigidity check is reduced, for bar frameworks, to check if the dimension of the kernel of the rigidity matrix verifies the following condition:

$$m = dim(Ker(\underline{\underline{R}}(\underline{p}))) \begin{cases} = \frac{d(d+1)}{2} - k \Rightarrow & \texttt{infinitesimally rigid} \\ > \frac{d(d+1)}{2} - k \Rightarrow & \texttt{infinitesimally flexible} \end{cases}, \quad (14)$$

that is, it matches the dimension of the space of rigid motions. If so, the only possible admisible motions are rigid ones.

Although this condition is sufficient for bar frameworks, for tensegrity frameworks to be infinitesimally rigid, this condition is necessary but not sufficient as shown later in this section (see the forces approach). When there are pinned nodes, and therefore constraints on the node coordinates, the corresponding columns of the rigidity matrix are eliminated, which leads to the reduction of the kernel's dimension.

Since the subspace spanned by the vectors of the kernel of the rigidity matrix is the subspace of velocities which do not instantly modify the length of the edges, the remaining vectors (i.e. the range of the rigidity matrix) span the subspace of velocities that modify the shape of the tensegrity framework. So,

$$\underline{\underline{R}}(\underline{p})\underline{p}'_i = \underline{d}, \quad (15)$$

is the general system of equations and $\underline{d}$ is the change in the length of the edges due to the velocity vector $\underline{p}'_i$.

In general, the velocity vector $\underline{p}'$ necessary to generate a given lengthening or stretching of the edges of the tensegrity framework $\underline{d}$ is the combination of the solution to the inhomogeneous system, eq. 15, $\underline{p}'_i$ and a linear combination of the trivial and orthogonal non-trivial infinitesimal flexes, which do not instantly change any length:

$$\underline{p}' = \underline{\alpha}^T(\underline{p}'_{h_1} \cdots \underline{p}'_{h_m}) + \underline{p}'_i \quad (16)$$

There are several ways to prove that infinitesimal rigidity implies rigidity.

Alexandrov and Gluck proved it using the implicit function theorem but just for bar frameworks, Roth and Whiteley [61] also proved it using some properties of the rigidity matrix. Finally, Connelly and Whiteley [19] used algebraic geometry to prove it.

The inverse implication, rigidity implies infinitesimal rigidity, is not valid in general. Asimow and roth [2, 3] proved that it is true for bar frameworks in regular placements. For tensegrity frameworks, Roth and Whiteley [61] proved that rigidity implies infinitesimal rigidity for general placements, which is a more restrictive condition than that of bar frameworks. They also managed to prove that infinitesimal rigidity is preserved by changing cables for struts and viceversa, and also that infinitesimal rigidity is only preserved under projective transformations if the roles of the members that intersect an hyperplane at the origin are interchanged (i.e. substitute cable for struts and viceversa). Several interesting results on rigidity of tensegrity frameworks, both in the plane and in the space, can be found in the work of Roth and Whiteley.

By successively taking derivatives of the length function it is possible to define higher order flexes, and therefore define other kinds of rigidity. Of special interest is the *second-order rigidity* introduced by Connelly [13]. The second derivative of the length function for each kind of edge is:

$$\left.\frac{d^2\|\underline{p}_i-\underline{p}_j\|^2}{dt^2}\right|_{t=0} =$$

$$= (\underline{p}_i'(0)-\underline{p}_j'(0))^T(\underline{p}_i'(0)-\underline{p}_j'(0)) + (\underline{p}_i-\underline{p}_j)^T(\underline{p}_i''(0)-\underline{p}_j''(0)) \begin{cases} = 0 \ \{ij\} \in B \\ \leqslant 0 \ \{ij\} \in C \\ \geqslant 0 \ \{ij\} \in S \end{cases}, \tag{17}$$

and a *second-order flex* is a pair $(\underline{p}', \underline{p}'')$ which verifies the above constraints. Since a second-order flex $(\underline{p}', \underline{p}'')$ is the solution of a inhomogeneous system of equations and inequations, given a trivial first-order flex $\underline{q}'$, the pair $(\underline{p}', \underline{p}''+\underline{q}')$ is also a second-order flex [22, 19] (This must be seen as a vectorial addition without any physical meaning).

A tensegrity framework is said to be second-order rigid if the only way for any second order flex pair $(\underline{p}', \underline{p}'')$ to comply with the constraints in eq. 17 with equality is based on a trivial first-order flex as $\underline{p}'$. Otherwise, the tensegrity framework is said to be *second-order flexible* [22]. Connelly and Whiteley [19] also extended this concept to higher order flexes, and proved that if a tensegrity framework is second-order rigid, then it is also rigid.

In order to clarify ideas about first and second order flexes, let's analyse the bar framework shown in Fig. 5. For tensegrity frameworks, the explanation is equivalent. It is obvious that the framework in Fig. 5 is not first order rigid since there exist a non trivial first order flex $\underline{p}'$ which comply with the constraints in eq. 11.

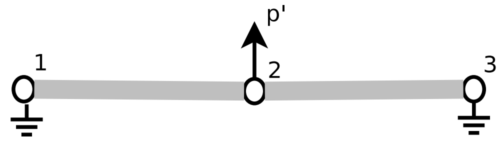


Fig. 5. A second order rigid bar framework which is not first order rigid.

Nevertheless, it is second order rigid since all the possible combinations of ($\underline{p}'$ and $\underline{p}''$) that comply with the constraints in eq. 17 with equality have a trivial first order flex ($\underline{p}'$). The system of constraint equations for both edges are shown in eq. 18:

$$\begin{array}{c} (\underline{p}_1 - \underline{p}_2)(-\underline{p}''_2) + \underline{p}'^2_2 = 0 \\ (\underline{p}_2 - \underline{p}_3)\underline{p}''_2 + \underline{p}'^2_2 = 0 \end{array}. \qquad (18)$$

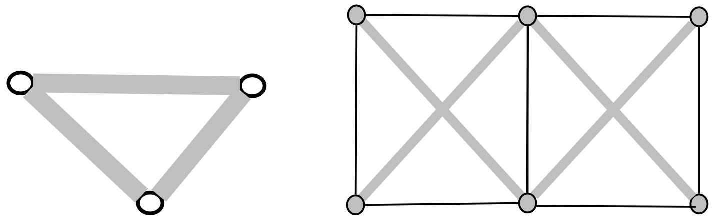
The only possible solution for this system of equations is the trivial one, that is $\underline{p}'_2 = \underline{p}''_2 = 0$.

The different rigidity definitions presented until now have only considered a single configuration, but a given tensegrity framework can have several non-congruent configurations. Therefore, stronger rigidity classes can be defined imposing conditions to all the possible non-congruent configurations.

If all the possible non-congruent configurations have exactly the same length for all edges ($\| \underline{q}_i - \underline{q}_j \| = \| \underline{p}_i - \underline{p}_j \| \quad \forall \{ij\} \in V, \forall \underline{q}$), then the tensegrity framework is called *unyielding*. A simple example of a tensegrity framework which is unyielding is the regular octahedron made of 3 bars and 9 cables shown in Fig. 1(b) since the regular prism configuration and the regular octahedral configuration have the same edge lengths.

If all the possible non-congruent configurations are actually the same, that is, there is only one possible realization of the framework in $\mathbb{R}^d$, then the tensegrity framework is called *globally rigid* [22] or *uniquely embedded* [13]. For globally rigid tensegrity frameworks, the substitution of cables for struts and viceversa do not preserve the global rigidity property of the framework [22]. An example of such tensegrity structure is the Snelson's X shown in Fig. 1(a) since given a set of compatible lengths, the realization is unique.

Finally, if the tensegrity framework is globally rigid for any euclidean space $\mathbb{R}^k \supset \mathbb{R}^d$ with $k \leqslant d$, then it is said to be *universally globally rigid*. This condition implies that a tensegrity framework keeps its shape when additional degrees of freedom are added as a result of increasing the dimension of the working space. For example, the bar triangle shown in Fig. 6(a) or the Snelson's X are universally globally rigid, but the framework shown in Fig. 6(b), while unyielding in $\mathbb{R}^2$, it is not even rigid in $\mathbb{R}^3$ since it can be folded like a book.



(a) A bar triangle framework.

(b) Tensegrity framework made from concatenating two Snelson's X.

Fig. 6. On the left an example of an universally globally rigid framework, and on the right an unyielding in $\mathbb{R}^2$ but not even rigid in $\mathbb{R}^3$ tensegrity framework.

Reviewing generic properties of bar frameworks, an abstract bar framework is *generically rigid* in $\mathbb{R}^d$ if all the realizations in regular points are infinitesimally rigid [61]. Since the set of regular points for a bar framework form a dense open set in $\mathbb{R}^{nd}$, just verifying that one realization is infinitesimally rigid it is enough to certify that the framework is generically rigid; otherwise the framework is generically flexible. In this case there exists what has been called a rigidity predictor [2] which has to do with the rank of the rigidity matrix:

$$\texttt{if } rank(\underline{\underline{R}}(\underline{p})) = nd - \frac{d(d+1)}{2} \Rightarrow G(\underline{p}) \texttt{ is infinitesimally rigid}. \quad (19)$$

Eq. 19 is equivalent to eq. 14, and is only valid if there are no coordinate constraints. In the case that some constraints were present, the rank of the rigidity matrix would be the same, and only the dimension of the kernel would be reduced by the number of constrained degrees of freedom as before.

Contrary to bar frameworks, for tensegrity frameworks the set of regular placements in $\mathbb{R}^{nd}$ is not dense [22], so the fact that a given placement be infinitesimally rigid in a regular point does not guarantee that the abstract framework would be. Therefore eq. 14 is necessary but not sufficient to guarantee generic rigidity. As explained later in this section, additional conditions are required so as to ensure generic rigidity (see the forces approach).

It is important to note that generic rigidity is a property of the abstract framework (and therefore of the underlying graph) but it has nothing to do with any particular configuration.

4.1.2 Forces approach

From a force point of view, in order to achieve a rigid configuration, the tensegrity framework must be in equilibrium, which implies that the equivalent force in each vertex must be null for each vertex j adjacent to it:

$$\sum_j \frac{\underline{p}_i - \underline{p}_j}{d_{ij0}} T_{ij} = \underline{F}_i, \tag{20}$$

where d_{ij0} and T_{ij} are the rest length and internal tension for each adjacent member respectively.

Expression 20 is non-linear since T_{ij} depend on both $\underline{p}_i$ and $\underline{p}_j$. Schek [65] introduced the force density coefficients ($q_{ij} = \frac{T_{ij}}{d_{ij0}}$) or stresses ($\omega_{ij}$) for each member. These coefficients represent the force per unit of length that vertex j applies to vertex i, and obviously $\omega_{ij} = \omega_{ji}$.

Using matrix notation and the linearization introduced by Sheck, the equilibrium condition for the whole framework can be stated as:

$$\underline{\underline{R}}(\underline{p})^T \underline{\omega} = \underline{F}_{eq\_ext}, \tag{21}$$

for any external equilibrium force F_{eq_ext} and proper stress $\underline{\omega}$. Such proper stress is known as an equilibrium stress. A given tensegrity framework is said to be *statically rigid* if every equilibrium force is resolvable [61, 22]. This means $\mathfrak{E} = \mathfrak{R}$, and that there exists a proper stress vector ($\underline{\omega}$) which compensates for the external equilibrium force without any structural deformation. Remember that in a general case, the set of resolvable forces is a subset of the equilibrium forces.

Similarly to what happened with velocities, the total stress in each edge for a given configuration is the combination of a self-stress of the framework (i.e. the solution to the homogeneous system of equations $\underline{\underline{R}}(\underline{p})^T \underline{\omega} = 0$) and the stresses introduced by external forces (i.e. the solution to the inhomogeneous system of eq. 21).

Any tensegrity framework can compensate for an external force in two different ways [59]: keeping the initial configuration but modifying the stress present in each member, or directly modifying the initial configuration. Only the external forces verifying the former condition are considered resolvable for a given tensegrity framework, since, in the second case, the framework itself is modified in order to withstand the external load. Any given external load is in general a combination of both kinds of forces. The set of resolvable forces ($\mathcal{R}$) form a convex cone in $\mathbb{R}^{nd}$, i.e., a convex set which is closed under multiplication by non-negative scalars [61].

The set of equilibrium forces ($\mathfrak{E}$), as explained in section 3, are those which do not accelerate the tensegrity framework, and therefore do not modify the position and orientation of the structure as a whole [61]. Intuitively, the equilibrium forces must be orthogonal to the rigid velocities to avoid undesired rigid motions. Crapo and Whiteley [22] defined the equilibrium forces as the set:

$$\mathfrak{E} = \{\underline{F} \in \mathbb{R}^{nd}, \underline{F}^T \underline{p}' = 0, \underline{p}' \in T(\underline{p})\}. \tag{22}$$

Therefore, equilibrium forces can be seen as a vectorial space spanned by the base of $T(\underline{p})^\perp$, that is, the set of directions which do not produce a rigid motion but may or may not infinitesimally modify the lengths of the members [61]. The concept of statically rigid can now be derived using linear algebra notation. A tensegrity framework is statically rigid if [22]:

$$span(Range(\underline{\underline{R}}(\underline{p})^T)) = span(\mathfrak{E}), \tag{23}$$

that is, all equilibrium forces are resolvable.

Using the approach proposed by Calladine [9] and S. Pellegrino [62, 63] based on a modified Gaussian reduction method, or the similar approach of Murakami [54] based on SVD decomposition, it is possible to obtain the dimension and a basis for both resolvable and non-resolvable external forces, as well as the corresponding basis for bar tensions (see Fig. 7). This method is based on the four subspaces associated to any rectangular matrix and it can find, both, the number of states of self stress and the number of infinitesimal flexes of a given tensegrity framework. With this method, it is possible to find out if the corresponding tensegrity framework is infinitesimally rigid or not using the above condition 14. Based on this method, several authors developed techniques to find the order of the flexes [59, 77, 32].

It has been proved by Crapo and Whiteley [22] that, for tensegrity frameworks, static rigidity is only preserved for affine transformations, but it is not preserved under projective transformations or orthogonal projections. Roth and Whiteley [61] and Crapo and Whiteley [22] proved that a tensegrity framework is statically rigid if and only if it is infinitesimally rigid.

Due to this result it is possible to connect the stresses with the rigidity characteristics of a tensegrity framework by means of the so called first and second order stress tests [19]. The first order stress test states that a given cable or strut of a tensegrity framework may modify its length if and only if its stress is null, that is, for any $\underline{p}'$ such that $(\underline{p}_i - \underline{p}_j)(\underline{p}'_i - \underline{p}'_j) \neq 0$ then $w_{ij} = 0$. So if the equivalent bar framework is infinitesimally rigid (using the condition presented in eq. 14) and there exist a strict proper self stress $\underline{\omega}$, then the tensegrity framework is infinitesimally rigid.

The second order stress test states that a first order flex $\underline{p}'$ extends to a second order flex $(\underline{p}', \underline{p}'')$ if and only if $\underline{p}'^T \underline{\underline{\Omega}} \underline{p}' \leqslant 0$ for all proper self stresses.

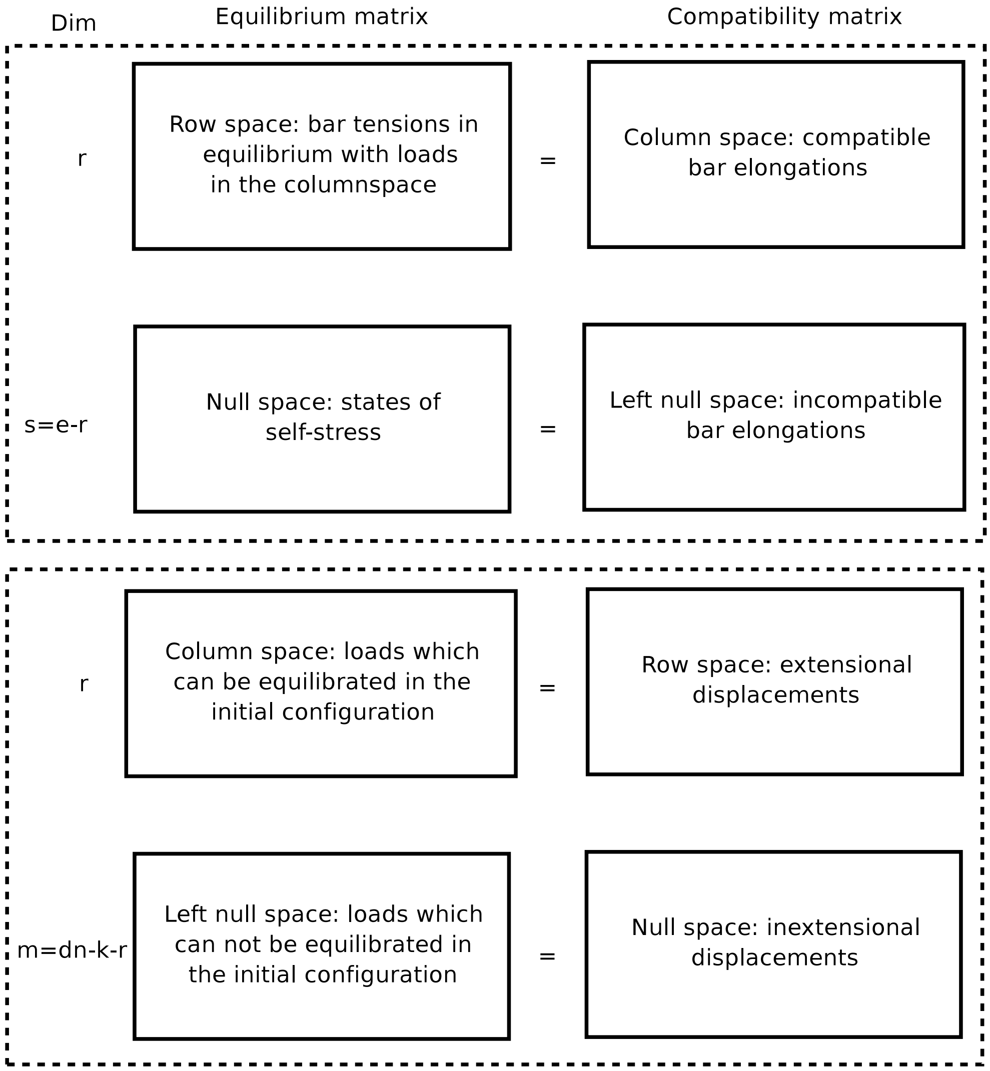


Fig. 7. The four subspaces associated to the equilibrium matrix and its transpose, the compatibility matrix.

This means that, in order to be second order rigid, the stress matrix $\underline{\underline{\Omega}}$ must be positive semi definite for any proper stress $\underline{\omega}$.

Due to the kinematic indeterminacy of tensegrity frameworks, there may exist mechanisms (flexes) of the structure that can make it to collapse. Pellegrino [59] and Calladine [10] developed a method to find out which flexes can be stiffened by a state of self-stress. Such method basically consists in computing the forces in the members (product force vector) that each flex would produce. Then, if these forces are orthogonal to the set of resolvable forces, the corresponding flex would contribute to the state of self-stress. Otherwise, the flex would modify the initial configuration.

4.1.3 Energy approach

Some of the concepts of rigidity presented so far for a tensegrity framework can also be stated in terms of energy [14, 19]. It is possible to define a function which models the energy of a tensegrity framework. Such a function has to take into account that the energy of a cable increases when stretched, the energy of a strut increases when shortened and that the energy of a bar increases under a length change. Some examples of the kind of functions that model the energetic behavior of each member are shown in Fig. 8.

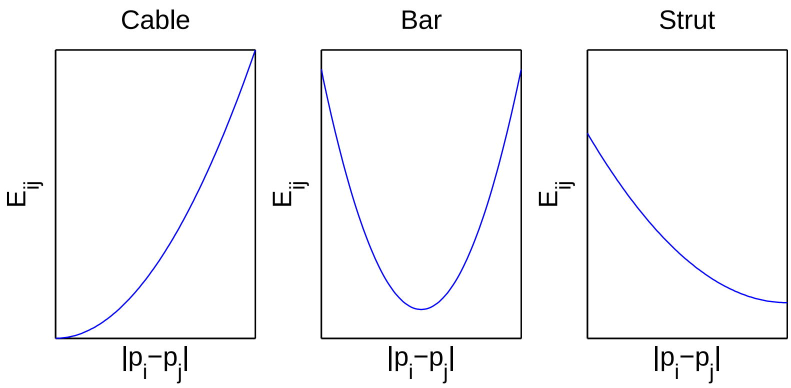


Fig. 8. Energy functions for cables, bars and struts.

First in Connelly [14] and later in Connelly and Whiteley [19], it has been proved that a rigid tensegrity framework $G(\underline{p})$ has a local minimum of the associated energy function at $\underline{p}$ and all other congruent placements. This statement proves that if a tensegrity framework is rigid, small inaccuracies in the length of the members result in an arbitrarily close placement ($|\underline{p} - \underline{q}| < \epsilon$) which is also rigid. So, this is an alternative condition for rigidity from an energy point of view. In terms of energy, each local minimum of the energy function associated to the tensegrity framework correspond to a different configuration. In the case there exists only one minimum (except for congruent configurations) the framework is globally rigid or uniquely embedded as defined before in this section.

The energy function can be arbitrarily chosen while verifying the above conditions; for instance, any function of ($|\underline{p}_i - \underline{p}_j|^2$) may be valid:

$$E(\underline{p}) = \sum_{ij} f_{ij}(|\underline{p}_i - \underline{p}_j|^2). \qquad (24)$$

In order to have a minimum in $\underline{p}$, it is necessary that the first derivative of the energy function in eq. 24 cancels at $\underline{p}$, but it is not sufficient since it only guarantees that a singular point has been found. It is also necessary that the second derivative of the energy function be positive at $\underline{p}$.

The first derivative of eq. 24 is:

$$\left.\frac{dE(\underline{p})}{dt}\right|_{t=0} = \sum_{ij} f'_{ij}(|\underline{p}_i - \underline{p}_j|^2)(2(\underline{p}_i - \underline{p}_j)(\underline{p}'_i - \underline{p}'_j)). \tag{25}$$

The term $f'_{ij}(|\underline{p}_i - \underline{p}_j|^2)$ can be seen as a force density coefficient equivalent to the stresses ω_{ij} defined earlier. So, with the definition of the energy functions as in Fig. 8, the stress $\underline{\omega}$ is always proper. Using matrix notation for eq. 25, the condition for the energy function to have a singular point in $\underline{p}$ can be stated as:

$$\left.\frac{dE(\underline{p})}{dt}\right|_{t=0} = 2\underline{\omega}^T \underline{\underline{R}}(\underline{p})\underline{p}' = 0. \tag{26}$$

Eq. 26 is verified, for all infinitesimal flexes $\underline{p}'$, if $\underline{\omega}^T \underline{\underline{R}}(\underline{p}) = 0$, which is only true if $\underline{\omega}$ is a self-equilibrium stress, and therefore proper (note that it may be 0 for some edge).

Now, given a proper self-stress for the tensegrity framework, in order to find out if the singular point is a minimum, the second derivative must be positive:

$$\begin{aligned} \left.\frac{d^2E(\underline{p})}{dt^2}\right|_{t=0} = & \textstyle\sum_{ij} f''_{ij}(|\underline{p}_i - \underline{p}_j|^2)(2(\underline{p}_i - \underline{p}_j)(\underline{p}'_i - \underline{p}'_j))^2 \\ & + \textstyle\sum_{ij} f_{ij}(|\underline{p}_i - \underline{p}_j|^2)2(|\underline{p}'_i - \underline{p}'_j|^2) \end{aligned}. \tag{27}$$

The term $f''_{ij}(|\underline{p}_i - \underline{p}_j|^2)$ is usually referred to as the stiffness coefficient c_{ij}, and it is closely related to physical properties (Young's modulus, rest length and cross-sectional area) of the used material for the cables, struts and bars.

Condition 27 for a single edge can be stated more compactly using matrix notation, and then the positive condition is transformed to positive semi definiteness of the resulting matrix. So, a tensegrity framework is rigid if the second derivative of the energy function associated to the whole structure (Hessian matrix $\underline{\underline{H}}$ from now on) verifies:

$$\underline{\underline{H}} = \underline{p}'^T \left[2\underline{\underline{\Omega}} + 4\underline{\underline{R}}(\underline{p})^T \underline{\underline{C}}\underline{\underline{R}}(\underline{p})\right] \underline{p}' \succcurlyeq 0 \tag{28}$$

for all infinitesimal flexes $\underline{p}'$, and $\underline{\underline{H}} = 0$ if and only if the infinitesimal flex is trivial, where $\underline{\underline{C}}$ is a diagonal matrix containing the stiffness coefficients for each member. Note that the second term of eq. 28 will be always positive semi-definite by construction.

This formulation puts together a lot of information about the framework itself and its environmental conditions. The second term in eq. 28, represents the physical stiffness of the framework which is an intrinsic parameter and depends only on the material of the cables, bars and struts. The first term, represents the self-stress given to the structure by design or caused by external forces applied to the structure.

So far, the energetic formulation has naturally lead to most of the rigidity definitions presented in both the motion and forces approaches. Furthermore, from an energetic point of view, it is possible to define new categories of rigidity and stability for tensegrity frameworks. One important concept is *pre-stress stability* [19], and the most known example of such structures is the spider web shown in fig. 9. It arises from the condition of positive semidefiniteness of the Hessian matrix by imposing that the proper stress must be also strict. Thus, the second terms in eq. 28 of those edges with null stress do not contribute to the positive semidefiniteness of the Hessian matrix.

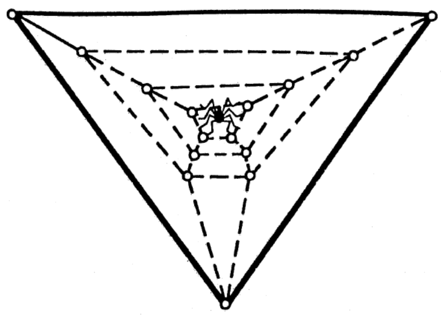


Fig. 9. A spider web, an example of a pre-stress stable tensegrity framework. This picture have been taken from Connelly and Whiteley [19].

Intuitively, the pre-stress stability property of tensegrity frameworks is related to the rigidifying effect of some stresses introduced by Pellegrino [59], Calladine [10]. However, as noted by Connelly [16], it is possible that even when the stress matrix has some negative eigenvalues, i.e. it is not positive semi-definite, the structure is still pre-stress stable, because the Hessian matrix in eq. 28 maintains the condition of positive semi-definiteness, i.e. the second term in eq. 28 dominates over the stress matrix. So, the tensegrity framework keeps its shape for certain values of pre-stress; but, it can happen, that an increase on the pre-stress makes the whole structure to collapse if the stiffness coefficients c_{ij} are unchanged, i.e., the second term in eq. 28 no longer dominates the stress matrix.

Following with this discussion, a new equivalent definition for unyielding tensegrtiy frameworks arise from an energy point of view. A tensegrity framework is said to be unyielding if it is pre-stress stable and additionally the stress matrix is positive semi-definite. This ensures the fact that increasing the pre-stress on the structure stiffens or rigidifies it.

Finally, Connelly [16] introduced an even stronger condition for stability called *super stability*. A tensegrity framework is super stable if it is unyielding and also the rank of the stress matrix ($\underline{\underline{\Omega}}$) is maximal. This second condition implies that the valid placement $\underline{p}$, and therefore the tensegrity framework shape, do not change whatever the dimension of the working space is.

To conclude this section, some comments regarding the hierarchical relationship presented in Fig. 4. Connelly and Whiteley [19] proved several implications about pre-stress stability: a pre-stress stable tensegrity framework is also rigid and second order rigid. An infinitesimally rigid tensegrity framework is also pre-stress stable and so, second-order rigid and rigid. It is worth to mention that the inverse implications are not always true. Connelly [16] showed that a super stable tensegrity framework is also globally rigid and pre-stress stable.

5 Formfinding methods

Parallely to the development of the first rigorous static analysis of tensegrity frameworks arose the problem of designing and building such structures. Fuller [29], Emmerich [24] (geometric approach) and even Snelson [68] (constructivism approach) presented simple methods to build tensegrity structures with high level of symmetry and based on convex polyhedra. Soon it became evident that the self-stressed shape of tensegrities is not identical to that of the polyhedron, therefore, a lot of new methods were introduced in this subject over the years.

Tibert and Pellegrino [76] presented a survey on form-finding methods for tensegrity structures in which they classify all the previous methods into two categories: kinematical methods and statical methods. The former methods are characterized by increasing (decreasing) the length of the struts (cables) and keeping the length of the cables (struts) constant until a maximum (minimum) is reached. These methods do not require the members to be in a state of prestress.

In this category, Connelly and Terrell [18] used an analytical method, mainly based on geometry, which states the coordinates of each node as a function of few parameters and then maximize (minimize) the length expressions for struts (cables) given the length of the cables (struts) starting from an arbitrary configuration. This approach is simple for highly symmetric structures, but it is infeasible for non-symmetric tensegrities due to the large number of variables needed.

Pellegrino [58] and Burkhardt [8] translated the form-finding problem into another one of constrained minimization using non-linear programming techniques. These methods require a valid configuration to start with and then try to minimize (maximize) the length of same struts (cables), but they don't take into account any stress constraints. Therefore, although geometrically correct, the resulting structure may not be stable.

Finally, Motro [50] and Belkacem [4] introduced the dynamic relaxation method for tensegrity structures. This method has been successfully used for membrane and cable net structures. In order to get the equilibrium configuration, this method solves a fictitious dynamic model in terms of the acceleration, velocity and displacement from the initial configuration like the one shown below:

$$M\ddot{d} + D\dot{d} + Kd = f \tag{29}$$

Motro [50] showed that this method is only useful for small size structures, but it takes into account equilibrium considerations and the existence of external forces.

In the statical methods, a relationship between equilibrium configurations and the forces in its members is analyzed using different approaches. For example, Kenner [39] used node equilibrium conditions and symmetry arguments to find the stable configurations of some simple tensegrity structures. This approach is similar to that of Connelly and Terrell for kinematic methods but it guarantees stability of the final structure without any external load.

Schek [65], Linkwitz [44] presented a method (called the force density method) which transforms the non-linear equilibrium equations into a set of linear equations. This method requires the a priory knowledge of the stress coefficients for all members which is one of its greatest drawbacks since some combinations of stresses may not have physical implementations in a given space. Another important problem of this method is that it is not possible to control the length of the members. In this case, the linearized system of equations is:

$$\underline{\underline{C}}^T \underline{\underline{Q}} \underline{\underline{C}} \underline{p}_i = \underline{f}_i, \tag{30}$$

where $\underline{\underline{C}}$ is the incidence matrix for a given topology, $\underline{\underline{Q}}$ is a diagonal matrix containing the force density coefficients and $\underline{p}_i$ and $\underline{f}_i$ are the coordinate vector and the external force applied to each node in the i^{th} direction respectively. The product $\underline{\underline{C}}^T \underline{\underline{Q}} \underline{\underline{C}}$ is called the force density matrix.

Connelly [15] also presented an energy based form-finding method which assigns an energy function to a tensegrity and searches the minimum of this function, which is equivalent to test the positive semi definiteness of the stress matrix ($\underline{\underline{\Omega}}$). This stress matrix is identical to the force density matrix used in the force density method, Schek [65] showed that these two methods are closely related.

Another approach proposed by Sultan et al. [70] is to identify a set of generalized coordinates for a particular tensegrity framework and use symbolic manipulation to obtain the equilibrium matrix ($\underline{\underline{R}}(\underline{p})$). The general solution to this problem is still complex due to the high dimension of the equilibrium matrix, therefore some authors have given particular solutions for highly symmetric structures. For example Sultan [69] and Sultan [69] particularized the pre-stress condition for a two stage SVD tensegrity structure which can be parametrized with just 3 parameters (i.e. the azimuth (α), the declination (δ) and the overlap (h)).

More recently, there have appeared some more sophisticated form finding methods which allow to simultaneously find the stresses on the members and the coordinates of the nodes in the space. Micheletti and Williams [47] propose a method based on solving a system of differential equations. These authors also propose a method to modify a given stable configuration to reach another one by modifying the length of a given edge and solving the system of differential equations to get the change in length of the other edges.

Zhang et al. [84] developed a method to first find a set of axial forces compatible with a given structure and then find the corresponding nodal coordinates under equilibrium conditions and structure constraints. This approach mainly finds the bases for the self-stress and placements subspaces and then needs to fix a number of stresses and coordinates equal to the dimension of those subspaces respectively in order to find the final solution. This method can take advantage of possible symmetries but it is not necessary.

Masic et al. [45] presented a modified version of the force density method introduced by Schek [65], Linkwitz [44] which explicitly includes shape constraints. They also studied how the symmetry properties can be used to systematically reduce the number of force density variables, equilibrium equations and geometrical variables. They also showed that the equilibrium of a tensegrity framework is invariant under an affine transformation (i.e. $\underline{\overline{p}} = \underline{\underline{S}}\underline{p} + \underline{r}$) of the nodal coordinates.

All the form finding methods presented so far assume a given topology and try to find a configuration which is stable in a given space verifying some constraints. A different approach is to find the topology which assures stability. In this direction Paul et al. [56] used genetic algorithms to evolute an initial arbitrary topology into a stable one in the work space. This approach is able to generate irregular structures.

Further developing the work started by Pellegrino [58] using non-linear programming techniques, Masic et al. [46] developed a procedure which seeks the topology, geometry and pre-stress of a structure under external forces, and taking into account strength, buckling and shape constraints. The use of a sequential quadratic programming (a penalty method) enables the algorithm to find stable configurations starting from an arbitrary one, but some singularities in the gradient function due to the use of the element length may cause the algorithm to diverge or alternatively, converge to a non-optimum solution. Additionally, their form-finding method handles some physical phenomena due to external loads.

Finally, Estrada et al. [25] presented an algorithm which only needs information about the type of each edge and about the topology, but does not account for external forces. Then, the equilibrium geometry and force densities for each edge are iteratively calculated using rank constraints on both the stress matrix ($\underline{\underline{\Omega}}$) and the rigidity matrix ($\underline{\underline{R}}(\underline{p})$).

In table 3 there is a summary of the form finding methods presented in this section. Most of them do not take into account any external forces or the self-weight of the structure so when built, the final configuration will be different from the solution obtained with the form finding methods.

Table 3
Summary of form-finding methods.

Method name	Class	Assures stability	Needs a valid initial configuration	Uses symmetry	Needs an initial topology	Uses external forces
Analytic solution (Connelly and Terrell [18])	Kinematic	No	No	Yes	Yes	No
Non-linear programming (Pellegrino [58])	Kinematic	No	Yes	No	Yes	No
Dynamic relaxation (Motro [50])	Kinematic	Yes	Yes	No	Yes	Yes
Analytic solution (Kenner [39] and Connelly and Terrell [18])	Static	Yes	No	Yes	Yes	No
Force density method (Linkwitz [44] and Schek Schek [65])	Static	Must be given	No	No	Yes	Yes
Energy method (Connelly [15])	Static	Yes	No	No	Yes	No
Reduced coordinates (Sultan et al. [70])	Static	Yes	No	Yes	Yes	No
Differential equations (Micheletti and Williams [47])	Static	Yes	Yes	No	Yes	No
Successive approximation (Zhang et al. [84])	Both	Some stresses have to be fixed	Some coordinates have to be fixed	No	Yes	No
Algebraic method (Masic et al. [45])	Static	Must be given	No	No	Yes	Yes
Genetic algorithm (Paul et al. [57])	Topologic	Yes	No	No	No	No
Sequential quadratic programming (Masic et al. [46])	Both	Yes	No	No	No	Yes
Numerical method (Estrada et al. [25])	Both	Yes	No	No	Yes	No

6 Conclusions

This paper hands in a review of the basic issues about the statics of tensegrity structures. Definitions and notation for the most important concepts, borrowed from a vast existing literature, have been summarized: an abstract tensegrity framework viewed as a graph, its placement in a d-dimensional space, whether general or regular, the rigidity and stress matrices, the concepts of self-stress and proper stress, or the equilibrium and resolvable forces. All of these concepts and definitions provided us a sufficient and complete mathematical framework so as to analyze the rigidity and stability of tensegrity structures from three different points of view: motion, force and energy approaches.

From a motion point of view, the concepts of rigid or flexible tensegrity were reviewed depending on whether all the possible flexes were trivial or not. A more restrictive rigidity concept is that of infinitesimal or first-order rigidity. This concept arises when introducing in the analysis the initial velocities of each node by means of using the first derivative of the length function. A tensegrity framework is infinitesimally rigid if all the admissible initial velocities are trivial, that is, if the space of admissible velocities is exactly the same as the space of congruent velocities. Yet, another important level of rigidity is that of second-order rigidity, which can be defined from second order flexes, that is, considering also the acceleration of the nodes. Other kinds of rigidity can also be considered by defining higher order flexes just by taking higher derivatives of the length function. Also methods to check for one or another type of rigidity have been reviewed, mainly depending on the dimension of the kernel of the rigidity matrix.

Using a force approach, we have seen that a tensegrity, in order to achieve a rigid configuration, must hold an equilibrium of forces in each of its nodes. A given tensegrity framework is said to be statically rigid if every equilibrium force is resolvable, that is, if there exist a proper stress vector which compensates for the external equilibrium force. By using the duality between the cones of velocities and forces it can be demonstrated that the concepts of statical and infinitesimal rigidity are equivalent.

The energy of a tensegrity structure can be obtained as the sum of the energies of each of its elements. An energy function can be chosen verifying that the energy of a cable increases when stretched, that of a strut increases when shortened and that of a bar increases whenever a change in its length is produced. It has been proved, that if this energy function has an overall minimum for a given tensegrity framework in a particular placement, then the tensegrity is globally rigid. In fact, every local minimum, if there are any, of the energy function corresponds to a different possible configuration. Infinitesimal rigidity can also be checked in terms of the energy function by analyzing its hessian. But other concepts of rigidity and stability can be defined from an energy point of view, those of pre-stress and super stability. There is a subtle difference between these two stability concepts. Pre-stress stability implies the positive semi-definiteness for the hessian of the energy function. Super stability is more restrictive because it also requires the stress matrix to be positive semi definite, a condition which does not necessarily hold for a pre-stress stable tensegrity.

From this analysis, an order for the different levels of rigidity and stability can be established, from most to less restrictive conditions, as: infinitesimal (also, first order or statical rigidity), super stable, pre-stress stable, second order rigid, third order rigid and so on to finally arrive to a simply rigid tensegrity. Implications go from the first to the last, the converse not being true in general. Examples of different tensegrity structures accomplishing with different levels of rigidity and stability have been presented and analyzed. Finally, also a review of existing form-finding methods has been done in order to complete the static study on tensegrity structures.

We hope this survey serve as a comprehensive study for the statics of such fascinating structures as tensegrity are, and will help and encourage new researchers to understand and contribute in this area.

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