Backstepping with virtual filtered command: Application to a 2D autonomous Vehicle

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***Abstract*— Through this work a deep understanding of the backstepping control technique is sought when applied over non-affine systems. It is shown that in this case appears the necessity to bound the value of internal states and that a modification over standard backstepping is mandatory. The principal goal of this study is to evaluate the effects of finite frequency filters, and the effects of saturation affecting intermediate states and control actions, in the tracking performance when using the command filtered backstepping. Some relations that bind the controller gains to maintain performance appear naturally. Finally simulations over a 2D steering robot model are given to illustrate the found results.**

I. INTRODUCTION

In the last years backstepping has appeared as a very promising control technique for solving the tracking and stabilization problem for a reduced class of non-linear systems in pure-feedback form [1]. Advances and modifications of the original technique simplify the control laws derivation, allowing to deal with uncertainty, parameter adaptation, and saturation in actuators and internal states [2], [3]. Under this framework, backstepping seems to be the solution of the tracking control problem for a wider variety of non-linear systems.

Backstepping has demonstrated its application over unmanned aerial vehicles [4], [5], ground robots [6], [7], [8] and underwater robots [9], [10].

The objective of this work is to explore the limitations and drawbacks of the method, its advantages and to clarify its implementability.

Through the paper we work with a scalar 3rd order system representing the steering model of a terrestrial mobile robot. The mathematical model is characterized to be in a non-strict feedback form for which backstepping can not be applied directly.

II. PROBLEM FORMULATION

Let the system under consideration be a non-linear dynamic system of order n, described by the model in strict feedback form

$$\begin{array}{ll} \dot{x}_1 = f_1(x_1) + g_1(x_1)x_2 & \\ \dot{x}_i = f_i(x_1,...,x_i) + g_i(x_1,...,x_i)x_{i+1} & \forall\ i = 2,..n-1\ . \\ \dot{x}_n = f_n(x_1,...,x_n) + g_n(x_1,...,x_n)u & \end{array} \tag{1}$$

Assume that functions f and g are completely determined and that $g_i \neq 0$ for all $i = 1,...,n$.

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The control objective is to obtain an implementable u such that x_1 follows continuous and bounded reference x_r, with known continuous and bounded derivative $\dot{x}_r$, while respecting possible limitations in the remaining states $x_i\,\forall\ i = 2,...,n$.

III. STANDARD BACKSTEPPING APPROACH

The standard backstepping approach, does not ensure to respect the intermediate nor the control action limitations. Nonetheless, it is the base of a more general technique adopted, and its final results will be used in next sections. To show the backstepping approach we follow the development in [1]. However we only show the key results and obviate some mathematical development.

Departing from tracking errors defined as $e_i = x_i - \alpha_{i-1}\,\forall\ i = 1,...,n$, being

$$\begin{array}{c} \alpha_0 = x_r \\ \alpha_1 = \frac{1}{g_1}\left(-f_1 - k_1 e_1 + \dot{x}_r\right) \\ \alpha_i = \frac{1}{g_i}\left(-f_i - k_i e_i + \dot{\alpha}_{i-1} - g_{i-1}e_{i-1}\right) \end{array} \tag{2}$$

for $i = 2,...,n$, the so-called virtual control signals, and $u = \alpha_n$. Under this control law, with $k_i > 0$, it can be mathematically proven that the errors tend asymptotically to the origin ensuring that $x_1 \to x_r$.

Although the time derivatives $\dot{\alpha}_{i-1}$ are analytically differentiable, its simple notation hides an explosion of terms that complicates terribly the derivation of the analytical expressions as n increases.

In the next, we select as control technique a modification of backstepping that appears in [3], [11], [12] and [13]. It avoids the analytical computation of derivatives and allows to introduce bounds for intermediate state and the control action.

IV. COMMAND FILTERED BACKSTEPPING CONTROL.

Assume that a system is described by the model in Eq. (1) and let a new state definition as

$$\begin{array}{l} z_1 = x_1 - x_r - \xi_1 \\ z_i = x_i - x_i^c - \xi_i \quad \forall\ i = 2,...,n \end{array} \tag{3}$$

The dynamics of the z_i variables can be expanded by using the model equations

$$\dot{z}_1 = f_1 + g_1 x_2 - \dot{x}_r - \dot{\xi}_1 \tag{4a}$$

$$\dot{z}_i = f_i + g_i x_{i+1} - \dot{x}_i^c - \dot{\xi}_i \tag{4b}$$

$$\dot{z}_n = f_n + g_3 u - \dot{x}_{n-1}^c - \dot{\xi}_n \tag{4c}$$

with $i=2,...,n-1$. If x_i terms are rewritten as

$$\begin{aligned} x_i = & \; x_i^0 + \left(x_i^c - x_i^0\right) + \left(x_i - x_i^c\right) = \\ & \; x_i^0 + \left(x_i^c - x_i^0\right) + \left(z_i + \xi_i\right) \\ u = & \; u^0 + \left(u - u^0\right) \end{aligned}$$

for $i=2,...,n-1$, and substituted in Eq. (4)

$$\dot{z}_1 = f_1 + g_1 x_2^0 + g_1\left(x_2^c - x_2^0\right) + g_1\left(z_2 + \xi_2\right) - \dot{x}_r - \dot{\xi}_1 \quad (5a)$$

$$\begin{aligned} \dot{z}_i = f_i + g_i(x_i) x_{i+1}^0 + g_i(x_i)\left(x_{i+1}^c - x_{i+1}^0\right) + \\ g_i(x_i)\left(z_{i+1} + \xi_{i+1}\right) - \dot{x}_i^c - \dot{\xi}_i \end{aligned} \quad (5b)$$

$$\dot{z}_n = f_n + g_n u^0 + g_n\left(u - u^0\right) - \dot{x}_n^c - \dot{\xi}_n. \quad (5c)$$

In this system, the variables virtual control signals, x_i^0 and the virtual control action, u^0 act as control actions for Eqs. (5a), (5b) and (5c). These virtual control signals/action are the base to obtain the commanded control signals/action x_i^c/u. In fact, the commanded control signals/action are a filtered version of the virtual ones that incorporate magnitude an rate saturations. To impose the limitations, a second order filter with saturated integrators is used, and representated by

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= 2\xi\omega_n\left(S_R\left(\frac{\omega_n^2}{2\xi\omega_n}\left(S_M\left(q^0\right) - q_1\right) - q_2\right)\right) \end{aligned} \quad (6)$$

In the design, n more degrees of freedom exist related with the choice of the ξ_i dynamics as has been stated before. Now it is clear that if

$$\begin{aligned} \dot{\xi}_i &= -k_i\xi_i + g_i\left(x_{i+1}^c - x_{i+1}^0 + \xi_{i+1}\right) \\ \dot{\xi}_n &= -k_n\xi_n + g_n\left(u - u^0\right) \end{aligned} \quad (7)$$

for $i=1,...,n-1$, and the virtual control actions are chosen as

$$\begin{aligned} g_1 x_2^0 &= -f_1 - k_1\left(z_1 + \xi_1\right) + \dot{x}_r \\ g_i x_{i+1}^0 &= -f_i - k_i\left(z_i + \xi_i\right) + \dot{x}_i^c - g_{i-1} z_{i-1} \\ g_n u^0 &= -f_n + \dot{x}_n^c - k_n\left(z_n + \xi_n\right) - g_{n-1} z_{n-1} \end{aligned} \quad (8)$$

with $i=2,...,n-1$, then the dynamic system in Eq. (5) becomes

$$\begin{aligned} \dot{z}_1 &= -k_1 z_1 + g_1 z_2 \\ \dot{z}_i &= -g_{i-1} z_{i-1} - k_i z_i + g_i z_{i+1} \ , \\ \dot{z}_n &= -g_{n-1} z_{n-1} - k_n z_n \end{aligned} \quad (9)$$

for $i=2,...,n-1$. Equivalently

$$\dot{\boldsymbol{z}} = \mathbf{A}\boldsymbol{z} \quad (10)$$

being

$$\boldsymbol{z} = \begin{pmatrix} z_1 & z_2 & \dots & z_n \end{pmatrix}^T \quad (11)$$

and

$$\mathbf{A} = \begin{pmatrix} -k_1 & g_1 & 0 & 0 & 0 & \cdots \\ -g_1 & -k_2 & g_2 & 0 & 0 & \cdots \\ 0 & -g_2 & -k_3 & g_3 & 0 & \ddots \\ 0 & 0 & \ddots & \ddots & \ddots & \\ \vdots & \vdots & & & & \\ 0 & \cdots & & -g_{n-2} & -k_{n-1} & g_{n-1} \\ 0 & \cdots & & & -g_{n-1} & -k_n \end{pmatrix}. \quad (12)$$

By selecting $k_i > 0$, $\forall\ i = 1, ...n$, the asymptotic stability of $\boldsymbol{z}$ is theoretically guaranteed, regardless the value of the filter frequency chosen, since for $\mathbf{P} = \mathbf{P}^T = \frac{1}{2}\mathcal{I} > 0$,

$$\mathbf{PA} + \mathbf{A}^T\mathbf{P} < 0. \quad (13)$$

If no limits are included in the filtering process, for $\omega_n \rightarrow \infty$ the filtered control actions x_i^c, and u tend to x_i^0,and u^0 respectively and the estimations of the derivatives by the filters tend to the analytical derivatives. Under this assumption, the solution of the presented control scheme reaches the solution of the standard backstepping where the asymptotic convergence of $x_1 \rightarrow x_r$ is guaranteed.

A. *Filter effects*

In order to evaluate the effects of filtering, assume that the system under study is not affected by physical or model limitations or, equivalently, that the saturation limits are fixed far away from the operation envelope. Assume in addition that a maximum value for the natural frequency of the filters exist e.g. due to the minimum sampling time of the real system under control.

When the filters are implemented with a finite natural frequency not necessarily much greater that the natural frequency of the input, the range of k_i gains is restricted to be

$$k_i \leq K_{li} < \omega_{ni} \quad (14)$$

being K_{li} a maximum value for k_i, if it is desired that x_{i+1}^c accurately track their unfiltered version x_{i+1}^0.

In [13] it is proven that, the control law in Eq. (8) drive the system variables to a bounded trajectory in the neighbourhood of the original backstepping solution. In particular it is said that

$$|x(t,\varepsilon) - \tilde{x}(t)| \leq l|\varepsilon|, \quad \forall|\varepsilon| < c, \quad t \geq 0 \quad (15)$$

for some positive constants $c > 0$ and $l > 0$ and $\varepsilon = \omega_n^{-1}$. Additionally it can be proven that the filtered backstepping solution becomes closer to the original one as k_i grows. When the constraint is violated, the performance is degraded and although the difference between solutions is bounded, in practice the bound can reach very large values.

B. *Saturation effects*

Saturation appears when any of the virtual control signals or the virtual control action, are outside of the bounds of their respective physical variables x_i and u.

Since saturation acts in the same way that filters errors, an equivalent result like the presented in Eq. (15) holds. However, when saturations are active, it exist a minimum gap between the standard backstepping solution and the filtered solution that can not be reduced by increasing the gain values.

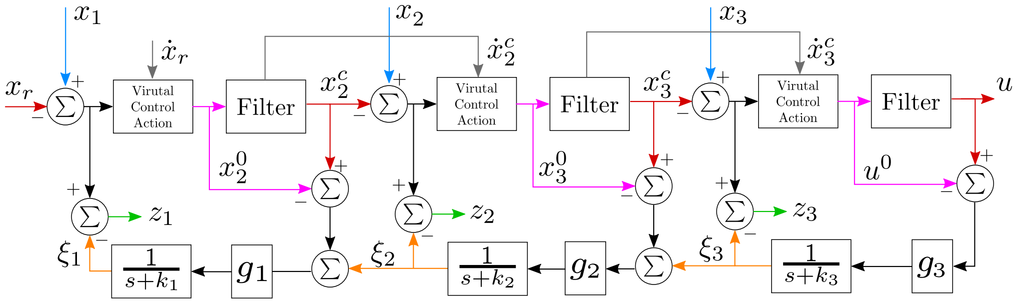

Fig. 1. Backstepping controller arrangement. Colours has been added to highlight the pattern repetitiveness in the construction.

V. Backstepping implementation

The goal of the mathematical derivations shown in Sec. IV is to understand the behaviour of the tracking error under the Command Filtered backstepping technique. However, its implementation can be confusing. Following the steps of [13], Fig. 1 presents the controller scheme construction for a third order system.

VI. Steering Model

The reference tracking problem for a 2D mobile robot is commonly reduced to the steering control problem in tricycle robots. Usually, it is assumed that a given velocity is provided from the engine of the mobile robot and that the lateral velocity is the target of control.

The equations that define the system motion are easily derivable from first principles and are presented for example in [14]

$$\dot{x} = -V \sin\theta \tag{16a}$$

$$\dot{\theta} = V\gamma \tag{16b}$$

where x represents the lateral displacement, V is the velocity of the robot, θ represents the angle between the vertical axis and the velocity and γ is the curvature radius.

A third equation is added, as usually, to account for the steering wheel dynamics

$$\dot{\gamma} = \frac{1}{T}\left(\gamma_r - \gamma\right), \tag{17}$$

where T is a time constant and γ_r the reference for the servo.

A. Model conditioning

The model for the lateral motion of the robot given by Eqs. (16a), (16b) and (17) is composed by a chain of integrators, nevertheless the non-linearity in the first equation makes the structure not affine and therefore the backstepping design does not apply directly. A variable transformation can be used to make the system affine with respect the inputs.

Let the next mapping between variables

$$x_1 = x \quad x_2 = \sin\theta$$
$$x_3 = \gamma \quad u = \gamma_r,$$

and let in addition

$$g_1 = -V \qquad g_2(\theta) = g_2 = V\cos(\theta)$$

$$f_3(x_3) = f_3 = -\frac{1}{T}x_3 \qquad g_3 = \frac{1}{T}$$

assuming V and T known values different from 0.

In the definition of g_2, the variable θ has not been substituted by $\sin^{-1} x_2$, avoiding the sign ambiguity and simplifying the design since θ will be a measured variable.

Then, the system in Eqs. (16) and (17) is converted to

$$\dot{x}_1 = g_1 x_2 \tag{18a}$$

$$\dot{x}_2 = g_2 x_3 \tag{18b}$$

$$\dot{x}_3 = f_3 + g_3 u. \tag{18c}$$

Eq. (18) is now expressed in a strict feedback form. However, the coordinate transformations have introduced limitations in the range of validity of the model due to the boundedness of the term $\sin\theta$, that it is not explicitly present on the variable x_2, and that must be taken in account by the controller.

$$-1 \leq x_2 \leq 1$$

Additional physical limitations appear in the servo subsystem represented in Eq. (17). The guidance system will have a limited range of operation represented e.g. $-1 \leq \gamma \leq 1$ that has to be considered in Eq. (18c) and in the generation of the control action.

VII. CFBS over a steering model

In this section some of the results presented along the paper are highlighted with examples that relate the command filtered control with the steering dynamics previously presented.

A. Filters effect

The model implemented in this section corresponds to the Eq. (18), to avoid the effect of limitations. The controller proposed in Fig. 1, has been implemented without rate nor magnitude saturations, for different sets of gains and two different natural frequencies.

Fig. 2, shows the lateral path for a constant reference and for very high filter frequencies

$$\omega_{n_1} = 8 \cdot 10^3\,\mathrm{rad\,s^{-1}} \quad \omega_{n_2} = 12 \cdot 10^3\,\mathrm{rad\,s^{-1}}$$
$$\omega_{n_3} = 16 \cdot 10^3\,\mathrm{rad\,s^{-1}}, \tag{19}$$

and for three different combinations of gains, presented in Table I.

	k_1	k_2	k_3
Set1	1	2	3
Set2	10	14	18
Set3	18	20	26

TABLE I
CONTROLLER GAINS.

	k_1	k_2	k_3
Set1	1	2	3
Set2	1.5	2.5	3.5
Set3	2	4	6

TABLE II
CONTROLLER GAINS.

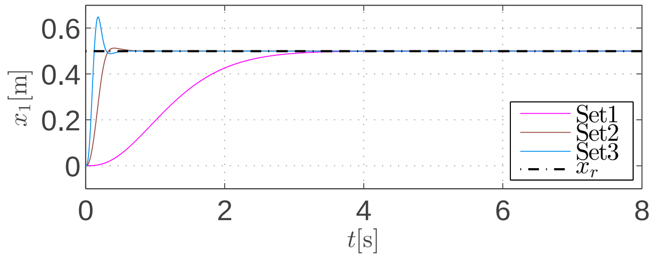

Fig. 2. Lateral path and tracking error for the gains in Table I and the filter natural frequencies in Eq. (19).

Fig. 3 shows the results for the produced lateral path for the same sets of gains presented in Table I and the same input, but with implementable filter frequencies

$$\omega_{n_1} = 200\,\mathrm{rad\,s^{-1}} \quad \omega_{n_2} = 250\,\mathrm{rad\,s^{-1}} \\ \omega_{n_3} = 300\,\mathrm{rad\,s^{-1}}. \tag{20}$$

Observing and comparing Figs. 2 and 3, it can be seen that for the same sets of gains, a change in the natural frequency of the filters leads to a change in the dynamics of the involved variables. In addition, as stated in Sec. IV-A the range of selectable gains to maintain performance becomes smaller as the natural frequency of the filters decrease. When the filters are implemented with the lower frequencies, and the gains of Set3, the tracking is completely degraded.

B. Saturation and filters effect

With the intention of proving that saturation decrease the selectable set of gains when maximum performance is sought, the CFBS controller is derived from the affine model in Eq. (18) considering the restrictions. Fig. 4 shows the lateral path for the steering model in Eqs. (16) and (17), a stairs like input, the three different sets of gains in Table II and the filter natural frequencies in Eq. (20).

Fig. 5, shows the saturation function over the different virtual filtered control actions for the three sets of gains. The saturation function is defined to be 1 when its variable is saturated and 0 otherwise.

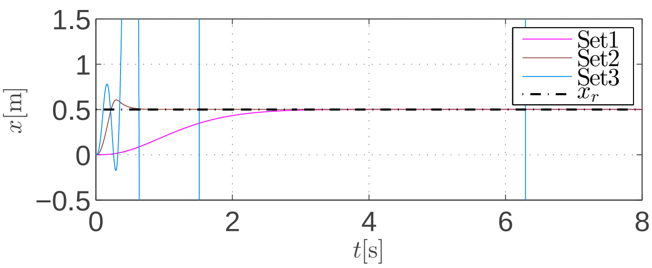

Fig. 3. Lateral path for the gains in Table I and the filter natural frequencies in Eq. (20).

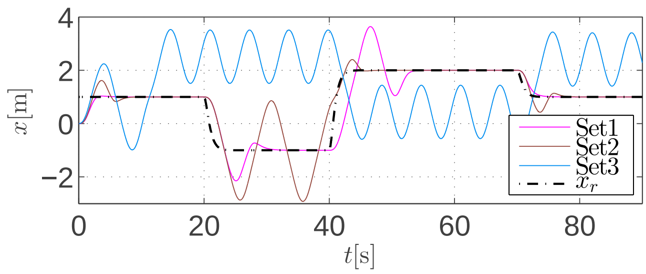

Fig. 4. Lateral path for the gains in Table II and the filter natural frequencies in Eq. (20).

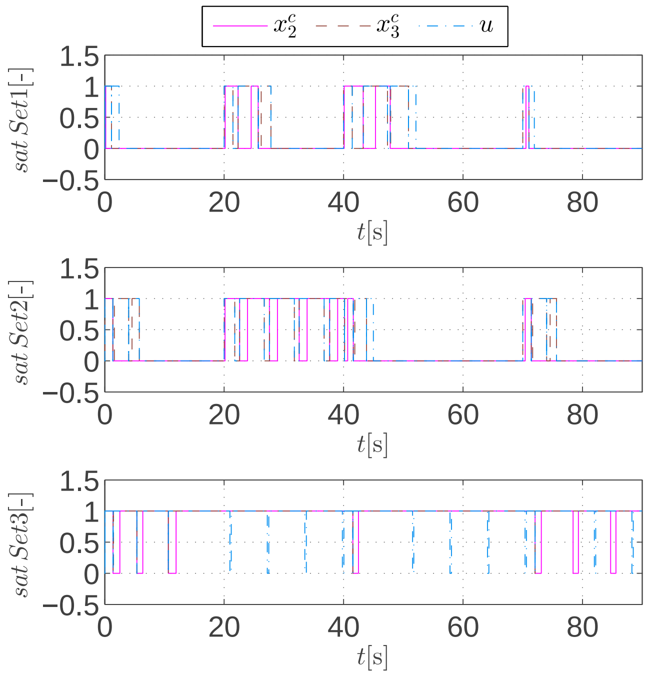

Fig. 5. Saturation of the commanded control signals for the three different sets of gains in Table II.

As can be observed the range of allowable gains have reduced with the increase of input spectral content. In Fig. 4 it is shown that the choice of gains of the Set2, does not lead to a correct tracking performance throughout the path. In Fig. 5, it can be seen that in these cases the saturation is active and then, the distance from the ideal solution that would provide the standard backstepping (if it was implementable), and the path generated by the command filtered backstepping is bounded but convergence is not achieved.

VIII. CONCLUSIONS

In this work the command filtered backstepping technique fundamentals have been deeply studied. Connections between performance, the effects of finite frequency filters, the effects of model and control action limitations and the tuning of the controller have been explicitly stated. Finally, simulations have been carried out over a model of a steering 2D robot to exalt the previous results.

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