

Hölder stability in Type III thermoelastodynamics

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To G.A. Maugin in his 70th birthday

Abstract

This note is concerned with the linear (and linearized) Type III thermoelastodynamic theory proposed by Green and Naghdi. We here assume that the mass density is positive and the thermal conductivity tensor is positive definite. However, we do not assume the positivity of any other tensor. In this situation, we obtain Hölder continuous dependence results on the supply terms. We also sketch how to prove the continuous dependence on the initial data.

keywords: Type III thermoelastodynamics, Hölder stability, Continuous dependence on initial data and supply terms, Lagrange identities method

1 Introduction

The thermoelastic theory proposed by Green and Naghdi [5–7] has deserved an intense investigation in the last years. Three sub-theories, where an entropy balance law replaces the customary entropy inequality, have been proposed. These theories were labelled as Type I, II and III. The linear version of Type I agrees with the usual classical theory of thermoelasticity. For Type II, the energy of the system is constant with respect to the time. For this reason it is also known as “thermoelasticity without energy dissipation”. Type III is the more general theory and it contains the other two as limiting cases. We can recall several papers (see Iesan [8,9]; Iesan and Quintanilla [10]; Lazzari and Nibbi [13]; Leseduarte *et al.* [14,15]; Liu and Quintanilla [16,17]; Liu and Lin [18]; Messaoudi and Soufyane [19]; Puri and Jordan [20]; Qin *et al.* [21]; Quintanilla [22–27]; Quintanilla and Racke [28]; Quintanilla and Straughan [29–31]; Yang and Wang [34], among others), where existence, uniqueness, continuous dependence, spatial and time behavior have been studied.

We recall that the linear system for the centrosymmetric Type III thermoelastodynamics can be written as

$$\begin{cases} \rho \ddot{u}_i = (a_{ijkh} u_{k,h} - a_{ij} \theta)_{,j} + \rho f_i, \\ c \dot{\theta} = -a_{ij} \dot{u}_{i,j} + (k_{ij} \theta_{,i})_{,j} + (b_{ij} \alpha_{,i})_{,j} + \rho r, \end{cases} \quad (1.1)$$

where u_i is the displacement vector, θ is the temperature, ρ is the mass density, c is the thermal capacity, (a_{ijkh}) is the elasticity tensor, a_{ij} is the coupling tensor, k_{ij} is the thermal

conductivity tensor and b_{ij} denotes a tensor which is typical for the Types II and III theories. The constitutive tensors ρ , c , a_{ijkh} , a_{ij} , b_{ij} and k_{ij} are smooth functions of the position. They satisfy the symmetries

$$a_{ijkh} = a_{khij} \quad (1.2)$$

and

$$k_{ij} = k_{ji}, \quad b_{ij} = b_{ji}. \quad (1.3)$$

The thermal displacement α is defined by

$$\alpha(\mathbf{x}, t) = \int_0^t \theta(\mathbf{x}, s) ds + \alpha_0(\mathbf{x}), \quad (1.4)$$

and f_i and r are the supply terms. To guarantee the well-posedness of the Type III thermoelasticity, we need to assume that the mass density and the heat capacity are positive as well as the elasticity tensor and the thermal conductivity. Lyapunov stability of the solutions is implied when b_{ij} is also positive definite.

In the case where we do not assume the positivity of the elasticity tensor a_{ijkh} , the problem determined by the system (1.1) with usual initial and boundary conditions becomes ill-posed. It is worth recalling that this condition can be present in the case of prestressed solids.

It is worth noting several references for this situation. For instance, results concerning uniqueness and growth of solutions have been obtained at [25, 29] under the condition that the tensors k_{ij} and b_{ij} are positive. It is also worth mentioning that the only contribution concerning the case when a_{ijkh} and b_{ij} are not definite was obtained recently (see [15]). There, it was showed the uniqueness of solutions. We here want to show how to obtain the continuous dependence with respect to the supply terms and initial data under similar restrictions.

In 1960, John [11] showed how to obtain continuous dependence results in the sense of Hölder. This concept is weaker than the usual definition of the continuous dependence. The basic idea consists to impose that the solutions belong to a suitable constraint class. Since this contribution, many investigations have been directed to this kind of results. We may cite the works of Ames and Payne [1] and Knops and Payne [12] concerning isothermal elastodynamics. We can also recall that Hölder stability results can be found in different frameworks from the thermoelasticity (see Cimmelli and dell'Isola [3] and dell'Isola [4]). For the classical thermoelastodynamics, we may recall the contributions of Wilkes [33], Ames and Straughan [2] and Rionero and Chirita [32]. It is worth recalling and comparing several related results. In [15], we proposed a uniqueness result under similar assumptions to the ones considered here and in [24] the author proposed a Hölder stability result by using the logarithmic convexity argument under the assumption that b_{ij} is positive definite. We here obtain a new result proving the Hölder stability of the solutions when we do not assume that the elasticity tensor neither the b_{ij} tensor are positive definite. It is known that two usual techniques to study ill-posed problems in thermoelasticity are the logarithmic convexity and the Lagrange identities method. We here consider the second one. We want to emphasize that since the tensor b_{ij} is not positive definite, it does not seem possible to apply the logarithmic convexity arguments. However, we here are able to use the Lagrange identities method. In fact, Lagrange identity method allows us to obtain equality (3.5) that implies the inequality (3.12). This inequality is the starting point to develop our approach. Therefore, Lagrange identity is the fundamental ingredient to obtain our results.

The plain of this note is the following. In the next section we recall the basic assumptions and the conditions defining the problem. A basic inequality is obtained in Section 3. Hölder's continuous dependence with respect to the supply terms is obtained in Section 4. In the last section we sketch how to extend the argument to prove continuous dependence with respect to the initial data.

2 Preliminaries

In this section we propose the basic assumptions where we are going to work with. We study smooth solutions of (1.1) on $B \times I$, where I is a bounded time interval and B is a regular domain with boundary Γ smooth enough to apply the Divergence Theorem.

In what follows we suppose that the constitutive tensors are bounded about and have the following properties:

(A1) The mass density ρ and the heat capacity c are positive functions. That is,

$$\rho(\mathbf{x}) \geq \rho_0 > 0, \quad c(\mathbf{x}) \geq c_0 > 0, \quad \mathbf{x} \in B.$$

(A2) The thermal conductivity tensor k_{ij} is positive definite. That is, there exists a positive constant C such that

$$k_{ij}\xi_i\xi_j \geq C\xi_i\xi_i, \tag{2.1}$$

for every vector (ξ_i) .

The physical meaning of condition (A1) is obvious. Condition (A2) guarantees that the dissipation of the system is not negative and then, the energy does not increase. It is an usual assumption in the thermomechanical studies. We also note that from (2.1), the following inequality

$$|b_{ij}\xi_i\xi_j| \leq C_1k_{ij}\xi_i\xi_j \tag{2.2}$$

is satisfied for every vector (ξ_i) , where C_1 is a calculable constant¹ which depends on the tensors k_{ij} and b_{ij} .

To the field equations we adjoin the boundary conditions

$$u_i(\mathbf{x}, t) = \bar{u}_i(\mathbf{x}, t), \quad \alpha(\mathbf{x}, t) = \bar{\alpha}(\mathbf{x}, t), \quad \mathbf{x} \in \Gamma, \quad t \in I, \tag{2.3}$$

together with the initial conditions

$$u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}), \quad \dot{u}_i(\mathbf{x}, 0) = v_i^0(\mathbf{x}), \quad \alpha(\mathbf{x}, 0) = \alpha^0(\mathbf{x}), \quad \dot{\alpha}(\mathbf{x}, 0) = \theta^0(\mathbf{x}), \quad \mathbf{x} \in B. \tag{2.4}$$

To study the continuous dependence of the solutions with respect to the supply terms, we denote by $(u_i^{(1)}, \alpha^{(1)})$ the solution corresponding to the external data $(f_i^{(1)}, r^{(1)})$ and by $(u_i^{(2)}, \alpha^{(2)})$, the solution corresponding to the supply terms $(f_i^{(2)}, r^{(2)})$.

We introduce the notation

$$u_i = u_i^{(2)} - u_i^{(1)}, \quad \alpha = \alpha^{(2)} - \alpha^{(1)}, \quad F_i = f_i^{(2)} - f_i^{(1)}, \quad q = r^{(2)} - r^{(1)}. \tag{2.5}$$

¹We note that $C_1 = b^*/k_*$, where b^* is the maximum of the absolute values of the eigenvalues of the matrix b_{ij} and k_* is the minimum of the eigenvalues of k_{ij} .

It follows that (u_i, α) satisfies the problem determined by the system

$$\begin{cases} \rho \ddot{u}_i = (a_{ijkh} u_{k,h} - a_{ij} \theta)_{,j} + \rho F_i, \\ c \dot{\theta} = -a_{ij} \dot{u}_{i,j} + (k_{ij} \theta_{,i})_{,j} + (b_{ij} \alpha_{,i})_{,j} + \rho q, \end{cases} \quad (2.6)$$

with the homogeneous boundary conditions

$$u_i(\mathbf{x}, t) = \alpha(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \Gamma, \quad t \in I, \quad (2.7)$$

and the null initial conditions

$$u_i(\mathbf{x}, 0) = \dot{u}_i(\mathbf{x}, 0) = \alpha(\mathbf{x}, 0) = \dot{\alpha}(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in B. \quad (2.8)$$

In our studies it will be useful the following inequality

$$\int_0^t k_{ij} \alpha_{,i} \alpha_{,j} ds \leq \frac{4C_2^2 t^2}{\pi^2} \int_0^t k_{ij} \dot{\alpha}_{,i} \dot{\alpha}_{,j} ds, \quad (2.9)$$

which is satisfied for every function α such that $\alpha(0) = 0$ and where C_2 is a calculable constant².

3 A basic inequality

In this section we obtain an inequality satisfied by the solutions of the problem defined by the system (2.6) with boundary and initial conditions (2.7) and (2.8), respectively. This inequality will be relevant to obtain our results.

As we assume null Dirichlet boundary conditions and null initial conditions, we obtain that the energy equality

$$\begin{aligned} & \int_B (\rho \dot{u}_i \dot{u}_i + c \theta^2 + a_{ijkh} u_{i,j} u_{k,h} + b_{ij} \alpha_{,i} \alpha_{,j}) dv \\ & + 2 \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds - 2 \int_0^t \int_B (\rho F_i \dot{u}_i + \rho q \theta) dv ds = 0 \end{aligned} \quad (3.1)$$

is satisfied for every solution of our problem.

On the other hand, from the equalities

$$\frac{d}{ds} [\rho \dot{u}_i(s) \dot{u}_i(2t - s)] = \rho \ddot{u}_i(s) \dot{u}_i(2t - s) - \rho \dot{u}_i(s) \ddot{u}_i(2t - s), \quad (3.2)$$

$$\frac{d}{ds} [c \theta(s) \theta(2t - s)] = c \dot{\theta}(s) \theta(2t - s) - c \theta(s) \dot{\theta}(2t - s) \quad (3.3)$$

and the null initial and boundary conditions, we obtain

$$\begin{aligned} & \int_B (\rho \dot{u}_i \dot{u}_i + b_{ij} \alpha_{,i} \alpha_{,j} - a_{ijkh} u_{i,j} u_{k,h} - c \theta^2) dv \\ & = \int_0^t \int_B \rho (F_i(s) \dot{u}_i(2t - s) - F_i(2t - s) \dot{u}_i(s)) dv ds \\ & - \int_0^t \int_B \rho (q(s) \theta(2t - s) - q(2t - s) \theta(s)) dv ds. \end{aligned} \quad (3.4)$$

²We note that $C_2^2 = k^*/k_*$, where k^* is the maximum of the eigenvalues of the symmetric matrix k_{ij} and k_* is the minimum of the eigenvalues of k_{ij} .

From the equality (3.1) and the relation (3.4), we conclude that

$$\begin{aligned}
& \int_B (\rho \dot{u}_i \dot{u}_i + b_{ij} \alpha_{,i} \alpha_{,j}) dv + \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds \\
&= \int_0^t \int_B (\rho F_i \dot{u}_i + \rho q \theta) dv ds + \frac{1}{2} \int_0^t \int_B (\rho F_i(s) \dot{u}_i(2t-s) - \rho F_i(2t-s) \dot{u}_i(s)) dv ds \\
& \quad + \frac{1}{2} \int_0^t \int_B (\rho q(2t-s) \theta(s) - \rho q(s) \theta(2t-s)) dv ds.
\end{aligned} \tag{3.5}$$

We now consider several estimates:

$$\begin{aligned}
\int_0^t \int_B [\rho F_i \dot{u}_i + \rho q \theta] dv ds &\leq \left(\int_0^t \int_B \rho F_i F_i dv ds \right)^{1/2} \left(\int_0^t \int_B \rho \dot{u}_i \dot{u}_i dv ds \right)^{1/2} \\
& \quad + \left(\int_0^t \int_B \rho q^2 dv ds \right)^{1/2} \left(\int_0^t \int_B \rho \theta^2 dv ds \right)^{1/2} \\
&\leq \left(\int_0^t \int_B [\rho F_i F_i + \rho q^2] dv ds \right)^{1/2} \left(\int_0^t \int_B [\rho \dot{u}_i \dot{u}_i + \rho \theta^2] dv ds \right)^{1/2},
\end{aligned} \tag{3.6}$$

where we have used the inequality

$$\sqrt{a} \sqrt{b} + \sqrt{c} \sqrt{d} \leq \sqrt{a+c} \sqrt{b+d}. \tag{3.7}$$

In a similar way, we have that

$$\begin{aligned}
& \int_0^t \int_B [\rho F_i(s) \dot{u}_i(2t-s) - \rho F_i(2t-s) \dot{u}_i(s)] dv ds \\
&\leq \left(\int_0^t \int_B \rho F_i F_i dv ds \right)^{1/2} \left(\int_t^{2t} \int_B \rho \dot{u}_i \dot{u}_i dv ds \right)^{1/2} \\
& \quad + \left(\int_t^{2t} \int_B \rho F_i F_i dv ds \right)^{1/2} \left(\int_0^t \int_B \rho \dot{u}_i \dot{u}_i dv ds \right)^{1/2} \\
&\leq \left(\int_0^{2t} \int_B \rho F_i F_i dv ds \right)^{1/2} \left(\int_0^{2t} \int_B \rho \dot{u}_i \dot{u}_i dv ds \right)^{1/2}.
\end{aligned} \tag{3.8}$$

We can also obtain that

$$\begin{aligned}
& \int_0^t \int_B [\rho q(2t-s) \theta(s) - \rho q(s) \theta(2t-s)] dv ds \\
&\leq \left(\int_0^{2t} \int_B \rho q^2 dv ds \right)^{1/2} \left(\int_0^{2t} \int_B \rho \theta^2 dv ds \right)^{1/2}.
\end{aligned} \tag{3.9}$$

Therefore, we see that

$$\begin{aligned}
& \int_B (\rho \dot{u}_i \dot{u}_i + b_{ij} \alpha_{,i} \alpha_{,j}) dv + \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds \\
&\leq \frac{3}{2} \left(\int_0^{2t} \int_B [\rho F_i F_i + \rho q^2] dv ds \right)^{1/2} \left(\int_0^{2t} \int_B [\rho \dot{u}_i \dot{u}_i + \rho \theta^2] dv ds \right)^{1/2}.
\end{aligned} \tag{3.10}$$

Let us assume that

$$\sup_{t \in [0, T]} \int_B (\rho \dot{u}_i \dot{u}_i + \rho \theta^2) dv \leq N_1^2 \quad (3.11)$$

and that $t \leq T/2$. We obtain that the inequality

$$\begin{aligned} & \int_B (\rho \dot{u}_i \dot{u}_i + b_{ij} \alpha_{,i} \alpha_{,j}) dv + \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds \\ & \leq \frac{3}{2} T^{1/2} N_1 \left[\int_0^T \int_B (\rho F_i F_i + \rho q^2) dv ds \right]^{1/2} \end{aligned} \quad (3.12)$$

is satisfied for every $t \leq T/2$. Estimate (3.12) is fundamental in our approach. However, as we do not assume that b_{ij} is a positive definite matrix, we do not have a positive quadratic form on the left side of the estimate. Thus, we will need to manipulate this term to work with a positive quadratic form. This will be the main aim of the next sections.

4 The main result

The aim of this section is to obtain an estimate for the solutions of the problem determined by (2.6), (2.7) and (2.8). From the inequality (3.12) and the use of the Poincaré type inequality (2.9) we will be able to get such kind of estimate.

We are going to decompose the interval $[0, T/2]$ in a finite sequence of subintervals $I_0, I_1, I_2, \dots, I_m$ such that $[0, T/2] \subset \bigcup_{j=0}^m I_j$ and such that each intersection $I_i \cap I_{i+1}$ has a unique element, denoted by $\{t_i\}$, for $i = 1, \dots, m-1$. Then, we will obtain that

$$\int_B \rho \dot{u}_i \dot{u}_i dv + \frac{1}{2} \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds \leq E_n, \quad (4.1)$$

whenever $t \in I_n \cap [0, T/2]$ where the estimate E_n is defined by the recurrence

$$E_0 = \frac{3}{2} T^{1/2} N_1 \left[\int_0^t \int_B (\rho F_i F_i + \rho q^2) dv ds \right]^{1/2} \quad (4.2)$$

and

$$E_{n+1} = \frac{16C_1 C_2 T}{\pi} E_n + \frac{3}{2} T^{1/2} N_1 \left[\int_0^t \int_B (\rho F_i F_i + \rho q^2) dv ds \right]^{1/2}. \quad (4.3)$$

Our first step is to obtain that

$$\int_B \rho \dot{u}_i \dot{u}_i dv + \frac{1}{2} \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds \leq E_0, \quad (4.4)$$

whenever $t \in I_0 = \left[0, \frac{\pi}{16C_1 C_2}\right]$.

From (2.2), the arithmetic-geometric mean inequality and (2.9), we know that

$$\begin{aligned}
& \int_B b_{ij}\alpha_{,i}\alpha_{,j} dv + \int_0^t \int_B k_{ij}\theta_{,i}\theta_{,j} dv ds \\
& \geq -C_1 \int_B k_{ij}\alpha_{,i}\alpha_{,j} dv + \int_0^t \int_B k_{ij}\theta_{,i}\theta_{,j} dv ds \\
& = -2C_1 \int_0^t \int_B k_{ij}\alpha_{,i}\theta_{,j} dv ds + \int_0^t \int_B k_{ij}\theta_{,i}\theta_{,j} dv ds \\
& \geq -2C_1 \left(\int_0^t \int_B k_{ij}\alpha_{,i}\alpha_{,j} dv ds \right)^{1/2} \left(\int_0^t \int_B k_{ij}\theta_{,i}\theta_{,j} dv ds \right)^{1/2} + \int_0^t \int_B k_{ij}\theta_{,i}\theta_{,j} dv ds \\
& \geq \left(1 - \frac{4C_1C_2t}{\pi} \right) \int_0^t \int_B k_{ij}\theta_{,i}\theta_{,j} dv ds.
\end{aligned} \tag{4.5}$$

If we assume that $t < \frac{\pi}{16C_1C_2}$, we see that

$$\int_B b_{ij}\alpha_{,i}\alpha_{,j} dv + \int_0^t \int_B k_{ij}\theta_{,i}\theta_{,j} dv ds \geq \frac{1}{2} \int_0^t \int_B k_{ij}\theta_{,i}\theta_{,j} dv ds. \tag{4.6}$$

Thus, in view of the estimate (3.12), we obtain the inequality (4.4).

Let us assume that we have obtained an estimate of the type

$$\int_B \rho \dot{u}_i \dot{u}_i dv + \frac{1}{2} \int_0^t \int_B k_{ij}\theta_{,i}\theta_{,j} dv ds \leq E_n, \tag{4.7}$$

for $\frac{n\pi}{16C_1C_2} < t < \frac{(n+1)\pi}{16C_1C_2} < \frac{T}{2}$, and we want to get a similar bound for

$$\frac{(n+1)\pi}{16C_1C_2} < t < \min \left\{ \frac{(n+2)\pi}{16C_1C_2}, \frac{T}{2} \right\}.$$

The analysis starts by considering the relations

$$\begin{aligned}
& \int_B \rho \dot{u}_i \dot{u}_i dv + \int_0^t \int_B k_{ij}\theta_{,i}\theta_{,j} dv ds \\
& = \int_B \rho \dot{u}_i \dot{u}_i dv + \int_B b_{ij}\alpha_{,i}\alpha_{,j} dv + \int_0^t \int_B k_{ij}\theta_{,i}\theta_{,j} dv ds - \int_B b_{ij}\alpha_{,i}\alpha_{,j} dv \\
& \leq \frac{3}{2} T^{1/2} N_1 \left[\int_0^T \int_B (\rho F_i F_i + \rho q^2) dv ds \right]^{1/2} + C_1 \int_B k_{ij}\alpha_{,i}\alpha_{,j} dv.
\end{aligned} \tag{4.8}$$

Here we have applied the estimates (2.2) and (3.12). We note that

$$\begin{aligned}
C_1 \int_B k_{ij}\alpha_{,i}\alpha_{,j} dv & \leq 2C_1 \int_B k_{ij} (\alpha_{,i}(t) - \alpha_{,i}(t_{n+1})) (\alpha_{,j}(t) - \alpha_{,j}(t_{n+1})) dv \\
& \quad + 2C_1 \int_B k_{ij}\alpha_{,i}(t_{n+1})\alpha_{,j}(t_{n+1}) dv,
\end{aligned} \tag{4.9}$$

where $t_{n+1} = \frac{(n+1)\pi}{16C_1C_2}$. We have that

$$\begin{aligned}
& C_1 \int_B k_{ij} \alpha_{,i} \alpha_{,j} dv \\
& \leq 4C_1 \int_{t_{n+1}}^t \int_B k_{ij} (\alpha_{,i}(s) - \alpha_{,i}(t_{n+1})) \theta_{,j} dv ds + 4C_1 \int_0^{t_{n+1}} \int_B k_{ij} \alpha_{,i} \theta_{,j} dv ds \\
& \leq 4C_1 \left(\int_{t_{n+1}}^t \int_B k_{ij} (\alpha_{,i}(s) - \alpha_{,i}(t_{n+1})) (\alpha_{,j}(s) - \alpha_{,j}(t_{n+1})) dv ds \right)^{1/2} \left(\int_{t_{n+1}}^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds \right)^{1/2} \\
& \quad + 4C_1 \left(\int_0^{t_{n+1}} \int_B k_{ij} \alpha_{,i} \alpha_{,j} dv ds \right)^{1/2} \left(\int_0^{t_{n+1}} \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds \right)^{1/2} \\
& \leq \frac{8C_1C_2(t-t_{n+1})}{\pi} \int_{t_{n+1}}^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds + \frac{8C_1C_2t_{n+1}}{\pi} \int_0^{t_{n+1}} \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds.
\end{aligned} \tag{4.10}$$

From (4.8) and (4.9), it follows that

$$\begin{aligned}
& \int_B \rho \dot{u}_i \dot{u}_i dv + \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds \\
& \leq \frac{3}{2} T^{1/2} N_1 \left[\int_0^T \int_B (\rho F_i F_i + \rho q^2) dv ds \right]^{1/2} \\
& \quad + \frac{8C_1C_2(t-t_{n+1})}{\pi} \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds + \frac{16C_1C_2t_{n+1}}{\pi} E_n.
\end{aligned} \tag{4.11}$$

If we assume that $t - t_{n+1} < \frac{\pi}{16C_1C_2}$, we obtain the desired estimate (4.1)–(4.3). We note that from the recurrence (4.3), we have

$$\begin{aligned}
E_m & \leq \frac{16C_1C_2T}{\pi} \left[\frac{16C_1C_2T}{\pi} E_{m-2} + E_0 \right] + E_0 \\
& \leq \dots \leq \left(\frac{16C_1C_2T}{\pi} \right)^m E_0 + \left(\frac{16C_1C_2T}{\pi} \right)^{m-1} E_0 + \dots + E_0.
\end{aligned} \tag{4.12}$$

Hence,

$$E_m \leq \left[\sum_{k=0}^m \left(\frac{16C_1C_2T}{\pi} \right)^k \right] E_0. \tag{4.13}$$

Thus, we can conclude that the estimate

$$\begin{aligned}
& \int_B \rho \dot{u}_i \dot{u}_i dv + \frac{1}{2} \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv \\
& \leq \frac{3}{2} T^{1/2} N_1 \left[\sum_{k=0}^m \left(\frac{16C_1C_2T}{\pi} \right)^k \right] \left(\int_0^T \int_B (\rho F_i F_i + \rho q^2) dv ds \right)^{1/2}
\end{aligned} \tag{4.14}$$

is satisfied whenever m is the first natural number such that $m > \frac{8C_1C_2T}{\pi}$. Therefore, we have proved the following result.

Theorem 4.1 Let $(u_i^{(1)}, \alpha^{(1)})$ and $(u_i^{(2)}, \alpha^{(2)})$ be the solutions of the system (1.1) corresponding to the supply terms $(f_i^{(1)}, r^{(1)})$ and $(f_i^{(2)}, r^{(2)})$, respectively. Then, the difference denoted by (2.5) satisfies the estimate

$$\begin{aligned} & \int_B \rho \dot{u}_i \dot{u}_i \, dv + \frac{1}{2} \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} \, dv \\ & \leq \frac{3}{2} T^{1/2} N_1 \left[\frac{1 - \left(\frac{16C_1 C_2 T}{\pi} \right)^{m+1}}{1 - \frac{16C_1 C_2 T}{\pi}} \right] \left(\int_0^T \int_B (\rho F_i F_i + \rho q^2) \, dv \, ds \right)^{1/2}, \end{aligned} \quad (4.15)$$

where m is the first natural number such that $m > 8C_1 C_2 T / \pi$.

5 Continuous dependence on initial data

The analysis proposed in Section 4 can be adapted to study the stability with respect to the initial data. Let us assume that we have two solutions $(u_i^{(1)}, \alpha^{(1)})$ and $(u_i^{(2)}, \alpha^{(2)})$ to the homogeneous version ($f_i = 0, r = 0$) of the system (1.1) with the same boundary conditions, but with different initial conditions. Using the notation proposed previously, we denote by (u_i, α) the solution of the homogeneous version of the system (2.6) with homogeneous boundary conditions (2.7) and the initial conditions

$$\begin{aligned} u_i(\mathbf{x}, 0) &= u_i^{(2)}(\mathbf{x}, 0) - u_i^{(1)}(\mathbf{x}, 0) = u_i^*(\mathbf{x}), \\ \dot{u}_i(\mathbf{x}, 0) &= \dot{u}_i^{(2)}(\mathbf{x}, 0) - \dot{u}_i^{(1)}(\mathbf{x}, 0) = v_i^*(\mathbf{x}), \\ \alpha(\mathbf{x}, 0) &= \alpha^{(2)}(\mathbf{x}, 0) - \alpha^{(1)}(\mathbf{x}, 0) = \alpha^*(\mathbf{x}), \\ \dot{\alpha}(\mathbf{x}, 0) &= \dot{\alpha}^{(2)}(\mathbf{x}, 0) - \dot{\alpha}^{(1)}(\mathbf{x}, 0) = \theta^*(\mathbf{x}). \end{aligned} \quad (5.1)$$

In this case, the energy equation gives

$$E(t) = \int_B (\rho \dot{u}_i \dot{u}_i + c \theta^2 + a_{ijkl} u_{i,j} u_{k,l} + b_{ij} \alpha_{,i} \alpha_{,j}) \, dv + 2 \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} \, dv \, ds = E(0), \quad (5.2)$$

where

$$E(0) = \int_B (\rho v_i^* v_i^* + c (\theta^*)^2 + a_{ijkl} u_{i,j}^* u_{k,l}^* + b_{ij} \alpha_{,i}^* \alpha_{,j}^*) \, dv. \quad (5.3)$$

As we consider the homogeneous system, the Lagrange identities argument implies that

$$\begin{aligned} & \int_B (\rho \dot{u}_i \dot{u}_i + b_{ij} \alpha_{,i} \alpha_{,j} - a_{ijkl} u_{i,j} u_{k,l} - c \theta^2) \, dv \\ & = \int_B (\rho v_i^* \dot{u}_i(2t) + b_{ij} \alpha_{,i}^* \alpha_{,j}(2t) - a_{ijkl} u_{i,j}^* u_{k,l}(2t) - c \theta^* \theta(2t)) \, dv. \end{aligned} \quad (5.4)$$

From (5.2) and (5.4), we obtain that

$$\begin{aligned} & \int_B (\rho \dot{u}_i \dot{u}_i + b_{ij} \alpha_{,i} \alpha_{,j}) \, dv + \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} \, dv \, ds \\ & = \frac{E(0)}{2} + \frac{1}{2} \int_B (\rho v_i^* \dot{u}_i(2t) + b_{ij} \alpha_{,i}^* \alpha_{,j}(2t) - a_{ijkl} u_{i,j}^* u_{k,l}(2t) - c \theta^* \theta(2t)) \, dv. \end{aligned} \quad (5.5)$$

We have that

$$\left| \int_B \rho v_i^* \dot{u}_i(2t) dv \right| \leq \left(\int_B \rho v_i^* v_i^* dv \right)^{1/2} \left(\int_B \rho \dot{u}_i(2t) \dot{u}_i(2t) dv \right)^{1/2} \quad (5.6)$$

$$\left| \int_B c \theta^* \theta(2t) dv \right| \leq \left(\int_B c (\theta^*)^2 dv \right)^{1/2} \left(\int_B c \theta^2(2t) dv \right)^{1/2} \quad (5.7)$$

$$\left| \int_B b_{ij} \alpha_{,i}^* \alpha_{,j}(2t) dv \right| \leq C_3 \left(\int_B \alpha_{,i}^* \alpha_{,i}^* dv \right)^{1/2} \left(\int_B \alpha_{,j}(2t) \alpha_{,j}(2t) dv \right)^{1/2} \quad (5.8)$$

$$\left| \int_B a_{ijkh} u_{i,j}^* u_{k,h}(2t) dv \right| \leq C_4 \left(\int_B u_{i,j}^* u_{i,j}^* dv \right)^{1/2} \left(\int_B u_{k,h}(2t) u_{k,h}(2t) dv \right)^{1/2}. \quad (5.9)$$

Here, C_3 and C_4 are two calculable positive constants depending on b_{ij} and a_{ijkh} , respectively.

If we assume that

$$\sup_{t \in [0, T]} \int_B (\rho \dot{u}_i \dot{u}_i + C_4 u_{i,j} u_{i,j} + C_3 \alpha_{,i} \alpha_{,i} + c \theta^2) dv < N_2^2 \quad (5.10)$$

and we take $T \geq 2t$, we see that

$$\int_B (\rho \dot{u}_i \dot{u}_i + b_{ij} \alpha_{,i} \alpha_{,j}) dv + \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds \leq \frac{E(0)}{2} + \frac{1}{2} N_2 T I^*, \quad (5.11)$$

where

$$I^* = \left[\int_B (\rho v_i^* v_i^* + C_3 \alpha_{,i}^* \alpha_{,i}^* + C_4 u_{i,j}^* u_{i,j}^* + c |\theta^*|^2) dv \right]^{1/2}. \quad (5.12)$$

Estimate (5.11) plays a similar role to the estimate (3.12) in the corresponding case.

We note that

$$\begin{aligned} \int_B b_{ij} \alpha_{,i} \alpha_{,j} dv &\leq C_1 \int_B k_{ij} \alpha_{,i} \alpha_{,j} dv \leq 2C_1 \int_B k_{ij} (\alpha_{,i} - \alpha_{,i}^*) (\alpha_{,j} - \alpha_{,j}^*) dv + 2C_1 \int_B k_{ij} \alpha_{,i}^* \alpha_{,j}^* dv \\ &\leq 4C_1 \int_0^t \int_B k_{ij} (\alpha_{,j} - \alpha_{,j}^*) \theta_{,j} dv ds + 2C_1 \int_B k_{ij} \alpha_{,i}^* \alpha_{,j}^* dv \\ &\leq \frac{8C_1 C_2 t}{\pi} \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds + 2C_1 \int_B k_{ij} \alpha_{,i}^* \alpha_{,j}^* dv. \end{aligned} \quad (5.13)$$

We can determine a recurrence similar to (4.2) and (4.3). We define H_n by means of the recurrence

$$H_0 = \frac{E(0)}{2} + 2C_1 \int_B k_{ij} \alpha_{,i}^* \alpha_{,j}^* dv + \frac{1}{2} N_2 T I^* \quad (5.14)$$

and

$$H_{n+1} = \frac{E(0)}{2} + \frac{1}{2} N_2 T I^* + \frac{16C_1 C_2 T}{\pi} H_n. \quad (5.15)$$

If we consider the estimate (5.13) and we choose $t_1 = \frac{\pi}{16C_1 C_2}$, we see that the estimate

$$\int_B \rho \dot{u}_i \dot{u}_i dv + \frac{1}{2} \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds \leq H_0 \quad (5.16)$$

is satisfied whenever $0 < t < t_1 < T/2$.

Let us assume that the estimate

$$\int_B \rho \dot{u}_i \dot{u}_i dv + \frac{1}{2} \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds \leq H_n \quad (5.17)$$

holds whenever $\frac{n\pi}{16C_1C_2} < t \leq \frac{(n+1)\pi}{16C_1C_2} < \frac{T}{2}$. We want to obtain a similar estimate in the interval

$$\frac{(n+1)\pi}{16C_1C_2} < t < \min \left\{ \frac{(n+2)\pi}{16C_1C_2}, \frac{T}{2} \right\}.$$

As in (4.8), it turns out

$$\begin{aligned} & \int_B \rho \dot{u}_i \dot{u}_i dv + \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds \\ & \leq \frac{E(0)}{2} + \frac{1}{2} N_2 T I^* + 2C_1 \int_B k_{ij} (\alpha_{,i}(t) - \alpha_{,i}(t_{n+1})) (\alpha_{,j}(t) - \alpha_{,j}(t_{n+1})) dv \\ & \quad + 2C_1 \int_B k_{ij} \alpha_{,i}(t_{n+1}) \alpha_{,j}(t_{n+1}) dv. \end{aligned} \quad (5.18)$$

Moreover,

$$\begin{aligned} & \int_B k_{ij} (\alpha_{,i}(t) - \alpha_{,i}(t_{n+1})) (\alpha_{,j}(t) - \alpha_{,j}(t_{n+1})) dv \\ & = 2 \int_{t_{n+1}}^t \int_B k_{ij} (\alpha_{,i}(s) - \alpha_{,i}(t_{n+1})) \theta_{,j}(s) dv ds \\ & \leq 2 \left(\int_{t_{n+1}}^t \int_B k_{ij} (\alpha_{,i}(s) - \alpha_{,i}(t_{n+1})) (\alpha_{,j}(s) - \alpha_{,j}(t_{n+1})) dv ds \right)^{1/2} \left(\int_{t_{n+1}}^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds \right)^{1/2} \\ & \leq \frac{4C_2(t_n - t_{n+1})}{\pi} \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds \end{aligned} \quad (5.19)$$

and

$$\int_B k_{ij} \alpha_{,i}(t_{n+1}) \alpha_{,j}(t_{n+1}) dv \leq \frac{2C_2 t_{n+1}}{\pi} \int_B k_{ij} \theta_{,i}(t_{n+1}) \theta_{,j}(t_{n+1}) ds \leq \frac{4C_2 t_{n+1}}{\pi} H_n. \quad (5.20)$$

From (5.18)-(5.20), it follows that

$$\begin{aligned} & \int_B \rho \dot{u}_i \dot{u}_i dv + \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds \\ & \leq \frac{E(0)}{2} + \frac{1}{2} N_2 T I^* + \frac{8C_1 C_2 (t_n - t_{n+1})}{\pi} \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds + \frac{16C_1 C_2 t_{n+1}}{\pi} H_n. \end{aligned} \quad (5.21)$$

So, we get that

$$\int_B \rho \dot{u}_i \dot{u}_i dv + \frac{1}{2} \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} dv ds \leq \frac{E(0)}{2} + \frac{1}{2} N_2 T I^* + \frac{16C_1 C_2 T}{\pi} H_n, \quad (5.22)$$

which is the desired estimate. In a similar way to the argument proposed at the end of Section 4, we can obtain the estimate

$$\begin{aligned} & \int_B \rho \dot{u}_i \dot{u}_i \, dv + \frac{1}{2} \int_0^t \int_B k_{ij} \theta_{,i} \theta_{,j} \, dv \, ds \\ & \leq \left[\frac{1 - \left(\frac{16C_1 C_2 T}{\pi} \right)^{m+1}}{1 - \frac{16C_1 C_2 T}{\pi}} \right] \left(\frac{E(0)}{2} + 2C_1 \int_B k_{ij} \alpha_{,i}^* \alpha_{,j}^* \, dv + \frac{1}{2} N_2 T T^* \right), \end{aligned} \quad (5.23)$$

where m is the first natural number such that $m > \frac{8C_1 C_2 T}{\pi}$. Thus, we conclude the Hölder stability with respect to the initial data.

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