LD-graphs and global location-domination in bipartite graphs ⋆

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Abstract. A dominating set S of a graph G is a *locating-dominating-set*, *LD-set* for short, if every vertex v not in S is uniquely determined by the set of neighbors of v belonging to S. Locating-dominating sets of minimum cardinality are called *LD-codes* and the cardinality of an LD-code is the *location-domination number*, $\lambda(G)$. An LD-set S of a graph G is *global* if it is an LD-set for both G and its complement, $\overline{G}$. One of the main contributions of this work is the definition of the *LD-graph*, an edge-labeled graph associated to an LD-set, that will be very helpful to deduce some properties of location-domination in graphs. Concretely, we use LD-graphs to study the relation between the location-domination number in a bipartite graph and its complement.

Key words: domination, location, complement graph, bipartite graph.

1 Introduction

In this work, $G = (V, E)$ stands for a simple, finite graph. The *open neighborhood* of a vertex $v \in V$ is $N_G(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* is $N_G[v] = \{u \in V : uv \in E\} \cup \{v\}$. We write $N(v)$ or $N[v]$ if the graph G is clear from the context. The *complement* of a graph G, denoted by $\overline{G}$, is the graph on the same vertices such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

A set $D \subseteq V$ is a *dominating set* if for every vertex $v \in V \setminus D$, $N(v) \cap D \neq \emptyset$. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set is *global* if it is a dominating set for both G and its complement graph, $\overline{G}$. If D is a subset of V and $v \in V \setminus D$, we say that v *dominates* D if $D \subseteq N(v)$.

A dominating set $S \subseteq V$ is a *locating-dominating set*, *LD-set* for short, if for every two different vertices $u, v \in V \setminus S$, $N(u) \cap S \neq N(v) \cap S$. The

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location-domination number of G, denoted by $\lambda(G)$, is the minimum cardinality of a locating-dominating set. A locating-dominating set of cardinality $\lambda(G)$ is an *LD-code* [11]. LD-codes and the location-domination parameter have been intensively studied during the last decade; see [1,2,5,6,7] A complete and regularly updated list of papers on locating dominating codes can be found in [9].

In the following section we introduce the LD-graph associated to an LD-set. After that, we study the relation between LD-sets and the location-domination number in a graph and its complement. Finally, we consider this parameter for connected bipartite graphs. We omit proofs due to space limitations.

2 The LD-graph associated to an LD-set

We introduce in this section the so-called *LD-graph*, an edge-labeled graph associated to an LD-set. This graph will allow us to deduce some properties of LD-sets and the location-domination number of graphs.

Let S be an LD-set of a graph G of order n. Consider $z \notin V(G)$ and define $N_G(z) = \emptyset$. Let $\triangle$ denote the symmetric difference set operation. The *LD-graph* associated to S, denoted by G^S, is the edge-labeled graph defined as follows:

i) $V(G^S) = (V \setminus S) \cup \{z\}$;
ii) $E(G^S) = \{xy \,|\, x, y \in V(G^S)\,,\, |\big(N_G(x) \cap S\big) \triangle \big(N_G(y) \cap S\big)| = 1\}$;
iii) The label of edge $xy \in E(G^S)$ is $\ell(xy) = \big(N_G(x) \cap S\big) \triangle \big(N_G(y) \cap S\big) \in S$.

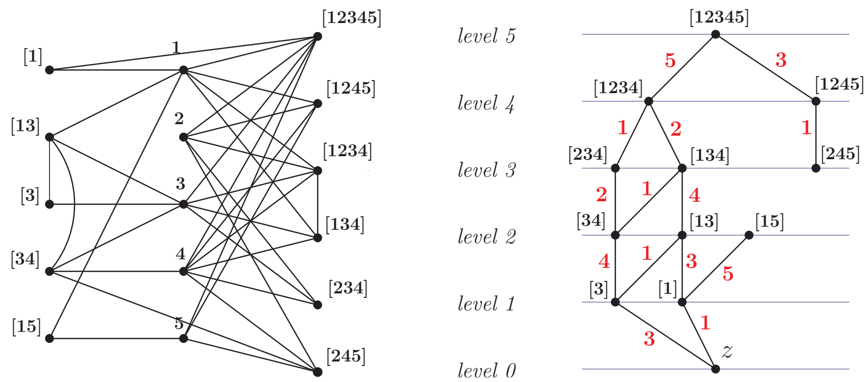

Fig. 1: Left: a graph G. Right: the LD-graph G^S associated to the LD-set $S = \{1, 2, 3, 4, 5\}$.

Notice that two vertices of $V \setminus S$ are adjacent in G^S if their neighborhood in S differ in exactly one vertex, the label of the edge, and z is adjacent to vertices of $V \setminus S$ with exactly a neighbor in S. Therefore, we can represent the graph G^S with the vertices lying on $|S|+1$ levels, from bottom (level 0) to top (level $|S|$), in such a way that vertices with exactly k neighbors in S are at level k. There is at most one vertex at level $|S|$ and, if it is so, this vertex is adjacent to all vertices of S. The vertices at level 1 are those with exactly one neighbor in S and z is the unique vertex at level 0. An edge of G^S has its endpoints at consecutive levels. Moreover, if $e = xy \in E(G^S)$, with $\ell(e) = u \in S$, and x is at exactly one level higher than y, then $N(x) \cap S = (N(y) \cap S) \cup \{u\}$, i.e., x and y have the same neighborhood in $S \setminus \{u\}$. Therefore, the existence of an edge in G^S with label $u \in S$ means that $S \setminus \{u\}$ is not an LD-set. Hence, if S is an LD-code, then for every $u \in S$ there exists at least an edge in G^S with label u. See an example of an LD-graph in Figure 1.

The following proposition states some properties of the LD-graph.

Proposition 1. *Let S be an LD-set with exactly r vertices of a connected graph $G = (V, E)$ of order n. Let G^S be the LD-graph associated to S. Then:*

i) $|V(G^S)| = n - r + 1$.
ii) *G^S is bipartite.*
iii) *Incident edges of G^S have different labels.*
iv) *Every cycle of G^S contains an even number of edges labeled v, for all $v \in S$.*
v) *Let ρ be a walk with no repeated edges in G^S. If, for every $v \in S$, ρ contains an even number of edges labeled v, then ρ is a closed walk.*
vi) *If $\rho = x_i x_{i+1} \ldots x_{i+h}$ is a path satisfying that vertex x_j lies at level j, for any $j \in \{i, i+1, \ldots, i+h\}$, then*
    (a) the labels of the edges of ρ are different;
    (b) for all $j \in \{i+1, i+2, \ldots, i+h\}$, $N(x_j) \cap S$ contains the vertex $\ell(x_k x_{k+1})$, for any $k \in \{i, i+1, \ldots, j-1\}$.

3 Global location domination

This section is devoted to approach the relationship between $\lambda(G)$ and $\lambda(\overline{G})$, for any arbitrary graph G.

Notice that $N_{\overline{G}}(x) \cap S = S \setminus N_G(x)$ for any set $S \subseteq V$ and any vertex $x \in V \setminus S$. A straightforward consequence of this fact are the following results.

Proposition 2 ([8]). *If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$, then S is an LD-set of $\overline{G}$ if and only if S is a dominating set of $\overline{G}$.*

Proposition 3 ([7]). *If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$, then S is an LD-set of $\overline{G}$ if and only if there is no vertex in $V \setminus S$ dominating S in G.*

Proposition 4 ([7]). *If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$ then there is at most one vertex $u \in V \setminus S$ dominating S, and in the case it exists, $S \cup \{u\}$ is an LD-set of $\overline{G}$.*

Theorem 1 ([7]). *For every graph G, $|\lambda(G) - \lambda(\overline{G})| \leq 1$.*

According to the preceding result, $\lambda(\overline{G}) \in \{\lambda(G) - 1, \lambda(G), \lambda(G) + 1\}$ for every graph G, all cases being feasible for some connected graph G. For example, it is straightforward to check that the complete graph K_n of order $n \geq 2$ satisfies $\lambda(\overline{K_n}) = \lambda(K_n) + 1$; the star $K_{1,n-1}$ of order $n \geq 2$ satisfies $\lambda(\overline{K_{1,n-1}}) = \lambda(K_{1,n-1})$, and the bi-star $K_2(r, s)$, $r, s \geq 2$, obtained by joining the central vertices of two stars $K_{1,r}$ and $K_{1,s}$, satisfies $\lambda(K_2(r, s)) = \lambda(\overline{K_2(r, s)}) + 1$.

We intend to obtain either necessary or sufficient conditions for a graph G to satisfy $\lambda(\overline{G}) > \lambda(G)$, i.e., $\lambda(\overline{G}) = \lambda(G) + 1$. After noticing that this fact is closely related to the existence or not of sets that are simultaneously locating-dominating sets in both G and its complement $\overline{G}$, the following definition was introduced in [8].

A set S of vertices of a graph G is a *global LD-set* if S is an LD-set for both G and its complement $\overline{G}$ and it is a *global LD-code* if it is an LD-code of G and an LD-set of $\overline{G}$. Next results follow immediately from the definition of global LD-set and global LD-code.

Proposition 5 ([8]). *If G is a graph with a global LD-code, then $\lambda(\overline{G}) \leq \lambda(G)$.*

Proposition 6 ([8]). *Let S be an LD-set of a graph G. Then, S is a non-global LD-set if and only if there exists a (unique) vertex $u \in V \setminus S$ such that $S \subseteq N(u)$.*

In Table 1, the location-domination number of some families of graphs is displayed, along with the location-domination number of their complement graphs. Concretely, we consider the path P_n of order $n \geq 7$; the cycle C_n of order $n \geq 7$; the wheel W_n of order $n \geq 8$, obtained by joining a new vertex to all vertices of a cycle of order $n-1$; the complete graph K_n of order $n \geq 2$; the complete bipartite graph $K_{r,n-r}$ of order $n \geq 4$, with $2 \leq r \leq n-r$ and stable sets of order r and $n-r$, respectively; the star $K_{1,n-1}$ of order $n \geq 4$, obtained by joining a new vertex to $n-1$ isolated vertices; and finally, the bi-star $K_2(r, n-r)$ of order $n \geq 6$ with $2 \leq r \leq n-r$, obtained by joining the central vertices of two stars $K_{1,r}$ and $K_{1,n-r}$ respectively.

4 The bipartite case

In this section we study the relation between $\lambda(G)$ and $\lambda(\overline{G})$ in bipartite connected graphs. Bipartite connected graphs of order at most 3 are the path

G	P_n	C_n	W_n	K_n	$K_{1,n-1}$	$K_{r,n-r}$	$K_2(r,n-r)$
n	$n \geq 7$	$n \geq 7$	$n \geq 8$	$n \geq 2$	$n \geq 4$	$2 \leq r \leq n-r$	$2 \leq r \leq n-r$
$\lambda(G)$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$	$n-1$	$n-1$	$n-2$	$n-2$
$\lambda(\overline{G})$	$\lceil \frac{2n-2}{5} \rceil$	$\lceil \frac{2n-2}{5} \rceil$	$\lceil \frac{2n+1}{5} \rceil$	n	$n-1$	$n-2$	$n-3$

Table 1: The values of $\lambda(G)$ and $\lambda(\overline{G})$ for some families of graphs.

graphs P_1, P_2 and P_3, and for these graphs, $\lambda(P_1) = \lambda(\overline{P_1}) = 1$; $1 = \lambda(P_2) < \lambda(\overline{P_2}) = 2$; $\lambda(P_3) = \lambda(\overline{P_3}) = 2$. In the sequel, $G = (V, E)$ stands for a bipartite connected graph of order $n = r + s \geq 4$, such that $V = U \cup W$, being U,W its stable sets and $1 \leq |U| = r \leq s = |W|$.

Proposition 7. *Let S be an LD-code of G. Then, $\lambda(\overline{G}) \leq \lambda(G)$ if any of the following conditions holds*

i) $S \cap U \neq \emptyset$ *and* $S \cap W \neq \emptyset$*;*
ii) $r < s$ *and* $S = W$*.*
iii) $2^r \leq s$*.*

Corollary 1. *If $\lambda(\overline{G}) = \lambda(G) + 1$, then $r \leq s \leq 2^r - 1$. Moreover, if $r < s$ then U is the unique LD-code of G, and if $r = s$ we may assume that U is a non-global LD-code of G.*

Proposition 8. *If $1 \leq r \leq 2$, then $\lambda(\overline{G}) \leq \lambda(G)$.*

Notice that bipartite connected graphs G of order at least 4 such that $\lambda(G) \leq 2$ are P_4, P_5 and C_4. These graphs satisfy $\lambda(\overline{G}) \leq \lambda(G)$.

Next, we approach the case $\lambda(\overline{G}) = \lambda(G) + 1$, when $\lambda(G) \geq 3$, using LD-graphs. We may assume that $r \geq 3$ by Proposition 8. First we give some properties of LD-graphs for bipartite graphs satisfying the preceding equality.

Lemma 1. *If $\lambda(\overline{G}) = \lambda(G) + 1$ and U is an LD-code of G, then G^U contains at least two edges with label u, for all $u \in U$.*

In the study of LD-sets of a connected bipartite graph, a family of graphs is particularly useful, the *cactus graphs*. A *block* of a graph is a maximal connected subgraph with no cut vertices. A connected graph G is a *cactus* if all its blocks are cycles or edges. Cactus are also characterized as those connected graphs with no edge shared by two different cycles. The following lemma gives some properties relating parameters of bipartite graphs having cactus as connected components.

Lemma 2. *Let H be a bipartite graph of order at least 4 such that all its connected components are cactus having $cc(H)$ connected components and $cy(H)$ cycles. Let $ex(H) = |E(H)| - 4\,cy(H)$. Then H satisfies:*

i) $|V(H)| = |E(H)| - cy(H) + cc(H)$.
ii) $ex(H) \geq 0$ *and* $|V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}ex(H) + cc(H)$.
iii) $|V(H)| \geq \frac{3}{4}|E(H)| + 1$.
iv) $|V(H)| = \frac{3}{4}|E(H)| + 1$ *if and only if* H *is connected and all blocks are cycles of order 4.*

Lemma 3. *Let* $\lambda(\overline{G}) = \lambda(G) + 1$ *and assume that* U *is an LD-code of* G*. Consider a subgraph* H *of* G^U *induced by a set of edges containing exactly two edges with label* u*, for each* $u \in U$*. Then, all the connected components of* H *are cactus.*

Proposition 9. *If* $r \geq 3$ *and* $\lambda(\overline{G}) = \lambda(G) + 1$*, then* $\frac{3r}{2} \leq s \leq 2^r - 1$*.*

Lemma 4. *If* $\lambda(\overline{G}) = \lambda(G) + 1$ *and* U *is an LD-code of* G*, let* z *be the vertex of* G^U *introduced in the definition of this graph and let* H *be a subgraph of* G^U *with exactly two edges with label* u*, for each* $u \in U$*. Then:*

i) If H *has at least two connected components, then* $s \geq \frac{3r}{2} + 1$*.*
ii) If z *is an isolated vertex in* G^U*, then* $s \geq \frac{3r}{2} + 1$*.*
iii) z *is an non-isolated vertex in* G^U *if and only if there is at least a vertex in* $V \setminus U$ *of degree 1 in* G*.*
iv) If G *has no vertex of degree 1 in* W*, then* $s \geq \frac{3r}{2} + 1$*.*

Proposition 10. *There are no bipartite graphs* G *satisfying* $\lambda(\overline{G}) = \lambda(G) + 1$ *if* $\frac{3r}{2} \leq s < \frac{3r}{2} + 1$*.*

Proposition 11. *For every pair* (r, s)*,* $r, s \in \mathbb{N}$*, such that* $3 \leq r$ *and* $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$*, there exists a bipartite graph* $G(r, s)$ *such that* $\lambda(\overline{G}) = \lambda(G) + 1$*.*

Graphs satisfying the conditions of Proposition 11 can be constructed from the LD-graph described in Figure 2 associated to the LD-code $U = \{1, 2, \ldots, r\}$ when $s = \left\lceil \frac{3r}{2} + 1 \right\rceil$.

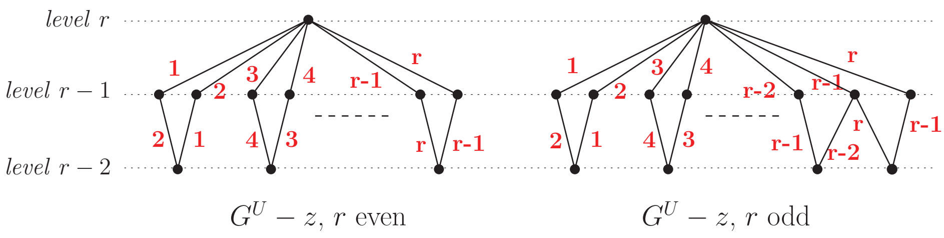

Fig. 2: The labeled graph $G^U - z$, for $G = G(r, \left\lceil \frac{3r}{2} + 1 \right\rceil)$ and $U = \{1, \ldots, r\}$.

For $s > \lceil \frac{3r}{2} + 1 \rceil$, we can add up to $2^r - 1 - r$ vertices to the set W of the graph $G(r, \left\lceil \frac{3r}{2} + 1 \right\rceil)$ taking into account that the neighborhoods in U of the vertices of W must be different and non-empty.

Proposition 12. *Let G be a bipartite connected graph with $V(G) = U \cup W$, $|U| = r$, $|W| = s$, $3 \leq r \leq s$.*

i) If $r \leq s < \frac{3r}{2} + 1 \Rightarrow \lambda(G) - \lambda(\overline{G}) \in \{0, 1\}$ and there are examples of both cases.

ii) If $\frac{3r}{2} + 1 \leq s \leq 2^r - 1 \Rightarrow \lambda(G) - \lambda(\overline{G}) \in \{-1, 0, 1\}$ there are graphs satisfying all cases.

iii) If $2^r - 1 \leq s \Rightarrow \lambda(G) - \lambda(\overline{G}) \in \{0, 1\}$ and there are examples of both cases.

Examples of graphs satisfying the different cases of Proposition 12 are the following. For every r, s, $3 \leq r \leq s$, complete bipartite graphs $K_{r,s}$, satisfy

$$\lambda(K_{r,s}) = \lambda(\overline{K_{r,s}}) = r + s - 2$$

and bistars $K_2(r - 1, s - 1)$ satisfy

$$r + s - 2 = \lambda(K_2(r, s)) > \lambda(\overline{K_2(r, s)}) = r + s - 3.$$

By Proposition 9, we know that the equality $\lambda(G) - \lambda(\overline{G}) = -1$ is possible only in the case that $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$ and by Proposition 11 we know examples for any s satisfying $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$.

References

[1] N. Bertrand, I. Charon, O. Hudry, A. Lobstein. Identifying and locating-dominating codes on chains and cycles. *Eur. J. Combin.*, 25:969–987, 2004.

[2] M. Blidia, M. Chellali, F. Maffray, J. Moncel, A. Semri. Locating-domination and identifying codes in trees. *Australas. J. Combin.*, 39:219–232, 2007.

[3] R. C. Brigham, J. R. Carrington. Global domination, in: T.W. Haynes, S.T. Hedetniemi, P.J. Slater (Eds.). *Domination in Graphs, Advanced Topics.* Marcel Dekker, New York, 1998, pp. 30–320.

[4] R. C. Brigham, R. D. Dutton. Factor domination in graphs. *Discrete Math.*,86(1-3):127–136, 1990.

[5] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, M. L. Puertas. Locating-dominating codes: Bounds and extremal cardinalities. *Appl. Math. Comput.*, 220:38–45, 2013.

[6] C. Chen, R. C. Lu, Z. Miao. Identifying codes and locating-dominating sets on paths and cycles. *Discrete Appl. Math.*, 159(15):1540–1547, 2011.

[7] C. Hernando, M. Mora, I. M. Pelayo. Nordhaus-Gaddum bounds for locating domination. *Eur. J. Combin.*, 36:1–6, 2014.

[8] C. Hernando, M. Mora, I. M. Pelayo, On global location-domination in graphs, submitted.

[9] A. Lobstein. Watching systems, identifying, locating-dominating and discriminating codes in graphs.
`http://www.infres.enst.fr/~lobstein/debutBIBidetlocdom.pdf`

[10] E. Sampathkumar. The global domination number of a graph. *J. Math. Phys. Sci.* 23:377–385, 1989.

[11] P. J. Slater. Dominating and reference sets in a graph. *J. Math. Phys. Sci.* 22:445–455, 1988.