On perfect and quasiperfect domination in graphs *

J. Cáceres[1], C. Hernando[2], M. Mora[2], I. M. Pelayo[2], and M. L. Puertas[1]

[1] University of Almería {jcaceres,mpuertas}@ual.es
[2] Polytechnical University of Cataluña
{carmen.hernando,merce.mora,ignacio.m.pelayo}@upc.edu

Abstract. Given a graph G, a set $D \subset V(G)$ is a dominating set of G if every vertex not in D is adjacent to at least one vertex of D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G.

If moreover, every vertex not in D is adjacent to exactly one vertex of D, then D is called a perfect dominating set of G. The perfect domination number $\gamma_{11}(G)$ is the minimum cardinality of a perfect dominating set of G. In general, for every integer $k \geq 1$, a dominating set D is called a k-quasiperfect dominating set if every vertex not in D is adjacent to at most k vertices of D. The k-quasiperfect domination number $\gamma_{1k}(G)$ is the minimum cardinality of a k-quasiperfect dominating set of G. These parameters are related in the following general way (Δ the maximum degree of G and by n the number of vertices): $\gamma(G) = \gamma_{1\Delta}(G) \leq \cdots \leq \gamma_{12}(G) \leq \gamma_{11}(G) \leq n$.

In this work we study the perfect domination number, with the help of this decreasing chain of domination parameters, in the following graph families: graphs with extremal maximum degree, that is, graphs with $\Delta \geq n-3$ or $\Delta = 3$, and also in cographs, claw-free graphs and trees. We also study the behavior of these parameters under some usual product operations.

Key words: Perfect domination, quasiperfect domination, claw-free graphs, cographs.

1 Introduction

All the graphs considered are finite, undirected, simple, and connected. Given a graph $G = (V, E)$, the *open neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V | uv \in E\}$ and the *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* $deg(v)$ of a vertex $v \in V(G)$ is the number of neighbors of v, i.e., $deg(v) = |N(v)|$. The *maximum degree* of G, denoted by $\Delta(G)$, is the largest degree among all vertices of G. For undefined basic concepts we refer the reader to introductory graph theoretical literature, e.g., [3].

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Given a graph G, a set $D \subseteq V(G)$ is a *dominating set* of G if every vertex v not in D is adjacent to at least one vertex of D, i.e., if $N(v) \cap D \neq \emptyset$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set of cardinality $\gamma(G)$ is called a γ-*code*[5].

If moreover, every vertex not in D is adjacent to exactly one vertex of D, then D is called a *perfect dominating set* of G [1,7]. The *perfect domination number* $\gamma_{11}(G)$ is the minimum cardinality of a perfect dominating set of G. A dominating set of cardinality $\gamma_{11}(G)$ is called a γ_{11}-*code*. This definition can be generalized in the following way.

Definition 1 ([4]). *For $k \geq 1$, we define a dominating subset $S \subseteq V$ in a graph $G = (V, E)$ to be a* k-quasiperfect dominating set *if every vertex not in D is adjacent to at most k vertices of D.*

Definition 2 ([4]). *For $k \geq 1$, The* k-quasiperfect domination number $\gamma_{1k}(G)$ *is the minimum cardinality of a k-quasiperfect dominating set of G. A dominating set of cardinality $\gamma_{1k}(G)$ is called a* γ_{1k}-code.

Certainly, 1-quasiperfect dominating sets and Δ-quasiperfect dominating sets are precisely the perfect dominating sets and dominating sets, respectively. There is an obvious relationship among these domination paramenters. If G is a graph of order n and maximum degree Δ, then

$$\gamma(G) = \gamma_{1\Delta}(G) \leq \ldots \gamma_{12}(G) \leq \gamma_{11}(G) \leq n$$

In this work we study this decreasing chain of domination parameters. We present our main contributions when restricting ourselves to the following graph families:

- Graphs with maximum degree $\Delta \geq n - 3$ or $\Delta = 3$.
- Cographs.
- Claw-free graphs.
- Trees.

We also study the behavior of these parameters under product operations.

2 Results

Theorem 1 ([4]). *If G is a graph of order n that satisfies some of the following conditions, then $\gamma(G) = \gamma_{12}(G)$:*

- *$\Delta(G) \geq n - 3$.*
- *$\Delta(G) \leq 2$.*
- *G is a P_4-free graph (cograph).*
- *G is a $K_{1,3}$-free graph (claw-free graph).*
- *Every vertex of G is either a support vertex or has degree at most 2.*

As a result of Theorem above, in graphs that satisfy some of its conditions the chain of quasiperfect domination parameters is shorter that in the general case: $\gamma(G) = \gamma_{12}(G) \leq \gamma_{11}(G) \leq n$, and it is interesting to consider what happen with the parameter γ_{11}. We have obtained the following results.

2.1 Graphs with maximum degree $\Delta(G) \geq n-3$

In this case we have obtained realization results for the parameter γ_{11}, that show that it can achieve all values in the interval between 2 and n, with a small number of exceptions.

Theorem 2. *Let k, n be integers such that $n \geq 4$, $2 \leq k \leq n$ and $(n,k) \notin \{(5,5),(5,4),(4,4),(4,3)\}$. Then, there exists a graph $G=(V,E)$ of order n such that $\Delta(G) = n-2$ and $\gamma_{11}(G) = k$.*

Theorem 3. *Let k, n be positive integers such that $n \geq 8$ and $2 \leq k \leq n$. Then, there exists a graph G of order n such that $\Delta(G) = n-3$ that satisfies $\gamma_{11}(G) = k$.*

2.2 Graphs with small maximum degree

The family of connected graphs with maximum degree $\Delta = 2$ contains just paths and cycles, and in both cases parameter γ_{11} is completely determined: $\gamma_{11}(P_n) = \lceil \frac{n}{3} \rceil$ and $\gamma_{11}(C_n) = \lceil \frac{2n}{3} \rceil - \lfloor \frac{n}{3} \rfloor$. So we focus on graphs with maximum degree $\Delta = 3$ and we have obtained the following result that provide an upper bound for γ_{11}.

Theorem 4. *If $\Delta(G) = 3$ and G is other than the bull graph, then $\gamma_{11}(G) \leq n-3$. Note also that the bull graph H has 5 vertices and $\gamma_{11}(H) = 3 = n-2$*

2.3 Cographs

In the family of P_4-free graphs, we have calculated the exact values of γ_{11}, depending on the value of de domination parameter γ.

Theorem 5. *Let G be a cograph of order n. Then:*

- *If $\gamma(G) = 2$, then $\gamma_{11}(G) \in \{2, n\}$.*
- *Cographs such that $\gamma(G) = \gamma_{11}(G) = 2$ are completely characterized.*
- *If $\gamma(G) \geq 3$, then $\gamma_{11}(G) = n$.*

2.4 Claw-free graphs

In this family of graphs, we have also studied the values of γ_{11} in relationship with the values of γ. But in contrast with the case above, the family of cographs, in this occasion a wider range of values can be achieved.

Theorem 6. *Let h, k, n be integers such that $2 \leq h \leq k < n$ and $h + k \leq n$. Then, there exists a claw-free graph G of order n such that $\gamma(G) = h$ and $\gamma_{11}(G) = k$.*

Proposition 1. *Let n be an integer such that $n \geq 6$. Then,*

- *there exists a claw-free graph G of order n and such that $\gamma(G) = 2$ and $\gamma_{11}(G) = n - 1$,*
- *there exists a claw-free graph G of order n and such that $\gamma(G) = 2$ and $\gamma_{11}(G) = n$.*

Proposition 2. *Let h, n be integers such that $n \geq 7$, $2 \leq h \leq \lfloor \frac{n-1}{3} \rfloor$. Then, there exists a claw-free graph G of order n such that $\gamma(G) = h$ and $\gamma_{11}(G) = n$.*

3 Trees

The following result about trees is known.

Theorem 7 ([2]).
Let T be a tree of order $n \geq 3$ with k leaves. Then,

- *Every $[1, 1]$-set contains all its strong support vertices.*
- $\gamma_{11}(T) \leq \frac{n}{2}$.
- $\gamma_{11}(T) = \frac{n}{2}$ *if and only if* $T = T' \odot K_1$, *for some tree* T'.
- $\gamma_{11}(T) \leq n - k$.
- $\gamma_{11}(T) = n - k$ *if and only if T contains a $[1, 1]$-code D such that $V \setminus D$ induces a coclique.*

So we focus our attention on the relationship between γ and γ_{11}. We have obtained a complete result in the particular case of caterpillars and a general inequality between both parameters that is satisfied for any tree.

Proposition 3. *Let T be a caterpillar. Then*

$$\gamma(T) = \gamma_{12}(T) \leq \gamma_{11}(T) < 2\gamma(T)$$

Proposition 4. *Let $\{h, k, n\}$ be integers with $1 \leq h \leq k \leq \frac{n}{2}$ and $h < 2k$. Then there exists a caterpillar T of order n such that $\gamma_{12}(T) = h$, $\gamma_{11}(T) = k$.*

Theorem 8. *For every tree T, $\gamma(T) \leq \gamma_{11}(T) \leq 2\gamma(T) - 1$. Moreover, both bounds are tight.*

4 Product graphs

Finally we present some results on the behavior of the quasiperfect domination parameters with standard product operations.

We begin with the *cartesian product* [6] of two connected graphs G and H, denoted by $G\Box H$, which is the graph with the vertex set $V(G)\times V(H)$ in which vertices (g,h) and (g',h') are adjacent whenever $gg'\in E(G)$ and $h=h'\in E(H)$ or $g=g'\in E(G)$ and $hh'\in E(H)$. The following result is known.

Proposition 5 ([4]). *For every grid graph* $G=P_h\Box P_k$, $\gamma_{13}(G)=\gamma(G)$.

We have obtained a general upper bound for this product-type operation.

Theorem 9. *Let G and H be two graphs and let r be an integer. Then,* $\gamma_{1r}(G\Box H)\leq\min\{\gamma_{1r}(G)|V(H)|,|V(G)|\gamma_{1r}(H)\}$. *Moreover, this bound is tight.*

On the other hand, the *strong product* [6] of graphs two connected G and H, denoted by $G\boxtimes H$, is the graph such that $V(G\boxtimes H)=(V(G)\times V(H)$ and $E(G\boxtimes H)=E(G\times H)\cup E(G\Box H)$. In this case, the following result is proved.

Proposition 6. *Let G be a graph and let k be an integer such that* $\gamma_{1k}(G)=|V(G)|$. *Then,* $\gamma_{1k}(G\boxtimes H)=|V(G\boxtimes H)|$, *for any graph H.*

Finally we have calculated exact values of parameters γ_{11} and γ_{12} for strong product of paths, cycles and complete graphs.

Proposition 7. $\gamma_{11}(P_r\boxtimes P_s)=\gamma(P_r\boxtimes P_s)=\gamma(P_r)\cdot\gamma(P_s)$

Proposition 8.

- $\gamma_{12}(C_r\boxtimes C_s)=\gamma(C_r\boxtimes C_s)=\gamma(C_r)\gamma(C_s)=\lceil\frac{r}{3}\rceil\lceil\frac{s}{3}\rceil$.
- $\gamma_{11}(C_r\boxtimes C_s)=\gamma(C_r\boxtimes C_s)$, *if* $r=3a$ *and* $s=3b$.
- $\gamma_{11}(C_r\boxtimes C_s)=rs=n$, *if* $r\neq 3a$ *or* $s\neq 3b$.

Proposition 9. $\gamma_{11}(K_r\boxtimes P_s)=\gamma(K_r\boxtimes P_s)=\lceil\frac{s}{3}\rceil$

Proposition 10.

- $\gamma_{12}(K_r\boxtimes C_s)=\gamma(K_r\boxtimes C_s)=\gamma(C_s)=\lceil\frac{s}{3}\rceil$.
- $\gamma_{11}(K_r\boxtimes C_s)=\gamma(K_r\boxtimes C_s)$, *if* $s=3a$.
- $\gamma_{11}(K_r\boxtimes C_s)=rs=n$, *if* $s\neq 3a$.

Proposition 11.

- $\gamma_{12}(C_r\boxtimes P_s)=\gamma(C_r\boxtimes P_s)=\gamma(C_r)\gamma(P_s)=\lceil\frac{r}{3}\rceil\lceil\frac{s}{3}\rceil$.
- $\gamma_{11}(C_r\boxtimes P_s)=\gamma(C_r\boxtimes P_s)$, *if* $r=3a$.
- $\gamma_{11}(C_r\boxtimes P_s)=rs=n$, *if* $r\neq 3a$.

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