One-dimensional T-preorders

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Abstract. This paper studies T-preorders by using their Representation Theorem that states that every T-preorder on a set X can be generated in a natural way by a family of fuzzy subsets of X. Especial emphasis is made on the study of one-dimensional T-preorders (i.e.: T-preorders that can be generated by only one fuzzy subset). Strong complete T-preorders are characterized.

Keywords. T-preorder, Representation Theorem, generator, dimension, strong complete T-preorder

Introduction

T-preorders were introduced by Zadeh in [8] and are very important fuzzy relations, since they fuzzify the concept of preorder on a set. Although there are many works studying their properties and applications to different fields, starting with [8], [7] [1], authors have not paid much attention to their relationship with the very important Representation Theorem. Roughly speaking the Representation Theorem states that every fuzzy subset μ of a set X generates a T-preorder P_μ on X in a natural way and that every T-preorder can be generated by a family of such special T-preorders.

The Representation Theorem provides us with a method to generate a T-preorder from a family of fuzzy subsets. These fuzzy subsets can measure the degrees in which different features are fulfilled by the elements of a universe X or can be the degrees of compatibility with different prototypes. Reciprocally, from a T-preorder a family (in fact many families) of fuzzy subsets can be obtained providing thus semantics to the relation.

This paper provides some results of T-preorders related to the Representation Theorem. Special attention is paid to one-dimensional T-preorders (i.e.: T-preorders generated by a single fuzzy subset) because they are the bricks out of which T-preorders are built. The fuzzy subsets that generate the same T-preorder are determined (Propositions 2.1, 2.3 and 2.5) and a characterization of one-dimensional T-preorders by the use of Sincov-like functional equations is provided in Propositions 2.12 and 2.13. Also the relation with T-preorders and reciprocal matrices [6] will allow us to find a one-dimensional T-preorder close to a given one as explained in Example 2.17.

A strong complete T-preorder P on a set X is a T-preorder satisfying that for $x, y \in X$, either $P(x, y) = 1$ or $P(y, x) = 1$. These are interesting fuzzy relations used in fuzzy preference structures [4]. It is a direct consequence of Lemma 1.6 that one-dimensional T-preorders are strong complete, but there are strong complete T-preorders that are not one-dimensional. In Section 3 strong complete T-preorders are characterized using the Representation Theorem (Propositions 3.4 and 3.5).

The last section of the paper contains some concluding remarks and an interesting open problem: Which conditions must a couple of fuzzy subsets μ and ν fulfill in order to exist a t-norm T with $P_\mu = P_\nu$. Also the possibility of defining two dimensions (right and left) is discussed.

A section of preliminaries with the results and definitions needed in the rest of the paper follows.

1. Preliminaries

This section contains the main definitions and properties related mainly to T-preorders that will be needed in the rest of the paper.

Definition 1.1. [8] *Let T be a t-norm. A fuzzy T-preorder P on a set X is a fuzzy relation $P : X \times X \to [0, 1]$ satisfying for all $x, y, z \in X$*

- $P(x, x) = 1$ *(Reflexivity)*
- $T(P(x, y), P(y, z)) \leq P(x, z)$ *(T-transitivity).*

Definition 1.2. *The inverse or dual R^{-1} of a fuzzy relation R on a set X is the fuzzy relation on X defined for all $x, y \in X$ by*

$$R^{-1}(x, y) = R(y, x).$$

Proposition 1.3. *A fuzzy relation R on a set X is a T-preorder on X if and only if R^{-1} is a T-preorder on X.*

Definition 1.4. *The residuation $\overrightarrow{T}$ of a t-norm T is defined for all $x, y \in [0, 1]$ by*

$$\overrightarrow{T}(x|y) = \sup\{\alpha \in [0, 1] \text{ such that } T(\alpha, x) \leq y\}.$$

Example 1.5.

1. *If T is a continuous Archimedean t-norm with additive generator t, then $\overrightarrow{T}(x|y) = t^{[-1]}(t(y) - t(x))$ for all $x, y \in [0, 1]$.*
   As special cases,
   - *If T is the Łukasiewicz t-norm, then $\overrightarrow{T}(x|y) = \min(1 - x + y, 0)$ for all $x, y \in [0, 1]$.*
   - *If T is the Product t-norm, then $\overrightarrow{T}(x|y) = \min(\frac{y}{x}, 1)$ for all $x, y \in [0, 1]$.*
2. *If T is the minimum t-norm, then $\overrightarrow{T}(x|y) = \begin{cases} y \text{ if } x > y \\ 1 \text{ otherwise.} \end{cases}$*

Lemma 1.6. *Let μ be a fuzzy subset of X. The fuzzy relation P_μ on X defined for all $x, y \in X$ by*

$$P_\mu(x,y) = \overrightarrow{T}(\mu(x)|\mu(y))$$

is a T-preorder on X.

Theorem 1.7. *Representation Theorem* [7]. *A fuzzy relation R on a set X is a T-preorder on X if and only if there exists a family $(\mu_i)_{i\in I}$ of fuzzy subsets of X such that for all $x, y \in X$*

$$R(x,y) = \inf_{i\in I} R_{\mu_i}(x,y).$$

Definition 1.8. *A family $(\mu_i)_{i\in I}$ in Theorem 1.7 is called a* generating family *of R and an element of a generating family is called a* generator *of R. The minimum of the cardinalities of such families is called the* dimension *of R (dim$R$) and a family with this cardinality a* basis *of R.*

A generating family can be viewed as the degrees of accuracy of the elements of X to a family of prototypes. A family of prototypes with low cardinality, especially a basis, simplifies the computations and gives clarity to the structure of X.

The next proposition states a trivial but important result.

Proposition 1.9. *μ is a generator of R if and only if $R_\mu \geq R$.*

Definition 1.10. *Two continuous t-norms T, T' are isomorphic if and only if there exists a bijective map $\varphi : [0,1] \to [0,1]$ such that $\varphi \circ T = T' \circ (\varphi \times \varphi)$.*

Isomorphisms φ are continuous and increasing maps.

It is well known that all strict continuous Archimedean t-norms T are isomorphic. In particular, they are isomorphic to the Product t-norm and $T(x,y) = \varphi^{-1}(\varphi(x) \cdot \varphi(y))$

Also, all non-strict continuous Archimedean t-norms T are isomorphic. In particular, they are isomorphic to the Łukasiewicz t-norm and $T(x,y) = \varphi^{-1}(\max(\varphi(x) + \varphi(y) - 1), 0)$.

Proposition 1.11. *If T, T' are two isomorphic t-norms, then their residuations $\overrightarrow{T}, \overrightarrow{T'}$ also are isomorphic (i.e. there exists a bijective map $\varphi : [0,1] \to [0,1]$ such that $\varphi \circ \overrightarrow{T} = \overrightarrow{T'} \circ (\varphi \times \varphi)$).*

2. One-dimensional T-preorders

Let us recall that according to Definition 1.8 a T-preorder P on X is one dimensional if and only if there exists a fuzzy subset μ of X such that for all $x, y \in X$, $P(x,y) = \overrightarrow{T}(\mu(x)|\mu(y))$.

2.1. Generators of One-dimensional T-preorders

For a one-dimensional T-preorder P it is interesting to find all fuzzy subsets μ that are a basis of P (i.e.: $P = P_\mu$). The two next propositions answer this question for continuous Archimedean t-norms and for the minimum t-norm.

Proposition 2.1. *Let T be a continuous Archimedean t-norm, t an additive generator of T and μ, ν fuzzy subsets of X. $P_\mu = P_\nu$ if and only if $\forall x \in X$ the following condition holds:*

$$t(\mu(x)) = t(\nu(x)) + k \text{ with } k \geq \sup\{-t(\nu(x))|x \in X\}$$

Moreover, if T is non-strict, then $k \leq \inf\{t(0) - t(\nu(x)) \mid x \in X\}$.

Proof.
$\Rightarrow$)
If $\mu(x) \geq \mu(y)$, then

$$P_\mu(x,y) = \overrightarrow{T}(\mu(x)|\mu(y)) = t^{-1}(t(\mu(y)) - t(\mu(x)))$$
$$P_\nu(x,y) = t^{-1}(t(\nu(y) - t(\nu(x)))$$

where $t^{[-1]}$ is replaced by t^{-1} because all the values in brackets are between 0 and $t(0)$.
If $P_\mu = P_\nu$, then

$$t(\mu(y)) - t(\mu(x)) = t(\nu(y)) - t(\nu(x)).$$

Let us fix $y_0 \in X$.Then

$$t(\mu(y_0)) - t(\mu(x)) = t(\nu(y_0)) - t(\nu(x)).$$

and

$$t(\mu(x)) = t(\nu(x)) + t(\mu(y_0)) - t(\nu(y_0)) = t(\nu(x)) + k$$

$\Leftarrow$) Trivial thanks to Example 1.5.1. $\square$

Example 2.2. *With the previous notations,*

- *If T is the Łukasiewicz t-norm, then*

$$\mu(x) = \nu(x) + k \text{ with } \inf_{x \in X}\{1 - \nu(x)\} \geq k \geq \sup_{x \in X}\{-\nu(x)\}.$$

- *If T is the product t-norm, then*

$$\mu(x) = \frac{\nu(x)}{k} \text{ with } k \geq \sup_{x \in X}\{\nu(x)\}.$$

Proposition 2.3. *Let T be the minimum t-norm, μ a fuzzy subset of X and x_M an element of X with $\mu(x_M) \geq \mu(x)$ $\forall x \in X$. Let $Y \subset X$ be the set of elements x of X with $\mu(x) = \mu(x_M)$ and $s = \sup\{\mu(x) \text{ such that } x \in X - Y\}$. A fuzzy subset ν of X generates the same T-preorder than μ if and only if*

$$\forall x \in X - Y \ \mu(x) = \nu(x) \textit{ and } \nu(y) = t \textit{ with } s \leq t \leq 1 \ \forall y \in Y.$$

Proof. It follows easily from the fact that

$$P_\mu(x,y) = \begin{cases} \mu(y) & \text{if } \mu(x) > \mu(y) \\ 1 & \text{otherwise.} \end{cases}$$

. ∎

At this point, it seems that the dimension of P and of P^{-1} should coincide, but this is not true in general as we will show in the next example. Nevertheless, for continuous Archimedean t-norms they do coincide in most of the cases as will be proved in Proposition 2.5.

Example 2.4. *Consider the one-dimensional* min*-preorder P of $X = \{x_1, x_2, x_3\}$ generated by the fuzzy subset $\mu = (0.8, 0.7, 0.4)$. Its matrix is*

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 \\ 0.7 & 1 & 1 \\ 0.4 & 0.4 & 1 \end{pmatrix} \end{array}$$

while the matrix of P^{-1} is

$$\begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 1 & 0.7 & 0.4 \\ 1 & 1 & 0.4 \\ 1 & 1 & 1 \end{pmatrix} \end{array}$$

which clearly is not one-dimensional.

Proposition 2.5. *Let T be a continuous Archimedean t-norm, t an additive generator of T, μ a fuzzy subset of X and P_μ the T-preorder generated by μ. Then P_μ^{-1} is generated by the fuzzy subset ν of X such that $t(\nu(x)) = -t(\mu(x)) + k$.*

Proof.

$$\begin{aligned} P_\mu^{-1}(x,y) = P_\mu(y,x) &= t^{[-1]}(t(\mu(x)) - t(\mu(y))) \\ &= t^{[-1]}(-t(\mu(y)) + k + t(\mu(x)) - k) \\ &= t^{[-1]}(t(\nu(y)) - t(\nu(x))) = P_\nu(x,y). \end{aligned}$$

∎

Example 2.6.

- *If T it the t-norm of Łukasiewicz, μ a fuzzy subset of X and P the T-preorder on X generated by μ (i.e.: $P = P_\mu$), then P^{-1} is generated by $k - \mu$, with* $\sup_{x\in X}\{\mu(x)\} \le k \le 1 + \inf_{x\in X}\{\mu(x)\}$.
- *If T it the product t-norm, μ a fuzzy subset of X such that $\inf_{x\in X}\{\mu(x)\} > 0$ and P the T-preorder on X generated by μ (i.e.: $P = P_\mu$), then P^{-1} is generated by $\frac{k}{\mu}$, with $0 < k \le \inf_{x\in X}\{\mu(x)\}$.*

Hence, the dimensions of a T-preorder P and its inverse P^{-1} coincide when T is the t-norm of Łukasiewicz (and any other continuous non-strict Archimedean t-norm) while for the product t-norm (and any other continuous strict t-norm) coincide when $\inf_{x,y\in X}\{P(x,y)\} \ne 0$.

2.2. Sincov Functional Equation and AHP

Definition 2.7. [3] *A mapping $F : X \times X \to \mathbb{R}$ satisfies the Sincov functional equation if and only if for all $x, y, z \in X$ we have*

$$F(x,y) + F(y,z) = F(x,z).$$

The following result characterizes the mappings satisfying Sincov equation.

Proposition 2.8. [3] *A mapping $F : X \times X \to \mathbb{R}$ satisfies the Sincov functional equation if and only if there exists a mapping $g : X \to \mathbb{R}$ such that*

$$F(x,y) = g(y) - g(x)$$

for all $x, y \in X$.

Proposition 2.9. *The real line $\mathbb{R}$ with the operation $*$ defined by $x * y = x + y - 1$ for all $x, y \in \mathbb{R}$ is an Abelian group with 1 as the identity element. The opposite of x is $-x + 2$.*

Replacing the addition by this operation $*$ we obtain a Sincov-like functional equation:

Proposition 2.10. *Let $F : X \times X \to \mathbb{R}$ be a mapping. F satisfies the functional equation*

$$F(x,y) * F(y,z) = F(x,z) \tag{1}$$

if and only if there exists a mapping $g : X \to \mathbb{R}$ such that

$$F(x,y) = g(y) - g(x) + 1$$

for all $x, y \in X$.

Proof. The mapping $G(x,y) = F(x,y) - 1$ satisfies the Sincov functional equation and so $G(x,y) = g(y) - g(x)$. □

Replacing the addition by multiplication we obtain another Sincov-like functional equation:

Proposition 2.11. *Let $F : X \times X \to \mathbb{R}^+$ be a mapping. F satisfies the functional equation*

$$F(x, y) \cdot F(y, z) = F(x, z) \tag{2}$$

if and only if there exists a mapping $g : X \to \mathbb{R}^+$ such that

$$F(x, y) = \frac{g(y)}{g(x)}$$

for all $x, y \in X$.

Proof. Simply calculate the logarithm of both hand sides of the functional equation to transform it to Sincov functional equation. □

If μ is a fuzzy subset of X, we can consider μ as a mapping from X to $\mathbb{R}$ or to $\mathbb{R}^+$. This will allow us to characterize one-dimensional T-preorders on X when T is a continuous Archimedean t-norm. For this purpose we will use the isomorphism φ between T and the Łukasiewicz or the Product t-norm.

Proposition 2.12. *Let $T(x, y) = \varphi^{-1}(\max(\varphi(x) + \varphi(y) - 1), 0)$ be a non-strict Archimedean t-norm and X a set. $F : X \times X \to \mathbb{R}$ satisfies equation (1) if and only if $\varphi \circ P = \min(\varphi \circ F, 1)$ is a one-dimensional T-preorder on X.*

Proposition 2.13. *Let $T(x, y) = \varphi^{-1}(\varphi(x) \cdot \varphi(y))$ be a non-strict Archimedean t-norm and X a set. $F : X \times X \to \mathbb{R}^+$ satisfies equation (2) if and only if $\varphi \circ P = \min(\varphi \circ F, 1)$ is a T-preorder on X.*

Definition 2.14. [6] *An $n \times n$ real matrix A with entries $a_{ij} > 0,\ 1 \le i, j \le n$, is reciprocal if and only if $a_{ij} = \frac{1}{a_{ji}}\ \forall i, j = 1, 2, ..., n$. A reciprocal matrix is consistent if and only if $a_{ik} = a_{ij} \cdot a_{jk}\ \forall i, j = 1, 2, ..., n$.*

If the cardinality of X is finite (i.e.: $X = \{x_1, x_2, ..., x_n\}$), then we can associate the matrix $A = (a_{ij})$ with entries $a_{ij} = F(x_i, x_j)$ to every map $F : X \times X \to \mathbb{R}^+$. Then F satisfies equation (2) if and only if A is a reciprocal consistent matrix as defined in [6].

Proposition 2.11 can be rewritten in this context by

Proposition 2.15. [6] *An $n \times n$ real matrix A is reciprocal and consistent if and only if there exists a mapping g of X such that*

$$a_{ij} = \frac{g(x_i)}{g(x_j)}\ \forall i, j = 1, 2, ..., n.$$

For a given reciprocal matrix A, Saaty obtains a consistent matrix A' close to A [6]. A' is generated by an eigenvector associated to the greatest eigenvalue of A and fulfills the following properties.

1. If A is already consistent, then $A = A'$.
2. If A is a reciprocal positive matrix, then the sum of its eigenvalues is n.

3. If A is consistent, then there exist a unique eigenvalue $\lambda_{\max} = n$ different from zero.
4. Slight modifications of the entries of A produce slight changes to the entries of A'.

We will use the third property to obtain a one-dimensional T-preorder close to a given one.

Definition 2.16. [6] *The consistent matrix $A = (a_{ij})$ associated to a T-preorder $P = (p_{ij})$ $i, j = 1, 2, ..., n$ is defined by*

$$a_{ij} = p_{ij} \; if \; p_{ij} \leq p_{ji}$$
$$a_{ij} = \frac{1}{p_{ji}} \; if \; p_{ij} > p_{ji}.$$

Then in order to obtain a one dimensional T-preorder P' close to a given one P (T the Product t-norm), the following procedure can be used:

- Calculate the consistent reciprocal matrix A associated to P.
- Find an eigenvector μ of the greatest eigenvalue of A.
- $P' = P_\mu$.

Example 2.17. *Let T be the Product t-norm and P the T-preorder on a set X of cardinality 5 given by the following matrix.*

$$P = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.74 & 1 & 1 & 1 & 1 \\ 0.67 & 0.87 & 1 & 1 & 1 \\ 0.50 & 0.65 & 0.74 & 1 & 1 \\ 0.41 & 0.53 & 0.60 & 0.80 & 1 \end{pmatrix}.$$

Its associated reciprocal matrix A is

$$A = \begin{pmatrix} 1 & 1.3514 & 1.4925 & 2.0000 & 2.4390 \\ 0.7400 & 1 & 1.1494 & 1.5385 & 1.8868 \\ 0.6700 & 0.8700 & 1 & 1.3514 & 1.6667 \\ 0.5000 & 0.6500 & 0.7400 & 1 & 1.2500 \\ 0.4100 & 0.5300 & 0.6000 & 0.8000 & 1 \end{pmatrix}.$$

Its greatest eigenvalue is 5.0003 and an eigenvector for 5.0003 is

$$\mu = (1, 0.76, 0.67, 0.50, 0.40).$$

This fuzzy set generates P_μ which is a one dimensional T-preorder close to P.

$$P_\mu = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0.76 & 1 & 1 & 1 & 1 \\ 0.67 & 0.88 & 1 & 1 & 1 \\ 0.50 & 0.66 & 0.74 & 1 & 1 \\ 0.40 & 0.53 & 0.60 & 0.81 & 1 \end{pmatrix}.$$

The results of this section can be easily generalized to continuous strict Archimedean t-norms.

If T' is a continuous strict Archimedean t-norm, then it is isomorphic to the Product t-norm T. Let φ be this isomorphism. If P is a T'-preorder, then $\varphi \circ P$ is a T-preorder. We can find P' one-dimensional close to $\varphi \circ P$ as before. Since isomorphisms between continuous t-norms are continuous and preserve dimensions, $\varphi^{-1} \circ P'$ is a one-dimensional T'-preorder close to P.

3. Strong Complete T-preorders

Definition 3.1. [4] *A T-preorder P on a set X is a strong complete T-preorder if and only if for all $x, y \in X$,*

$$\max(P(x,y), P(y,x)) = 1.$$

Of course every one-dimensional fuzzy T-preorder is a strong complete T-preorder, but there are strong complete T-preorders that are not one-dimensional. In Propositions 3.3 and 3.5 these fuzzy relations will be characterized exploiting the fact that they generate crisp linear orderings.

Lemma 3.2. *Let μ be a generator of a strong complete T-preorder P on X. If $P(x,y) = 1$, then $\mu(x) \leq \mu(y)$.*

Proof. Trivial, since $\overrightarrow{T}(\mu(x)|\mu(y)) = P_\mu(x,y) \geq P(x,y) = 1$. □

Proposition 3.3. *Let μ, ν be two generators of a strong complete T-preorder P on X. Then for all $x, y \in X$, $\mu(x) \leq \mu(y)$ if and only if $\nu(x) \leq \nu(y)$.*

Proof. Given $x, y \in X$, $x \neq y$, let us suppose that $P(x,y) = 1$ (and $P(y,x) < 1$). Then $\mu(x) \leq \mu(y)$ and $\nu(x) \leq \nu(y)$. □

Proposition 3.4. *Let P be a strong complete T-preorder on a set X. The elements of X can be totally ordered in such a way that if $x \leq y$, then $P(x,y) = 1$.*

Proof. Consider the relation $\leq$ on X defined by $x \leq y$ if and only if $\mu(x) \leq \mu(y)$ for any generator μ of P. (If for $x \neq y$, $\mu(x) = \mu(y)$ for any generator, then chose either $x < y$ or $y < x$). □

Reciprocally,

Proposition 3.5. *If for any couple of generators μ and ν of a T-preorder P on a set X $\mu(x) \leq \mu(y)$ if and only if $\nu(x) \leq \nu(y)$, then P is strong complete.*

Proof. Trivial. □

4. Concluding Remarks

T-preorders have been studied with the help of its Representation Theorem. The different fuzzy subsets generating the same T-preorder have been characterized and the relation between one-dimensional T-preorders, Sincov-like functional equations and Saaty's reciprocal matrices has been studied. Also strong complete T-preorders have been characterized.

We end pointing at two directions toward a future work.

- We can look at the results of the Subsection 2.1 from a different point of view: Let us suppose that we obtain two different fuzzy subsets μ and ν of a universe X by two different measurements or by two different experts. It would be interesting to know in which conditions we could assure the existence of a (continuous Archimedean) t-norm for which $P_\mu = P_\nu$.
- A fuzzy subset μ of a set X generates a T-preorder P_μ by $P_\mu(x, y) = \overrightarrow{T}(\mu(x)|\mu(y))$, but also another T-preorder ${}_\mu P(x, y) = \overrightarrow{T}(\mu(y)|\mu(x))$ (in fact, the inverse of P_μ). In this way we could define two dimensions of a T-preorder according weather we consider it generated by families $(P_{\mu_i})_{i\in I}$ or by families $({}_{\mu_i}P)_{i\in I}$. For instance the $\min$-preorder of Example 2.4 would have right dimension 1 and left dimension 2.

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