Abstract

We prove that the genus $g$, the relative irregularity $q_f$ and the Clifford index $c_f$ of a non-isotrivial fibration $f$ satisfy the inequality $q_f \leq g - c_f$. This gives in particular a proof of Xiao’s conjecture for fibrations whose general fibres have maximal Clifford index.

1 Introduction

In the classification of smooth algebraic surfaces it is natural to study its possible fibrations over curves, trying to relate the geometry of the surface to the properties of the fibres and the base. In this article we focus on the relations between the isotriviality of a fibration and some of its numerical invariants, proving Xiao’s conjecture for fibrations whose general fibres have maximal Clifford index.

Let $f : S \rightarrow B$ be a fibration from a compact surface $S$ to a compact curve $B$, (that is, a surjective morphism with connected fibres), and let $F$ be a general (smooth) fibre of $f$. The fibration is called isotrivial if all the smooth fibres are mutually isomorphic, and it is trivial if $S$ is birational to $B \times F$ and the given fibration corresponds to the first projection.

We first consider the genus $g$ of $F$ (also called the genus of $f$) and the relative irregularity $q_f = q(S) - g(B)$. Beauville showed in its Appendix to [4] that

$$0 \leq q_f \leq g,$$

and the equality $q_f = g$ holds if and only if $f$ is trivial. As a consequence of the work of Serrano [11], non-trivial isotrivial fibrations satisfy

$$q_f \leq g + 1.$$

For non-isotrivial fibrations, the only known general upper bound for $q_f$ is

$$q_f \leq \frac{5g + 1}{6},$$

proven by Xiao in [12]. However, according to some comments made by Xiao himself in his later work [13], there is little hope for the inequality [3] to be sharp, since the methods used to prove it are not very accurate. In fact, in [13] he considers the special case in which the base is $B \cong \mathbb{P}^1$, obtaining the same upper-bound [3] known for non-trivial isotrivial fibrations. In view of this result, Xiao conjectured in [14] that the inequality [4] should hold for every non-trivial fibration, and he provided several examples attaining the equality. This conjecture was shown to be false by Pirola in [10], where he provided a non-isotrivial fibration with fibres of genus $g = 4$ and relative irregularity $q_f = \frac{3}{2} = \frac{g - 1}{2}$. The
same method has been recently applied to other cases by Albano and Pirola in [1], giving different counterexamples for even $g$ and satisfying

$$q_f = \frac{g}{2} + 1 = \frac{g + 1}{2} + \frac{1}{2}.$$ 

The fact that the only known counterexamples fail by just $\frac{1}{2}$ motivates the following version of the conjecture.

**Conjecture 1.1** (Xiao’s conjecture). *For any non-trivial fibration $f : S \to B$ one has*

$$q_f \leq \frac{g}{2} + 1,$$

*or equivalently*

$$q_f \leq \left\lceil \frac{g + 1}{2} \right\rceil.$$ 

Note that for odd values of $g$, the bound in Conjecture 1.1 is equivalent to the inequality (2) originally conjectured by Xiao.

In this article we prove the following

**Theorem 3.2.** Let $f : S \to B$ be a fibration of genus $g \geq 2$, relative irregularity $q_f$ and Clifford index $c_f$. If $f$ is non-isotrivial, then

$$q_f \leq g - c_f.$$ 

The Clifford index of $f$, $c_f$, was introduced by Konno in [7] as the Clifford index of a general fibre (which is in fact the maximum of the Clifford indexes of the smooth fibres). The Clifford index leads to several improvements of the *slope-inequality*, as those obtained by Konno himself, and by Barja and Stoppino in [2].

Theorem 3.2 above can in fact be interpreted as the proof of Conjecture 1.1 in the general case, in which $c_f = \left\lfloor \frac{g - 1}{2} \right\rfloor$.

In order to prove Theorem 3.2 we first need the existence of a supporting divisor for $f$, which is guaranteed by the more general study of families of irregular varieties carried out by one of the authors in [6]. By a *supporting divisor* for $f$, we mean a divisor on $S$ whose restriction to a general fibre supports the corresponding first-order deformation induced by $f$ (see Definitions 2.2 and 2.5). Once this divisor is obtained, we must consider whether its restriction to a general fibre is rigid or not. In the former case we conclude using a structure result also proved in [6], while in the latter case we need a result on the rank of first-order deformations of curves (Theorem 2.4). This result was stated by Ginensky as part of a more general theorem in [5], whose original proof contains some inaccuracies. Though the part of the proof we need can be completed and slightly shortened, we have decided to include here a different, much shorter proof, suggested to us by G.P. Pirola.

**Acknowledgements:** We would like to thank Prof. Gian Pietro Pirola for the many stimulating discussions around this topic, especially for presenting to us several counterexamples to the original conjecture of Xiao and for suggesting the new proof of the result of Ginensky (Theorem 2.4).

**Basic assumptions and notation:** Throughout the whole article, all varieties are assumed to be smooth and defined over $\mathbb{C}$. Unless otherwise is explicitly said, $f : S \to B$ will be a fibration (a surjective morphism with connected fibres) from a compact surface $S$ to a compact curve $B$. The *genus* of $f$, defined as the genus of any smooth fibre, will be denoted by $g$, and assumed to be at least 2. The *relative irregularity* of $f$ is by definition the difference $q_f = q(S) - g(B)$.

## 2 Preliminaries

We will use some notions about infinitesimal deformations of curves, as well as some results on fibred surfaces developed in the previous work [6].
2.1 Infinitesimal deformations

Let $C$ be a smooth curve of genus $g \geq 2$. A first-order infinitesimal deformation of $C$ is a proper flat morphism $C \to \Delta$ over the spectrum of the dual numbers $\Delta = \text{Spec } \mathbb{C}[\epsilon]/(\epsilon^2)$, such that the special fibre (over $0 = \text{Spec } \mathbb{C}(\epsilon)$) is isomorphic to $C$. A first-order infinitesimal deformation is uniquely determined by its Kodaira-Spencer class $\xi \in H^1(C, T_C)$, defined as the extension class of the sequence defining the conormal bundle,

$$0 \to N^\vee_{\mathcal{C}/\mathcal{C}} \cong T^\vee_{\Delta,0} \otimes \mathcal{O}_C \to \Omega^1_{\mathcal{C}/\mathcal{C}} \to \omega_C \to 0,$$

(4)

after choosing an isomorphism $T^\vee_{\Delta,0} \cong \mathbb{C}$. We will assume that the deformation is not trivial, that is $C \not\cong C \times \Delta$, or equivalently, $\xi \neq 0$.

Cup-product with $\xi$ gives a map

$$\partial_\xi = \cup \xi : H^0(C, \omega_C) \to H^1(C, \mathcal{O}_C)$$

which coincides with the connecting homomorphism in the exact sequence of cohomology of $(4)$.

**Definition 2.1.** The rank of $\xi$ is

$$\text{rk } \xi = \text{rk } \partial_\xi.$$

If $C$ is non-hyperelliptic, the map $H^1(C, T_C) \to \text{Hom} \left( H^0(C, \omega_C), H^1(C, \mathcal{O}_C) \right)$ given by $\xi \mapsto \partial_\xi$ is injective, hence no information is lost when considering $\partial_\xi$ instead of $\xi$. However, if $C$ is hyperelliptic, the above map is not injective, and we may have $\text{rk } \xi = 0$ even if $\xi \neq 0$. This exception is a manifestation of the failure of the infinitesimal Torelli Theorem for hyperelliptic curves.

From now on, until the end of the section, $D$ will denote an effective divisor on $C$ of degree $d$. We will also denote by $r = r(D) = h^0(C, \mathcal{O}_C(D)) - 1$ the dimension of its complete linear series.

**Definition 2.2.** The deformation $\xi$ is supported on $D$ if and only if

$$\xi \in \ker \left( H^1(C, T_C) \to H^1(C, T_C(D)) \right),$$

where the map is induced by the injection of line bundles $T_C \xrightarrow{+D} T_C(D)$. Furthermore, if $\xi$ is not supported on any strictly smaller effective divisor $D' < D$, we say that $\xi$ is minimally supported on $D$.

As far as we are aware, the notion of supporting divisor was introduced in [3], while the minimality was first considered in [5]. The use of the word “support” has two motivations. On the one hand, $\xi$ is supported on $D$ if and only if it is the image of a Laurent tail of a meromorphic section $\eta \in H^0 \left( D, T_C(D) \right)$, which is obviously supported on $D$. On the other hand, $\xi$ is supported on $D$ if and only if, in the bicanonical space of $C$, the line $C(\xi)$ corresponds to a point in the span of $D$.

If $D$ has the smallest degree among the divisors supporting $\xi$, then $\xi$ is minimally supported on $D$, but not conversely. Indeed, $\xi$ being minimally supported on $D$ means that it is not possible to remove some point of $D$ and still support $\xi$, but there is no reason for $D$ to have minimal degree.

One could equivalently define $\xi$ to be supported on the divisor $D$ if and only if the top row in the following pull-back diagram is split.

$$\begin{array}{ccccccccc}
\xi_D : & 0 & \to & N^\vee_{\mathcal{C}/\mathcal{C}} & \xrightarrow{} & \mathcal{F}_D & \xleftarrow{} & \omega_C (-D) & \to & 0 \\
\xi : & 0 & \to & N^\vee_{\mathcal{C}/\mathcal{C}} & \xrightarrow{} & \Omega^1_{\mathcal{C}/\mathcal{C}} & \xrightarrow{} & \omega_C & \xrightarrow{} & 0
\end{array}$$

Indeed, the map $H^1(C, T_C) \to H^1(C, T_C(D))$ is naturally identified with the pull-back of extensions $\text{Ext}^1_{\mathcal{O}_C} (\omega_C, \mathcal{O}_C) \to \text{Ext}^1_{\mathcal{O}_C} (\omega_C (-D), \mathcal{O}_C)$.

The following is a first relation between the rank of a deformation and the invariants of a supporting divisor.
Lemma 2.3. Suppose $\xi$ is supported on $D$. Then $H^0 (C, \omega_C (-D)) \subseteq \ker \partial_\xi$. In particular,

$$\text{rk } \xi \leq \deg D - r (D).$$

Proof. The fact that $\xi_D$ is split implies that all the sections of $\omega_C (-D)$ lift to sections of $\Omega^1_{\mathbb{C}/\mathbb{C}}$, and hence belong to the kernel of $\partial_\xi$. The inequality is an easy consequence of Riemann-Roch, because

$$\text{rk } \xi = g - \dim \ker \partial_\xi \leq g - h^0 (C, \omega_C (-D)) = g - (r (D) - d + g) = d - r (D).$$

We will need also a lower-bound on $\text{rk } \xi$, which was first proved by Ginensky in [5]. We include here a different (and shorter) proof, suggested to us by G.P. Pirola. Recall that the Clifford index of any divisor $D$ is defined as

$$\text{Cliff } (D) = \deg D - 2r (D).$$

Recall also that the Clifford index of the curve $C$ is

$$\text{Cliff } (C) = \min \{ \text{Cliff } (D) \mid h^0 (C, \mathcal{O}_C (D)) , h^1 (C, \mathcal{O}_C (D)) \geq 2 \}.$$

Theorem 2.4. If $\xi$ is minimally supported on $D$, then

$$\text{rk } \xi \geq \deg D - 2r (D) = \text{Cliff } (D).$$

Proof. Since $\xi$ is supported on $D$, the inclusion $\omega_C (-D) \hookrightarrow \omega_C$ factors through $i_D : \omega_C (-D) \hookrightarrow \Omega^1_{\mathbb{C}/\mathbb{C}}$.

Claim: If $D$ supports $\xi$ minimally, the cokernel $K_D$ of $i_D$ is torsion-free.

Assuming the claim for a moment, the proof finishes as follows. On the one hand, comparing determinants one has $K_D \cong \mathcal{O}_C (D)$, giving the exact sequence of sheaves

$$0 \rightarrow \omega_C (-D) \rightarrow \Omega^1_{\mathbb{C}/\mathbb{C}} \rightarrow \mathcal{O}_C (D) \rightarrow 0,$$

from which the inequality

$$h^0 \left( C, \Omega^1_{\mathbb{C}/\mathbb{C}} \right) \leq h^0 (C, \omega_C (-D)) + h^0 (C, \mathcal{O}_C (D))$$

follows. On the other hand, from the exact sequence of cohomology of $\xi$, one gets

$$g - \text{rk } \xi = \dim \ker \partial_\xi = h^0 \left( C, \Omega^1_{\mathbb{C}/\mathbb{C}} \right) - 1. \quad (6)$$

Combining the inequality [5] and the equality [6] with Riemann-Roch, one finally obtains

$$g - \text{rk } \xi \leq h^0 (C, \omega_C (-D)) + h^0 (C, \mathcal{O}_C (D)) - 1 = 2r (D) - \deg D + g.$$

Proof of the claim: We will show in fact that if $K_D$ has torsion, then $D$ does not support $\xi$ minimally. Indeed, if $T \neq 0$ is the torsion subsheaf of $K_D$, there is a line bundle $\mathcal{M}$ such that

$$0 \rightarrow \mathcal{M} \rightarrow \Omega^1_{\mathbb{C}/\mathbb{C}} \rightarrow K_D / T \rightarrow 0 \quad \text{and} \quad 0 \rightarrow \omega_C (-D) \rightarrow \mathcal{M} \rightarrow T \rightarrow 0.$$

The image of the composition $\phi : \mathcal{M} \hookrightarrow \Omega^1_{\mathbb{C}/\mathbb{C}} \rightarrow \omega_C$ contains $\omega_C (-D)$ by construction. Therefore $\phi$ is injective, and gives an isomorphism $\mathcal{M} \cong \omega_C (-E)$ for some $0 \leq E < D$. Since the inclusion $\mathcal{M} \hookrightarrow \omega_C$ factors through $\Omega^1_{\mathbb{C}/\mathbb{C}}$, this contradicts the minimality of $D$, as wanted. \qed
2.2 Fibred surfaces

We now recall some results on fibred surfaces proved in the previous work [6] of one of the authors. Let \( f : S \to B \) be a fibration of genus \( g \) and relative irregularity \( q_f \). We say that \( f \) is isotrivial if all the smooth fibres are isomorphic. For any smooth fibre \( C_b \), the kernel of the restriction map \( H^0(S, \Omega^1_S) \to H^0(C_b, \omega_{C_b}) \) is exactly \( f^*H^0(B, \omega_B) \). Therefore, there is an injection

\[
V_f := H^0(S, \Omega^1_S) / f^*H^0(B, \omega_B) \to H^0(C_b, \omega_{C_b})
\]

which implies the inequality \( q_f \leq g \).

For any finite map \( \pi : B' \to B \), let \( S' = \tilde{S} \times_B B' \) be the minimal desingularization of the fibred product, and \( f' : S' \to B' \) the induced fibration. The fibres of \( f' \) obviously have the same genus as the fibres of \( f \), but for the relative irregularity only the inequality \( q_{f'} \geq q_f \) can be proved (which might be strict). Indeed, for any \( b \in B \) where \( \pi \) is not ramified, the injection \( V_f \) factors as

\[
V_f \to V_{f'}, \to H^0(C_b, \omega_{C_b}),
\]

which forces the first map to be injective, and hence \( q_{f'} \leq q_f \).

For any smooth fibre \( C_b \), denote by \( \xi_b \in H^1(C_b, T_{C_b}) \) the class of the first order deformation induced by \( f \).

**Definition 2.5** ([6] Definition 2.13). Let \( D \subset S \) be an effective divisor. The fibration \( f \) is supported on \( D \) if for a general \( b \in B \), \( \xi_b \) is supported on \( D|_{C_b} \).

Note that this definition is local around the smooth fibres. As a consequence, if \( f \) is supported on \( D \) and we perform a change of base \( \pi : B' \to B \) as above, then \( f' \) is supported on \( \pi'^*D \subset S' \) (where \( \pi' : S' \to S \) is the induced map between the surfaces).

The existence of supporting divisors is investigated in [6]. For our current purposes, the most useful result is the following.

**Theorem 2.6** ([6] Corollary 3.15). If \( f : S \to B \) is a fibration of genus \( g \) such that \( q_f > \frac{g+1}{2} \), then after a base change as above, \( f' \) is supported on a divisor \( D \) such that \( D \cdot C_b < 2g - 2 \) for any fibre \( C_b \). Furthermore, if \( f \) is relatively minimal with reduced fibres, then \( D \cdot C \leq 2g(C) - 2 - C^2 \) for any component \( C \) of a fibre.

If a supporting divisor \( D \) is rigid on a general fibre, then there is a strong result on the structure of the fibred surface:

**Theorem 2.7** ([6] Theorem 2.27). Suppose that \( f \) is stable with \( q_f \geq 2 \) and that it is supported on an effective divisor \( D \) without components contained in fibres. Suppose also that \( D \cdot C \leq 2g(C) - 2 - C^2 \) for any component \( C \) of a fibre, and that \( b^0(C_b, \mathcal{O}_{D|_{C_b}}(D|_{C_b})) = 1 \) for some smooth fibre \( C_b \). Then there is another fibration \( h : S \to B' \) over a curve of genus \( g(B') = q_f \). In particular \( S \) is a covering of the product \( B \times B' \), and both surfaces have the same irregularity.

2.3 Adjoint images

We will give an alternative proof of part of Theorem 2.27 with more local flavor, using the adjoint images (introduced by Collino and Pirola in [3]) and the Volumetric Theorem (proved by Pirola and Zucconi in [10]). In fact, the adjoint images are already a fundamental tool in the proof of Theorem 2.27. Although the theory can be developed for varieties of any dimension, we will recall only the simplest case of curves, which is enough for our objective.

Let \( C \) be a smooth curve of genus \( g \geq 2 \), and \( 0 \neq \xi \in H^1(C, T_C) \) a non-trivial first order deformation, corresponding to the extension

\[
0 \to \mathcal{O}_C \to \mathcal{E} \to \omega_C \to 0.
\]

Suppose that the kernel \( K_\xi \) of \( \partial_\xi = \cup \xi : H^0(C, \omega_C) \to H^1(C, \mathcal{O}_C) \) has dimension at least 2, and let \( \eta_1, \eta_2 \in K_\xi \) be two linearly independent 1-forms. Take \( s_i \in H^0(C, \mathcal{E}) \) arbitrary preimages of the \( \eta_i \), and
let \( w \in H^0(C,\omega_C) \) be the 1-form corresponding to \( s_1 \wedge s_2 \) by the natural isomorphism \( \wedge^2 \mathcal{E} \cong \omega_C \). It turns out that the class \([w]\) of \( w \) modulo the span \( W \) of \( \{\eta_1,\eta_2\} \) is well-defined, independently of the choice of the preimages \( s_i \).

**Definition 2.8.** The class \([w] \in H^0(C,\omega_C)/W\) is the adjoint class of \( \{\eta_1,\eta_2\} \).

Changing \( \{\eta_1,\eta_2\} \) by another basis of \( W \) amounts to multiply \([w]\) by the determinant of the change of basis. Therefore, whether \([w]\) vanishes or not is an intrinsic property of the subspace \( W \), and not only of the chosen basis. Moreover, the Adjoint Theorem (\[3\] Th. 1.1.8) says that if \([w]=0\), then the deformation \( \xi \) is supported on the base divisor of the linear system \([W] \subseteq [\omega_C] \).

In this work we will use another result about adjoint images: the Volumetric Theorem, which we introduce now in the case of a family of curves. Let \( \pi : C \rightarrow U \) be a smooth family of curves over an open disc \( U \), and for every \( u \in U \), let \( \xi_u \) be the induced first order deformation of the fibre \( C_u \). Let \( A \) be an Abelian variety, and let \( \Phi : C \rightarrow A \times U \) be a morphism such that \( p_2 \circ \Phi = \pi \) where \( p_2 \) denotes the second projection of the product \( A \times U \), that is, a family of morphisms \( \phi_u : C_u \rightarrow A \) from the fibres of \( \pi \) onto a fixed Abelian variety \( A \). Given a 2-dimensional subspace \( W \subseteq H^0(A,\Omega^1_A) \), denote by \( W_u = \phi_u^*W \subseteq H^0(C_u,\omega_{C_u}) \) its pull-back to \( C_u \). Since the elements of \( W_u \) extend to all the fibres by construction, \( W_u \) is contained in the kernel of \( \partial_{\xi_u} \), so it is possible to define \([w_u]\), the adjoint class of \( W_u \) corresponding to some chosen basis of \( W \).

**Theorem 2.9** (Volumetric Theorem(\[11\], Theorem 1.5.3)). Keeping the above notations, assume that \( \pi \) is not isotrivial. Suppose also that for some \( u_0 \in U \), \( \phi_{u_0} : C_{u_0} \rightarrow A \) is birational onto its image \( Y_{u_0} \), and that \( Y_{u_0} \) generates \( A \) as a group. Then, for general 2-dimensional \( W \subseteq H^0(A,\Omega^1_A) \) and general \( u \in U \), the adjoint class \([w_u]\) is non-zero.

### 3 The Main Theorem

We are now ready to prove our main result. The statement involves the Clifford index of the fibration, which was defined by Konno as follows.

**Definition 3.1** (\[7\] Def. 1.1). Given a fibration \( f : S \rightarrow B \), its Clifford index is defined as 
\[
c_f = \max\{\text{Cliff}(C_b) \mid C_b = f^{-1}(b) \text{ is smooth}\},
\]
which is attained for \( b \) ranging in a non-empty Zariski-open set.

With this definition, the announced main result is the following

**Theorem 3.2.** Let \( f : S \rightarrow B \) be a fibration of genus \( g \geq 2 \), relative irregularity \( q_f \) and Clifford index \( c_f \). If \( f \) is non-isotrivial, then
\[
q_f \geq g - c_f.
\]

**Proof of Theorem 3.2.** Suppose, looking for a contradiction, that the fibration \( f : S \rightarrow B \) is non-isotrivial and that \( q_f > g - c_f \). Furthermore, after a suitable base change and blowing down the \((-1)\)-curves contained in the fibres, we may also assume that \( f \) is stable. In particular, since \( c_f \leq \lfloor \frac{2g+1}{2} \rfloor \), we have \( q_f > \frac{2g+1}{2} \). Hence we can apply Theorem 2.10 and assume, possibly after a change of base, that \( f \) is supported on a divisor \( D \subset S \) such that \( D \cdot C < 2g - 2 \) for any fibre \( C \), and also \( D \cdot C \leq 2g(C) - 2 - C^2 \) for any component \( C \) of a fibre. Note that the inequality \( q_f > \frac{2g+1}{2} \) combined with \( g \geq q_f \) implies that \( g \geq 2 \).

We consider now two cases:

Case 1: The divisor \( D \) is relatively rigid, that is \( h^0(C,\mathcal{O}_C(D)) = 1 \) for some smooth fibre \( C = C_b \). In this case we can apply Theorem 2.7 to obtain a new fibration \( h : S \rightarrow B' \) over a curve of genus \( g(B') = q_f \). Let \( \phi : C \rightarrow B' \) be the restriction of \( h \) to the smooth fibre \( C \). Applying Riemann-Hurwitz we obtain
\[
2g - 2 \geq \deg \phi(2q_f - 2).
\]
At the beginning of the proof we obtained that $q_f > \frac{g+1}{2}$, so that $2q_f - 2 > g - 1$, and thus

$$2(g - 1) > \deg \phi (g - 1).$$

It follows that $\deg \phi = 1$, so every smooth fibre is isomorphic to $B'$ and hence $f$ is isotrivial.

**Case 2:** The divisor $D$ moves on any smooth fibre, i.e. $h^0 (C_b, \mathcal{O}_{C_b} (D)) \geq 2$ for every regular value $b \in B$.

After a further change of base, we may assume that $D$ consists of $d$ sections of $f$ (possibly with multiplicities), and the new fibration is still supported on $D$. Then we can replace $D$ by a minimal sub divisor $D' \leq D$ such that $\xi$ is still supported on $D'$. Since the components of $D$ are sections of $f$, this implies that for general $b \in B$, the deformation $\xi_b$ is minimally supported on $D_b$. Note that this might not be true if the supporting divisors were not a union of sections, as different points of $D_b$ lying on the same irreducible component of $D$ may be redundant.

If this new $D$ is rigid on the general fibres, the proof finishes as in Case 1. Otherwise, if it still holds $h^0 (C_b, \mathcal{O}_{C_b} (D)) \geq 2$ for general $b \in B$, we may use Theorem 2.4 to obtain

$$\text{rk} \xi_b \geq \text{Cliff} (D_b) = c_f. \tag{8}$$

But $V_f \subseteq \ker \partial \xi_b = K_\xi_b$, so that $\text{rk} \xi_b = g - \dim K_\xi_b \leq g - q_f$, and the inequality (8) implies that

$$g - q_f \geq c_f,$$

contradicting our very first hypothesis.

\[\square\]

**Corollary 3.3.** If $f$ is not isotrivial and has maximal Clifford index, i.e., $c_f = \left\lfloor \frac{g-1}{2} \right\rfloor$, then $q_f \leq \left\lfloor \frac{g+1}{2} \right\rfloor$.

**Remark 3.4.** Note that, whenever we can produce a relatively rigid divisor $D$ supporting the fibration, the inequality $q_f > \frac{g+1}{2}$ is enough to prove that the fibration $f$ is isotrivial (together with the structure Theorem 2.7), while the stronger inequality $q_f > g - c_f$ is used only if it is impossible to find such a $D$ (even allowing arbitrary changes of base). Hence, all possible counterexamples to Xiao’s original conjecture must fall into this second case.

**Remark 3.5.** Although in general our bound is better than the general one proved by Xiao, for small $c_f$ our Theorem is worse. As a extremal case, if the general fibres are hyperelliptic, $c_f = 0$ and Theorem 3.3 has no content at all. But in this special case, the strong inequality $q_f \leq \frac{g+1}{2}$ has been recently proved by Lu Xin and Kang Zuo in [15] (Theorem 1.4). A different proof of the same result can be carried out using the results of Pirola in [8] about rigidity of rational curves on Kummer varieties.

We wish to close this final section with Proposition 3.6 which gives an alternative proof of Case 1 in the proof of Theorem 3.2. This Proposition uses the Volumetric Theorem 2.4 instead of Theorem 2.7 hence applies for non-necessarily compact families. On the contrary, the compactness of the surface is crucial in Theorem 2.7 since its proof uses the Castelnuovo-de Franchis Theorem (see [9]).

**Proposition 3.6.** Suppose that $f : S \to B$ is a fibration where the base $B$ is a smooth, not necessarily compact curve. Assume that there is an Abelian variety $A$ of dimension $a$, and a morphism $\Phi : S \to A \times B$ respecting the fibres of $f$ and such that the image of any restriction to a fibre $\phi_b : C_b \to A$ generates $A$. Suppose also that the deformation is supported on a divisor $D \subseteq S$ such that $h^0 (C_b, \mathcal{O}_{C_b} (D)) = 1$ for general $b \in B$. If $a > \frac{g+1}{2}$, then $f$ is isotrivial.

**Remark 3.7.** If we start from a fibration with compact $B$, we may take $A$ to be the kernel of the map induced between the Albanese varieties $\alpha_f : \text{Alb} (S) \to J (B)$, which has dimension $a = q_f$. After replacing $B$ by an open disk, the Albanese map gives a morphism $\Phi$ as in the hypothesis. Hence, Proposition 3.6 gives indeed a new proof of the first case in the proof of Theorem 3.2 above.
Proof of Proposition 3.6. Take any \( b \in B \) such that \( C_b \) is smooth, and let \( \overline{C}_b \) be the image of \( \phi_b : C_b \to A \). Since \( \overline{C}_b \) generates \( A \), it has genus \( g' \geq \dim A = a > \frac{2a+1}{3} \). This implies, by Riemann-Hurwitz, that \( \phi_b \) is birational onto its image for any regular value \( b \in B \).

If \( f \) is not isotrivial, the Volumetric Theorem 2.9 implies that, for a general fibre \( C = C_b \), the adjoint class of a generic 2-dimensional subspace

\[
W \subseteq V := H^0(A, \Omega^1_A) \subseteq H^0(C, \omega_C)
\]

is non-zero.

However, we will now show that, for every fibre, the adjoint class of every 2-dimensional subspace of \( V \) vanishes, which finishes the proof. Fix any regular value \( b \in B \) and denote by \( C = C_b \) the corresponding fibre, by \( \xi = \xi_b \) the infinitesimal deformation induced by \( f \), and by \( D = D_C \) the restriction of the global divisor. Let also \( K = K_\xi \) be the kernel of \( \partial_\xi \). Since \( \xi \) is supported on \( D \), Lemma 2.3 gives the inclusion

\[ H^0(C, \omega_C(-D)) \subseteq K \]

which is in fact an equality. Indeed, on the one hand we have

\[ \dim H^0(C, \omega_C(-D)) = g - \deg D \]

because \( D \) is rigid, while on the other hand it holds

\[ \dim K = g - \rk \xi = g - \deg D \]

because of the combination of Lemma 2.3 and Theorem 2.4. Therefore, \( V \subseteq K = H^0(C, \omega_C(-D)) \).

Now, since \( \xi \) is supported on \( D \), the upper sequence in

\[
\begin{array}{cccccc}
\xi_D : & 0 & \to & \mathcal{O}_C & \to & \mathcal{F}_D \\
\xi : & 0 & \to & \mathcal{O}_C & \to & \Omega^1_{S|C} \\
\end{array}
\]

is split, giving a lifting \( \omega_C(-D) \rightarrow \Omega^1_{S|C} \) such that every pair of elements of \( H^0(C, \omega_C(-D)) \subseteq H^0(C, \Omega^1_{S|C}) \) wedge to zero (they are sections of the same sub-line bundle of \( \Omega^1_{S|C} \)), which finishes the proof.

Remark 3.8. In the above proof, to show that the images \( \overline{C}_b \) are all isomorphic it is only necessary to use the Volumetric Theorem 2.9. The inequality \( a > \frac{2a+1}{3} \) is only used, combined with Riemann-Hurwitz, to show that the maps \( \phi_b \) are birational. Therefore, if we drop the inequality \( a > \frac{2a+1}{3} \) from the hypothesis (but still keep that the deformations are supported on rigid divisors), the same proof shows that the fibres \( C_b \) are coverings of a fixed curve \( \overline{C}_b \).


Miguel Ángel Barja
Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya (UPC-BarcelonaTECH)
Av. Diagonal 647, 08028 Barcelona, Spain
miguel.angel.barja@upc.edu

Víctor González-Alonso
Institut für Algebraische Geometrie, Leibniz Universität Hannover
Welfengarten 1, 30167 Hannover, Germany
gonzalez@math.uni-hannover.de
Previous address: Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya (UPC-BarcelonaTECH), Av. Diagonal 647, 08028 Barcelona, Spain.

Juan Carlos Naranjo
Departament d’Àlgebra i Geometria, Universitat de Barcelona
Gran Via de les Corts Catalanes 585, 08007 Barcelona, Spain
jcnaranjo@ub.edu