J. ALGEBRAIC GEOMETRY
00 (XXXX) 000–000
S 1056-3911(XX)0000-0

BEILINSON-FLACH ELEMENTS AND EULER SYSTEMS I: SYNTOMIC REGULATORS AND p-ADIC RANKIN L-SERIES

MASSIMO BERTOLINI, HENRI DARMON, VICTOR ROTGER

Abstract

This article is the first in a series devoted to the Euler system arising from p-adic families of *Beilinson-Flach elements* in the first K-group of the product of two modular curves. It relates the image of these elements under the p-adic syntomic regulator (as described by Besser [Bes3]) to the special values at the near-central point of Hida's p-adic Rankin L-function attached to two Hida families of cusp forms.

Contents

1. Introduction

This article is the first in a series devoted to the Euler system of *Beilinson-Flach elements* in the motivic cohomology of a product of two modular curves. Its main result (see Theorem 4.2 and Corollary 4.4 of §4.2) is a p-adic analogue of the formula of Beilinson [Bei, Ch. 2, §6] expressing special values of Rankin

L-series in terms of complex regulators. Beilinson's theorem (cf. §4.1 for an explicit version) relates:

(1) the Rankin L-series $L(f \otimes g, s)$ attached to the convolution of weight 2 newforms f and g on $\Gamma_1(N)$, evaluated at the *near-central point* $s = 2$;
(2) the image under the complex regulator of certain explicit elements in the motivic cohomology group $H^3_{\mathcal{M}}(X_1(N)^2, \mathbb{Q}(2))$, or, equivalently, in the higher Chow group $\mathrm{CH}^2(X_1(N)^2, 1) \otimes \mathbb{Q}$. These elements, whose definition is recalled in Section 3.1, are constructed from modular units and are referred to in the sequel as *Beilinson-Flach elements*.

In the p-adic setting, the complex L-series $L(f \otimes g, s)$ is replaced by Hida's p-adic Rankin L-series attached to two ordinary families of modular forms interpolating f and g, whose definition is briefly recalled in Section 2.2. The role of the complex regulator is played by the p-adic syntomic regulator on K_1 of a surface. Besser's description of it in terms of Coleman integration [Bes3], which is summarised in §3.3, is a key ingredient in the proof of Theorem 4.2.

Our approach also relies crucially on techniques developed in [DR] for relating p-adic Abel-Jacobi images of diagonal cycles to values of the Garrett-Rankin triple product p-adic L-function attached to a triple $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ of Hida families of cusp forms. Corollary 4.4 deals with the setting where the cuspidal family $\mathbf{h}$ in the triple $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ is replaced by a Hida family of Eisenstein series. The reader will also note the close parallel between Theorem 4.2 and the main result of [BD], in which the p-adic regulators of certain elements in $K_2(X_1(N))$ are related to the value at $s = 2$ of the Mazur-Swinnerton-Dyer p-adic L-functions attached to weight two cusp forms. The results of the present article are in fact intermediate between those of [DR] and [BD], the latter treating the case where *both* $\mathbf{g}$ and $\mathbf{h}$ are replaced by Hida families of Eisenstein series—a setting in which the resulting p-adic Rankin L-function factors as a product of two Mazur-Kitagawa L-functions attached to $\mathbf{f}$.

We also remark that a function field analogue of Beilinson's Theorem involving Drinfeld modular curves is described in [Sre2], based on a description of non-archimedean regulators given in [Sre1]. See also the related work of Ambrus Pàl in the setting of the K_2 of Mumford curves [Pa].

Let us conclude this introduction by briefly discussing some eventual arithmetical applications of the main result of this paper.

I. The Euler system of Beilinson-Flach elements. The image of Beilinson-Flach elements under the p-adic étale regulator map gives rise to classes in the global cohomology group $H^1(\mathbb{Q}, V_f \otimes V_g(2))$, where V_f and V_g are the p-adic Galois representations attached to f and g, respectively. The work in preparation [BDR] explores the theme of the p-adic variation of the Beilinson-Flach classes attached to Hida families of cusp forms $\mathbf{f}$ and $\mathbf{g}$. In particular, when $\mathbf{g}$ specialises in weight one to a classical cusp form attached to an odd irreducible Artin representation ρ, and $\mathbf{f}$ specialises in weight two to the cusp form associated with an elliptic curve E over $\mathbb{Q}$, we expect the associated cohomology class to yield new cases of the Birch and Swinnerton-Dyer conjecture for the complex L-series $L(E, \rho, s)$, proving in particular that ρ does not occur in the representation $E(\bar{\mathbb{Q}}) \otimes \mathbb{C}$ when $L(E, \rho, 1) \neq 0$.

The idea of using Beilinson elements in Euler system arguments occurs much earlier in the work of Flach [Fl], who used them to construct classes in $H^1(\mathbb{Q}, \mathrm{Sym}^2(E)(2))$ which are cristalline at p but ramified at a single prime $\ell \neq p$. Applying Kolyvagin's method to these classes leads to the finiteness of the Shafarevich-Tate group of $\mathrm{Sym}^2(E)(2)$ and an explicit annihilator of this group related to the special value $L(\mathrm{Sym}^2(E), 2)$, which is critical in the sense of Deligne, unlike the special values $L(f \otimes g, 2)$ when f and g are distinct normalised newforms.

II. Hida's L-function for the symmetric square of a modular form. Theorem 4.2, specialised to the case $f = g$, is exploited by S. Dasgupta [Das] to study the Hida L-function $L(f \otimes f, s)$ and express it as the product of the Coates-Schmidt p-adic L-function attached to $\mathrm{Sym}^2(f)$ and a Kubota-Leopoldt p-adic L-function. This factorisation, which can be viewed as another manifestation of the Artin formalism for p-adic L-series, is analogous to a formula of Gross [Gross] expressing the restriction to the cyclotomic line of the Katz two-variable p-adic L-function attached to an imaginary quadratic field as a product of two Kubota-Leopoldt L-functions. The Beilinson-Flach elements play the same role in Dasgupta's proof as elliptic units in the work of Gross.

III. p-adic L-functions and Euler systems over $\mathbb{Z}_p^2$-extensions. The paper in preparation [LLZ] of A. Lei, D. Loeffler and S.L. Zerbes builds on the methods of this paper, in the setting where g varies over a collection of theta series attached to Hecke characters of an imaginary quadratic field K, to construct an Euler system for V_f over the various layers of the *two variable* $\mathbb{Z}_p$-extension K_∞ of K, thus supplying the global input for their extension [LZ] of Perrin-Riou's machinery in which the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ is replaced by K_∞.

Acknowledgements: The authors are grateful to Amnon Besser, François Brunault, Samit Dasgupta, Antonio Lei, David Loeffler, Nicola Mazzari and Sarah Zerbes for their useful comments and stimulating conversations about their research in connection to this work.

2. Rankin L-series

Let f and g be normalised newforms of weights k, ℓ, levels N_f, N_g, and nebentypus characters χ_f, χ_g respectively. The p-adic representations V_f and V_g are part of a compatible system of representations which we continue to denote by the same symbol. Let

$$L(V_f \otimes V_g, s) = \prod_p \det((1 - \sigma_p p^{-s})|(V_f \otimes V_g)^{I_p})^{-1}$$

be the motivic L-function attached to the tensor product $V_f \otimes V_g$, where I_p denotes the inertia subgroup of a decomposition group at p, and σ_p a corresponding geometric Frobenius element.

The goal of this first chapter is to briefly recall the basic analytic properties of this L-series, describe Hida's construction of a p-adic avatar, and— in the special case where f and g are both of weight two—present parallel formulae for their special values at the near-central point $s = 2$, which is *not* critical in the sense of Deligne.

2.1. Complex L-series. Set $N := \mathrm{lcm}(N_f, N_g)$ and replace χ_f and χ_g by their counterparts of modulus N sending any prime $r|N$ to 0. It is also convenient to replace f, as well as g, by a normalised eigenform of level N which

1. has the same eigenvalues for the Hecke operators T_r with $r \nmid N$;
2. is also an eigenvector for the Hecke operators U_r attached to the primes r dividing N.

This substitution having been made, let

$$f(z) = \sum_{n=1}^{\infty} a_n(f) e^{2\pi i n z}, \qquad g(z) = \sum_{n=1}^{\infty} a_n(g) e^{2\pi i n z} \tag{2.1}$$

be the Fourier expansions of f and g, let K_f and $K_g \subset \bar{\mathbb{Q}}$ denote the subfields generated by the coefficients $a_n(f)$ and $a_n(g)$ respectively, and let K_{fg} denote the compositum of the two fields. The Hecke polynomials attached to f can be factored as

$$x^2 - a_p(f)x + \chi_f(p)p^{k-1} = (x - \alpha_p(f))(x - \beta_p(f)),$$

where $(\alpha_p(f), \beta_p(f)) = (a_p(f), 0)$ when $p|N$. Similar notations are adopted for g. The Rankin L-function attached to the pair (f, g) is defined by the formula

$$\begin{aligned} L(f\otimes g,s) &:= \prod_p L_{(p)}(f\otimes g,s), \qquad \text{where} \\ L_{(p)}(f\otimes g,s) &:= (1-\alpha_p(f)\alpha_p(g)p^{-s})^{-1}(1-\alpha_p(f)\beta_p(g)p^{-s})^{-1} \\ &\quad \times(1-\beta_p(f)\alpha_p(g)p^{-s})^{-1}(1-\beta_p(f)\beta_p(g)p^{-s})^{-1}. \end{aligned}$$

The Euler factors at p defining $L(V_f\otimes V_g,s)$ and $L(f\otimes g,s)$ agree for all $p\nmid N$, and hence the special values of $L(V_f\otimes V_g,s)$ and $L(f\otimes g,s)$ at integer points differ by elementary quantities in $K_{fg}^{\times}$. It will be more convenient, for the sequel, to focus attention on $L(f\otimes g,s)$. Assume without loss of generality that the forms f and g have been ordered in such a way that $k\geq \ell$.

2.1.1. Rankin's method. We begin by recalling the general formula for $L(f\otimes g,s)$ coming out of Rankin's method, involving the *non-holomorphic Eisenstein series*

$$\widetilde{E}_{k-\ell,\chi}(z,s)=\sum_{(m,n)\in N\mathbb{Z}\times\mathbb{Z}}{}^{\prime}\frac{\chi^{-1}(n)}{(mz+n)^{k-\ell}}\cdot\frac{y^s}{|mz+n|^{2s}} \tag{2.2}$$

of weight $k-\ell$, level N and character

$$\chi:=\chi_f^{-1}\chi_g^{-1},$$

where the superscript $'$ in (2.2) indicates that the sum is taken over the non-zero lattice vectors $(m,n)\in N\mathbb{Z}\times\mathbb{Z}$. For fixed complex s with $\mathrm{Re}(s)\gg 0$, the product $\widetilde{E}_{k-\ell,\chi}(z,s)\times g(z)$ is a real-analytic $\mathbb{C}$-valued function on the Poincaré upper half-plane $\mathcal{H}$ which transforms like a modular form of weight k, level N and character χ_f^{-1} and is of rapid decay at infinity. The space of such functions, denoted $S_k^{\mathrm{ra}}(N,\chi_f^{-1})$, is equipped with the Petersson scalar product

$$\langle\ ,\ \rangle_{k,N}: S_k^{\mathrm{ra}}(N,\chi_f^{-1})\times S_k^{\mathrm{ra}}(N,\chi_f^{-1})\longrightarrow\mathbb{C}$$

given by the formula

$$\langle f_1,f_2\rangle_{k,N}:=\int_{\Gamma_0(N)\backslash\mathcal{H}} y^k\overline{f_1(z)}f_2(z)\frac{dxdy}{y^2}, \tag{2.3}$$

which is hermitian linear in the first argument and $\mathbb{C}$-linear in the second. Let $f^*\in S_k(N,\chi_f^{-1})$ denote the modular form obtained from f by applying complex conjugation to its Fourier coefficients.

Proposition 2.1 (Shimura)**.** *For all $s\in\mathbb{C}$ with $\Re(s)\gg 0$,*

$$L(f\otimes g,s)=\frac{1}{2}\frac{(4\pi)^s}{\Gamma(s)}\left\langle f^*(z),\widetilde{E}_{k-\ell,\chi}(z,s-k+1)\cdot g(z)\right\rangle_{k,N}. \tag{2.4}$$

This well-known formula for the Rankin L-series is taken from equation (14) of [BD].

2.1.2. Critical values. Assume here and in §2.1.3 that $\ell < k$. The functional equation for $L(f \otimes g, s)$ arising from Proposition 2.1 reveals that the integer j is critical for $L(f \otimes g, s)$ if any only if it lies in the closed interval $[\ell, k-1]$. We now describe a further closed formula for the value at an integer j belonging to the "right half critical segment" $[\frac{\ell+k-1}{2}, k-1]$, which will be useful in deriving the algebraicity (up to periods) of $L(f \otimes g, j)$ predicted by the Deligne conjectures, and ultimately in constructing Hida's p-adic Rankin L-function by interpolating these quantities p-adically.

Having fixed an integer $j \in [\frac{\ell+k-1}{2}, k-1]$, let $t \geq 0$ and $m \geq 1$ be given by

$$t := k - 1 - j, \qquad m := k - \ell - 2t.$$

If $m \leq 2$, let us assume also that χ is nontrivial. Then the series

$$E_{m,\chi}(z) = 2^{-1} L(\chi, 1-m) + \sum_{n=1}^{\infty} \sigma_{m-1,\chi}(n) q^n, \tag{2.5}$$

where $\sigma_{m-1,\chi}(n) = \sum_{d|n} \chi(d) d^{m-1}$, is the q-expansion of a *holomorphic* Eisenstein series of weight m and character χ.

The Shimura-Maass derivative operator

$$\delta_m := \frac{1}{2\pi i} \left(\frac{d}{dz} + \frac{im}{2y} \right)$$

transforms modular forms of weight m into (real analytic) modular forms of weight $m+2$ which are *nearly holomorphic* in the sense of [Sh2], and its t-fold iterate $\delta_m^t := \delta_{m+2t-2} \cdots \delta_{m+2} \delta_m$ maps the space $M_m(N, \chi)$ to the space $M_{m+2t}^{\mathrm{nh}}(N, \chi)$ of nearly holomorphic modular forms of weight $m+2t$. Let

$$C(k, \ell, j) := \frac{(-1)^t 2^{k-1} (2\pi)^{k+m-1} \iota_\chi (iN)^{-m} \tau(\chi^{-1})}{(m+t-1)!(j-1)!}$$

be the elementary constant (in which $\iota_\chi = 1$ when χ is primitive) appearing in equation (18) of [BD]. The following formula for $L(f \otimes g, j)$, is obtained by setting $c = j$ in equation (18) of loc. cit. (See also Theorem 2 of [Sh1].)

Proposition 2.2. *The special value $L(f \otimes g, j)$ is given by the formula*

$$L(f \otimes g, j) = C(k, \ell, j) \left\langle f^*(z), \delta_m^t E_{m,\chi}(z) \times g(z) \right\rangle_{k,N}. \tag{2.6}$$

2.1.3. Algebraicity and Deligne's conjecture. Let $S_k^{\mathrm{nh}}(N, \chi_f^{-1}; K_{fg}) \subset S_k^{\mathrm{nh}}(N, \chi_f^{-1})$ denote the space of nearly-holomorphic cusp forms which are *defined over K_{fg}* in the sense of Shimura (cf. Section 2.4 of [DR]). The cusp form

$$\Xi(f, g, j) := \delta_m^t E_{m,\chi} \times g \in S_k^{\mathrm{nh}}(N, \chi_f^{-1}) \tag{2.7}$$

appearing in Prop. 2.2 belongs to the K_{fg}-rational structure $S_k^{\mathrm{nh}}(N, \chi_f^{-1}; K_{fg})$. Hence, its image

$$\Xi(f,g,j)^{\mathrm{hol}} := \Pi_N^{\mathrm{hol}}(\Xi(f,g,j)) \tag{2.8}$$

under the holomorphic projection Π_N^{hol} of loc. cit. belongs to $S_k(N, \chi_f^{-1}; K_{fg})$, the space of holomorphic cusp forms with Fourier coefficients in K_{fg}. In particular, the ratio

$$\begin{aligned} L^{\mathrm{alg}}(f\otimes g, j) &:= C(f,g,j)^{-1}\,\frac{L(f\otimes g,j)}{\langle f^*, f^*\rangle_{k,N}} \\ &= \frac{\langle f^*, \Xi(f,g,j)\ \rangle_{k,N}}{\langle f^*, f^*\rangle_{k,N}} = \frac{\left\langle f^*, \Xi(f,g,j)^{\mathrm{hol}}\right\rangle_{k,N}}{\langle f^*, f^*\rangle_{k,N}} \end{aligned} \tag{2.9}$$

belongs to K_{fg}. This algebraicity result is consistent with Deligne's conjecture which predicts that the period $C(f,g,j)\langle f^*, f^*\rangle_{k,N}$ is the 'transcendental part" of the special value $L(f\otimes g, j)$. The associated "algebraic part" appearing in (2.9) will later be interpolated p-adically to obtain Hida's p-adic Rankin L-function attached to f and g.

In order to do this, it will be convenient to give a more geometric description of the quantity $L^{\mathrm{alg}}(f\otimes g, j)$ appearing in (2.9), in terms of the Poincaré duality on the de Rham cohomology of the modular curve $X_1(N)$ with values in appropriate sheaves with connection, as described in [DR, § 2.3]. To lighten the notations, denote by Y and by X the open modular curve $Y_1(N)$ and the complete modular curve $X_1(N)$ respectively, classifying (generalised) elliptic curves equipped with an embedding of the finite flat group scheme μ_N of N-th roots of unity.

Let K be any field containing K_{fg}. Denote by $\mathcal{E} \longrightarrow Y$ the universal elliptic curve over Y, and by ω the sheaf of relative differentials on $\mathcal{E}$ over Y, extended to X as in [BDP, § 1.1]. Recall the Kodaira-Spencer isomorphism $\omega^2 \simeq \Omega^1_X(\log \mathrm{cusps})$, where $\Omega^1_X(\log \mathrm{cusps})$ is the sheaf of regular differentials on Y with log poles at the cusps.

A modular form ϕ on $\Gamma_1(N)$ of weight $k = r+2$ with Fourier coefficients in K corresponds to a global section of the sheaf $\omega^{r+2} = \omega^r \otimes \Omega^1_X(\log \mathrm{cusps})$ over the base-change X_K of X to K. The sheaf ω^r can be viewed as a subsheaf of $\mathcal{L}_r := \mathrm{Sym}^r\,\mathcal{L}$, where

$$\mathcal{L} := R^1\pi_*(\mathcal{E} \longrightarrow Y)$$

is the relative de Rham cohomology sheaf on Y, extended to X as in loc. cit., equipped with the filtration

$$0 \longrightarrow \omega \longrightarrow \mathcal{L} \longrightarrow \omega^{-1} \longrightarrow 0 \tag{2.10}$$

arising from the Hodge filtration on the fibers. The sheaf $\mathcal{L}_r$ is a coherent sheaf over X of rank $r+1$, endowed with the Gauss-Manin connection

$$\nabla : \mathcal{L}_r \longrightarrow \mathcal{L}_r \otimes \Omega^1_X(\log \text{cusps}).$$

Let $H^1_{\text{dR}}(X_K, \mathcal{L}_r, \nabla)$ be the de Rham cohomology of $\mathcal{L}_r$. It is equipped with the perfect Poincaré pairing

$$\langle \ , \ \rangle_{k,X} : H^1_{\text{dR}}(X_K, \mathcal{L}_r, \nabla) \times H^1_{\text{dR}}(X_K, \mathcal{L}_r, \nabla) \longrightarrow K, \tag{2.11}$$

which is compatible with the exact sequence
(2.12)

$$0 \longrightarrow H^0(X_K, \omega^r \otimes \Omega^1_X) \longrightarrow H^1_{\text{dR}}(X_K, \mathcal{L}_r, \nabla) \longrightarrow H^1(X_K, \omega^{-r}) \longrightarrow 0,$$

in the sense that $H^0(X_K, \omega^r \otimes \Omega^1_X)$ is an isotropic subspace. (Cf. Sections 2 and 3 of [Col], for a more detailed account.) In particular, Poincaré duality induces a perfect pairing

$$\langle \ , \ \rangle_{k,X} : H^1(X_K, \omega^{-r}) \times H^0(X_K, \omega^r \otimes \Omega^1_X) \longrightarrow K, \tag{2.13}$$

which is denoted by the same symbol by a slight abuse of notation.

The antiholomorphic differential η_f^{ah} defined by

$$\eta_f^{\text{ah}} := \frac{\bar{f}^*(z) d\bar{z}}{\langle f^*, f^* \rangle_{k,N}} \ . \tag{2.14}$$

gives rise to a class in $H^1_{\text{dR}}(X_{\mathbb{C}}, \mathcal{L}_r, \nabla)$, whose image η_f in $H^1(X_{\mathbb{C}}, \omega^{-r})$ belongs to $H^1(X_K, \omega^{-r})$ (cf. Corollary 2.13 of [DR]). The following expression for the algebraic part $L^{\text{alg}}(f \otimes g, j)$ in terms of the class η_f follows directly from (2.9) in light of the discussion above:

Proposition 2.3. *The algebraic part* $L^{\text{alg}}(f \otimes g, j)$ *is equal to*

$$L^{\text{alg}}(f \otimes g, j) = \left\langle \eta_f, \Xi(f, g, j)^{\text{hol}} \right\rangle_{k,X}. \tag{2.15}$$

2.1.4. The value at the near central point. Consider now the case where $k = \ell = 2$ and $\chi_f \neq \chi_g^{-1}$, so that the character $\chi = \chi_f^{-1}\chi_g^{-1}$ is not the trivial one. The functional equation for $L(f \otimes g, s)$ relates $L(f \otimes g, s)$ to $L(f^* \otimes g^*, 3-s)$ and this L-series has no critical points in the sense of Deligne. Proposition 2.5 below describes its value at the near-central point $s = 2$ in terms of logarithms of *modular units.*

Enlarge K so that it contains the field which is cut out by all the Dirichlet characters of modulus N, and let F be the field generated over K by the values of these characters. Let $\text{Eis}_\ell(\Gamma_1(N); F)$ denote the subspace of $M_\ell(\Gamma_1(N); F)$ spanned by the weight ℓ Eisenstein series with coefficients in F. The logarithmic derivative gives a surjective homomorphism

$$\mathcal{O}(Y_K)^\times \otimes F \xrightarrow{\text{dlog}} \text{Eis}_2(\Gamma_1(N); F), \tag{2.16}$$

whose kernel is the subspace $K^\times \otimes F$ spanned by the nonzero constant functions.

Definition 2.4. Let u_χ be the modular unit satisfying

$$\mathrm{dlog}(u_\chi) = E_{2,\chi} \tag{2.17}$$

whose value at ∞ is 1 in the sense of [Br, §5].

Proposition 2.5. *Given weight two eigenforms f and g as above,*

$$L(f \otimes g, 2) = 16\pi^3 N^{-2}\tau(\chi^{-1}) \langle f^*(z), \log|u_\chi(z)| \cdot g(z)\rangle_{2,N}. \tag{2.18}$$

Proof. By Proposition 2.1,

$$L(f \otimes g, 2) = \frac{1}{2}(4\pi)^2 \left\langle f^*(z), \widetilde{E}_{0,\chi}(z,1) \cdot g(z)\right\rangle_{2,N}. \tag{2.19}$$

A direct calculation (cf. equation (26) of [BD]) shows that

$$\frac{1}{2\pi i}\frac{d}{dz}\widetilde{E}_{0,\chi}(z,1) = -\frac{1}{4\pi}\widetilde{E}_{2,\chi}(z) = 2\pi N^{-2}\tau(\chi^{-1})E_{2,\chi}(z). \tag{2.20}$$

Having normalized u_χ as in Definition 2.4, one obtains the equality

$$\widetilde{E}_{0,\chi}(z,1) = 2\pi N^{-2}\tau(\chi^{-1}) \log|u_\chi(z)|, \tag{2.21}$$

which is compatible with (2.17). Combining (2.19) with (2.21) completes the proof of the proposition. □

2.2. p-adic L-series. Let $p \geq 3$ be a prime, and fix an embedding of K into $\mathbb{C}_p$. This section recalls the definition of the Rankin p-adic L-function associated by Hida [Hi] to the convolution of two Hida families of cusp forms. For the sake of brevity, we proceed here—just as in [BD]— by specialising the approach and notations of [DR], which constructs the p-adic L-function associated to a triple product of three Hida families $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ of cusp forms. The setting considered here consists, essentially, in letting $\mathbf{h}$ be a Hida family of Eisenstein series.

2.2.1. Ordinary projections. Let f, g be eigenforms of level N, weights $k > \ell$ and nebentypus χ_f, χ_g as in (2.1). Let also $j \in [\frac{\ell+k-1}{2}, k-1]$ be an integer and set $t = k-1-j \geq 0$ and $m = k-\ell-2t \geq 1$ as in §2.1.2. The following ordinariness assumption is important for the constructions described in this section.

Assumption 2.6. *The cuspidal eigenforms f and g are ordinary at p, and $p \nmid N$.*

Under this assumption, the f-isotypic part of the exact sequence (2.12) with $K = \mathbb{C}_p$ admits a canonical *unit root* splitting, arising from the action of Frobenius on de Rham cohomology. Let η_f^{ur} be the lift of η_f to the unit root subspace $H^1_{\mathrm{dR}}(X_{\mathbb{C}_p}, \mathcal{L}_r, \nabla)^{f,\mathrm{ur}}$. The right-hand side of (2.15) is then equal to

$$\left\langle \eta_f, \Xi(f,g,j)^{\mathrm{hol}}\right\rangle_{k,X} = \left\langle \eta_f^{\mathrm{ur}}, \Xi(f,g,j)^{\mathrm{hol}}\right\rangle_{k,X}. \tag{2.22}$$

Now let $e_{\rm ord}$ be Hida's ordinary projector to $H^1_{\rm dR}(Y_K, \mathcal{L}_r, \nabla)^{\rm ord}$. By Proposition 2.11 of [DR], the right-hand side of (2.22) can be re-written, after viewing $\Xi(f,g,j)^{\rm hol}$ as an overconvergent p-adic modular form and setting $\Xi(f,g,j)^{\rm ord} := e_{\rm ord}\Xi(f,g,j)^{\rm hol}$, as

$$\left\langle \eta_f^{\rm ur}, \Xi(f,g,j)^{\rm hol}\right\rangle_{k,X} = \left\langle \eta_f^{\rm ur}, \Xi(f,g,j)^{\rm ord}\right\rangle_{k,X}. \tag{2.23}$$

By Proposition 2.8 of [DR],

$$\Xi(f,g,j)^{\rm ord} = e_{\rm ord}(d^t E_{m,\chi}\cdot g), \tag{2.24}$$

where $d = q\frac{d}{dq}$ is Serre's derivative operator on p-adic modular forms.

Given a p-adic modular form $\phi = \sum c_n q^n$, let $\phi^{[p]} := \sum_{p\nmid n} c_n q^n$ denote its "p-depletion", and set

$$\Xi(f,g,j)^{{\rm ord},p} := e_{\rm ord}(d^t E^{[p]}_{m,\chi}\cdot g). \tag{2.25}$$

Proposition 2.7. *Let e_{f^*} be the projector to the f^*-isotypic subspace of $H^1_{\rm dR}(Y_K, \mathcal{L}_{k-2}, \nabla)$. Then*

$$e_{f^*}\Xi(f,g,j)^{{\rm ord},p} = \frac{\mathcal{E}(f,g,j)}{\mathcal{E}(f)}\cdot e_{f^*}\Xi(f,g,j)^{\rm ord},$$

where

$$\begin{aligned}\mathcal{E}(f,g,j) &= (1-\beta_p(f)\alpha_p(g)p^{t-k+1})(1-\beta_p(f)\beta_p(g)p^{t-k+1})\\ &\quad\times(1-\beta_p(f)\alpha_p(g)\chi(p)p^{t-k+m})(1-\beta_p(f)\beta_p(g)\chi(p)p^{t-k+m}),\\ \mathcal{E}(f) &= 1-\beta_p(f)^2\chi_f^{-1}(p)p^{-k}.\end{aligned}$$

Proof. This follows from Corollary 4.17 of [DR], in light of Proposition 2.8 of loc. cit. □

2.2.2. Hida's p-adic L-series. Let $\mathbf{f}$ and $\mathbf{g}$ be Hida families of ordinary p-adic modular forms of tame level N, indexed by weight variables k and ℓ in suitable neighborhoods $U_{\mathbf{f}}$ and $U_{\mathbf{g}}$ of $\mathbb{Z}/(p-1)\mathbb{Z}\times\mathbb{Z}_p$, contained in a single residue class modulo $p-1$. (These families may be obtained, as shall be the case considered in §4.2, by deforming two given ordinary classical eigenforms f and g of possibly equal weights.) Assume likewise that the parameter $j = k-1-t$ belongs to a single residue class modulo $p-1$, so that the same holds true for the weight $m = k-\ell-2t$ of the Eisenstein series $E_{m,\chi}$.

For $k \in U_{\mathbf{f}}\cap\mathbb{Z}^{\geq 2}$ and $\ell\in U_{\mathbf{g}}\cap\mathbb{Z}^{\geq 2}$, let

$$f_k \in S_k(N,\chi_f), \qquad g_\ell\in S_\ell(N,\chi_g)$$

be the classical cusp forms whose p-stabilisations are the weight k and ℓ specialisations of $\mathbf{f}$ and $\mathbf{g}$ respectively. (We denote by χ_f, resp. χ_g the common character of the modular forms f_k, resp. g_k.)

The collection of p-adic modular forms $\Xi(f_k, g_\ell, j)^{\mathrm{ord},p}$ (defined as in equation (2.25)) indexed by

$$\{(k,\ell,j), \qquad k \in U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}, \quad \ell \in U_{\mathbf{g}} \cap \mathbb{Z}^{\geq 2}, \quad \frac{\ell+k-1}{2} \leq j \leq k-1\} \tag{2.26}$$

has Fourier coefficients which extend analytically to $U_{\mathbf{f}} \times U_{\mathbf{g}} \times \mathbb{Z}_p$, as functions in k, ℓ and j. Hence, they can be viewed as a (three-variable) Λ-adic family of modular forms of level N in the sense of [DR, §2.7].

Set

$$\mathcal{E}^*(f_k) := 1 - \beta_p(f_k)^2 \chi_f^{-1}(p) p^{1-k}.$$

Proposition 4.10 of loc. cit. shows that the expression

$$L_p(\mathbf{f},\mathbf{g})(k,\ell,j) := \frac{1}{\mathcal{E}^*(f_k)} \left\langle \eta_{f_k}^{\mathrm{ur}}, \Xi(f_k, g_\ell, j)^{\mathrm{ord},p} \right\rangle_{k,X}, \tag{2.27}$$

defined on the triples (k,ℓ,j) in the set in (2.26) extends to an analytic function $L_p(\mathbf{f},\mathbf{g})$ on $U_{\mathbf{f}} \times U_{\mathbf{g}} \times \mathbb{Z}_p$, which we refer to as the Hida *p-adic Rankin L-function* attached to $\mathbf{f}$ and $\mathbf{g}$. This appelation is justified by noting that, for all triples (k,ℓ,j) in the range of "classical interpolation", i.e, belonging to (2.26), the function $L_p(\mathbf{f},\mathbf{g})(k,\ell,j)$ satisfies the interpolation property

$$L_p(\mathbf{f},\mathbf{g})(k,\ell,j) = \frac{\mathcal{E}(f_k, g_\ell, j)}{\mathcal{E}^*(f_k)\mathcal{E}(f_k)} L^{\mathrm{alg}}(f_k \otimes g_\ell, j).$$

This follows from a direct calculation combining (2.27), Proposition 2.7, (2.23), (2.22) and (2.15).

Note that the point $(2,2,2)$ lies outside the region of classical interpolation for this function. (In fact, there are no critical values for the pair of weights $(2,2)$.) Corollary 4.4 of Section 4.2 relates the value of $L_p(\mathbf{f},\mathbf{g})$ at $(2,2,2)$ to the p-adic regulator attached in Section 3.3 to the triple of modular forms $(f = f_2, g = g_2, E_{2,\chi})$.

Generalising our setting somewhat, we do not assume now that the modular form $g \in S_2(N, \chi_g)$ is ordinary, so that g may not necessarily be viewed as the weight 2 specialisation of a Hida family.

In this case, the above construction still allows us to define a two-variable p-adic L-function $L_p(\mathbf{f}, g)(k,j)$ on $U_{\mathbf{f}} \times \mathbb{Z}_p$, by the equation

$$L_p(\mathbf{f},g)(k,j) := \frac{1}{\mathcal{E}^*(f_k)} \left\langle \eta_{f_k}^{\mathrm{ur}}, \Xi(f_k, g, j)^{\mathrm{ord},p} \right\rangle_{k,X}, \tag{2.28}$$

for $k \in U_{\mathbf{f}} \cap \mathbb{Z}^{\geq 2}$ and $(k+1)/2 \leq j \leq k-1$. Theorem 4.2 relates $L_p(\mathbf{f}, g)(2,2)$ to the p-adic regulator attached to $(f, g, E_{2,\chi})$.

3. Beilinson-Flach elements

3.1. Definition and basic properties. Let S be a quasi-projective variety over a field K, and $K_j(S)$ denote Quillen's algebraic K-groups of S. The motivic cohomology groups $H^i_{\mathcal{M}}(S,\mathbb{Q}(n)) = K^{(n)}_{2n-i}(S)$ of S were defined by Beilinson [Bei, §2] as the n-th graded piece of the Adams filtration on $K_{2n-i}(S)\otimes\mathbb{Q}$. In parallel with Beilinson's motivic cohomology groups, Bloch [Bl] introduced the higher Chow groups $\mathrm{CH}^i(S,n)$ of S.

In this note we shall focus on the smooth projective surface $S := X \times X$, where X is the modular curve over the field K of §2.1.4. For $i = 3$ and $n = 2$, $H^3_{\mathcal{M}}(S,\mathbb{Q}(2)) = K^{(2)}_1(S)$ is identified with $\mathrm{CH}^2(S,1)\otimes\mathbb{Q}$. The higher Chow group $\mathrm{CH}^2(S,1)$ may be explicitly described (cf. also [Sc]) as the first homology of the *Gersten complex*

$$K_2(K(S)) \xrightarrow{\ \partial\ } \bigoplus_{Z\subset S} K(Z)^\times \xrightarrow{\ \mathrm{div}\ } \bigoplus_{P\in S}\mathbb{Z}, \tag{3.1}$$

where

(1) $K_2(K(S))$ denotes the second Milnor K-group of the rational function field $K(S)$, and ∂ is the map whose "component at Z" is the tame symbol attached to the valuation ord_Z;

(2) the group
$$\Theta := \bigoplus_{Z\subset S} K(Z)^\times$$
is the set of finite formal linear combinations $\sum_i (Z_i, u_i)$, where the Z_i are irreducible curves in S and u_i is a rational function on Z_i;

(3) the map div is the divisor map and the direct sum defining its target is taken over all closed points $P \in S_K$.

Given a closed point $P \in X$ and a rational function u on X, an element of Θ of the form $(\{P\}\times X, u)$ (resp. of the form $(X\times\{P\}, u)$) is said to be *vertical* (resp. *horizontal*). A linear combination of vertical and horizontal terms is said to be *negligible*. Similar definitions apply to the tensor product $\Theta\otimes F$ over any field F.

Let $\Delta\subset S$ be a copy of the curve X diagonally embedded in S. Let F denote the field introduced in §2.1.4, $u\in\mathcal{O}(Y_K)^\times\otimes F$ be a modular unit with coefficients in F, and consider the element $(\Delta, u)\in\Theta\otimes F$.

Lemma 3.1. *There exists a negligible element $\theta_u\in\Theta\otimes F$ satisfying*

$$\mathrm{div}(\theta_u) = \mathrm{div}(\Delta, u).$$

Proof. Let $D_u = \mathrm{div}(\Delta,u) \in \coprod_{P\in S} F$ be the image of the element $(\Delta,u)\in\Theta$ under the divisor map. Since D_u is an F-linear combination of elements of the form $(c_1, c_1) - (c_2, c_2)$ where c_1 and c_2 are cusps of the modular curve X_K, it is enough to construct a negligible element $\theta \in \Theta \otimes \mathbb{Q}$ satisfying

$$\mathrm{div}(\theta) = (\mathrm{c}_1, \mathrm{c}_1) - (\mathrm{c}_2, \mathrm{c}_2). \tag{3.2}$$

By the Manin-Drinfeld theorem, there is an element $\alpha \in \mathcal{O}(Y_K)^\times \otimes \mathbb{Q}$ whose divisor is $c_1 - c_2$, and the negligible element given by

$$\theta = (\{c_1\} \times X, \alpha) + (X \times \{c_2\}, \alpha)$$

satisfies (3.2). The lemma follows. □

Thanks to Lemma 3.1, we can associate to any element of the form $(\Delta, u) \in \Theta \otimes F$ the element

$$\Delta_u := \text{ class of } \quad (\Delta, u) - \theta_u \quad \text{ in } H^3_{\mathcal{M}}(S, F(2)). \tag{3.3}$$

These elements were introduced by Beilinson in [Bei, Ch. 2, §6]. A variant ([Fl, Prop 2.1]) of the above construction was later exploited by Flach in loc. cit. to prove the finiteness of the Tate-Shafarevic group of the symmetric square of an elliptic curve, using the method of Kolyvagin. We call Δ_u the *Beilinson-Flach element* attached to the modular unit $u \in \mathcal{O}(Y_K)^\times \otimes F$. Strictly speaking, Δ_u is not a well-defined element in $H^3_{\mathcal{M}}(S, F(2))$, as it is only well-defined modulo the F-vector space generated by the classes of negligible elements. However, this inherent ambiguity will not lead to problems because the image of Δ_u under the relevant piece of the regulator maps will turn out to depend only on u and not on the choice of θ_u made in defining Δ_u. See Proposition 3.3 below for more details.

3.2. Complex regulators. Fix an embedding of K into the field of complex numbers. Following the definitions in [Bei, §2], [DS, §2], the *complex regulator* on $H^3_{\mathcal{M}}(S_{\mathbb{C}}, \mathbb{Q}(2))$ may be regarded as a map

$$\mathbf{reg}_{\mathbb{C}} : H^3_{\mathcal{M}}(S_{\mathbb{C}}, \mathbb{Q}(2)) \longrightarrow (\mathrm{Fil}^1 H^2_{\mathrm{dR}}(S/\mathbb{C}))^\vee, \tag{3.4}$$

where here the superscript $\vee$ denotes the complex linear dual. It sends the class of $\theta = \sum_i (Z_i, u_i)$ to the element $\mathbf{reg}_{\mathbb{C}}(\theta)$ defined by

$$\mathbf{reg}_{\mathbb{C}}(\theta)(\omega) = \frac{1}{2\pi i} \sum_i \int_{Z_i - Z_i^{\mathrm{sing}}} \omega \log |u_i|.$$

Recall the modular unit u_χ associated to the Dirichlet character χ, and the class $\eta_f^{\mathrm{ah}} \in H^1_{\mathrm{dR}}(X/\mathbb{C})$ attached to the cusp form f. Moreover, write as customary $\omega_g \in \mathrm{Fil}^1 H^1_{\mathrm{dR}}(X/\mathbb{C})$ for the class associated to the regular differential $2\pi i g(z) dz$.

The tensor product $\omega_g \otimes \eta_f^{\mathrm{ah}}$ of these classes gives rise, via the Künneth decomposition of $H^2_{\mathrm{dR}}(S/\mathbb{C})$, to an element of $\mathrm{Fil}^1 H^2_{\mathrm{dR}}(S/\mathbb{C})$.

Proposition 3.2. *With notations as above, we have*

$$\begin{aligned}\mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})(\omega_g\otimes\eta_f^{\mathrm{ah}}) &= (-2i)[\Gamma_0(N):\Gamma_1(N)(\pm 1)]\langle f^*,f^*\rangle_{2,N}^{-1}\\ &\quad\times\langle f^*(z),\log|u_\chi(z)|\cdot g(z)\rangle_{2,N}.\end{aligned}$$

Proof. Since the differential $\omega_g\otimes\eta_f^{\mathrm{ah}}$ vanishes identically on the horizontal and vertical curves on S, the negligible element θ_{u_χ} arising in the definition of Δ_{u_χ} does not contribute to the value of the regulator at that class. Hence

$$\mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})(\omega_g\otimes\eta_f^{\mathrm{ah}}) = \int_{X(\mathbb{C})}\frac{\bar f^*(z)}{\langle f^*,f^*\rangle_{2,N}}g(z)\log|u_\chi(z)|dzd\bar z$$

$$= (-2i)[\Gamma_0(N):\Gamma_1(N)(\pm 1)]\langle f^*,f^*\rangle_{2,N}^{-1}\,\langle f^*(z),\log|u_\chi(z)|\cdot g(z)\rangle_{2,N},$$

where the last equality follows from the explicit formula for the Petersson scalar product on $S_k^{\mathrm{ra}}(N,\chi_f^{-1})$. □

3.3. p-adic regulators. Let K_p be a finite extension of $\mathbb{Q}_p$ containing K and fix an embedding of K_p in $\mathbb{C}_p$. Write $\mathcal{O}_p$, resp. k_p for the ring of integers, resp. the residue field of K_p. Let $\mathcal{X}$ denote the (Deligne-Rapoport) smooth model of X over $\mathcal{O}_p$, and $\tilde{\mathcal{X}}/k_p$ its special fiber. Define $\mathcal{S}=\mathcal{X}\times\mathcal{X}$, which is a smooth projective model of S_{K_p} over $\mathcal{O}_p$.

In analogy with the complex regulator (3.4), there is a p-adic syntomic regulator map

(3.5)
$$\mathbf{reg}_p: H^3_{\mathcal{M}}(S_{K_p},\mathbb{Q}(2))\longrightarrow(\mathrm{Fil}^1H^2_{\mathrm{dR}}(S/K_p))^\vee:=\mathrm{Hom}(\mathrm{Fil}^1H^2_{\mathrm{dR}}(S/K_p),K_p)$$

arising from the syntomic Chern character in K-theory (cf. [Gros], [Ni], [Bes3]).

After possibly enlarging the field K_p, let $\{P_1,...,P_t\}\subset\mathcal{X}(\mathcal{O}_p)$ be a set of points consisting of the cusps and of a choice of a lift of every supersingular point in $\tilde{\mathcal{X}}(\bar{\mathbb{F}}_p)$. Set

$$\mathcal{X}'=\mathcal{X}\setminus\{P_1,...,P_t\},\qquad X'=\mathcal{X}'\times_{\mathrm{spec}\,\mathcal{O}_p}\mathrm{spec}\,K_p.$$

Let $\mathrm{red}:\mathcal{X}(\mathcal{O}_p)\longrightarrow\tilde{\mathcal{X}}(k_p)$ denote the reduction map and let $\mathcal{A}\subset X(K_p)$ be the affinoid subspace of the rigid analytic variety underlying X defined by

$$\mathcal{A}:=X(K_p)-\mathrm{red}^{-1}(\{\tilde P_1,\dots,\tilde P_t\}),\qquad \tilde P_j:=\mathrm{red}(P_j).$$

Fix a system $\{\mathcal{W}_\epsilon\}_{\epsilon>0}$ of wide open neighborhoods of $\mathcal{A}$ as in [DR, §2.1] and denote Φ the canonical lift of Frobenius on X as in [DR, §2.2]. As explained in loc. cit., restriction from X' to $\mathcal{W}_\epsilon$ gives rise to an isomorphism

$$H^1_{\mathrm{dR}}(X')\xrightarrow{\mathrm{comp}_\epsilon}H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)\tag{3.6}$$

between the de Rham cohomology of the open curve X' and the rigid cohomology $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ of $\mathcal{W}_\epsilon$. The inclusion $X'\subset X$ yields by restriction a monomorphism

$$H^1_{\mathrm{dR}}(X) \hookrightarrow H^1_{\mathrm{dR}}(X'),$$

and the image of $H^1_{\mathrm{dR}}(X)$ under comp_ϵ consists of those classes in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ whose annular residues about all the points $\{P_i\}$ vanish. The lift Φ of Frobenius induces a linear endomorphism of $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ which preserves the subspace $H^1_{\mathrm{dR}}(X)$.

Label now two copies of X as X_1 and X_2, denote by Φ_1 and Φ_2 the corresponding canonical lifts of Frobenius on the system of wide open neighborhoods $\mathcal{W}_\epsilon$, and write $\Phi_{12} := (\Phi_1, \Phi_2)$ for the associated lift of Frobenius on the product $X_1 \times X_2$.

Choose a polynomial $P(x) \in \mathbb{C}_p[x]$ such that

(i) $P(\Phi_{12})$ annihilates the class of $\omega_g \otimes \frac{du_\chi}{u_\chi}$ in $H^2_{\mathrm{rig}}(\mathcal{W}_\epsilon^2)$;

(ii) $P(\Phi)$ is an invertible endomorphism on $H^1_{\mathrm{dR}}(X')$.

Such a polynomial exists, since the eigenvalues of Φ_{12} acting on the space spanned by the Frobenius translates of $\omega_g \otimes \frac{du_\chi}{u_\chi}$ have complex absolute value $p^{3/2}$, while Φ acts on $H^1_{\mathrm{dR}}(X')$ with eigenvalues of complex absolute value $p^{1/2}$ and p.

Thanks to (i), there exists a rigid analytic one-form

$$\varrho_P \in \Omega^1(\mathcal{W}_\epsilon^2) \text{ such that } d(\varrho_P) = P(\Phi_{12}) \left(\omega_g \otimes \frac{du_\chi}{u_\chi} \right). \tag{3.7}$$

This form, which depends on the choice of P, is only determined up to *closed* forms in $\Omega^1(\mathcal{W}_\epsilon^2)$ by (3.7).

In order to adapt our calculations to Besser's in [Bes2] and [Bes3], it will be convenient to fix a particular choice of polynomial P and form ϱ_P. (In the next section we shall exploit the fact that the computations performed there hold independently of the choice of P, and will work with a different polynomial so that we can take advantadge of the results obtained in [DR].)

Let $P_g(t) \in \mathbb{C}_p[t]$ be a polynomial such that $P_g(\Phi)$ annihilates the class of ω_g in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$. Specifically, we may set $P_g(t) := t^2 - a_p(g)t + \chi_g(p)p$, and let $F_g \in \mathcal{O}_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ be a Coleman integral of ω_g, that is to say, a rigid analytic function such that

$$p dF_g = p\omega_{g^{[p]}} = P_g(\Phi)\omega_g \tag{3.8}$$

(cf. for example equation (127) of [DR]). Likewise, let $P_{E_\chi}(t) \in \mathbb{C}_p[t]$ be a polynomial such that $P_{E_\chi}(\Phi)$ annihilates the class of the Eisenstein series $E_\chi = \frac{du_\chi}{u_\chi}$ in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$. Here we make the specific choice $P_{E_\chi}(t) := t^h - p^h$, where h is the order of the root of unity $\chi(p)$ (in other words, Φ^h/p^h fixes the class of E_χ). Although a more optimal choice for $P_{E_\chi}(t)$ would have been the linear polynomial $t - \chi(p)p$, we made here a choice corresponding to the one made in the definition of the modified syntomic regulator $\mathrm{reg}(u_\chi)$ of the function u_χ (cf. [Bes2, Prop. 10.3]). The rigid analytic function

$$F_{E_\chi} := p^{-h} P_{E_\chi}(\Phi) \log(u_\chi) \in \mathcal{O}_{\mathrm{rig}}(\mathcal{W}_\epsilon) \tag{3.9}$$

is a Coleman integral of E_χ, satisfying

$$p^h dF_{E_\chi} = P_{E_\chi}(\Phi) E_\chi.$$

Given two choices as above of polynomials $P_g(t) = \prod_i (t - \alpha_i)$ and $P_{E_\chi}(t) = \prod_j (t - \beta_j)$, it is clear that the polynomial

$$P(t) := P_g(t) \star P_{E_\chi}(t) := \prod_{i,j} (t - \alpha_i \beta_j) \tag{3.10}$$

satisfies (i) above. Moreover, as explained in [Bes1, Lemma 4.2, (4)], there exist polynomials $a(t_1,t_2)$, $b(t_1,t_2)$ such that $P(t_1 \cdot t_2) = p^{-1} a(t_1,t_2) P_g(t_1) + p^{-h} b(t_1,t_2) P_{E_\chi}(t_2) \in \mathbb{C}_p[t_1,t_2]$ and one checks that

$$\varrho_{P} \quad = \quad a(\Phi_1,\Phi_2)\left(F_g \otimes \frac{du_\chi}{u_\chi}\right) \; + \; b(\Phi_1,\Phi_2)(\omega_g \otimes F_{E_\chi}) \quad \in \quad \Omega^1(\mathcal{W}_\epsilon^2) \tag{3.11}$$

then satisfies (3.7).

There is a certain degree of ambiguity in (3.11): neither the Coleman primitives F_g, F_{E_χ} nor the polynomials $a(t_1,t_2)$, $b(t_1,t_2)$ are unique. But all solutions of the differential equation (3.7) are of the form (3.11); moreover, given one such ϱ_P, all them can be written as $\varrho_P + \varrho_0$ with ϱ_0 a closed 1-form on $\mathcal{W}_\epsilon^2$.

We can single out a canonical choice of ϱ_P (up to exact 1-forms on $\mathcal{W}_\epsilon^2$) by setting $F_g(\infty) = F_{E_\chi}(\infty) = 0$ in (3.11); more precisely, in doing this, two different choices of pairs $(a(t_1,t_2), b(t_1,t_2))$, $(a'(t_1,t_2), b'(t_1,t_2))$ allowed by [Bes1, Lemma 4.2, (4)] give rise to forms $\varrho_{P,a,b}$, $\varrho_{P,a',b'}$ such that $\varrho_0 = \varrho_{P,a,b} - \varrho_{P,a',b'}$ is exact on $\mathcal{W}_\epsilon^2$ and therefore the class of ϱ_0 in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon^2)$ vanishes.

Imposing $F_g(\infty) = 0$ amounts to normalizing the q-expansion of F_g to be

$$F_g(q) = \sum_{p \nmid n} \frac{a_n(g)}{n} q^n, \tag{3.12}$$

and the condition $F_{E_\chi}(\infty) = 0$ is equivalent to normalizing the modular unit u_χ as was done in Definition 2.4. This way F_{E_χ} also equals the modified syntomic regulator $\mathrm{reg}(u_\chi)$ of u_χ defined in [Bes2, Prop. 10.3].

Let $\Delta \subset \mathcal{W}_\epsilon^2$ denote the diagonal and define

$$\xi'_P := [\varrho_{P|\Delta}] \in H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon) \simeq H^1_{\mathrm{dR}}(X'). \tag{3.13}$$

The above discussion shows that the class ξ'_P in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon) = \frac{\Omega^1(\mathcal{W}_\epsilon)}{d\mathcal{O}(\mathcal{W}_\epsilon)}$ is well-defined. Moreover, in view of condition (ii), we can now set

$$\xi' := P(\Phi)^{-1} \cdot \xi'_P \in H^1_{\mathrm{dR}}(X'), \tag{3.14}$$

which is directly seen to be independent of the choice of P.

Finally, let $\mathrm{spl}_X : H^1_{\mathrm{dR}}(X') \longrightarrow H^1_{\mathrm{dR}}(X)$ denote the Frobenius equivariant splitting of the short exact sequence

$$0 \to H^1_{\mathrm{dR}}(X) \longrightarrow H^1_{\mathrm{dR}}(X') \longrightarrow K_p(-1)^{t-1} \to 0 \tag{3.15}$$

and set $\xi := \mathrm{spl}_X(\xi') \in H^1_{\mathrm{dR}}(X)$.

Proposition 3.3. *With notations as above, we have*

$$\mathbf{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\mathrm{ur}}) = \langle \eta_f^{\mathrm{ur}}, \xi \rangle,$$

where $\langle\ ,\ \rangle$ is the pairing on $H^1_{\mathrm{dR}}(X)$ induced by Poincaré duality.

Proof. Thanks to the work of Besser [Bes3], the p-adic syntomic regulator (3.5) admits the following description in terms of Coleman integration. Let $\theta = \sum_i (Z_i, u_i)$ be an element in $K_1^{(2)}(S)$ and write $\iota_i : Z_i \hookrightarrow S$ for the embedding of Z_i in S given by inclusion. Assume for simplicity that the curves Z_i are all non-singular, and that θ is *integral*, by what we mean that for each i:

- the curve Z_i admits a smooth integral model $\mathcal{Z}_i$ over $\mathcal{O}_p$, and
- the divisor of u_i, when regarded as a function on $\mathcal{Z}_i$, does not contain the special fiber.

Note that these conditions are satisfied in our setting.

Under this assumption, θ lies in the image of the natural restriction map $K_1(\mathcal{S}) \longrightarrow K_1(S)$.

Let $\Omega^{\mathrm{II}}(X_{K_p})$ denote the space of differential forms of the second kind on X_{K_p}, that is to say, the space of meromorphic 1-forms whose residue at any point of the curve is zero. There is an exact sequence

$$0 \to K_p(X)^\times \xrightarrow{d} \Omega^{\mathrm{II}}(X_{K_p}) \longrightarrow H^1_{\mathrm{dR}}(X/K_p) \to 0$$

and for any $\eta \in \Omega^{\mathrm{II}}(X_{K_p})$ we write $[\eta]$ for its class in $H^1_{\mathrm{dR}}(X/K_p)$.

Instead of invoking the description of the p-adic syntomic regulator in terms of Besser-de Jeu's global triple index as stated in the main theorem of [Bes3], it will be more convenient for us to exploit [Bes3, Prop. 6.3], which provides a formula for (3.5) in the language of Besser's finite polynomial cohomology [Bes1]. In order to state this formula, let H^*_{ms} and H^*_{fp} denote, respectively, Besser's modified version of syntomic cohomology and finite polynomial cohomology: cf. e.g. [Bes3, §2] for a quick review of both and their interactions.

Let $\omega \in \Omega^1(X_{K_p})$ be a regular form on X and $\eta \in \Omega^{\mathrm{II}}(X_{K_p})$ be a differential of the second kind, regular on some affine curve $X^0 \subset X$. Write

$$\omega_1 = \pi_1^*(\omega) \in \Omega^1(S), \qquad \eta_2 = \pi_2^*(\eta) \in \Omega^{\mathrm{II}}(S)$$

for the pull-back of ω and η under the projection of S into the first and second component, respectively.

Then the class $\omega_1 \wedge [\eta_2]$ is an element of $\mathrm{Fil}^1 H^2_{\mathrm{dR}}(S)$ and, according to [Bes3, Theorem 1.1, Proposition 6.3]:

$$\mathbf{reg}_p(\theta)(\omega_1 \otimes [\eta_2]) = \sum_i \langle \iota_i^* \tilde{\eta}_2, \iota_i^* \tilde{\omega}_1 \cup \mathrm{reg}(u_i) \rangle_{\mathcal{Z}_i^0, \mathrm{fp}}, \tag{3.16}$$

where

- $Z_i^0 = Z_i \cap (X \times X^0)$, $\mathcal{Z}_i^0$ is the model for Z_i^0 deduced from $\mathcal{Z}_i$,
- $\mathrm{reg}(u_i) \in H^1_{\mathrm{ms}}(\mathcal{Z}_i^0, 1) \subseteq H^1_{\mathrm{fp}}(\mathcal{Z}_i^0, 1, 2)$ is the regulator of the function u_i as defined in [Bes2, Prop. 10.3],
- $\iota_i^* \tilde{\omega}_1 \in H^1_{\mathrm{fp}}(\mathcal{Z}_i, 1, 1)$ is a Coleman primitive of $\iota_i^* \omega \in \Omega^1(Z_i)$,
- $\iota_i^* \tilde{\eta}_2 \in H^1_{\mathrm{fp},c}(\mathcal{Z}_i^0, 0, 1)$ is the single lift of $\iota_i^*([\eta_2])$ under the isomorphism

$$\mathrm{p} : H^1_{\mathrm{fp},c}(\mathcal{Z}_i^0, 0, 1) \stackrel{\sim}{\to} H^1_{\mathrm{dR}}(Z_i) \tag{3.17}$$

  of [Bes3, Lemma 6.2], and

$$\langle\, , \rangle_{\mathcal{Z}_i^0, \mathrm{fp}} : H^1_{\mathrm{fp},c}(\mathcal{Z}_i^0, 0, 1) \times H^2_{\mathrm{fp}}(\mathcal{Z}_i^0, 2, 3) \longrightarrow H^3_{\mathrm{fp},c}(\mathcal{Z}_i^0, 2, 4) \simeq H^2_{\mathrm{dR},c}(Z_i^0) \stackrel{\mathrm{tr}}{\simeq} K_p \tag{3.18}$$

is the pairing induced by Poincaré duality in finite polynomial cohomology. Here $H^*_{\mathrm{fp},c}$ stands for finite polynomial cohomology with compact support, as introduced in [Bes3, §4]. The cup-product (3.18) is constructed in loc. cit., where it is also shown that it satisfies the projection formula.

At the time [Bes3] was written, the results were subject to the compatibility of pushforward maps in syntomic and motivic cohomology, as specified in [Bes3, Conjecture 4.2]. At present this compatibility has been checked by Déglise and Mazzari [DM], and thus (3.16) holds unconditionally.

Let us now apply (3.16) to the Beilinson-Flach element Δ_{u_χ} that was introduced in (3.3). Recall that the curves in $X \times X$ on which Δ_{u_χ} is supported are the images of X under the diagonal embedding $\iota_{12}(x) = (x, x)$ and the various horizontal and vertical embeddings $\iota_{1,c}(x) = (x, c)$ and $\iota_{2,c}(x) = (c, x)$, where c is a cusp on the modular curve X.

We firstly claim that the terms on the right-hand side of (3.16) corresponding to $\iota_{1,c}$ and $\iota_{2,c}$ vanish and the one corresponding to ι_{12} is independent of the choices of lifts to finite polynomial cohomology.

To see that, put $\omega = \omega_g$ and $\eta = \eta_f^{\mathrm{ur}}$ and recall $X' = \mathcal{X}' \times K_p$ is the curve obtained from X by removing a finite set of points including all the cusps. Note first that $\iota_{1,c}^*([\eta_2]) = 0 \in H^1_{\mathrm{dR}}(X)$, because the composition $\pi_2 \circ \iota_{1,c}$ is the constant function c on X. Hence, since the map p in (3.17) is an isomorphism, the class of the lift $\iota_{1,c}^*(\tilde{\eta}_2)$ is also trivial and

$$\langle \iota_{1,c}^*\tilde{\eta}_2, \iota_{1,c}^*\tilde{\omega}_1 \cup \mathrm{reg}(u)\rangle_{\mathcal{X}',\mathrm{fp}} = 0,$$

for any rational function u.

We similarly have $\iota_{2,c}^*(\omega_1) = 0 \in \Omega^1(X)$ because $\pi_1 \circ \iota_{2,c} = c$. Notice however that a lift of 0 to $H^1_{\mathrm{fp}}(\mathcal{X}, 1, 1)$ is not necessarily trivial, but represented by a pair in $\mathcal{O}_{\mathrm{rig}}(\mathcal{W}_\epsilon) \oplus \Omega^1(X)$ of the form $[(\lambda, 0)]$, where λ is a constant. Then, if u is a modular unit on X, the cup-product

$$\iota_{2,c}^*\tilde{\omega}_1 \cup \mathrm{reg}(u) \in H^2_{\mathrm{fp}}(\mathcal{X}', 2, 3) \simeq H^1_{\mathrm{dR}}(X') \simeq H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$$

may be represented by the pair $(\lambda \frac{du}{u}_{|\mathcal{W}_\epsilon}, 0)$. But then

$$\langle \iota_{2,c}^*\tilde{\eta}_2, \iota_{2,c}^*\tilde{\omega}_1 \cup \mathrm{reg}(u)\rangle_{\mathcal{X}',\mathrm{fp}} = \lambda \langle \eta_f^{\mathrm{ur}}, \frac{du}{u}\rangle_{\mathrm{dR}} = 0 \tag{3.19}$$

because the cusp form f is orthogonal to the Eisenstein series $\frac{du}{u}$. This accounts for the vanishing of the horizontal and vertical terms, and explains why we call them negligible.

As for the diagonal term, let us show that $\langle \iota_{12}^*\tilde{\eta}_2, \iota_{12}^*\tilde{\omega}_1 \cup \mathrm{reg}(u_\chi)\rangle_{\mathcal{X}',\mathrm{fp}}$ is independent of the choices of lifts to finite polynomial cohomology. Since $\pi_1 \circ \iota_{12}$ and $\pi_2 \circ \iota_{12}$ are both the identity map on X, this is just $\langle \tilde{\eta}_f^{\mathrm{ur}}, \tilde{\omega}_g \cup \mathrm{reg}(u_\chi)\rangle_{\mathcal{X}',\mathrm{fp}}$. Again there is a single choice for $\tilde{\eta}_f^{\mathrm{ur}}$, but the Coleman integral F_g of ω_g is only well-defined up to a constant. The difference between any two choices is then equal to

$$\langle \tilde{\eta}_f^{\mathrm{ur}}, [(\lambda, 0)] \cup \mathrm{reg}(u_\chi)\rangle_{\mathcal{X}',\mathrm{fp}} = \lambda \left\langle \eta_f^{\mathrm{ur}}, \frac{du_\chi}{u_\chi} \right\rangle_{\mathrm{dR}}$$

for some $\lambda \in K_p$, and the same orthogonality argument between cusp and Eisenstein forms again shows that this is 0. The claim follows.

Summing up, we obtain from (3.16) that

$$\mathbf{reg}_p(\Delta_{u_\chi})(\omega_g \otimes \eta_f^{\mathrm{ur}}) = \langle \tilde{\eta}_f^{\mathrm{ur}}, \tilde{\omega}_g \cup \mathrm{reg}(u_\chi)\rangle_{\mathcal{X}',\mathrm{fp}}. \tag{3.20}$$

Recall that $\tilde{\omega}_g$ may be represented by the pair (F_g, ω_g) where $F_g \in \mathcal{O}_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ is a Coleman integral of ω_g, which in light of the above claim we are entitled to normalize as it was done in (3.12). Besides, by [Bes2, Prop. 10.3] the class $\mathrm{reg}(u_\chi)$ is represented by the pair $(F_{E_\chi}, \frac{du_\chi}{u_\chi}) \in \mathcal{O}_{\mathrm{rig}}(\mathcal{W}_\epsilon) \oplus \Omega^1(X')$ where F_{E_χ} is the Coleman integral of $\frac{du_\chi}{u_\chi}$ introduced in (3.9) and normalized as we explained right after (3.12).

By definition, $\tilde{\omega}_g \cup \mathrm{reg}(u_\chi)$ is the restriction to the diagonal of $\pi_1^*\tilde{\omega}_g \wedge \pi_2^*\mathrm{reg}(u_\chi)$. Note that the polynomial P defined in equation (3.10) satisfies the properties (i) and (ii) above. The class $\pi_1^*\tilde{\omega}_g \wedge \pi_2^*\mathrm{reg}(u_\chi)$ in $H^2_{\mathrm{fp}}(\mathcal{X}'^2, 2, 3)$ may then be represented by the pair

$$(\varrho_P, \pi_1^*\omega_g \wedge \pi_2^* \frac{du_\chi}{u_\chi}) \in \Omega^1_{\mathrm{rig}}(\mathcal{W}^2_\epsilon) \oplus \Omega^2(X'^2) \tag{3.21}$$

where ϱ_P is the form introduced in (3.11), which satisfies

$$d\varrho_P = P(\Phi_{12})(\pi_1^*\omega_g \wedge \pi_2^* \frac{du_\chi}{u_\chi}). \tag{3.22}$$

Let us again remark that this differential equation does not determine ϱ_P uniquely, but that the above normalizations of F_g and F_{E_χ} completely determine it up to exact 1-forms on $\mathcal{W}^2_\epsilon$. Obviously, when we restrict (3.21) to the diagonal, this ambiguity does not affect the class we obtain in $H^2_{\mathrm{fp}}(\mathcal{X}', 2, 3)$, because exact 1-forms on $\mathcal{W}_\epsilon$ vanish in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$.

In conclusion, the class $\tilde{\omega}_g \cup \mathrm{reg}(u_\chi)$ in $H^2_{\mathrm{fp}}(\mathcal{X}', 2, 3)$ may be represented by the pair

$$(\iota_{12}^*(\varrho_P), 0) \in \Omega^1_{\mathrm{rig}}(\mathcal{W}_\epsilon) \oplus \Omega^2(X'),$$

where ϱ_P is as above and $\iota_{12}^*(\varrho_P)$ is the form denoted ξ'_P in (3.13).

As in [Bes1, (14)] there is a commutative diagram

$$\begin{array}{ccc} H^1_{\mathrm{fp},c}(\mathcal{X}', 0, 1) \times H^1_{\mathrm{dR}}(X') & \xrightarrow{\mathrm{Id}\times \mathrm{i}} & H^1_{\mathrm{fp},c}(\mathcal{X}', 0, 1) \times H^2_{\mathrm{fp}}(\mathcal{X}', 2, 3) \\ \downarrow {\scriptstyle \mathrm{p}\times\mathrm{Id}} & & \downarrow {\scriptstyle \langle\,,\rangle_{\mathrm{fp}}} \\ H^1_{\mathrm{dR},c}(X')^{w=1} \times H^1_{\mathrm{dR}}(X') & \xrightarrow{\langle\,,\rangle_{\mathrm{dR}}} & H^2_{\mathrm{dR},c}(X') \simeq H^3_{\mathrm{fp},c}(\mathcal{X}', 2, 4), \end{array} \tag{3.23}$$

where $H^1_{\mathrm{dR},c}(X')^{w=1}$ stands for the pure submodule of weight 1 of $H^1_{\mathrm{dR},c}(X')$. In fact both maps

$$H^1_{\mathrm{dR}}(X') \xrightarrow{\mathrm{i}} H^1_{\mathrm{fp}}(\mathcal{X}', 2, 3) \ \text{ and } \ H^1_{\mathrm{fp},c}(\mathcal{X}', 0, 1) \xrightarrow{\mathrm{p}} H^1_{\mathrm{dR},c}(X')^{w=1}$$

are isomorphisms, as it follows from [Bes3, (2.7) and the first assertion of Lemma 2.8].

By definition of i, the preimage of $\tilde{\omega}_g \cup \mathrm{reg}(u_\chi) = [(\xi'_P, 0)]$ under i is the class in $H^1_{\mathrm{rig}}(\mathcal{W}_\epsilon)$ of the 1-form $P(\Phi)^{-1}(\xi'_Q) = \xi'$. To conclude, we now deduce from the commutativity of (3.23) that

$$\langle \eta_f^{\mathrm{ur}}, \xi' \rangle_{\mathrm{dR}} = \langle \tilde{\eta}_f^{\mathrm{ur}}, \mathrm{i}(\xi') \rangle_{\mathrm{fp}} = \langle \tilde{\eta}_f^{\mathrm{ur}}, \tilde{\omega}_g \cup \mathrm{reg}(u_\chi) \rangle_{\mathrm{fp}}.$$

Since the class η_f^{ur} is orthogonal to the complement of $H^1_{\mathrm{dR}}(X)$ in $H^1_{\mathrm{dR}}(X')$ under the Frobenius equivariant splitting of (3.15), we have $\langle \eta_f^{\mathrm{ur}}, \xi' \rangle_{\mathrm{dR}} = \langle \eta_f^{\mathrm{ur}}, \xi \rangle_{\mathrm{dR}}$ and the proposition follows. □

4. The Beilinson formula

Let $f \in S_2(N, \chi_f)$, $g \in S_2(N, \chi_g)$ be eigenforms of weight 2 as in Section 2.1.4. Recall that f and g are not assumed to be newforms. Moreover, we insist on the condition $\chi_f \neq \chi_g^{-1}$, which implies that $\chi = \chi_f^{-1}\chi_g^{-1}$ is non-trivial.

4.1. The complex setting. In [Bei, Ch. 2, § 6], Beilinson relates the image of Δ_{u_χ} under the complex regulator map to the value at $s = 2$ of the Rankin L-series attached to $f \otimes g$. The following explicit version of Beilinson's theorem is a slight generalisation of the results of [BaSr].

Proposition 4.1. *For cusp forms f and g of weight two as in Section 2.1.4, we have*

$$\frac{L(f\otimes g,2)}{\langle f^*, f^*\rangle_{2,N}} = (8i)\pi^3[\Gamma_0(N):\Gamma_1(N)(\pm 1)]^{-1}N^{-2}\tau(\chi^{-1})\mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})(\omega_g\otimes\eta_f^{\mathrm{ah}}).$$

Proof. This follows by combining the explicit formula for $L(f \otimes g, 2)$ obtained in Proposition 2.5 with the explicit expression for $\mathbf{reg}_{\mathbb{C}}(\Delta_{u_\chi})$ given in Proposition 3.2. □

4.2. The p-adic setting. Let $p \geq 3$ be a prime which does not divide N. Assume that f is ordinary at p (with respect to a fixed embedding of the field K_f in $\mathbb{C}_p$). Let $\mathbf{f}$ be the Hida family whose specialisation in weight 2 is the p-stabilisations of f, and let $L_p(\mathbf{f}, g)(k, j)$ be the p-adic L-function defined in Section 2.2.2.

Let $\mathcal{E}(f)$, $\mathcal{E}^*(f)$ and $\mathcal{E}(f, g, 2)$ be the p-adic multipliers defined in Sections 2.2.1 and 2.2.2. Recall that

$$\begin{aligned}\mathcal{E}(f,g,2) \quad = \quad & (1-\beta_p(f)\alpha_p(g)p^{-2})(1-\beta_p(f)\beta_p(g)p^{-2})\\ & \times(1-\beta_p(f)\alpha_p(g)\chi(p)p^{-1})(1-\beta_p(f)\beta_p(g)\chi(p)p^{-1}).\end{aligned}$$

The following p-adic Beilinson formula is the main result of this paper.

Theorem 4.2. *For cusp forms f and g of weight two as in Section 2.1.4, we have*

$$L_p(\mathbf{f},g)(2,2) = \frac{\mathcal{E}(f,g,2)}{\mathcal{E}(f)\cdot\mathcal{E}^*(f)}\times\mathbf{reg}_p(\Delta_{u_\chi})(\omega_g\otimes\eta_f^{\mathrm{ur}}).$$

Proof. By the description of the p-adic L-function given in equation (2.28),

$$L_p(\mathbf{f},g)(k,j) = \frac{1}{\mathcal{E}^*(f_k)}\left\langle \eta_{f_k}^{\mathrm{ur}}, \Xi(f_k,g,j)^{\mathrm{ord},p}\right\rangle_{k,X}$$

for all triples (k, ℓ, j) belonging to the set (2.26). Since the terms in the above expression vary analytically, taking the limit to $k = \ell = j = 2$ yields, in light of equation (2.25),

$$L_p(\mathbf{f},g)(2,2) = \frac{1}{\mathcal{E}^*(f)} \left\langle \eta_f^{\mathrm{ur}}, e_{\mathrm{ord}}(d^{-1}E_{2,\chi}^{[p]}\cdot g)\right\rangle_{2,X}. \tag{4.1}$$

On the other hand, by Proposition 3.3

$$\mathbf{reg}_p(\Delta_{u_\chi})(\omega_g\otimes\eta_f^{\mathrm{ur}}) = \langle \eta_f^{\mathrm{ur}}, \xi\rangle.$$

Since

$$\Phi(\eta_f^{\mathrm{ur}}) = \alpha_p(f)\eta_f^{\mathrm{ur}}, \qquad \langle\Phi(\eta_f^{\mathrm{ur}}),\Phi(\xi)\rangle = p\langle\eta_f^{\mathrm{ur}},\xi\rangle, \qquad \alpha_p(f)\beta_p(f)=\chi_f(p)p,$$

we deduce by multi-linearity that

$$\langle\eta_f^{\mathrm{ur}},\xi\rangle = P\big(\chi_f^{-1}(p)\beta_p(f)\big)^{-1}\langle\eta_f^{\mathrm{ur}},\xi'_P\rangle.$$

Since f is an ordinary eigenform, the quantity $\langle\eta_f^{\mathrm{ur}},\xi'_P\rangle$ only depends on the f^*-isotypical ordinary projection of ξ'_P, that is to say, $\langle\eta_f^{\mathrm{ur}},\xi'_P\rangle = \langle\eta_f^{\mathrm{ur}}, e_{f^*}e_{\mathrm{ord}}\xi'_P\rangle$.

Choose the polynomial $P(x)$ satisfying conditions (i) and (ii) to be

$$P(x) := (x-\alpha_p(g))\cdot(x-\alpha_p(g)\chi(p)p)\cdot(x-\beta_p(g))\cdot(x-\beta_p(g)\chi(p)p).$$

This choice of P has the advantage of allowing us to directly invoke the calculations already performed in [DR, Prop. 5.4]. They give

$$e_{f^*}e_{\mathrm{ord}}\xi'_P = \chi_f(p)^{-2}p^4\mathcal{E}(f)\cdot e_{f^*}e_{\mathrm{ord}}(d^{-1}E_{2,\chi}^{[p]}\cdot g).$$

A direct calculation shows that

$$\mathcal{E}(f,g,2) = p^{-4}\chi_f(p)^{-2}P\big(\chi_f^{-1}(p)\beta_p(f)\big).$$

By combining the above remarks, we find the following expression for the p-adic regulator:

$$\mathbf{reg}_p(\Delta_{u_\chi})(\omega_g\otimes\eta_f^{\mathrm{ur}}) = \frac{\mathcal{E}(f)}{\mathcal{E}(f,g,2)}\times\left\langle\eta_f^{\mathrm{ur}}, e_{f^*}e_{\mathrm{ord}}(d^{-1}E_{2,\chi}^{[p]}\cdot g)\right\rangle_{2,X}. \tag{4.2}$$

The theorem follows by comparing equations (4.1) and (4.2). □

Remark 4.3. Note that the modular form g that arises in Theorem 4.2 is fixed throughout the argument, and is thus not required to be ordinary at p. Assume now that both f and g are ordinary at p (with respect to a fixed embedding of the field K_{fg} in $\mathbb{C}_p$). Let $\mathbf{f}$ and $\mathbf{g}$ be the Hida families whose specialisations in weight 2 are the p-stabilisations of f and g, respectively, and let $L_p(\mathbf{f},\mathbf{g})(k,\ell,j)$ be the p-adic L-function defined in Section 2.2.2. The following corollary is an immediate consequence of Theorem 4.2.

Corollary 4.4. *For cusp forms f and g of weight two as in Section 2.1.4, we have*

$$L_p(\mathbf{f},\mathbf{g})(2,2,2) = \frac{\mathcal{E}(f,g,2)}{\mathcal{E}(f)\cdot\mathcal{E}^*(f)}\times\mathbf{reg}_p(\Delta_{u_\chi})(\omega_g\otimes\eta_f^{\mathrm{ur}}).$$

References

BaSr. S. Baba, R. Sreekantan, *An analogue of cyclotomic units for products of elliptic curves*, Proc. Edinb. Math. Soc. (2) **47** (2004), no. 1, 35–51.

BD. M. Bertolini, H. Darmon, *Kato's Euler system and rational points on elliptic curves I: A p-adic Beilinson formula*, Israel J. Math. **199** (2014), 163-188.

BDP. M. Bertolini, H. Darmon, K. Prasanna, *Generalised Heegner cycles and p-adic Rankin L-series*, Duke Math. J. **162** (2014), 1033–1148.

BDR. M. Bertolini, H. Darmon, V. Rotger, *Beilinson-Flach elements and Euler systems II: the Birch and Swinnerton-Dyer conjecture for Hasse-Weil-Artin L-series*, submitted.

Bei. A.A. Beilinson, *Higher regulators and values of L-functions*, in *Current problems in mathematics* **24**, 181–238, Akad. Nauk SSSR, Vsesoyuz, Inst. Nauchn. Tekhn. Inform., Moscow, 1984.

Bes1. A. Besser, *A generalization of Coleman's p-adic integration theory*, Invent. Math. **142** (2000), 397–434.

Bes2. A. Besser, *Syntomic regulators and p-adic integration I: rigid syntomic regulators*, Israel J. Math. **120** (2000), 291–334.

Bes3. A. Besser, *On the syntomic regulator for K_1 of a surface*, Israel J. Math. **190** (2012), 29–66.

Bl. S. Bloch, *Algebraic cycles and higher K-theory*, Advances Math. **61** (1986) 267–304

Br. Brunault, François, *Valeur en* 2 *de fonctions L de formes modulaires de poids* 2*: théorème de Beilinson explicite.* Bull. Soc. Math. France **135** (2007), no. 2, 215-246.

Col. R.F. Coleman, *A p-adic Shimura isomorphism and p-adic periods of modular forms*, in p-adic monodromy and the Birch and Swinnerton-Dyer conjecture (Boston, MA, 1991), 21–51, Contemp. Math. **165**, Amer. Math. Soc., Providence, RI, 1994.

DR. H. Darmon, V. Rotger, *Diagonal cycles and Euler systems I: A p-adic Gross-Zagier formula*, Ann. Sc. Ecole Norm. Sup. **47** (2014) 779–832.

Das. S. Dasgupta, *Greenberg's conjecture on symmetric square p-adic L-functions*, preprint 2014.

DM. F. Déglise, N. Mazzari, *The rigid syntomic spectrum*, preprint 2012.

DS. C. Deninger, A.J. Scholl, *The Beilinson conjectures*, in *L-functions and Arithmetic*, Proceedings of the Durham Symposium, July, 1989 (J. Coates and M. J. Taylor, eds.), London Math. Soc. Lecture Notes Series **153**, Cambridge University Press, 1991, 173–209.

Fl. M. Flach, *A finiteness theorem for the symmetric square of an elliptic curve*, Invent. Math. **109** (1992), 307–328.

Gros. M. Gros, *Regulateurs syntomiques et valeurs de fonctions L p-adiques II*, Invent. Math. **115** (1994), 61–79.

Gross. B.H. Gross, *On the factorization of p-adic L-series*, Invent. Math. **57** (1980), 83–95.

Hi. H. Hida, Elementary theory of L-functions and Eisenstein series. London Mathematical Society Student Texts **26**, 1993.

LLZ. A. Lei, D. Loeffler, S. L. Zerbes, *Euler systems for Rankin-Selberg convolutions of modular forms*, Annals Math. **180** (2014), no. 2, 653–771.

LZ. D. Loeffler and S.L. Zerbes, *Iwasawa theory and p-adic L-functions over $\mathbb{Z}_p^2$-extensions*, to appear in Int. J. Number Theory.

Ni. W. Nizioł, *On the image of p-adic regulators*, Invent. Math. **127** (1997), 375–400.

Pa. A. Pàl. *The rigid analytical regulator for the K_2 of Mumford curves.* Publications of the Research Institute for Mathematical Sciences **46** (2010), 219–253.

Sc. A. Scholl, *Integral elements in K-theory and products of modular curves*, preprint 2007.

Sh1. G. Shimura. *The special values of the zeta functions associated with cusp forms*, Communications on Pure and Applied Mathematics, vol. XXIX, (1976) 783–795.

Sh2. G. Shimura, *On a class of nearly holomorphic automorphic forms,* Annals Math. **123** (1986), 347-406.

Sre1. R. Sreekantan, *Non-Archimedean regulator maps and special values of L-functions*, Proceedings of the International Colloquium on Motives and Shimura Varieties, TIFR Mumbai, January 2008.

Sre2. R. Sreekantan, K_1 *of products of Drinfeld modular curves and special values of L-functions*, Compositio Math. **146:4** (2010), 886–918.

M.B.: Universität Duisburg-Essen, Institut für Experimentelle Mathematik, Ellernstr. 29 45326 Essen, Germany
E-mail address: massimo.bertolini@uni-due.de

H. D.: McGill University, Burnside Hall, Room 1111, Montréal, Canada
E-mail address: darmon@math.mcgill.ca

V.R.: Universitat Politècnica de Catalunya, MA II, Despatx 413, C. Jordi Girona 1-3, 08034 Barcelona, Spain
E-mail address: victor.rotger@upc.edu