The Geometry of the Set of Conditioned Invariant Flags

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1 Preliminaries

We briefly summarize the results on the orbit structure of conditioned invariant subspaces, as described in [**?**].

Let $(C,A) \in F^{p\times n} \times F^{n\times n}$ be an observable pair of matrices in dual Brunovsky canonical form. Let $\mu = (\mu_1,\ldots,\mu_p)$, $\mu_1 \geq \ldots \geq \mu_p > 0$ denote the observability indices of (C,A). Recall that a subspace $V \subset F^n$ is (C,A)-invariant if there exists an output injection matrix J such that V is $(A+JC)$-invariant. The restriction of (C,A) to V is the pair $(C,A)|_V := (C|_V, (A+JC)|_V)$. Although is not uniquely determined some basic properties of the restricted system are invariant under output injection. If (C,A) is observable then the restriction $(C,A)|_V$ is observable. Moreover the observability indices $\lambda_1 \geq \ldots \geq \lambda_p \geq 0$ of the restricted subsystem are well defined and independent of the choice of J. We refer to λ as the *restricted indices* of (C,A) on V. We note without proof that the restricted indices $\lambda_1,\ldots,\lambda_p$ of a (C,A)-invariant subspaces satisfy the inequalities $\lambda_i \leq \mu_i$ for $i = 1,\ldots,l$. Moreover for any $\lambda = (\lambda_1,\ldots,\lambda_p)$ with $\lambda_1 \geq \ldots \geq \lambda_p \geq 0$ and $\lambda_i \leq \mu_i$ there exists a (C,A)-invariant subspace V so that λ_i are the restricted observability indices.

Let

$$Inv_k(C,A) := \{V \in G_k(F^n) \quad | \quad V \text{ is } (C,A)\text{-invariant } \}$$

and

$$Inv_\lambda(C,A) := \{V \in Inv_k(C,A) \quad | \quad (C,A)|_V \text{ has restricted indices } \lambda\}.$$

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$Inv_\lambda(C,A)$ defines a finite partition of $Inv_k(C,A)$ into disjoint subsets

$$Inv_k(C,A) = \bigcup_{\lambda\le\mu} Inv_\lambda(C,A).$$

The space $Inv_\lambda(C,A)$ has a description as an orbit space similar to that of $Inv_\lambda(A)$; see [?],[?]. Let $(\bar{C},\bar{A})$ denote the Brunovsky canonical form of $(C,A)|_V$, $V \in Inv_\lambda(C,A)$.

Let $M(\lambda,\mu)$ be the set of matrices $Z \in F^{n\times k}$ with

$$(1) \quad \begin{cases} AZ = Z\bar{A} + AZ\bar{C}^t\bar{C} \\ CZ = CZ\bar{C}^t\bar{C} \end{cases}$$

and such that $CZ\bar{C}^t$ has maximal rank. Let $G(\lambda) := M(\lambda,\lambda)$ denote the set of matrices $S \in Gl_k(F)$ with

$$(2) \quad \begin{cases} \bar{A}S = S\bar{A} + \bar{A}S\bar{C}^t\bar{C} \\ \bar{C}S = \bar{C}S\bar{C}^t\bar{C} \end{cases}$$

$M(\lambda,\mu)$ and $G(\lambda)$ consists of blocks of *upper triangular* Toeplitz matrices. $G(\lambda)$ forms a Lie subgroup of $Gl_k(F)$ that acts freely on $M(\lambda,\mu)$ from the right. It is in fact the largest subgroup of $Gl_k(F)$ that acts on $M(\lambda,\mu)$ via $Z \longmapsto ZS$. Thus, if $S \in Gl_k(F)$ is a transformation such that for $Z \in M(\lambda,\mu)$, $ZS \in M(\lambda,\mu)$, then $S \in G(\lambda)$. The orbit space of the above group action defines a smooth quotient manifold

$$M(\lambda,\mu)/G(\lambda),$$

and the inclusion

$$i : M(\lambda,\mu) \longrightarrow V_k(F^n)$$

induces an embedding of $M(\lambda,\mu)/G(\lambda)$ into the Grassmann manifold $G_k(F^n)$ with image $Inv_\lambda(C,A)$. Thus, $Inv_\lambda(C,A)$ is a smooth submanifold of $G_k(F^n)$ and the map $Z \longmapsto ImZ$ defines a diffeomorphism

$$M(\lambda,\mu)/G(\lambda) \cong Inv_\lambda(C,A).$$

In the sequel we will identify $Inv_\lambda(C,A)$ with its equivalent homogeneous space description as $M(\lambda,\mu)/G(\lambda)$.

We are intrested in generalizing the above construction to describe the strata of conditioned invariant flags. We now recall briefly the orbit space structure of the flag manifolds. as well as some basic definitions.

Let $k = (k_1, \ldots, k_d)$ a sequence of integers with $k_1 \leq \ldots \leq k_d \leq n$. We define

$$Flag(k) := \{(V_1, \ldots, V_d) \in \prod_{i=1}^{d} G_{k_i}(F^n) \quad | \quad V_1 \subset \ldots \subset V_d\}.$$

$Flag(k)$ is called a flag manifold and is indeed a compact manifold of dimension $(k_0 := 0)$

$$\sum_{i=1}^{d}(k_i - k_{i-1})(n - k_i).$$

$Flag(k)$ has the following description as an orbit space. Let $M^*_{n\times k_d}$ be the set of full rank $n \times k_d$-matrices, $P(k)$ is the parabolic subgroup of $Gl_{k_d}(F)$ consisting on matrices $X \in Gl_{k_d}(F)$ such that the last $n-k_i$ entries in columns $k_{i-1}+1, \ldots, k_d$ are equal to 0, $i = 1, \ldots, d$. Then $Flag(k)$ is diffeomorphic to the orbit space $M^*_{n\times k_d}/P(k)$.

2 Conditioned invariant flags

In this section we introduce the definition of conditioned invariant flag, as well as some basic properties.

Definition 1 *A flag $V = (V_1, \ldots, V_d) \in Flag(k)$ is a* conditioned invariant flag *with regard to a pair (C, A) or, simply, a (C, A)-*invariant flag *if each V_i is a (C, A)-invariant subspace.*

Let

$$Flag_{(C,A)}(k) := \{V \in Flag(k) \quad | \quad V \text{ is } (C, A)\text{-invariant }\}.$$

Given a (C, A)-invariant flag $(V_1, \ldots, V_d) \in Flag_{(C,A)}(k)$, by definition there exists a sequence of output injections $J_1, \ldots, J_d$ such that $(A + J_iC)V_i \subset V_i$. We next show that, actually, there exists an output injection J such that $(A + JC)V_i \subset V_i$ for all $i = 1, \ldots, d$; moreover we parametrize explicitely the set of all these J.

Proposition 2 *Given a (C, A)-invariant flag $V = (V_1, \ldots, V_d)$ there exists an output injection J such that $(A+JC)V_i \subset V_i$ for all $i = 1, \ldots, d$. Moreover, if C has full rank the set of all these J is a non-empty linear space of dimension ($V_0 := \{0\}, V_{d+1} := F^n$)*

$$\sum_{i=1}^{d+1} \dim V_i(\dim V_i + \dim V_{i-1} \cap KerC - \dim V_i \cap KerC - \dim V_{i-1}).$$

Proof The existence of J making the spaces V_i $(A + JC)$-invariants simultaneously can be easily proved in the dual setting. However, since we are interested in parametrizing all these output injections J, we construct explicitely the maps

$$J : F^p \longrightarrow F^n$$

making $(A + JC)V_i \subset V_i$ for all $i = 1, \dots, d$.

Let $v^i = \{v^i_1, \dots, v^i_{r_i}\}$ a set of linear independent vectors such that ($V_0 := \{0\}$, $V_{d+1} := F^n$)

$$V_i = \text{span} v^i \oplus (V_{i-1} + V_i \cap KerC), \quad i = 1, \dots, d+1$$

Clearly $F^n = \text{span} \bigcup_{1=1}^{d+1} v^i \oplus KerC$. Therefore $\{Cv^i_j\}_{i,j}$ is a set of linearly independent vectors; moreover, if C has full rank, $\{Cv^i_j\}_{i,j}$ is a basis of F^p.

We define $J : F^p \longrightarrow F^n$ by $Cv^i_j \longmapsto -Av^i_j + u^i_j$ with u^i_j any vector of V_i.

We prove by induction that V_i are $(A + JC)$-invariants.

For V_1, let $v \in V_1$; we have that $v = \sum_i v^1_i + w$ with $w \in KerC$. Then $(A + JC)v = (A + JC)\sum_i v^1_i + (A + JC)w = u^1_i + Aw \in V_1$, because $A(V_1 \cap KerC) \subset V_1$ (V_1 is (C, A)-invariant).

Now suppose that $V_1, \dots, V_{l-1}$ are $(A + JC)$-invariants. We prove that V_l is $(A + JC)$-invariant as well. Let $v \in V_l$; we have that $v = \sum_i v^l_i + v' + w$ with $v' \in V_{l-1}$ and $w \in KerC$. Then, $(A + JC)v = (A + JC)\sum_i v^l_i + (A + JC)v' + (A + JC)w = u^l_i + (A + JC)v' + Aw \in V_1$, by the induction hipothesis and that $A(V_i \cap KerC) \subset V_i$.

We have therefore proved that the map J makes $(A + JC)$-invariants all V_i for $i = 1, \dots, d$. On the other hand if in the definition of J we take u^i_j arbitrary and we suppose V_i $(A + JC)$-invariants, following the former induction we conclude that it is necessary that $u^i_j \in V_i$. Hence, if C has full rank, this is to say, if $\{Cv^i_j\}_{i,j}$ is a basis of F^p, all the possible J such that $(A + JC)V_i \subset V_i$ for all $i = 1, \dots, d$ are in the set

$$\{J : F^p \longrightarrow F^n \quad Cv^i_j \longmapsto -Av^i_j + u^i_j \quad | \quad u^i_j \in V_i\}$$

which is a linear space of dimension

$$\sum_{i=1}^{d+1} (\dim V_i - \dim(V_{i-1} + V_i \cap KerC)) \dim V_i.$$

Using the Grassmann formula we obtain the formula of the proposition.

We now particularize the construction of J in the proof of the former proposition when we have the elements of the flag $V = (V_1, \dots, V_d)$ described, as in the previous section, as $V_i = ImZ_i$ being Z_i a solution of the equations

$$\begin{cases} AZ_i = Z_iA_i + AZ_iC_i^tC_i \\ CZ_i = CZ_iC_i^tC_i \end{cases}$$

having $CZ_iC_i^t$ full rank, where (C_i, A_i) is the Brunovsky canonical form of the restriction $(C, A)|_{V_i}$. Let $p_i = \text{rk}C_iq$.

Notice that $V_i = ImZ_iC_i^t \oplus V_i \cap KerC$. Then, considering V_i individually and following the construction of the proof of 2 for $d = 1$, we have that the set of the output injections J_i making V_i $(A + J_iC)$-invariant is

$$\{J_i \in F^{n\times p} \quad | \quad J_iCZ_iC_i^t = -AZ_iC_i^t + Z_iS, \quad S \in F^{d\times p_i}\}.$$

In particular, taking $S = 0$ we can define an output injection J_i by the equation (CZ_iC^t has full rank)

$$J_iCZ_iC_i^t = -AZ_iC_i^t$$

For this particular J_i, the pair $(C|_{V_{i-1}}, (A + J_iC)|_{V_i})$ is reducible by similarty to (C_i, A_i) (see [...]).

However, since in general $ImZ_iC_i^t \cap (V_i + V_i \cap KerC) \neq \{0\}$, the system of equations $JCZ_iC_i^t = -AZ_iC_i^t$, $i = 1 \dots d$, don't need to have solution (see the proof of 2). Certainly, there exists an output injection J such that $(A + JC)V_i \subset V_i$ for all $i = 1, \dots, d$; it is indeed a solution of the equations

$$JCZ_iC_i^t = -AZ_iC_i^t + Z_iS_i, \quad i = 1 \dots d$$

for some convenient matrices $S_i \in F^{d\times p_i}$, which exists thanks to the proposition 2. But for this J, the pairs $(C|_{V_i}, (A+JC)|_{V_i})$ don't have to be similar to (C_i, A_i). In fact, the Brunovsky canonical form of $(C|_{V_i}, (A+JC)|_{V_i})$, (C_i, A_i), is obtained from $(C|_{V_i}, (A + JC)|_{V_i})$ by an aditional output injection J'_i (depending on V_i) defined by the equation

$$J'_iCZ_iC_i^t = -Z_iS_i.$$

Example 3 *We consider the pair*

$$C = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and the flag

$V_1 = span\{(0,1,0,0,1)\} \subset V_2 = span\{(1,0,0,0,0),(0,1,0,0,0),(0,0,1,0,0),(0,0,0,0,1),\}.$

We have that (Z_i is the matrix obtained arranging the basis of V_i in columns)

$$CZ_1C_1^t = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad AZ_1C_1^t = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad CZ_2C_2t = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad AZ_2C_2^t = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Hence, the system of equations $JCZ_iC_i^t = -AZ_iC_i^t$, $i = 1,2$ has not solution. However one can obtain a J making V_i $(A + JC)$-invariants following the contruction given in 2 (we use the notation of the proof of 2):

$V_1 \cap KerC = \{0\}$. *Then* $v^1 = \{(0,1,0,0,1)\}$, $Cv^1 = \{(0,1)\}$ *and* J *maps* $(0,1) \mapsto -(0,0,1,0,0) + (0,a,0,0,a)$.

$V_1 + V_2 \cap KerC = span\{(0,1,0,0,1),(1,0,0,0,0),(0,1,0,0,0)\}$. *Then* $v^2 = \{(0,0,1,0,0)\}$, $Cv^2 = \{(1,0)\}$ *and* J *maps* $(1,0) \mapsto (0,0,0,0,0) + (b,c,d,0,e)$.

Hence, we obtain the output injections

$$J = \begin{pmatrix} b & a \\ c & 0 \\ d & -1 \\ 0 & 0 \\ e & a \end{pmatrix}$$

making V_1 and V_2 simultaneously $(A + JC)$-invariants. Notice that taking for example $a = b = c = d = e = 0$, the pair

$$C|_{V_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (A + JC)|_{V_2} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is not reducible without an aditional output injection (which will not be 'compatible' with V_1) to its Brunovsky canonical form.

The previous example suggest the following conjecture

Conjecture 4 *Given a (C, A)-invariant flag $V = (V_1, \dots, V_d)$ there exists an output injection J making V_i $(A + JC)$-invariants for $i = 1, \dots, d$ with $(A + JC)|_{V_i}$ having a prescribed spectra provided the natural condition that $\sigma(A + JC)|_{V_i} \subset \sigma(A + JC)|_{V_{i+1}}$.*

In spite that, in general, there is not an output injection J making the elements V_i of a flag $(A + JC)$-invariants and reducing $(C_{V_i}, (A + JC)|_{V_i})$ to the Brunovsky form at the same time, given for each flag a J with $(A + JC)V_i \subset V_i$ for all V_i, our aim is to stratify the set $Flag_{(C,A)}(k)$ in a finite set of smooth manifolds by fixing the Brunovsky indices of the restrictions $(C_{V_i}, (A+JC)|_{V_i})$ in a similar way as we do with the set conditioned invariant subspaces. In order to do this we introduce the following notation. Given a flag $(V_1, \dots, V_d) \in Flag_{(C,A)}(k)$, we call to the d-tuple $\lambda = (\lambda^1, \dots, \lambda^d)$ where λ^i are the Brunovsky indices of the restrictions $(C|_{V_i}, (A+JC)|_{V_i})$, the *restricted indices* of the flag.

We consider a d-tuple $\lambda = (\lambda^1, \dots, \lambda^d)$ of partitions $\lambda^i = (\lambda^i_1, \dots, \lambda^i_p)$ with $\lambda^i_1 \geq, \dots, \geq \lambda^i_p \geq 0$, $\lambda^i \leq \lambda^{i+1} \leq \mu$, $k_i = \lambda^i_1 + \dots + \lambda^i_p$. Let

$$Flag_{(C,A)}(k, \lambda) := \{V \in Flag_{(C,A)}(k) \quad | \quad V_i \text{ has restricted indices } \lambda^i\}.$$

$Flag_{(C,A)}(k, \lambda)$ defines a finite partition of $Flag_{(C,A)}(k)$ into disjoint subsets

$$Flag_{(C,A)}(k) = \bigcup_{\lambda^i \leq \mu} Flag_{(C,A)}(k, \lambda).$$

In the next section we define a differentiable structure on $Flag_{(C,A)}(k, \lambda)$ by describing it as an homogeneous space.

3 Orbit space structure of conditioned invariant flags

The space $Flag_{(C,A)}(k, \lambda)$ has a description as an orbit space which generalizes the description for $d = 1$ given in the first section. Let $V = (V_1, \dots, V_d) \in Flag_{(C,A)}(k, \lambda)$ and let (C_i, A_i) denote the Brunovsky canonical form of $(C|_{V_i}, (A+JC)_{V_i})$ with J such that V_i are $(A + JC)$-invariants; $(C_{d+1}, A_{d+1}) := (C, A)$.

Since $V_1 \subset \dots \subset V_d$, we can represent V_i as $Im Z_d \cdots Z_i$ with $Z_i \in F^{k_{i+1} \times k_i}$ $(k_{d+1} = n)$. Our goal is to identify $V = (V_1, \dots, V_d)$ with the orbit of a d-tuple $Z = (Z_1, \dots, Z_d)$ with $Z_i \in F^{k_{i+1} \times k_i}$, defined by a convenient group action.

Let $M(\lambda, \mu)$ be the set of d-tuples $Z = (Z_1, \dots, Z_d) \in F^{k_2 \times k_1} \times \dots \times F^{k_d \times k_{d-1}} \times$

$F^{n\times k_d}$ with

$$(3) \quad \begin{cases} A_{i+1}Z_i = Z_iA_i + A_{i+1}Z_iC_i^tC_i \\ C_{i+1}Z_i = C_{i+1}Z_iC_i^tC_i \end{cases}$$

and such that $C_{i+1}Z_iC_i^t$ has full rank. Let $G(\lambda) := M(\lambda,\lambda)$ denote the set of d-tuples $S = (S_1,\dots,S_d) \in Gl_{k_1}(F)\times\cdots\times Gl_{k_d}(F)$ with

$$(4) \quad \begin{cases} A_iS_i = S_iA_i + A_iS_iC_i^tC_i \\ C_iS_i = C_iS_iC_i^tC_i \end{cases}$$

The solutions of (3) and (4) consist of blocks of *upper triangular* Toeplitz matrices.

We will make use without explicit mention that $C_{i+1}Z_iC_i^t$ has full rank implies that Z_i has full rank (see [...]).

We consider in $Gl_{k_1}(F)\times\cdots\times Gl_{k_d}(F)$ the product $(S_1,\dots,S_d)\cdot(T_1,\dots,T_d) := (T_2^{-1}S_1T_1,\dots,T_d^{-1}S_{d-1}T_{d-1},S_dT_d)$. The following proposition is a consequence of 4.4,[...].

Proposition 5 *$G(\lambda)$ forms a Lie subgroup of $Gl_{k_1}(F)\times\cdots\times Gl_{k_d}(F)$ and $G(\lambda)$ acts freely on $M(\lambda,\mu)$ via*

$$(Z_1,\dots,Z_d) \longmapsto (S_2^{-1}Z_1S_1,\dots,S_d^{-1}Z_{d-1}S_{d-1},Z_dS_d)$$

Moreover if $S \in Gl_{k_1}(F)\times\cdots\times Gl_{k_d}(F)$ is a transformation such that for $Z \in M(\lambda,\mu)$, $ZS \in M(\lambda,\mu)$, then $S \in G(\lambda)$.

Thus, one can prove in a similar way as in 4.5[...] that the orbit space of the above group action defines a smooth quotient manifold

$$M(\lambda,\mu)/G(\lambda).$$

We consider the map

$$\phi : M(\lambda,\mu) \longrightarrow Flag(k)$$

defined by $\phi(Z_1,\dots,Z_d) = (ImZ_d\cdots Z_1,\dots,ImZ_{d-1}Z_d,ImZ_d)$. One can easily check that ϕ induces a map

$$\tilde{\phi} : M(\lambda,\mu)/G(\lambda) \longrightarrow Flag(k)$$

defined by $\tilde{\phi}([Z]_{G(\lambda)}) := \phi(Z)$. The goal of this section is to prove the following result

Theorem 6 *$\tilde{\phi}$ maps bijectively $M(\lambda,\mu)/G(\lambda)$ on $Flag_{(C,A)}(k,\lambda)$. Moreover $\tilde{\phi}$ is an embedding of $M(\lambda,\mu)/G(\lambda)$ in $Flag(k)$.*

Proof

We split the proof of this theorem in two parts. We first prove that ϕ induces a bijective map between $M(\lambda,\mu)/G(\lambda)$ and $Flag_{(C,A)}(k,\lambda)$. And second we prove that ϕ induces an embedding by proving that the differentiable structures of $M(\lambda,\mu)/G(\lambda)$ and $Flag(k)$ are compatibles.

For the first part we prove that $\tilde{\phi}$ maps bijectively $M(\lambda,\mu)/G(\lambda)$ on $Flag_{(C,A)}(k,\lambda)$.

We first prove that $Im\phi = Flag_{(C,A)}(k,\lambda)$. In fact, let $(Z_1,\dots,Z_d) \in M(\lambda,\mu)$; reacall that $p_i = \mathrm{rk}C_i$; we note that $\lambda^i_{p_i} > \lambda^i_{p_i+1} = \dots = \lambda^i_p = 0$. The equations (3) are equivalent to the commutation of

$$(5)\quad \begin{array}{ccc} F_1^k & \overset{(A_1^t,C_1^t)^t}{\longrightarrow} & F^{k_1+p_1} \\ \downarrow Z_1 & & \downarrow {Z_1}' \\ F^{k_2} & \overset{(A_2^t,C_2^t)^t}{\longrightarrow} & F^{k_2+p_2} \\ \vdots & \vdots & \vdots \\ \downarrow Z_{d-1} & & \downarrow {Z_{d-1}}' \\ F^{k_d} & \overset{(A_d^t,C_d^t)^t}{\longrightarrow} & F^{k_d+p_d} \\ \downarrow Z_d & & \downarrow {Z_d}' \\ F^n & \overset{(A^t,C^t)^t}{\longrightarrow} & F^{n+p} \end{array} \quad \text{where } {Z_i}' = \begin{pmatrix} Z_i & A_{i+1}Z_iC_i^t \\ 0 & C_{i+1}Z_iC_i^t \end{pmatrix}$$

Using (3) one can check that

$$\begin{pmatrix} Z_i & A_{i+1}Z_iC_i^t \\ 0 & C_{i+1}Z_iC_i^t \end{pmatrix} \begin{pmatrix} Z_{i-1} & A_iZ_{i-1}C_{i-1}^t \\ 0 & C_iZ_{i-1}C_{i-1}^t \end{pmatrix} =$$

$$= \begin{pmatrix} Z_iZ_{i-1} & (Z_iA_i + A_{i+1}Z_iC_i^tC_i)Z_{i-1}C_{i-1}^t \\ 0 & C_{i+1}Z_iC_i^tC_iZ_{i-1}C_{i-1}^t \end{pmatrix} = \begin{pmatrix} Z_iZ_{i-1} & A_{i+1}Z_iZ_{i-1}C_{i-1}^t \\ 0 & C_{i+1}Z_iZ_{i-1}C_{i-1}^t \end{pmatrix}$$

and by induction,

$$Z_d' \cdots Z_i' = \begin{pmatrix} Z_d\cdots Z_i & AZ_d\cdots Z_iC_i^t \\ 0 & CZ_d\cdots Z_iC_i^t \end{pmatrix}$$

Therefore, from the commutation of

$$(6)\quad \begin{array}{rcl} F_i^k & \overset{(A_i^t,C_i^t)^t}{\longrightarrow} & F^{k_i+p_i} \\ Z_d\cdots Z_i \downarrow & & \downarrow Z_d'\cdots Z_i' \\ F^n & \overset{(A^t,C^t)^t}{\longrightarrow} & F^{n+p} \end{array}$$

we obtain that $Z_d \cdots Z_i$ satisfy the equations

$$(7) \qquad \begin{cases} AZ_d \cdots Z_i = Z_d \cdots Z_i A_i + AZ_d \cdots Z_i C_i^t C_i \\ CZ_d \cdots Z_i = CZ_d \cdots Z_i C_i^t C_i \end{cases}$$

for $1 \leq i \leq d$.

Moreover, since $CZ_d \cdots Z_i C_i^t = CZ_d C_d^t C_d Z_{d-1} C_{d-1}^t \cdots C_2 Z_1 C_1^t$, we have that $C_{i+1} Z_i C_i^t$ has full rank for $1 \leq i \leq d$ implies that $CZ_d \cdots Z_i C_i^t$ has full rank. Hence, we have that $ImZ_d \cdots Z_i \in Inv_{\lambda_i}(C, A)$, this is to say, $\phi(Z) \in Flag_{(C,A)}(k, \lambda)$.

Notice that (C_i, A_i) is the Brunovsky canonical form of the restriction of (C, A) to $ImZ_d \cdots Z_i$.

Conversely, if $(V_1, \ldots, V_d) \in Flag_{(C,A)}(k, \lambda)$ then $V_i = ImZ_d \cdots Z_i$ where $Z_d \cdots Z_i$ are solutions of (7) having $CZ_d \cdots Z_i C_i^t$ full rank. Then, since (6) commutes, using the injectivity of Z_i we have that the diagram (5) commutes and since $CZ_d \cdots Z_i C_i^t$ has full rank we have that $C_{i+1} Z_i C_i^t$ has full rank for $1 \leq i \leq d$. This proves that $(V_1, \ldots, V_d) = \phi(Z)$ with $Z \in M(\lambda, \mu)$ and therefore $Im\phi = Flag_{(C,A)}(k, \lambda)$.

To complet the proof of the first part of the theorem we prove that $\tilde{\phi}$ is injective.

In fact, suppose that $\phi(Z) = \phi(Z')$, this is to say, suppose that $ImZ_d \cdots Z_i = ImZ_d{}' \cdots Z_i{}'$ for $1 \leq i \leq d$. Then $Z_d{}' \cdots Z_i{}' = Z_d \cdots Z_i S_i$ with $S_i \in Gl_{k_i}(F)$ for $1 \leq i \leq d$. In particular, $Z_d{}' = Z_d S_d$. Moreover $Z_d{}' Z_{d-1}{}' = Z_d Z_{d-1} S_{d-1}$ implies that $Z_d S_d Z_{d-1}{}' = Z_d Z_{d-1} S_{d-1}$, and from the injectivity of Z_d we have that $S_d Z_{d-1}{}' = Z_{d-1} S_{d-1}$ and therefore $Z_{d-1}{}' = S_d^{-1} Z_{d-1} S_{d-1}$. Applying this reasoning recurrently we obtain that $Z_i{}' = S_{i+1}^{-1} Z_i S_{i+1}$ for $1 \leq i \leq d$.

Now, applying the proposition 5 we have that $(S_1, \ldots, S_d) \in G(\lambda)$. Hence Z and Z' belong to the same orbit and therefore the injectivity of $\tilde{\phi}$ holds.

We prove now that $\tilde{\phi}$ is an embedding of $M(\lambda, \mu)/G(\lambda)$ in $Flag(k)$.

Recall that $Flag(k) \cong M^*_{n \times k_d}/P(k)$. For our proof we need an alternative description of $Flag(k)$ as a quotient manifold.

Consider the map

$$\alpha : M^*_{n \times k_d} \longrightarrow M^*_{k_2 \times k_1} \times \cdots \times M^*_{k_d \times k_{d-1}} \times M^*_{n \times k_d}$$

defined by $\alpha(Z) = (U_1, \ldots, U_{d-1}, Z)$ where

$$U_i = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & & \cdots & 1 \\ 0 & & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & 0 \end{pmatrix} \in M^*_{k_{i+1}\times k_i}.$$

We consider in $M^*_{k_2\times k_1} \times \cdots \times M^*_{n\times k_d}$ the action of the group $Gl_{k_1}(F) \times \cdots \times Gl_{k_d}(F)$ defined by

$$(Z_1, \ldots, Z_d) \longmapsto (S_2^{-1} Z_1 S_1, \ldots, S_d^{-1} Z_{d-1} S_{d-1}, Z_d S_d)$$

We claim that α induces a diffeomorphism

$$\tilde{\alpha} : M^*_{n\times k_d}/P(k) \longrightarrow (M^*_{k_2\times k_1} \times \cdots \times M^*_{n\times k_d})/(Gl_{k_1}(F) \times \cdots \times Gl_{k_d}(F))$$

We first show that $\tilde{\alpha}$ is well defined, this is to say, that given $Z \in M^*_{n\times k_d}$ and $S \in P(k)$, we have that $(U_1, \ldots, U_{d-1}, Z)$ and $(U_1, \ldots, U_{d-1}, ZS)$ are in the same orbit. In fact, notice that

$$S^{-1} U_{d-1} = \begin{pmatrix} X \\ 0 \end{pmatrix} \text{ with } X \in P((k_1, \ldots, k_{d-1})).$$

Hence, defining $S_{d-1} := X^{-1}$ we have that $S^{-1} U_{d-1} S_{d-1} = U_{d-1}$. Since $S_{d-1} \in P((k_1, \ldots, k_{d-1}))$, it is clear that we can define recurrently a sequence $S_i \in P((k_1, \ldots, k_i))$, $1 \leq i \leq d-1$, such that $S_{i+1}^{-1} U_i S_i = U_i$. Therefore $(U_1, \ldots, U_{d-1}, Z)$ and $(U_1, \ldots, U_{d-1}, ZS)$ are in the same orbit and $\tilde{\alpha}$ is well defined.

We now prove the injectivity of $\tilde{\alpha}$. Suppose that $\tilde{\alpha}([Z]_{P(k)}) = \tilde{\alpha}([Z']_{P(k)})$, this is to say, that $(U_1, \ldots, U_{d-1}, Z')$ is in the orbit of $(U_1, \ldots, U_{d-1}, Z)$. This means that there exists a sequence $S_1, \ldots, S_d$ such that $Z' = Z S_d$ and $S_{i+1}^{-1} U_i S_i = U_i$ for $1 \leq i \leq d-1$. But the former equalities imply that $S_d \in P(k)$. Therefore, Z and Z' are in the same orbit and the injectivity of $\tilde{\alpha}$ follows.

Finally we prove the surjectivity of $\tilde{\alpha}$. Let $(Z_1, \ldots, Z_d) \in M^*_{k_2\times k_1} \times \cdots \times M^*_{n\times k_d}$. We claim that there exist an element of the form $(U_1, \ldots, U_{d-1}, Z)$ in the orbit of $(Z_1, \ldots, Z_d)$. In fact, it is clear that we can define recurrently from $i = 1$ to

d a sequence $S_1, \dots, S_d$ such that

$$S_{i+1}^{-1} Z_i S_i = \begin{pmatrix} T_i \\ 0 \end{pmatrix} \quad \text{with } T_i \text{ an upper triangular square matrix.}$$

Let $Z := Z_d S_d$. We have that $(S_d^{-1} Z_{d-1} S_{d-1}) T_{d-1}^{-1} = U_{d-1}$ and

$$T_{d-1}(S_{d-1}^{-1} Z_{d-2} S_{d-2}) = \begin{pmatrix} \tilde{T}_{d-2} \\ 0 \end{pmatrix}$$

with $\tilde{T}_{d-2}$ an upper triangular square matrix as well as T_{d-2}. Hence,

$$T_{d-1}(S_{d-1}^{-1} Z_{d-2} S_{d-2}) \tilde{T}_{d-2}^{-1} = U_{d-2}.$$

Proceding recurently in this way we obtain that $(U_1, \dots, U_{d-1}, Z)$ is in the orbit of $(Z_1, \dots, Z_d)$ and therefore the surjectivity of $\tilde{\alpha}$ holds.

The smothness of $\tilde{\alpha}$ follows form the smothness of α and the local triviality of the principal bundles defined by the projections $M^*_{n \times k_d} \longrightarrow M^*_{n \times k_d}/P(k)$ and $M^*_{k_2 \times k_1} \times \cdots \times M^*_{n \times k_d} \longrightarrow (M^*_{k_2 \times k_1} \times \cdots \times M^*_{n \times k_d})/(Gl_{k_1}(F) \times \cdots \times Gl_{k_d}(F))$ Therfore $\tilde{\alpha}$ is a diffeomorphism.

We now consider the inclusion $j : M(\lambda, \mu) \hookrightarrow M^*_{k_2 \times k_1} \times \cdots \times M^*_{n \times k_d}$ and the commutative diagram

$$\begin{array}{ccc} M(\lambda, \mu) & \xrightarrow{j} & M^*_{k_2 \times k_1} \times \cdots \times M^*_{n \times k_d} \\ p_1 \downarrow & & p_2 \downarrow \\ M(\lambda, \mu)/G(\lambda) & \xrightarrow{\tilde{j}} & (M^*_{k_2 \times k_1} \times \cdots \times M^*_{n \times k_d})/(Gl_{k_1}(F) \times \cdots \times Gl_{k_d}(F)) \end{array}$$

where p_1 and p_2 are the natural projections of the principal bundles.

Thanks to the diffeomorphism $\tilde{\alpha}$ we prove that $\tilde{\phi}$ is an embedding by proving that $\tilde{j}$ is an embedding.

To prove that $\tilde{j}$ is an embedding it is sufficient to prove that a local representation of $\tilde{j}$, $\sigma_2 \tilde{j} \sigma_1^{-1}$ with σ_i local sections of p_i, is an embedding.

In fact, let $Z = (Z_1, \dots, Z_d) \in M(\lambda, \mu)$, L a linear space in $\overline{M(\lambda, \mu)}$ transverse to $[Z]_{G(\lambda)}$, and σ_1 a local section of p_1 having it image in L. We claim that L is transverse to $[Z]_{Gl_{k_1}(F) \times \cdots \times Gl_{k_d}(F)}$. Let π be the natural projection

$$\pi : M^*_{k_2 \times k_1} \times \cdots \times M^*_{n \times k_d} \longrightarrow M^*_{n \times k_d}$$

It is sufficient to show that $\pi(L)$ is transverse to $\pi([Z]_{Gl_{k_1}(F) \times \cdots \times Gl_{k_d}(F)})$. But this follows from that $\pi([Z]_{Gl_{k_1}(F)\times\cdots\times Gl_{k_d}(F)}) = Z_d Gl_{k_d}(F)$ is a linear space intersecting $\pi(L)$ in a single point $\{Z_d\}$.

Now we consider a linear space V containing L, transverse to $[Z]_{Gl_{k_1}(F)\times\cdots\times Gl_{k_d}(F)}$ such that

$$T_Z V \oplus T_Z([Z]_{Gl_{k_1}(F)\times\cdots\times Gl_{k_d}(F)}) = T_Z(M^*_{k_2\times k_1} \times \cdots \times M^*_{n\times k_d})$$

It is clear that there exists a local section σ_2 of p_2 having it image in V and therefore $\sigma_2 \tilde{j} \sigma_1^{-1}$ is a local embedding representing $\tilde{j}$.

This complets the proof of that $\tilde{\phi}$ is an embedding and therefore the theorem is proved.

As a corollary of the theorem 6 and the proposition 5 we have the following formula for the dimension of $Flag_{(C,A)}(k,\lambda)$.

Corollary 7

$$\dim Flag_{(C,A)}(k,\lambda) = \dim Inv_{\lambda^1}(C_2,A_2) + \cdots + \dim Inv_{\lambda^{d-1}}(C_d,A_d) + \dim Inv_{\lambda^d}(C,A)$$

$$= \sum_{1\le j\le d} \sum_{1\le i\le p_j} (\mu_i - \lambda_i^j)(p_j - i + 1) + \sum_{1\le j\le d} \sum_{\substack{1\le i,l\le p_j \\ \lambda_i^j = \lambda_l^j}} (\mu_i - \lambda_l^j)$$

Proof Since the group action of $G(\lambda)$ on $M(\lambda,\mu)$ is free (5) by theorem (6) and [...] we have that $(\lambda^{i+1} := \mu)$

$$\dim Flag_{(C,A)}(k,\lambda) = \dim M(\lambda,\mu) - \dim G(\lambda) = \sum_{i=1}^{d} \dim M(\lambda^i,\lambda^{i+1}) - \sum_{i=1}^{d} \dim G(\lambda^i) =$$

$$\sum_{i=1}^{d} (\dim M(\lambda^i,\lambda^{i+1}) - \dim G(\lambda^i)) = \sum_{i=1}^{d} \dim Inv_{\lambda^i}(C_{i+1},A_{i+1}).$$

Now, applying 5.3[...] we have the second part of the formula.

Example 8 *The set of flags of the same type of the example* **??** *are defined by $\mu = (3,2)$ and $\lambda = ((3,1),(1,0))$. We have that*

$$\dim Flag_{(C,A)}(k,\lambda) = (6-5) + (4-1) = 4.$$

4 Retraction onto a biflag manifold

We recall that the strata of A-invariant flag manifolds can be retracted onto a biflag manifold (see [...]). On the other hand, for $d = 1$ the strata of (C, A)-invariant subspaces can be retracted onto a generalized flag manifold which is a particular class of biflag manifold (see [...]). The goal of this section is to generalize the last result extending the work of Helmke and Shaymann [...] to (C, A)-invariant flag manifolds.