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On the ascending subgraph decomposition problem for bipartite graphs

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Abstract

The Ascending Subgraph Decomposition (ASD) Conjecture asserts that every graph G with $\binom{n+1}{2}$ edges admits an edge decomposition $G = H_1 \oplus \cdots \oplus H_n$ such that H_i has i edges and is isomorphic to a subgraph of H_{i+1}, $i = 1, \ldots, n-1$. We show that every bipartite graph G with $\binom{n+1}{2}$ edges such that the degree sequence $d_1, \ldots, d_k$ of one of the stable sets satisfies $d_i \geq n - i + 2$, $1 \leq i < k$, admits an ascending subgraph decomposition with star forests. We also give a necessary condition on the degree sequence which is not far from the above sufficient one.

Keywords: Ascending subgraph deocmposition, Sumset partition problem.

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1 Introduction

A graph G with $\binom{n+1}{2}$ edges has an Ascending Subgraph Decomposition (ASD) if it admits an edge–decomposition $G = H_1 \oplus \cdots \oplus H_n$ such that H_i has i edges and is isomorphic to a subgraph of H_{i+1}, $1 \leq i < n$. It was conjectured by Alavi, Boals, Chartrand, Erdős and Oellerman [1] that every graph of size $\binom{n+1}{2}$ admits an ASD. The conjecture has been proved for a number of particular cases, including forests [5], regular graphs [9], complete multipartite graphs [8] or graphs with maximum degree $\Delta \leq n/2$ [6].

In the same paper Alavi et al. [1] conjectured that every star forest of size $\binom{n+1}{2}$ in which each connected component has size at least n admits an ASD in which every graph in the decomposition is a star. This conjecture was proved by Ma, Zhou and Zhou [13], and the condition was later on weakened to the effect that the second smaller component of the star forest has size at least n by Chen, Fu , Wang and Zhou [4]. The above two results are connected to the Sumset Partition Problem (SPP): given an integer sequence $d_1 \geq \cdots \geq d_k > 0$ such that $\sum_i d_i = \binom{n+1}{2}$, the SPP asks for a partition $\{X_1, \ldots, X_k\}$ of the integer interval $[1, n]$ such that the sum of the elements in X_i is precisely d_i. If the answer is positive we say that the sequence $d_1, \ldots, d_k$ is n*–realizable*. The result of Chen et al. [4] states that every sequence with $\sum_i d_i = \binom{n+1}{2}$ and $d_{k-1} \geq n$ is n–realizable.

This result can also be reformulated in terms of ASD of bipartite graphs. Let G be a bipartite graph with bicoloration $\{A, B\}$ and size $\binom{n+1}{2}$. Suppose that the degree sequence $d_1 \geq \cdots \geq d_k$ of the vertices in A is n–realizable. Then G admits a star ASD. This motivates the study of ASD for bipartite graphs in terms of the degree sequence of one of the stable sets, which is the purpose of this paper.

The condition $d_{k-1} \geq n$ given by Chen et al. [4] for a sequence to be n–realizable can not be weakened in the sense that the result fails to be true for sequences with $d_{k-1} = d_k = 1$ or $d_{k-1} = d_k = 2$, say. The two examples above belong to a family of natural obstructions for a sequence to be n–realizable. A sequence $a_1 \geq \cdots \geq a_t$ is a *forbidden sequence* if there is no family of pairwise disjoint subsets $Y_1, \ldots, Y_t$ of the integer interval $[1, a_1]$ such that $\sum_{x \in Y_1} = a_1, \ldots, \sum_{x \in Y_t} = a_t$. Clearly, a sequence $d_1, \ldots, d_k$ which contains a forbidden sequence can not be n–realizable. The main result in [12] states that, for n large enough, a sequence $d_1 \geq \cdots \geq d_k$ with $\sum_i d_i = \binom{n+1}{2}$ is n–realizable if and only if its subsequence of terms smaller than n does not contain a forbidden subsequence. The full characterization of forbidden sequences seems to be a difficult problem. In this paper we give some sufficient conditions which ensure the existence of a star ASD of bipartite graphs in terms of the degree sequence.

Theorem 1.1 *Let G be a bipartite graph with $\binom{n+1}{2}$ edges. Let $d_1 \geq \cdots \geq d_k$ be the degree sequence of the vertices in one stable set of G. If $k \leq (n+1)/4$ and $d_k \geq 4k$ then G admits a star ASD.* □

Most of the results on the ASD conjecture use families of graphs for which the isomorphic contention is easily checked, like matchings or stars. The main contribution of the paper is to study the conjecture for bipartite graphs in terms of the dfegree sequence of one of the stable sets by using star forests as building blocks. Our main result is the following one.

Theorem 1.2 *Let G be a bipartite graph with $\binom{n+1}{2}$ edges. Let $d_1 \geq d_2 \geq \cdots \geq d_k$ be the degree sequence of one of the stable sets of G. If $d_i \geq n-i+2$ then there is a star forest ASD of G.* □

The proof of Theorem 1.2 is based in a reduction lemma, which uses a deep result of Häggkvist [10] on edge–colorings, and on results on the Sumset Partition Problem [12]. The sufficient condition in Theorem 1.2 is not far from necessary.

Lemma 1.3 *Let G be a bipartite graph with $\binom{n+1}{2}$ edges. Let $d_1 \geq d_2 \geq \cdots \geq d_k$ be the degree sequence of the stable set A of G. If G admits a star forest ASD with the centers of the stars in A then $\sum_{i=1}^{t} d_i \geq \sum_{i=1}^{t}(n-i+1)$, $t = 1, \ldots, k$.*

The paper is organized as follows. We first consider star ASD in section 2, where we recall some results derived from the Sumset Partition Problem and prove Theorem 1.1. Section 3 contains a reduction lemma for the star forest ASD of bipartite graphs and the proof of Lemma 1.3. The proof of Theorem 1.2 is contained in section 4.

2 Star ASD of bipartite graphs

A sequence $d_1 \geq \cdots \geq d_k$ of nonnegative integers with $d_1 + \cdots + d_k = \binom{n+1}{2}$ is *n–realizable* if there is a partition $\{X_1, \ldots, X_k\}$ of $[n]$ such that the sum of the elements in X_i is d_i for each i.

Let G be a bipartite graph with stable sets $A = \{a_1, \ldots, a_k\}$ and B. Let $d_1 \geq \ldots \geq d_k$ be the degrees of the vertices $a_1, \ldots, a_k$. If the sequence $(d_1, \ldots, d_k)$ is n–realizable then G clearly admits a star ASD. Known results on the Sumset Partition Problem [4,12] imply the following results on the ASD problem

Proposition 2.1 *Let $G(A,B)$ be a bipartite graph with $\binom{n+1}{2}$ edges. If $|A| \leq 4$ and $n \geq 7$ then $G(A,B)$ admits an ASD.* □

Chen et al. [4] proved that a sequence $(d_1 \geq \cdots \geq d_k)$ such that $\sum_i d_i = \binom{n+1}{2}$ and $d_{k-1} \geq n$ is n–realizable. The result can be rephrased in terms of ASD of bipartite graphs as follows.

Corollary 2.2 *Let $G(A,B)$ be a bipartite graph with $\binom{n+1}{2}$ edges. Let $d(A) = (d_1 \geq \cdots \geq d_k)$ be the degree sequence of the elements in A. If $d_{k-1} \geq n$ then $G(A,B)$ admits a star ASD_n.*

Proposition 2.3 *Let $G(A,B)$ be a bipartite graph with $\binom{n+1}{2}$ edges. If $k \leq (n+1)/4$ and $d_k \geq 4k$ then $G(A,B)$ admits a star ASD_n.*

3 A reduction Lemma

Let G be a bipartite graph with stable sets $A = \{a_1, \ldots, a_k\}$ and B. Let $d = (d_1 \geq \cdots \geq d_k)$ be the degree sequence of the vertices in A, $d_i = d(a_i), i = 1, \ldots, k$. We denote by G_0 the bipartite graph with stable sets A and $B' = \{b_1, \ldots, b_{d_1}\}$ where a_i is adjacent to the vertices $b_1, \ldots, b_{d_i}$, $i = 1, \ldots, k$, and call G_0 the *compression* of G. In this section we prove that, if G_0 admits a star forest decomposition then so does G. This reduces the problem of giving sufficient conditions on the degree sequence of one stable set to ensure the existence of a star forest ASD to bipartite compressed graphs. For the proof of our reduction Lemma we use the following result [10] on edge list–colorings of bipartite multigraphs.

Theorem 3.1 ([10]) *Let H be a bipartite multigraph with stable sets A and B. If H admits a proper edge–coloring such that each vertex $a \in A$ is incident with edges colored $\{1, 2, \ldots, d(a)\}$, then H can be properly edge–colored for an arbitrary assignment of lists $\{L(a) : a \in A\}$ such that $|L(a)| = d(a)$ for each $a \in A$.* □

Lemma 3.2 (Reduction Lemma) *Let G be a bipartite graph with bipartition $A = \{a_1, \ldots, a_k\}$ and B and degree sequence $d = (d_1 \geq \cdots \geq d_k)$, $d = d(a_i)$, of the vertices in A. If the compression G_0 of G admits a decomposition $G_0 = F'_1 \oplus \cdots \oplus F'_t$, where each F'_i is a star forest, then G has an edge decomposition $G = F_1 \oplus \cdots \oplus F_t$, where $F_i \cong F'_i$ for each $i = 1, \ldots, t$.* □

Proof. Let C be the $(k \times t)$ matrix whose entry c_{ij} is the number of edges incident to a_i in the star forest F'_j of the edge decomposition of G_0.

Consider the bipartite multigraph H with A and $U = \{u_1, \ldots, u_t\}$ as stable sets, where a_i is joined with u_j with c_{ij} parallel edges. Now, for each pair (i, j), color the c_{ij} parallel edges of H with the neighbors of a_i in the forest F'_j bijectively. Note that in this way we get a proper edge–coloring of H: two edges incident with a vertex a_i receive different colors since the bipartite graph G_0 has no multiple edges, and two edges incident to a vertex u_j receive different colors since F'_j is a star forest.

By the definition of the bipartite graph G_0, each vertex $a_i \in A$ is incident in the bipartite multigraph H with edges colored $1, 2, \ldots, d_i$. Let $L(a_i)$ be the list of neighbours of a_i in the original bipartite graph G. By Theorem **??**, there is a proper edge–coloring χ' of H in which the edges incident to vertex a_i in A receive the colors from the list $L(a_i)$ for each $i = 1, \ldots, k$. Now construct F_s by letting the edge $a_i b_j$ be in F_s whenever the edge $a_i u_s$ is colored b_j in the latter edge–coloring of H. Thus F_s has the same number of edges than F'_s and the degree of a_i in F_s is c_{is}, the same as in F'_s. Moreover, since the coloring is proper, F'_s is a star forest. This concludes the proof. □

4 Ascending Star forest decompositions

Let $G = G(A, B)$ be a bipartite graph with $\binom{n+1}{2}$ edges. We denote by $d = (d_1 \geq \cdots \geq d_k)$ the degree sequence of the vertices in the stable set A of G.

We focus on star forest ASD with the stars of the decomposition centered at the vertices in A. We say that a degree sequence $d = (d_1 \geq \cdots \geq d_k)$ with $\sum_i d_i = \binom{n+1}{2}$ is *good* if every bipartite graph $G(A, B)$ with $A = \{a_1, \ldots, a_k\}$ and $d(a_i) = d_i$, $1 \leq i \leq k$, admits a star forest ASD with the centers of the stars in A. We next give a necessary condition for a sequence to be good.

Lemma 4.1 *If the sequence* $(d_1 \geq \cdots \geq d_k)$ *is good then*

$$\sum_{i=1}^{t} d_i \geq \sum_{i=1}^{t} (n - i + 1) \ \text{ for each } \ t = 1, \ldots, k. \qquad (1)$$

Proof. Consider the compressed bipartite graph $G = G(A, B)$ such that $a_i \in A$ is adjacent to $\{b_j : \ j = 1, \ldots, d_i\}$, $i = 1, \ldots, k$. Let $G = F_1 \oplus \cdots \oplus F_n$ be a star forest ASD of G. Since F_n has n leaves in B we clearly have $|B| = d_1 \geq n$. Thus (1) is satisfied for $t = 1$. Suppose that (1) is satisfied for some $t = j - 1 < k$. If $d_j \geq n - j + 1$ then the inequality extends to $t = j$. Suppose that $d_j \leq n - j$. Since G is compressed, the neighborhhod of the vertices $a_{j+1}, \ldots, a_k$ is contained in the neighborhood of a_j. It follows that the forest F_n has at least $(n - d_j)$ end–vertices adjacent only to vertices from $\{a_1, \ldots, a_j\}$. Likewise, F_{n-i} has at least $(n - i + 1 - d_j)$ end–vertices adjacent only to vertices from $\{a_1, \ldots, a_t\}$, $i = 1, \ldots, t$. Hence, $\sum_{i=1}^{j} d_i \geq jd_j + (n-d_j) + (n-1-d_j) + \cdots + (n-t+1-d_j) = n + (n-1) + \cdots + (n-j+1)$, and (1) is satisfied for $t = j$. This concludes the proof. □

We next obtain a close sufficient condition for a sequence to be good. We first introduce some definitions. Given two k–dimensional vectors $c = (c_1, \ldots, c_k)$ and $c' = (c'_1, \ldots, c'_k)$, we say that $c \preceq c'$ if there is a permutation $\sigma \in Sym(k)$ such that $c_i \leq c'_{\sigma(i)}$ for $i = 1, 2, \ldots, k$. In other words, after reordering the components of each vector in nonincreasing order, the i–th component of c is not larger than the i–th component of c'. This definition is motivated by the following remark.

Remark 4.2 Let F, F' be two forests of stars with centers $x_1, \ldots, x_k$ and $x'_1, \ldots, x'_k$ respectively. Then F is isomorphic to a subgraph of F' if and only if $(d_F(x_1), \ldots, d_F(x_k)) \preceq (d_{F'}(x'_1), \ldots, d_{F'}(x'_k))$. □

Given a sequence $d = (d_1 \geq \cdots \geq d_k)$ of positive integers with $\sum_i d_i = \binom{n+1}{2}$, we say that a $(k \times n)$ matrix C with nonnegative integer entries is *d–ascending* if it satisfies the following three properties:

(A1) $\sum_j c_{ij} = d_i$, $i = 1, \ldots, k$,

(A2) $\sum_i c_{ij} = n - j + 1$, $j = 1, \ldots, n$,

(A3) $c^j \succeq c^{j+1}$, $j = 1, \ldots, n-1$, where c^j denotes the j–th column of C.

Next Lemma gives a sufficient condition for a degree sequence to be good assuming the existence of an appropriate ascending matrix.

Lemma 4.3 *Let $d = (d_1 \geq \cdots \geq d_k)$ be a sequence of positive integers with $\sum_i d_i = \binom{n+1}{2}$. Suppose that there is a d–ascending matrix C such that $c_{ij} \geq 1$ for each pair (i, j) with $i + j \leq k + 1$. If $d_i \geq n - i + 1$, $i = 1, \ldots, k - 1$, then d is good.*

Proof. By the reduction Lemma it suffices to show that the compressed graph G with degree sequence d admits a star forest decomposition. Let H be the bipartite multigraph with stable sets $A = \{a_1, \ldots, a_k\}$ and $U = \{u_1, \ldots, u_n\}$ and with c_{ij} parallel edges joining $a_i \in A$ with $u_j \in U$. We next show that H can be properly edge–colored in such a way that the edges incident to a_i receive colors from the set $\{1, \ldots, d_i\}$, $i = 1, \ldots, k$. Now, for each $s = 1, \ldots, k$ denote by M_s the matching in H formed by the s edges $a_1u_s, a_2u_{s-1}, \ldots, a_su_1$. Such matchings exist by the condition $c_{ij} \geq 1$ for each pair (i, j) with $i + j \leq k + 1$. We color the edges of the matching M_s with $k - s + 1$. In this way the vertex a_i is incident in $M_1 \oplus \cdots \oplus M_k$ with edges colored $\{1, \ldots, k - i + 1\}$.

Let H' denote the bipartite multigraph obtained from H by removing the edges in $M_1 \oplus \cdots \oplus M_k$. Let $d'_A = (d'_1 \geq \cdots \geq d'_k)$ be the degree sequence of A in H'. Since $d_i \geq n-i+1$, we have $d'_i = d_i - (k-i+1) \geq n-k$, $i = 1, \ldots, k$. On the other hand, each vertex u_i has degree $n-i+1$ in H and, for $i \leq k$, it is incident to the matchings $M_1, \ldots, M_{k+1-i}$. Hence, every vertex in U has degree at most $n-k$ in H'.

Let $\Delta'(A)$ be the maximum degree in H' of the vertices in A. If $\Delta'(A) > n-k$ then there is a matching M'_1 in H' from the vertices of maximum degree in A to U. Color the edges of this matching with $\Delta'(A)$. By removing this matching from H' we obtain a bipartite multigraph in which the maximum degree of vertices in A is $\Delta'(A) - 1$. By iterating this process we reach a bipartite multigraph H'' with $\Delta''(A) = n-k$, while the maximum degree of the vertices in U still satisfies $\Delta''(U) \leq n-k$. By König's theorem, the edge–chromatic number of H'' is $n-k$. Hence H' can be properly edge–colored in such a way that vertex a_i is incident in H' with colors $\{1, \ldots, d'_i = d_i - (k-i+1)\}$. By Theorem 3, there is also a proper edge–coloring of H' in which each vertex a_i is incident with edges colored $\{k-i+1, k-i+2, \ldots, d_i\}$. By combining this coloring with the one of $M_1 \oplus \cdots \oplus M_k$ defined above we get a proper edge–coloring of the original bipartite multigraph H in which the vertex a_i is incident with edges colored $\{1, \ldots, d_i\}$ for each $i = 1, \ldots, k$.

We use this coloring to obtain a star forest decomposition $G = F_1 \oplus \cdots \oplus F_n$, of the compressed bipartite graph with stable sets A and $B = \{b_1, \ldots, b_{d_1}\}$ by letting F_s consist of the edges $a_i b_j$ such that $a_i u_s$ is colored b_j in the edge-colored multigraph H. Thus F_s has degree sequence $d_A(F_s) = (c_{s1}, \ldots, c_{sk})$. By the column sum property of the matrix C, the star forest F_s has $\sum_i c_{is} = n-s+1$ edges and, by the ascending column property, it is isomorphic to a subgraph of F_{s-1}. This completes the proof. □

We are now ready to prove our main result.

Proof. Consider the matrix C' whose first row vector is $(n-k+1, \ldots, n-k+1, n-k, n-k-1, \ldots 3, 2, 1)$ and, for $j > 1$ the i-th row of C' has $k-i+1$ ones followed by zeros. The sum of the entries of the j–th column of C' is $n-i+1$. Thus C' is an ascending matrix for the degree sequence $d'_1 = \binom{n-k}{2} + (n-k+1)k$ and $d'_i = k-i+1$ for $i = 2, \ldots, k$. Hence, $d_i - d'_i \geq n-i+2-(k-i+1) = n-k+1$, $i = 2, \ldots, k$. Now, consider the sequence $(d_2 - d'_2, \ldots, d_k - d'_k, \alpha)$ where α is such that $\sum_{i=2}^{k}(d_i - d'_i) + \alpha = \binom{n-k+1}{2}$. Since all but perhaps one elements of the sequence are larger than $n-k+1$, the sequence is $(n-k+1)$–realizable. Thus there is a partition $\{X_2, \ldots, X_k, X_{k+1}\}$ of the interval $[1, n-k+1]$ with $\Sigma(X_i) = d_i - d'_i$, $i = 2, \ldots, k$. By permuting the corresponding entries from the first column to the i–th we get an ascending matrix which has row sum d_A. □

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