One-dimensional $T$-preorders

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Abstract. This paper studies $T$-preorders by using their Representation Theorem that states that every $T$-preorder on a set $X$ can be generated in a natural way by a family of fuzzy subsets of $X$. Especial emphasis is made on the study of one-dimensional $T$-preorders (i.e.: $T$-preorders that can be generated by only one fuzzy subset). Strong complete $T$-preorders are characterized.

Keywords. $T$-preorder, Representation Theorem, generator, dimension, strong complete $T$-preorder

Introduction

$T$-preorders were introduced by Zadeh in [8] and are very important fuzzy relations, since they fuzzify the concept of preorder on a set. Although there are many works studying their properties and applications to different fields, starting with [8], [7] [1], authors have not paid much attention to their relationship with the very important Representation Theorem. Roughly speaking the Representation Theorem states that every fuzzy subset $\mu$ of a set $X$ generates a $T$-preorder $P_\mu$ on $X$ in a natural way and that every $T$-preorder can be generated by a family of such special $T$-preorders.

The Representation Theorem provides us with a method to generate a $T$-preorder from a family of fuzzy subsets. These fuzzy subsets can measure the degrees in which different features are fulfilled by the elements of a universe $X$ or can be the degrees of compatibility with different prototypes. Reciprocally, from a $T$-preorder a family (in fact many families) of fuzzy subsets can be obtained providing thus semantics to the relation.

This paper provides some results of $T$-preorders related to the Representation Theorem. Special attention is paid to one-dimensional $T$-preorders (i.e.: $T$-preorders generated by a single fuzzy subset) because they are the bricks out of which $T$-preorders are built. The fuzzy subsets that generate the same $T$-preorder are determined (Propositions 2.1, 2.3 and 2.5) and a characterization of one-dimensional $T$-preorders by the use of Sincov-like functional equations is provided in Propositions 2.12 and 2.13. Also the relation with $T$-preorders and reciprocal matrices [6] will allow us to find a one-dimensional $T$-preorder close to a given one as explained in Example 2.17.
A strong complete \( T \)-preorder \( P \) on a set \( X \) is a \( T \)-preorder satisfying that for \( x, y \in X \), either \( P(x, y) = 1 \) or \( P(y, x) = 1 \). These are interesting fuzzy relations used in fuzzy preference structures [4]. It is a direct consequence of Lemma 1.6 that one-dimensional \( T \)-preorders are strong complete, but there are strong complete \( T \)-preorders that are not one-dimensional. In Section 3 strong complete \( T \)-preorders are characterized using the Representation Theorem (Propositions 3.4 and 3.5).

The last section of the paper contains some concluding remarks and an interesting open problem: Which conditions must a couple of fuzzy subsets \( \mu \) and \( \nu \) fulfill in order to exist a \( t \)-norm \( T \) with \( P_\mu = P_\nu \). Also the possibility of defining two dimensions (right and left) is discussed.

A section of preliminaries with the results and definitions needed in the rest of the paper follows.

1. Preliminaries

This section contains the main definitions and properties related mainly to \( T \)-preorders that will be needed in the rest of the paper.

**Definition 1.1.** [8] Let \( T \) be a \( t \)-norm. A fuzzy \( T \)-preorder on a set \( X \) is a fuzzy relation \( P: X \times X \to [0, 1] \) satisfying for all \( x, y, z \in X \)

- \( P(x, x) = 1 \) (Reflexivity)
- \( T(P(x, y), P(y, z)) \leq P(x, z) \) (\( T \)-transitivity).

**Definition 1.2.** The inverse or dual \( R^{-1} \) of a fuzzy relation \( R \) on a set \( X \) is the fuzzy relation on \( X \) defined for all \( x, y \in X \) by

\[
R^{-1}(x, y) = R(y, x).
\]

**Proposition 1.3.** A fuzzy relation \( R \) on a set \( X \) is a \( T \)-preorder on \( X \) if and only if \( R^{-1} \) is a \( T \)-preorder on \( X \).

**Definition 1.4.** The residuation \( \overrightarrow{T} \) of a \( t \)-norm \( T \) is defined for all \( x, y \in [0, 1] \) by

\[
\overrightarrow{T}(x|y) = \sup\{\alpha \in [0, 1] \text{ such that } T(\alpha, x) \leq y\}.
\]

**Example 1.5.**

1. If \( T \) is a continuous Archimedean \( t \)-norm with additive generator \( t \), then

\[
\overrightarrow{T}(x|y) = t([-1](t(y) - t(x))) \text{ for all } x, y \in [0, 1].
\]

As special cases,

- If \( T \) is the \( \text{Łukasiewicz} \) \( t \)-norm, then \( \overrightarrow{T}(x|y) = \min(1 - x + y, 0) \) for all \( x, y \in [0, 1] \).
- If \( T \) is the Product \( t \)-norm, then \( \overrightarrow{T}(x|y) = \min(\frac{y}{x}, 1) \) for all \( x, y \in [0, 1] \).

2. If \( T \) is the minimum \( t \)-norm, then \( \overrightarrow{T}(x|y) = \begin{cases} y & \text{if } x > y \\ 1 & \text{otherwise.} \end{cases} \)
Lemma 1.6. Let $\mu$ be a fuzzy subset of $X$. The fuzzy relation $P_\mu$ on $X$ defined for all $x, y \in X$ by

$$P_\mu(x, y) = \overrightarrow{T}(\mu(x)|\mu(y))$$

is a $T$-preorder on $X$.

Theorem 1.7. Representation Theorem [7]. A fuzzy relation $R$ on a set $X$ is a $T$-preorder on $X$ if and only if there exists a family $(\mu_i)_{i \in I}$ of fuzzy subsets of $X$ such that for all $x, y \in X$

$$R(x, y) = \inf_{i \in I} R_{\mu_i}(x, y).$$

Definition 1.8. A family $(\mu_i)_{i \in I}$ in Theorem 1.7 is called a generating family of $R$ and an element of a generating family is called a generator of $R$. The minimum of the cardinalities of such families is called the dimension of $R$ ($\dim R$) and a family with this cardinality a basis of $R$.

A generating family can be viewed as the degrees of accuracy of the elements of $X$ to a family of prototypes. A family of prototypes with low cardinality, especially a basis, simplifies the computations and gives clarity to the structure of $X$.

The next proposition states a trivial but important result.

Proposition 1.9. $\mu$ is a generator of $R$ if and only if $R_\mu \geq R$.

Definition 1.10. Two continuous t-norms $T, T'$ are isomorphic if and only if there exists a bijective map $\varphi : [0, 1] \to [0, 1]$ such that $\varphi \circ T = T' \circ (\varphi \times \varphi)$.

Isomorphisms $\varphi$ are continuous and increasing maps.

It is well known that all strict continuous Archimedean t-norms $T$ are isomorphic. In particular, they are isomorphic to the Product t-norm and $T(x, y) = \varphi^{-1}(\varphi(x) \cdot \varphi(y))$.

Also, all non-strict continuous Archimedean t-norms $T$ are isomorphic. In particular, they are isomorphic to the Łukasiewicz t-norm and $T(x, y) = \varphi^{-1}(\max(\varphi(x) + \varphi(y) - 1), 0)$.

Proposition 1.11. If $T, T'$ are two isomorphic t-norms, then their residuations $\overrightarrow{T}, \overrightarrow{T'}$ also are isomorphic (i.e. there exists a bijective map $\varphi : [0, 1] \to [0, 1]$ such that $\varphi \circ \overrightarrow{T} = \overrightarrow{T'} \circ (\varphi \times \varphi)$).

2. One-dimensional $T$-preorders

Let us recall that according to Definition 1.8 a $T$-preorder $P$ on $X$ is one dimensional if and only if there exists a fuzzy subset $\mu$ of $X$ such that for all $x, y \in X$, $P(x, y) = \overrightarrow{T}(\mu(x)|\mu(y))$. 
2.1. Generators of One-dimensional $T$-preorders

For a one-dimensional $T$-preorder $P$ it is interesting to find all fuzzy subsets $\mu$ that are a basis of $P$ (i.e.: $P = P_\mu$). The two next propositions answer this question for continuous Archimedean t-norms and for the minimum t-norm.

**Proposition 2.1.** Let $T$ be a continuous Archimedean t-norm, $t$ an additive generator of $T$ and $\mu$, $\nu$ fuzzy subsets of $X$. $P_\mu = P_\nu$ if and only if $\forall x \in X$ the following condition holds:

$$t(\mu(x)) = t(\nu(x)) + k$$

with $k \geq \sup \{-t(\nu(x))|x \in X\}$.

Moreover, if $T$ is non-strict, then $k \leq \inf \{t(0) - t(\nu(x))|x \in X\}$.

**Proof.**

$\Rightarrow$) If $\mu(x) \geq \mu(y)$, then

$$P_\mu(x,y) = \overline{T}(\mu(x)|\mu(y)) = t^{-1}(t(\mu(y)) - t(\mu(x)))$$

$$P_\nu(x,y) = t^{-1}(t(\nu(y)) - t(\nu(x)))$$

where $t^{-1}$ is replaced by $t^{-1}$ because all the values in brackets are between 0 and $t(0)$.

If $P_\mu = P_\nu$, then

$$t(\mu(y)) = t(\nu(y)) - t(\nu(x)).$$

Let us fix $y_0 \in X$. Then

$$t(\mu(y_0)) = t(\nu(y_0)) - t(\nu(x)).$$

and

$$t(\mu(x)) = t(\nu(x)) + t(\mu(y_0)) - t(\nu(y_0)) = t(\nu(x)) + k$$

$\Leftarrow$) Trivial thanks to Example 1.5.1.

**Example 2.2.** With the previous notations,

- If $T$ is the Łukasiewicz t-norm, then

  $$\mu(x) = \nu(x) + k$$

  with $\inf \{1 - \nu(x)\} \geq k \geq \sup \{-\nu(x)\}$.

- If $T$ is the product t-norm, then

  $$\mu(x) = \frac{\nu(x)}{k}$$

  with $k \geq \sup \{\nu(x)\}$. 

\[\square\]
Proposition 2.3. Let $T$ be the minimum $t$-norm, $\mu$ a fuzzy subset of $X$ and $x_M$ an element of $X$ with $\mu(x_M) \geq \mu(x) \forall x \in X$. Let $Y \subset X$ be the set of elements $x$ of $X$ with $\mu(x) = \mu(x_M)$ and $s = \sup \{\mu(x) \text{ such that } x \in X - Y\}$. A fuzzy subset $\nu$ of $X$ generates the same $T$-preorder than $\mu$ if and only if

$$\forall x \in X - Y \mu(x) = \nu(x) \text{ and } \nu(y) = t \text{ with } s \leq t \leq 1 \forall y \in Y.$$ 

Proof. It follows easily from the fact that $P_{\mu}(x, y) = \begin{cases} \mu(y) \text{ if } \mu(x) > \mu(y) \\ 1 \text{ otherwise.} \end{cases}$.

At this point, it seems that the dimension of $P$ and of $P^{-1}$ should coincide, but this is not true in general as we will show in the next example. Nevertheless, for continuous Archimedean $t$-norms they do coincide in most of the cases as will be proved in Proposition 2.5.

Example 2.4. Consider the one-dimensional min-preorder $P$ of $X = \{x_1, x_2, x_3\}$ generated by the fuzzy subset $\mu = (0.8, 0.7, 0.4)$. Its matrix is

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & 1 & 1 & 1 \\ x_2 & 0.7 & 1 & 1 \\ x_3 & 0.4 & 0.4 & 1 \end{pmatrix}$$

while the matrix of $P^{-1}$ is

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_1 & 1 & 0.7 & 0.4 \\ x_2 & 1 & 1 & 0.4 \\ x_3 & 1 & 1 & 1 \end{pmatrix}$$

which clearly is not one-dimensional.

Proposition 2.5. Let $T$ be a continuous Archimedean $t$-norm, $t$ an additive generator of $T$, $\mu$ a fuzzy subset of $X$ and $P_{\mu}$ the $T$-preorder generated by $\mu$. Then $P_{\mu}^{-1}$ is generated by the fuzzy subset $\nu$ of $X$ such that $t(\nu(x)) = -t(\mu(x)) + k$.

Proof.

$$P_{\mu}^{-1}(x, y) = P_{\mu}(y, x) = t^{-1}(t(\mu(x)) - t(\mu(y))) = t^{-1}(-t(\mu(y)) + k + t(\mu(x)) - k) = t^{-1}(t(\nu(y)) - t(\nu(x))) = P_{\nu}(x, y).$$

Example 2.6.
• If $T$ is the $t$-norm of Łukasiewicz, $\mu$ a fuzzy subset of $X$ and $P$ the $T$-preorder on $X$ generated by $\mu$ (i.e.: $P = P_\mu$), then $P^{-1}$ is generated by $k - \mu$, with $\sup_{x \in X} \{\mu(x)\} \leq k \leq 1 + \inf_{x \in X} \{\mu(x)\}$.

• If $T$ is the product $t$-norm, $\mu$ a fuzzy subset of $X$ such that $\inf_{x \in X} \{\mu(x)\} > 0$ and $P$ the $T$-preorder on $X$ generated by $\mu$ (i.e.: $P = P_\mu$), then $P^{-1}$ is generated by $k \mu$, with $0 < k \leq \inf_{x \in X} \{\mu(x)\}$.

Hence, the dimensions of a $T$-preorder $P$ and its inverse $P^{-1}$ coincide when $T$ is the $t$-norm of Łukasiewicz (and any other continuous non-strict Archimedean $t$-norm) while for the product $t$-norm (and any other continuous strict $t$-norm) coincide when $\inf_{x,y \in X} \{P(x,y)\} \neq 0$.

2.2. Sincov Functional Equation and AHP

**Definition 2.7.** [3] A mapping $F : X \times X \rightarrow \mathbb{R}$ satisfies the Sincov functional equation if and only if for all $x, y, z \in X$ we have

$$F(x, y) + F(y, z) = F(x, z).$$

The following result characterizes the mappings satisfying Sincov equation.

**Proposition 2.8.** [3] A mapping $F : X \times X \rightarrow \mathbb{R}$ satisfies the Sincov functional equation if and only if there exists a mapping $g : X \rightarrow \mathbb{R}$ such that

$$F(x, y) = g(y) - g(x)$$

for all $x, y \in X$.

**Proposition 2.9.** The real line $\mathbb{R}$ with the operation $*$ defined by $x * y = x + y - 1$ for all $x, y \in \mathbb{R}$ is an Abelian group with 1 as the identity element. The opposite of $x$ is $-x + 2$.

Replacing the addition by this operation $*$ we obtain a Sincov-like functional equation:

**Proposition 2.10.** Let $F : X \times X \rightarrow \mathbb{R}$ be a mapping. $F$ satisfies the functional equation

$$F(x, y) * F(y, z) = F(x, z)$$

if and only if there exists a mapping $g : X \rightarrow \mathbb{R}$ such that

$$F(x, y) = g(y) - g(x) + 1$$

for all $x, y \in X$.

**Proof.** The mapping $G(x, y) = F(x, y) - 1$ satisfies the Sincov functional equation and so $G(x, y) = g(y) - g(x)$. \qed

Replacing the addition by multiplication we obtain another Sincov-like functional equation:
Proposition 2.11. Let $F : X \times X \to \mathbb{R}^+$ be a mapping. $F$ satisfies the functional equation

$$F(x, y) \cdot F(y, z) = F(x, z)$$  \hspace{1cm} (2)

if and only if there exists a mapping $g : X \to \mathbb{R}^+$ such that

$$F(x, y) = \frac{g(y)}{g(x)}$$

for all $x, y \in X$.

Proof. Simply calculate the logarithm of both hand sides of the functional equation to transform it to Sincov functional equation.

If $\mu$ is a fuzzy subset of $X$, we can consider $\mu$ as a mapping from $X$ to $\mathbb{R}$ or to $\mathbb{R}^+$. This will allow us to characterize one-dimensional $T$-preorders on $X$ when $T$ is a continuous Archimedean t-norm. For this purpose we will use the isomorphism $\varphi$ between $T$ and the Łukasiewicz or the Product t-norm.

Proposition 2.12. Let $T(x, y) = \varphi^{-1}(\max(\varphi(x) + \varphi(y) - 1), 0)$ be a non-strict Archimedean t-norm and $X$ a set. $F : X \times X \to \mathbb{R}$ satisfies equation (1) if and only if $\varphi \circ P = \min(\varphi \circ F, 1)$ is a one-dimensional $T$-preorder on $X$.

Proposition 2.13. Let $T(x, y) = \varphi^{-1}(\varphi(x) \cdot \varphi(y))$ be a non-strict Archimedean t-norm and $X$ a set. $F : X \times X \to \mathbb{R}^+$ satisfies equation (2) if and only if $\varphi \circ P = \min(\varphi \circ F, 1)$ is a $T$-preorder on $X$.

Definition 2.14. [6] An $n \times n$ real matrix $A$ with entries $a_{ij} > 0$, $1 \leq i, j \leq n$, is reciprocal if and only if $a_{ij} = \frac{1}{a_{ji}} \forall i, j = 1, 2, ..., n$. A reciprocal matrix is consistent if and only if $a_{ik} = a_{ij} \cdot a_{jk} \forall i, j = 1, 2, ..., n$.

If the cardinality of $X$ is finite (i.e.: $X = \{x_1, x_2, ..., x_n\}$), then we can associate the matrix $A = (a_{ij})$ with entries $a_{ij} = F(x_i, x_j)$ to every map $F : X \times X \to \mathbb{R}^+$. Then $F$ satisfies equation (2) if and only if $A$ is a reciprocal consistent matrix as defined in [6].

Proposition 2.11 can be rewritten in this context by

Proposition 2.15. [6] An $n \times n$ real matrix $A$ is reciprocal and consistent if and only if there exists a mapping $g$ of $X$ such that

$$a_{ij} = \frac{g(x_i)}{g(x_j)} \forall i, j = 1, 2, ..., n.$$ 

For a given reciprocal matrix $A$, Saaty obtains a consistent matrix $A'$ close to $A$ [6]. $A'$ is generated by an eigenvector associated to the greatest eigenvalue of $A$ and fulfills the following properties.

1. If $A$ is already consistent, then $A = A'$.
2. If $A$ is a reciprocal positive matrix, then the sum of its eigenvalues is $n$. 
3. If $A$ is consistent, then there exist a unique eigenvalue $\lambda_{\text{max}} = n$ different from zero.

4. Slight modifications of the entries of $A$ produce slight changes to the entries of $A'$.

We will use the third property to obtain a one-dimensional $T$-preorder close to a given one.

**Definition 2.16.** [6] The consistent matrix $A = (a_{ij})$ associated to a $T$-preorder $P = (p_{ij})$, $i, j = 1, 2, ..., n$ is defined by

\[ a_{ij} = p_{ij} \text{ if } p_{ij} \leq p_{ji} \]

\[ a_{ij} = \frac{1}{p_{ji}} \text{ if } p_{ij} > p_{ji}. \]

Then in order to obtain a one dimensional $T$-preorder $P'$ close to a given one $P$ ($T$ the Product t-norm), the following procedure can be used:

- Calculate the consistent reciprocal matrix $A$ associated to $P$.
- Find an eigenvector $\mu$ of the greatest eigenvalue of $A$.
- $P' = P_\mu$.

**Example 2.17.** Let $T$ be the Product t-norm and $P$ the $T$-preorder on a set $X$ of cardinality 5 given by the following matrix.

\[
P = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0.74 & 1 & 1 & 1 & 1 \\
0.67 & 0.87 & 1 & 1 & 1 \\
0.50 & 0.65 & 0.74 & 1 & 1 \\
0.41 & 0.53 & 0.60 & 0.80 & 1
\end{pmatrix}.
\]

Its associated reciprocal matrix $A$ is

\[
A = \begin{pmatrix}
1 & 1.3514 & 1.4925 & 2.0000 & 2.4390 \\
0.7400 & 1 & 1.1494 & 1.5385 & 1.8888 \\
0.6700 & 0.8700 & 1 & 1.3514 & 1.6667 \\
0.5000 & 0.6500 & 0.7400 & 1 & 1.2500 \\
0.4100 & 0.5300 & 0.6000 & 0.8000 & 1
\end{pmatrix}.
\]

Its greatest eigenvalue is 5.0003 and an eigenvector for 5.0003 is

\[ \mu = (1, 0.76, 0.67, 0.50, 0.40). \]

This fuzzy set generates $P_\mu$, which is a one dimensional $T$-preorder close to $P$.

\[
P_\mu = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
0.76 & 1 & 1 & 1 & 1 \\
0.67 & 0.88 & 1 & 1 & 1 \\
0.50 & 0.66 & 0.74 & 1 & 1 \\
0.40 & 0.53 & 0.60 & 0.81 & 1
\end{pmatrix}.
\]
The results of this section can be easily generalized to continuous strict Archimedean t-norms. If $T'$ is a continuous strict Archimedean t-norm, then it is isomorphic to the Product t-norm $T$. Let $\varphi$ be this isomorphism. If $P$ is a $T'$-preorder, then $\varphi \circ P$ is a $T$-preorder. We can find $P'$ one-dimensional close to $\varphi \circ P$ as before. Since isomorphisms between continuous t-norms are continuous and preserve dimensions, $\varphi^{-1} \circ P'$ is a one-dimensional $T'$-preorder close to $P$.

3. Strong Complete $T$-preorders

**Definition 3.1.** [4] A $T$-preorder $P$ on a set $X$ is a strong complete $T$-preorder if and only if for all $x, y \in X$,

$$\max(P(x, y), P(y, x)) = 1.$$ 

Of course every one-dimensional fuzzy $T$-preorder is a strong complete $T$-preorder, but there are strong complete $T$-preorders that are not one-dimensional. In Propositions 3.3 and 3.5 these fuzzy relations will be characterized exploiting the fact that they generate crisp linear orderings.

**Lemma 3.2.** Let $\mu$ be a generator of a strong complete $T$-preorder $P$ on $X$. If $P(x, y) = 1$, then $\mu(x) \leq \mu(y)$.

**Proof.** Trivial, since $T(\mu(x) | \mu(y)) = P(x, y) \geq P(x, y) = 1$. \hfill $\square$

**Proposition 3.3.** Let $\mu, \nu$ be two generators of a strong complete $T$-preorder $P$ on $X$. Then for all $x, y \in X$, $\mu(x) \leq \mu(y)$ if and only if $\nu(x) \leq \nu(y)$.

**Proof.** Given $x, y \in X$, $x \neq y$, let us suppose that $P(x, y) = 1$ (and $P(y, x) < 1$). Then $\mu(x) \leq \mu(y)$ and $\nu(x) \leq \nu(y)$. \hfill $\square$

**Proposition 3.4.** Let $P$ be a strong complete $T$-preorder on a set $X$. The elements of $X$ can be totally ordered in such a way that if $x \leq y$, then $P(x, y) = 1$.

**Proof.** Consider the relation $\leq$ on $X$ defined by $x \leq y$ if and only if $\mu(x) \leq \mu(y)$ for any generator $\mu$ of $P$. (If for $x \neq y$, $\mu(x) = \mu(y)$ for any generator, then chose either $x < y$ or $y < x$). \hfill $\square$

Reciprocally,

**Proposition 3.5.** If for any couple of generators $\mu$ and $\nu$ of a $T$-preorder $P$ on a set $X$ $\mu(x) \leq \mu(y)$ if and only if $\nu(x) \leq \nu(y)$, then $P$ is strong complete.

**Proof.** Trivial. \hfill $\square$
4. Concluding Remarks

$T$-preorders have been studied with the help of its Representation Theorem. The different fuzzy subsets generating the same $T$-preorder have been characterized and the relation between one-dimensional $T$-preorders, Sincov-like functional equations and Saaty’s reciprocal matrices has been studied. Also strong complete $T$-preorders have been characterized.

We end pointing at two directions toward a future work.

- We can look at the results of the Subsection 2.1 from a different point of view: Let us suppose that we obtain two different fuzzy subsets $\mu$ and $\nu$ of a universe $X$ by two different measurements or by two different experts. It would be interesting to know in which conditions we could assure the existence of a (continuous Archimedean) $t$-norm for which $P_{\mu} = P_{\nu}$.

- A fuzzy subset $\mu$ of a set $X$ generates a $T$-preorder $P_{\mu}$ by $P_{\mu}(x, y) = \overrightarrow{T}(\mu(x)|\mu(y))$, but also another $T$-preorder $\mu P(x, y) = \overrightarrow{T}(\mu(y)|\mu(x))$ (in fact, the inverse of $P_{\mu}$). In this way we could define two dimensions of a $T$-preorder according weather we consider it generated by families $(P_{\mu_i})_{i \in I}$ or by families $(\mu_i, P)^{i \in I}$. For instance the min-preorder of Example 2.4 would have right dimension 1 and left dimension 2.

References