Feedback Control of Limit Cycles: A Switching Control Strategy Based on Nonsmooth Bifurcation Theory

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***Abstract*—In this paper, we present a method to control limit cycles in smooth planar systems making use of the theory of nonsmooth bifurcations. By designing an appropriate switching controller, the occurrence of a corner-collision bifurcation is induced on the system and the amplitude and stability properties of the target limit cycle are controlled. The technique is illustrated through a representative example.**

***Index Terms*—Bifurcations, control, limit cycles, piecewise-smooth dynamical systems.**

I. INTRODUCTION

ONE OF THE most common sources of instability in applications is the onset of unwanted oscillatory behavior. For instance, recent progress in nonlinear dynamics has shown that so-called Hopf bifurcations can lead to the onset of such oscillatory motion in a variety of different systems. Undesirable stable oscillatory motion has been observed in aircraft systems [1], mechanical devices, control systems and electrical circuits [2], [3]. It has been shown that limit cycles associated to these oscillations are usually locally stable and can, at times, coexist with the desired steady-state behavior.

Classical control techniques can be used to suppress these unwanted oscillations by means of feedback control actions aimed at changing the system dynamics over the entire region of interest [4]. Thus, in the case where two or more different attractors exist, the controller objective is that of eliminating them, taming the system dynamics onto a desired stable equilibrium point. Many authors (see, for example, [5], [6] and the references therein) have studied the problem of controlling bifurcations within a smooth feedback framework. Examples include the method based on manifold reduction presented in [6] and the use of smooth nonlinear control laws discussed in [7] to tame a limit cycle occurring in a flutter problem.

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Recently, it has been proposed that results from bifurcation theory can be used to synthesize *ad hoc* control strategies for nonlinear systems [8]–[10]. The main aim of this paper is to present a novel approach to control limit cycles in planar dynamical systems. Namely, an appropriate switching controller is synthesised by using results from the theory of bifurcation in nonsmooth system. Rather than aiming at changing the entire dynamics of the system of interest, we shall seek to design a controller acting in a local neighborhood of the target limit cycle. (For a review of the theory of nonsmooth bifurcations in dynamical systems we refer the reader to [11], [12]). The analysis is based on the study of the Poincaré map associated to the limit cycle and relies on the theory of corner-collision bifurcations recently presented in [13]. Namely we will show that it is possible to control the amplitude of an oscillatory motion, or even suppress it (if unwanted) by means of a switching controller acting in a relatively small neighborhood of the limit cycle. By appropriately selecting the switching manifolds and the control action, it is possible to move the fixed point corresponding to the target limit cycle on the Poincaré map and hence control the cycle itself. In so doing, the control effort is low as control is only activated in a small neighborhood of the cycle. To synthesize the controller, we will proceed in two separate stages. A control law, based on cancellation, is used as a first step toward the synthesis of a controller which instead will not rely on cancellation. To select the features of the limit cycles in a controlled way, we will use the strategy to classify so-called border collisions (or C-bifurcations) of fixed points of nonsmooth maps recently presented in [14].

Note that we are not designing a controller to change the bifurcation properties of the system, but rather choosing a local control strategy to place the system close to a known bifurcation phenomenon. We will then use our analytical understanding of such phenomenon to achieve the control goal, i.e., suppress or modify the limit cycle of interest. Namely, the controller applied to the system flow will be based on a switching action which is designed by taking into account a nearby nonsmooth bifurcation of the cycle under control and then influence the properties of the associated Poincaré map in order to change its properties according to the classification strategy presented in [14]. In so doing, we will do explicit use of the technique to derive the approximate Poincaré map of the system analytically during the control design stage. It is worth to note that although the method also works to even suppress the limit cycle and obtain an equilibrium point (nonlocal action) our analytical understanding of the bifurcation phenomenon can only explain local changes (the cycle amplitude variation, for example).

The rest of the paper is outlined as follows. In Section II, the proposed method to control limit cycles is presented. In Section III, a brief description of corner collision is made. In Section IV controller synthesis is explained. In Section V, an example of application is shown. In the example the system presents a limit cycle and the technique was successful. Finally, the conclusions are presented in Section VI.

II. Controlling Limit Cycles Through Corner Collision

Let us consider a general system of the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) + \mathbf{B}(\mathbf{x})\mathbf{u} \tag{1}$$

where $\mathbf{F} := (f_1, f_2, \ldots, f_n) : \mathbb{R}^n \mapsto \mathbb{R}^n$ is a sufficiently smooth and differentiable vector field over the region of interest, say $D \subseteq \mathbb{R}^n$. For the sake of clarity, we assume that $\mathbf{B}(\mathbf{x})$ is the identity (note that this assumption is not necessary for the strategy presented here to be valid). Also, we suppose that, at some parameter value, the system exhibits a stable limit cycle of period T, i.e., $\mathbf{x}(t) = \mathbf{x}(t+T)$. We want to design a feedback controller to suppress such periodic oscillations or, alternatively, to select its characteristics (periodicity, amplitude etc.). Note that while the control action will be applied on the continuous-time system, the aim is to change the properties of its Poincaré map.

For this purpose our aim is to synthesise a controller based on the theory of nonsmooth bifurcations. Namely, we will select a switching feedback controller $\mathbf{u}$ in order to vary the main features of the local Poincaré map associated to the limit cycle of interest. This in turn will allow the variation of the properties of such local map and hence the local control of the cycle.

As it will be seen in Section III, locally to a corner collision bifurcation point the Poincaré map can be estimated analytically as a piecewise-linear one. Thus, the main idea is for the controller to put the system close to a corner collision bifurcation event of the cycle of interest with an appropriately defined switching strategy in state-space. In so doing, the controller will switch from one configuration to the other whenever the system trajectories cross the boundaries defining a corner-like switching manifold in phase space. By varying the functional form of the control signal, we will change the properties of the local map and hence the main features of the fixed point corresponding to the cycle of interest.

In so doing we need to:

1) choose an appropriate Poincaré section and define the Poincaré map for the system under investigation;
2) synthesize a feedback controller to change the properties of this map, i.e., choose (i) the corner region in phase space and (ii) the controller functional form;
3) validate the effectiveness of the controller through numerical simulations

First, we give a brief overview of the theory of corner-collision bifurcations.

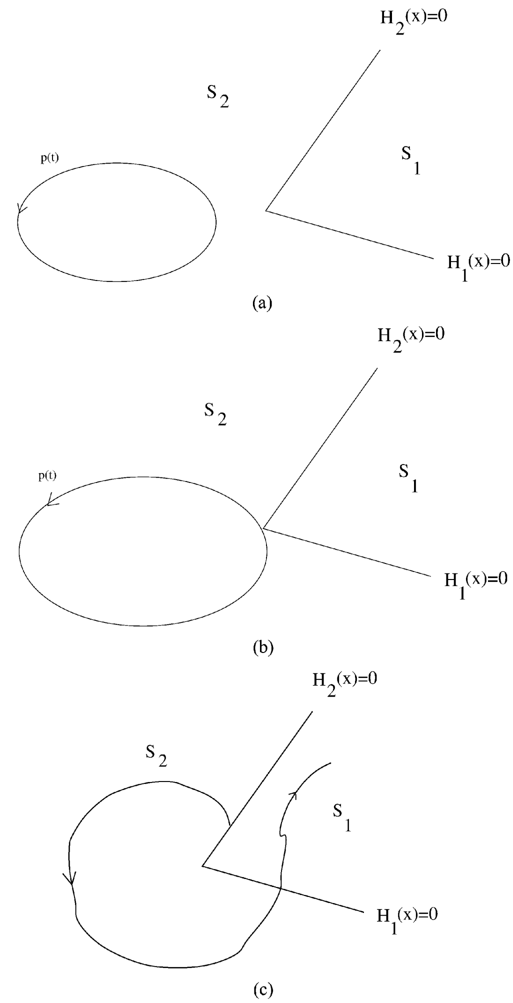


Fig. 1. Scheme of a border-collision bifurcation which destroys a limit cycle. (a) Original limit cycle. (b) Border-collision bifurcation. (c) Limit cycle destroyed.

III. Corner-Collision: A Brief Description

In many control systems and electronic switching devices, switching conditions may be governed by several overlapping inequalities. A generic feature of such examples is that the discontinuity boundary has a corner-type singularity formed by the intersection between smooth codimension one surfaces $\Gamma_1 := \{\mathbf{x} \in \mathbb{R}^n : H_1(\mathbf{x}) = 0\}$ and $\Gamma_2 := \{\mathbf{x} \in \mathbb{R}^n : H_2(\mathbf{x}) = 0\}$ at a nonzero angle.

The locus of corners $\mathcal{C}$ will in general be a $(n-2)$-dimensional subset of the phase space $\mathbb{R}^n$. The passage of a trajectory through a point in $\mathbf{c} \in \mathcal{C}$ is a nonsmooth bifurcation event because, in a neighborhood of the corner, there are distinct trajectories that do not behave similarly with respect to regions S_1 and S_2 on either side of Γ (the border of regions S_1 and S_2), which is a subset of $\Gamma_1 \cup \Gamma_2$. If such a corner-colliding trajectory is part of an isolated periodic orbit $\mathbf{p}(t)$, we shall refer to this as a *corner-collision grazing bifurcation*, or 'corner collision' for short (this is the case, for example, of dc/dc buck converters [2]). Fig. 1 illustrates the geometry that we are considering.

Here, there are two different regions, namely S_1 and S_2. In each zone the system presents a different dynamical behavior described by different vector fields. Whenever the boundary between the two regions is crossed (i.e., whenever the trajectory crosses the corner), the system vector field looses continuity.

We assume that the system of interest is planar and can be described as

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{G}(\mathbf{x}), & \text{if } \mathbf{x} \in S_1 \\ \mathbf{F}(\mathbf{x}), & \text{if } \mathbf{x} \in S_2. \end{cases} \tag{2}$$

As some parameter is varied, a corner collision can occur where a limit cycle hits the tip of the corner region. Further parameter variations can lead to several different scenarios. To classify the possible scenarios following a corner collision the key issue is to be able to construct the Poincaré normal form map of the cycle undergoing the bifurcation. Recently, it was shown that a local map describing the dynamics of the system close to a corner-collision point can be derived by using the concept of *discontinuity map* [13].

Namely, suppose we want to construct a Poincaré map for the cycle of interest. Then, in the absence of the corner, the map would be defined by considering the system flow from some suitable Poincaré section Σ back to itself. The discontinuity map is a local map that describes the correction that needs to be made to trajectories that pass through region S_2 close to the corner in order to solve the global Poincaré map. It was shown that such map is locally piecewise linear so that a corner-collision bifurcation of the flow implies a border collision of the associated fixed point of the map [15].

Next, we calculate the Poincaré section and the discontinuity map.

A. *Poincaré Map*

As our analysis is concerned with a planar dynamical system, the Poincaré map is one-dimensional. In the following, we will determine the behavior of the fixed point associated to a limit cycle undergoing a corner-collision event. Initially, we calculate the Poincaré map, assuming that the evolution of the whole system in state space is through the flow Φ_F associated to the vector field $\mathbf{F}$. The point at which corner collision occurs will be identified as x_c and corresponds to the intersection of Γ_1 and Γ_2. Without loss of generality, we assume that

$$\Gamma_1 := \{\mathbf{x} \in \mathbb{R}^2 : H_1(\mathbf{x}) := -x_2 = 0\} \tag{3}$$

$$\Gamma_2 := \{\mathbf{x} \in \mathbb{R}^2 : H_2(\mathbf{x}) := x_2 - mx_1 + b = 0\} \tag{4}$$

and that the flow Φ_F is transversal to Γ_1. A convenient subset $\Sigma_1 \subset \Gamma_1$ is therefore chosen as a suitable Poincaré section for the flow. We consider also $\Sigma_2 = \{\mathbf{x} \in \Gamma_2 : x_2 \geq 0\} \subset \Gamma_2$. Clearly, the tip of the corner is located at the point $x_c = (b/m, 0)$.

Say $(x_1^0, 0)$ a fixed point of the map defined on Σ_1 associated to a limit cycle of the whole flow in $\mathbb{R}^2$. To construct the map from Σ_1 back to itself, we consider a perturbation of such point $(x_1^0 + \delta, 0)$ for a small $|\delta|$. Note that if $x_1^0 + \delta < b/m$ the limit cycle evolves entirely in region S_2 while, if $x_1^0 + \delta > b/m$, then the cycle penetrates the corner for some time.

Let us first consider the case where the limit cycle lies entirely to the left of the corner (i.e., assume δ to be such that $x_1^0 + \delta < b/m$). Let $t_{\min} > 0$ be the minimum time for which the system evolves from $(x_1^0 + \delta, 0)$ until it hits Σ_1 again. Then

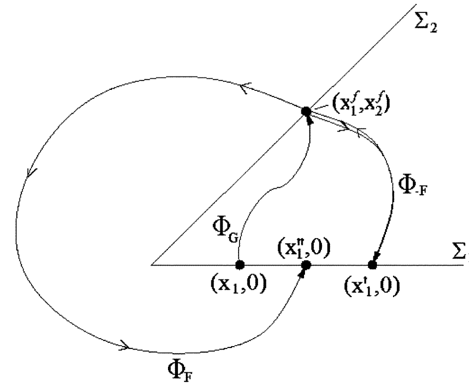


Fig. 2. Scheme of the discontinuity map.

$$H_1\left(\Phi_F\left(\left(x_1^0 + \delta, 0\right), t_{\min}\right)\right) = 0$$

is fulfilled. To leading order, we can then write that the Poincaré map Π_0 can be written as

$$\begin{aligned} \Pi_0 : \Sigma_1 &\rightarrow \Sigma_1 \\ (x_1, 0) &\rightarrow (Px_1, 0) \end{aligned} \tag{5}$$

where

$$P = \frac{\Phi_F\left(\left(x_1^0 + \delta, 0\right), t_{\min}\right)}{x_1^0 + \delta}. \tag{6}$$

Now we construct the Poincaré map when the cycle interacts with the corner, i.e., $x_1^0 + \delta > b/m$. We follow the same procedure first discussed in [13]. As it is schematically shown in Fig. 2, to obtain the Poincaré map in this case, we need to compose Π_0 as derived above with the so-called discontinuity map (i.e., a map that makes the appropriate corrections to Π_0 in order to take into account the fact that the system trajectory is now crossing into region S_1).

To obtain such correction, from initial conditions on Σ_1 we first solve the equations given by $\dot{\mathbf{x}} = \mathbf{G}(\mathbf{x})$ until Σ_2 is reached, at a point (x_1^f, x_2^f). This map from Σ_1 to Σ_2 will be Π_G. Then, from (x_1^f, x_2^f) we solve the equations for the reverse flow $\Phi_F(\mathbf{x}, -t)$ until we hit again Σ_1. In this case the map from Σ_2 to Σ_1 will be denoted by Π_{-F}. The correction that needs to be made to Π_0, i.e., the discontinuity mapping, is then given by $\Pi_{-F} \circ \Pi_G$.

Finally, to obtain the Poincaré map from Σ_1 back to itself, in this case, we compose the three maps, i.e.,

$$\Pi_0 \circ \Pi_{-F} \circ \Pi_G : \Sigma_1 \rightarrow \Sigma_1.$$

Concretely, let $t_1 > 0$ and $t_2 > 0$ be the minimum times verifying

$$H_2(\Phi_G((x_1, 0), t_1)) = 0$$

and

$$H_1\left(\Phi_{-F}\left(\left(x_1^f, x_2^f\right), t_2\right)\right) = 0.$$

Hence, we have

$$\Pi_G : \Sigma_1 \to \Sigma_2$$
$$(x_1, 0) \to \left(x_1^f, x_2^f\right) := \Phi_G((x_1, 0), t_1) \quad (7)$$

and

$$\Pi_{-F} : \Sigma_2 \to \Sigma_1$$
$$\left(x_1^f, x_2^f\right) \to (x_1', 0) := \Phi_{-F}\left(\left(x_1^f, x_2^f\right), t_2\right). \quad (8)$$

As shown in [13] and discussed below, it is possible to obtain an analytical estimate of the discontinuity mapping by considering a set of appropriate approximations.

B. Discontinuity Map

Equipped with the definition of the map given above, we can now compute the discontinuity map analytically in order to obtain the analytical expression of the global Poincaré map. Following the methodology presented in [13], when the trajectory in S_2 collides with the corner zone we solve the equations through a first order approximation of the flow. Hence, if g_1 and g_2 are the components of the vector field in region S_1 we have

$$\Phi_G = \mathbf{x} + \mathbf{G}^0 t + \text{h.o.t}$$

with $\mathbf{x} = (x_1, 0)$, and $\mathbf{G}^0 = (g_1^0, g_2^0)$ is $\mathbf{G}$ evaluated at the corner point. The intersection of the flow with Σ_2 yields

$$g_2^0 t - m x_1 - m g_1^0 t + b = 0. \quad (9)$$

This implies that

$$t = \frac{m x_1 - b}{g_2^0 - m g_1^0}. \quad (10)$$

With this time, the intersection point is

$$\begin{pmatrix} x_1^f \\ x_2^f \end{pmatrix} = \begin{pmatrix} x_1 + g_1^0 \frac{m x_1 - b}{g_2^0 - m g_1^0} \\ g_2^0 \frac{m x_1 - b}{g_2^0 - m g_1^0} \end{pmatrix}. \quad (11)$$

Now we proceed to find the time that has to be spent for the original system in reverse time starting at the point (x_1^f, x_2^f) until the surface Σ_1 is crossed. Doing an expansion to first order of the original flow, being f_1^0 and f_2^0 the components of the vector field $\mathbf{F}$ at the corner point, we have

$$\Phi_F = \mathbf{x}^f - \mathbf{F}^0 t + \text{h.o.t.}$$

Considering the second component only, we have

$$x_2^f - f_2^0 t = 0$$

then, taking into account that

$$x_2^f = g_2^0 \frac{m x_1 - b}{g_2^0 - m g_1^0}$$

the reverse time can be found as

$$t = \frac{g_2^0}{f_2^0} \frac{m x_1 - b}{g_2^0 - m g_1^0}. \quad (12)$$

Hence, the discontinuity map is given by

$$x_1 \to x_1 + g_1^0 \frac{m x_1 - b}{g_2^0 - m g_1^0} - \frac{f_1^0 g_2^0}{f_2^0} \frac{m x_1 - b}{g_2^0 - m g_1^0}$$

or equivalently as

$$x_1 \to x_1 + \frac{m x_1 - b}{g_2^0 - m g_1^0} \left(\frac{f_2^0 g_1^0 - f_1^0 g_2^0}{f_2^0} \right). \quad (13)$$

Note that the condition $f_2^0 \neq 0$ is always fulfilled with an appropriate choice of the Poincaré section Σ_1.

The discontinuity map can be written in a compact form as

$$x_1 \to \begin{cases} x_1 & \text{if } x_1 \leq b/m \\ x_1 + \frac{a}{m} H_2(x_1, 0) & \text{if } x_1 > b/m \end{cases} \quad (14)$$

where

$$\begin{aligned} a &= \frac{m}{g_2^0 - m g_1^0} \left(\frac{f_1^0 g_2^0 - f_2^0 g_1^0}{f_2^0} \right) \\ &= \frac{1}{\langle \nabla H_2^0, \mathbf{G}^0 \rangle} \left\langle \nabla H_2^0 - \mathbf{J}, \mathbf{G}^0 \right\rangle \end{aligned} \quad (15)$$

and

$$\mathbf{J} = \nabla H_1^0 \frac{\langle \nabla H_2^0, \mathbf{F}^0 \rangle}{\langle \nabla H_1^0, \mathbf{F}^0 \rangle}.$$

Hence, the Poincaré map is given by

$$x_1 \to \begin{cases} P x_1, & \text{if } x_1 \leq b/m \\ P\left(x_1 + \frac{a}{m} H_2(x_1, 0)\right), & \text{if } x_1 > b/m \end{cases} \quad (16)$$

where P is defined as in (6).

The derivation for the case of n-dimensional nonsmooth systems undergoing corner collisions can be found in [13].

IV. Control Synthesis

According to the theory of corner collisions, as confirmed by the derivations reported above, the Poincaré map of a limit cycle undergoing such a bifurcation is piecewise-linear and dependent on the vector fields inside and outside the corner. The corner is also supposed such that sliding on its boundaries is not possible. As before, we label S_1 the region inside the corner while S_2 the region outside of it.

Without loss of generality, we select $\mathbf{u}$ as the switching controller defined by

$$\mathbf{u} = \begin{cases} \mathbf{0}, & \text{if } \mathbf{x} \in S_2 \\ \phi(\mathbf{x}, t), & \text{if } \mathbf{x} \in S_1 \end{cases} \quad (17)$$

with $S_1 \subseteq \mathbb{R}^2$ being the region (corner) limited by the manifolds defined by $H_1(\mathbf{x}) = 0$ and $H_2(\mathbf{x}) = 0$ as depicted in Fig. 1.

With this choice of $\mathbf{u}$, the controlled system becomes

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{F}(\mathbf{x}), & \text{if } \mathbf{x} \in S_2 \\ \mathbf{F}(\mathbf{x}) + \phi(\mathbf{x}, t) := \mathbf{G}(\mathbf{x}), & \text{otherwise.} \end{cases} \quad (18)$$

The control effort can be calculated in analytical form. Since the rms value for a periodic signal is given by the $\mathcal{L}^2$-norm

$$f_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T \|f(\tau)\|^2 d\tau}$$

we can evaluate the rms value for the control signal in stationary state once the amplitude of the limit cycle has been changed and the system has stabilized. Taking into account that the signal control acts during a time given by (10), then the rms value is

$$u_{\rm rms} = \sqrt{\frac{1}{T}\int_0^t \|\phi(\mathbf{x},\tau)\|^2 d\tau} \tag{19}$$

where t is calculated from (10). Since $\phi(\mathbf{x},\tau)$ is independent of τ, the expression can be simplified to

$$u_{\rm rms} = \sqrt{\frac{1}{T}(\phi_1(\mathbf{x})^2 + \phi_2(\mathbf{x})^2)\left(\frac{mx_1 - b}{g_2^0 - mg_1^0}\right)}. \tag{20}$$

A. Step 1: Choosing the Switching Strategy

In order for the control to be effective we need to select the boundaries of regions S_1 and S_2, i.e., define the corner in phase space. According to the theory of corner collision, the corner must be such that: 1) sliding or Filippov solutions are not possible on its boundaries; 2) it penetrates the cycle to be controlled as one of its defining parameters is changed so that at some critical parameter value the target limit cycle undergoes a corner-collision bifurcation.

In order to avoid sliding mode [13] we choose $H_1(\mathbf{x})$ and $H_2(\mathbf{x})$ so that

$$\begin{aligned}\langle \nabla H_1(\mathbf{x}), \mathbf{F}(\mathbf{x})\rangle &< 0\\ \langle \nabla H_1(\mathbf{x}), \mathbf{G}(\mathbf{x})\rangle &< 0\\ \langle \nabla H_2(\mathbf{x}), \mathbf{F}(\mathbf{x})\rangle &> 0\\ \langle \nabla H_2(\mathbf{x}), \mathbf{G}(\mathbf{x})\rangle &> 0.\end{aligned} \tag{21}$$

For simplicity, we suppose counterclockwise direction of the vector field in neighborhood of corner collision point. We select H_1 and H_2 as in Section II, with m and b being real constants. It is easy to see that it is possible to rescale the system coordinates so that a corner collision occurs when $b = 0$ at the point (0, 0). Therefore, varying b, we can move the tip of the corner and hence yield a corner-collision bifurcation. Without loss of generality, we assume therefore that the corner collision bifurcation occurs at the point (0, 0).

From (14), we then have that the local interaction of the cycle with the corner can be described by using the local mapping given by

$$x_1 \to \begin{cases} x_1, & \text{if } x_1 \le b/m \\ x_1 + \frac{mx_1 - b}{g_2^0 - mg_1^0}\left(\frac{f_2^0 g_1^0 - f_1^0 g_2^0}{f_2^0}\right), & \text{if } x_1 > b/m \end{cases} \tag{22}$$

where f_i^0 and g_i^0 are the components of the vector field evaluated at the corner collision point (0, 0). Note that in order for the mapping to be well defined we must have

$$g_2^0 - mg_1^0 \neq 0 \tag{23}$$

$$f_2^0 \neq 0. \tag{24}$$

Also, in order for the control to have an influence on the map properties we must have

$$f_2^0 g_1^0 - f_1^0 g_2^0 \neq 0. \tag{25}$$

These conditions represent an important set of constraints on the control design. The quantities above can be computed analytically by means of any algebraic manipulation software (see Section V for a representative example of such computation). The next step is to choose the control signal $\phi(\mathbf{x},t)$.

B. Choosing $\phi(\mathbf{x},t)$

1) Perfect Knowledge of $\mathbf{F}(\mathbf{x})$: In the most general case, a first choice for $\phi(\mathbf{x},t)$ in (18) can be expressed as

$$\phi(\mathbf{x},t) = -\mathbf{F}(\mathbf{x}) + \mathbf{K}(\mathbf{x}) \tag{26}$$

where $\mathbf{K}(\mathbf{x})$ is a generic control function to be appropriately chosen. This means that the control signal contains two actions: the first compensates the nonlinear dynamic terms acting on the system; the second, instead, allow us to select the desired dynamics within the corner.

The main disadvantage of this controller is that it relies on perfect knowledge of the system vector field $\mathbf{F}(\mathbf{x})$. This is a rather strong assumption that can hardly be satisfied in realistic applications. Thus, in Section IV-B2), we will show that control can also be achieved successfully by removing the need for a perfect cancellation of the nonlinear dynamics.

In what follows, to illustrate the main idea, we detail the derivation of the controller, starting with the assumption that cancellation of nonlinear dynamic is possible. With this choice of the controller, the closed-loop system is given by

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{F}(\mathbf{x}), & \text{if } \mathbf{x} \in S_2 \\ \mathbf{K}(\mathbf{x}) := \mathbf{G}(\mathbf{x}), & \text{otherwise} \end{cases} \tag{27}$$

where $\mathbf{K}(\mathbf{x})$ must be chosen so that (23)–(25) are satisfied. Notice that this excludes the case where $\mathbf{K}(\mathbf{x})$ is chosen as a purely proportional action. In fact, in this case, $\mathbf{K}(\mathbf{0}) = \mathbf{0}$ and, therefore, the control would not affect the map to leading order but introduces higher order effects which are beyond the scope of this paper and will be discussed elsewhere.

In general, from the theory of corner collision and the expression of the local map (13), we see that $\mathbf{K}(\mathbf{x})$ must be chosen so that $\mathbf{K}(x_c) \neq 0$ if we want the control to cause first order variations of the map. Thus, we choose

$$\mathbf{K}(\mathbf{x}) = \mathbf{c} \tag{28}$$

where $\mathbf{c}$ is an appropriately selected constant vector. According to (17), the controller is then defined as

$$\mathbf{u} = \begin{cases} \mathbf{0}, & \text{if } \mathbf{x} \in S_2 \\ -\mathbf{F}(\mathbf{x}) + \mathbf{c}, & \text{if } \mathbf{x} \in S_1. \end{cases} \tag{29}$$

In this case

$$u_{\rm rms} = \sqrt{\frac{1}{T}\left({f_1^0}^2 + c_1^2 - 2f_1^0 c_1 + {f_2^0}^2 + c_2^2 - 2f_2^0 c_2\right)\left(\frac{mx_1 - b}{c_2 - mc_1}\right)}. \tag{30}$$

Then, with fixed c_1 and c_2, the amplitude of the limit cycle depends on the corner penetration and the point x_1 which corresponds to the first coordinate of the limit cycle in stationary state, when it enters S_1.

Notice that the control strategy given by (29) is indeed a feedback control strategy. Namely, even if the control action is determined by the addition of two state-independent constants c_1

and c_2, its switching is determined by state-dependent boundary conditions. As we will show in Section V, this results in a sequence of short carefully selected additions of constant perturbations of the vector field which steer the trajectory toward the desired goal.

In this case, each component of $\mathbf{u}$ is such that in the region S_1 the system evolves according to the following:

$$\begin{aligned}\dot{x}_1 &= c_1\\ \dot{x}_2 &= c_2\end{aligned} \tag{31}$$

where c_1 and c_2 are two suitably chosen constants. With this choice of the vector field, the system inside the corner will follow the trajectory given by

$$\begin{aligned}cx_1 &= c_1 t + x_1(0)\\ x_2 &= c_2 t + x_2(0).\end{aligned} \tag{32}$$

It is necessary to take into account that sliding needs to be avoided. According to conditions (21), we must then have:

1) $c_1 > 0, c_2 > 0$, and $c_2/c_1 > m$; or
2) $c_1 < 0, c_2 > 0$.

If we construct the map for this planar system, using (16), we have

$$x_1 \rightarrow \begin{cases} Px_1, & \text{if noncrossing}\\ P((1+qm)x_1 - qb), & \text{if crossing}\end{cases} \tag{33}$$

where $q = (f_1^0 c_2 - f_2^0 c_1)/(f_2^0(mc_1 - c_2)))$.

Here, we observe the explicit dependence on c_1 and c_2 of the Poincaré map, confirming that by varying the control constants we can effectively change the properties of the map and hence those of the fixed point associated to the limit cycle of interest. Namely, the fixed point of the map is

$$x_1^* = \frac{Pqb}{P(1+qm) - 1}$$

and so the amplitude of the limit cycle can be changed. An example will be discussed in Section V.

Now, let us assume that we have no *a priori* knowledge of the system vector field $\mathbf{F}(\mathbf{x})$ for feedback.

2) $\mathbf{F}(\mathbf{x})$ Unknown: We now remove the assumption that the vector field system $\mathbf{F}(\mathbf{x})$ is perfectly known. In this case, the control signal in a simplified form becomes

$$\mathbf{u} = \begin{cases} \mathbf{0}, & \text{if } \mathbf{x} \in S_2\\ \mathbf{c}, & \text{if } \mathbf{x} \in S_1\end{cases} \tag{34}$$

where S_1 is defined as above; and the closed-loop system takes the form

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{F}(\mathbf{x}), & \text{if } \mathbf{x} \in S_2\\ \mathbf{F}(\mathbf{x}) + \mathbf{c} := \mathbf{G}(\mathbf{x}), & \text{otherwise.}\end{cases} \tag{35}$$

In this case, the rms value of control effort can be calculated as

$$u_{\mathrm{rms}} = \sqrt{\frac{1}{T}\left(c_1^2 + c_2^2\right)\left(\frac{mx_1 - b}{f_2^0 + c_2 - mf_1^0 - mc_1}\right)}. \tag{36}$$

Now, it is possible to consider points near the origin and to proceed to analyze the vector field with the aim of guaranteeing a change in this field, according with the objectives. We consider the Poincaré map in the corner collision zone. In this case, according to (16), we have

$$x_1 \rightarrow \begin{cases} Px_1, & \text{if noncrossing}\\ P((1+qm)x_1 - qb), & \text{if crossing}\end{cases} \tag{37}$$

where

$$q = \frac{f_1^0\left(f_2^0 + c_2\right) - f_2^0\left(f_1^0 + c_1\right)}{f_2^0\left(m\left(f_1^0 + c_1\right) - \left(f_2^0 + c_2\right)\right)}.$$

Hence, the main features of the Poincaré map are again explicitly dependent on c_1 and c_2 and, therefore, it is possible to change the behavior of the fixed point associated to the limit cycle by appropriately selecting the control action. With the previous value for q, the fixed point of the map is

$$x_1^* = \frac{Pqb}{P(1+qm) - 1}$$

and so the amplitude of the limit cycle can also be changed in this case. The effectiveness of the control action presented above will be discussed using a representative example in what follows.

C. Changing the Properties of the Local Map: Feigin's Strategy

The next step is now to choose the control constants c_1 and c_2 in order to vary the properties of the local map associated to the corner collision of the cycle and hence change its features. The main idea is to use the fact that the corner collision of the cycle implies a border collision of the corresponding fixed point of the local map [15]. Thus, controlling the cycle can be achieved by changing the properties of the map in order to control the scenario following a border collision.

To this aim, we use the strategy for the classification of border collisions presented in [13]. Namely, according to such a strategy, different scenarios are possible at a border collision which can be classified using the slopes of the map on both sides of its discontinuity boundaries. In particular, if we say α the slope of the map when noncrossing and β its slope when crossing, according to Feigin's strategy we have the following three possible simplest scenarios (for a list of all possible scenarios we refer to [13]).

- **Persistence:** the bifurcating fixed point (limit cycle) crosses the boundary, changing continuously into a fixed point (limit cycle) lying on the other side of the boundary which may or may not have the same stability properties if $\alpha < 1$ and $\beta < 1$, or otherwise $\alpha > 1$ and $\beta > 1$.
- **Nonsmooth Saddle Node:** a stable fixed point (limit cycle) collides with an unstable point (cycle), on the boundary and they both disappear if α and β do not fulfill any of the conditions above.
- **Nonsmooth Period Doubling:** a two-periodic point of the map (x_1, x_2) characterized by having one iteration on

each side of the boundary is involved in the bifurcation scenario. Note that according to a given set of conditions this might either result into a period-two orbit arising from the bifurcation point or being annihilated through it. (This case is obviously impossible in planar cases as flip bifurcations of cycles are not possible in $\mathbb{R}^2$.)

Thus, by carefully choosing c_1 and c_2 in the controller equation, we can change the slope β of the map when crossing and in turns select the scenario following the border collision. For example, selecting a value for β such that a nonsmooth saddle node occurs at the corner collision, corresponds to selecting a controller that suppress locally the oscillatory motion in the system. On the other hand, selecting β so that persistence is observed correspond to changing the amplitude of the cycle etc.

We will now better illustrate the strategy by means of a representative example. In what follows we will indicate by A the fixed point associated to the original limit cycle we want to control and by B the cycle of the controlled system. We will use a and b to indicate unstable cycles and $\rightarrow$ to indicate the occurrence of a corner collision.

V. 2-D Representative Example

Next, we show a simple example to illustrate the stages of the control design presented above. We choose the planar normal form of a Hopf bifurcation, described by

$$\dot{x_1} = \varepsilon(x_1+1)\left(a - \sqrt{(x_1+1)^2 + x_2^2}\right) - x_2 := f_1$$
$$\dot{x_2} = \varepsilon x_2\left(a - \sqrt{(x_1+1)^2 + x_2^2}\right) + x_1 + 1 := f_2. \quad (38)$$

This system exhibits a limit cycle, which is a perfect circle of radius a centered in $(x_1^*, x_2^*) = (-1, 0)$. The system flow moves in an anticlockwise direction as time increases and hence crosses the line $\{x_2 = 0\}$ upwards.

We choose the region S_1 as the phase space set (corner) bounded by $H_1(\mathbf{x}) := -x_2 = 0$ and $H_2(\mathbf{x}) := x_2 - mx_1 + b = 0$, which satisfies the relations involving $\mathbf{F}(\mathbf{x})$ in (21). Note that when $a = 1$ and $b = 0$ a corner collision occurs, as the limit cycle of radius 1 hits the tip of the corner defined above at the point (0, 0). Varying the control parameter b will cause the corner to penetrate the limit cycle and hence change its properties.

In what follows, we suppose without loss of generality, that $m = 1, a = 1, \varepsilon = 0.1$, and $b = -0.1$.

A. $\mathbf{F}(\mathbf{x})$ *Perfectly Known,* $\phi(\mathbf{x}, \mathbf{t}) = -\mathbf{F}(\mathbf{x}) + \mathbf{c}$

In this case, according to the development made in Section IV.B.1, and in order to fully satisfy (21) we need to choose c_1 and c_2 such that:

1) $c_2 > 0, c_1 > 0$, and $(c_2)/(c_1) > 1$;
2) $c_2 > 0$ and $c_1 < 0$.

Taking into account that in the case under investigation $f_1^0 = 0, f_2^0 = 1, g_1^0 = c_1$, and $g_2^0 = c_2$, the equation describing the local piecewise-linear map, in a Poincaré section, is

$$x_1 \rightarrow \begin{cases} Px_1, & \text{if noncrossing} \\ P\left(x_1 - c_1\frac{x_1 - b}{c_1 - c_2}\right), & \text{if crossing} \end{cases} \quad (39)$$

with $P = 0.5335$, which is computed with (6) and δ small.

As discussed above, the fixed point of this map associated to the limit cycle undergoing a corner collision as b is varied, undergoes a border-collision bifurcation. Hence, we shall seek to control the cycle by varying c_1 and c_2 in order to change the properties of the map and hence affect the nature of the border collision of its fixed point.

Taking into account the slope of the map at the fixed point, and Feigin conditions [14], we can deduce some interesting results regarding the control design of the system.

The slope of the map at the fixed point (39) is given by $\alpha = P = 0.5335 < 1$ on one side and

$$\beta = P\frac{c_2}{c_2 - c_1}$$

on the other.

Thus, using Feigin's conditions we can distinguish two cases as follows.

1) $-1 < \beta < 1$.
2) $\beta > 1$.

1) Case I: $-1 < \beta < 1$*:* According to Feigin conditions, in this case we have persistence of the fixed point and hence of the cycle. As discussed above, to avoid sliding on the corner boundaries, we must choose either:

1) $c_1 > 0, c_2 > 0$, and $c_2/c_1 > 1$; or
2) $c_1 < 0, c_2 > 0$.

Note that for both cases, $\beta > 0$. Thus, to have persistence, we want $\beta < 1$, i.e.,

$$0 < \beta < 1 \Leftrightarrow 0 < P\frac{c_2}{c_2 - c_1} < 1.$$

As β is certainly positive, whatever the choice of c_1 and c_2, we want to have

$$P\frac{c_2}{c_2 - c_1} < 1 \Leftrightarrow P < \frac{c_2 - c_1}{c_2} = 1 - \frac{c_1}{c_2}$$
$$\Leftrightarrow \frac{c_1}{c_2} < 1 - P = 0.4665. \quad (40)$$

Thus, the limit cycle will persist with a different amplitude if c_1 and c_2 are chosen so that (40) is satisfied. For example, if $c_1 = 0.2$ and $c_2 = 1$, and thus $\beta = 0.6669$, these conditions are fulfilled, and the possibilities according to [14] (with Feigin notation) are

$A \rightarrow b$ Stable fixed point to unstable fixed point
$a \rightarrow b$ Unstable fixed point to unstable fixed point
$A \rightarrow B$ Stable fixed point to stable fixed point
$a \rightarrow B$ Unstable fixed point to stable fixed point. (41)

Since before the bifurcation, we have a stable fixed point in the Poincaré map (corresponding to the stable limit cycle in the system), and after the bifurcation, the slope β in this case is such that $0 < \beta < 1$ (which corresponds to a stable fixed point), we can deduce that the bifurcation scenario is

$$A \rightarrow B.$$

Fig. 3 shows the analytical piecewise linear map [only P is numerically computed through (6)], the numerically computed Poincaré map, and the line $x_1(k+1) = x_1(k)$ for $c_1 = 0.2$ and $c_2 = 1$, and for $c_1 = -1$ and $c_2 = 1$. It can be observed

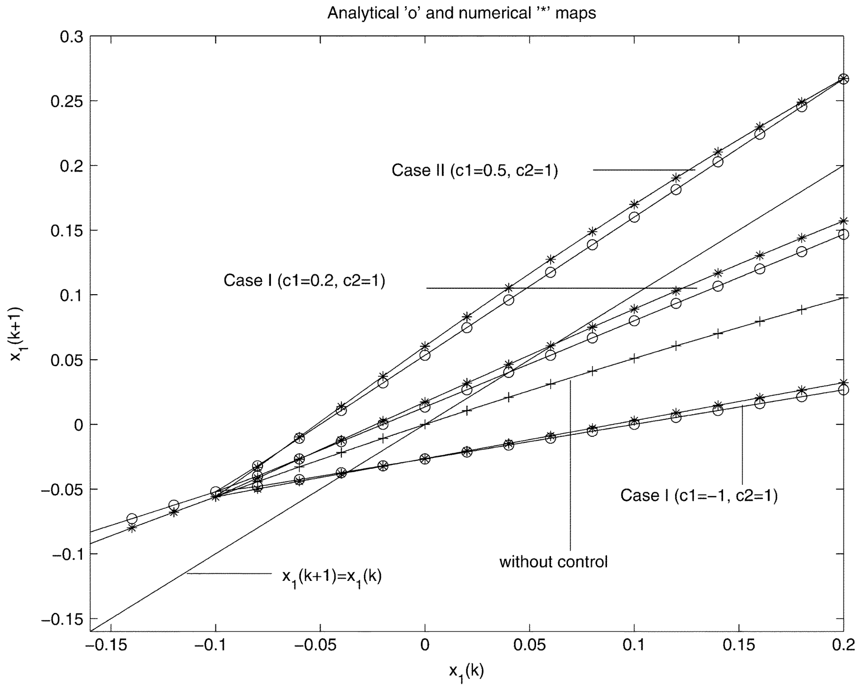


Fig. 3. Case $\mathbf{F}(\mathbf{x})$ perfectly known. Evolution of fixed point for several constant values. The corner collision point is $(-0.1, 0)$. Other bifurcation scenarios are not possible.

that the piecewise-linear map is a very good approximation of the numerically computed one, specially at the corner point. By varying b, c_1, and c_2 we can move the fixed point of the map and therefore change the amplitude of the limit cycle. Namely the fixed point of the map when trajectories cross the corner can be derived from (39) as

$$x_1^* = \frac{Pc_1 b}{Pc_2 + c_1 - c_2}.$$

Note that the location of x_1^* is related to the amplitude of the corresponding cycle in phase space. Thus, varying b, c_1, and c_2 one can control locally the amplitude of the cycle. For example, for $b = 0.1, c_1 = -1, c_2 = 1$, the limit cycle is shown in Fig. 4(a). The cycle exhibited by the close-loop system has a smaller amplitude. Note that a further exploration of the effects of varying the control parameters on the amplitude of the limit cycle will be reported in Section B for the more realistic case of $\mathbf{F}(\mathbf{x})$ being unknown.

2) Case II: $\beta > 1$: Following a similar derivation, we can now select c_1 and c_2 in order to have $\beta > 1$. In this case, according to Feigin's condition, we should observe the occurrence of a nonsmoothsaddle node and hence the disappearance of the limit cycle when control is activated. If we choose $c_1 > 0, c_2 > 0$, and $c_2 > c_1$, we then have that at the border collision induced by the controller we have

$$A, b \rightarrow \emptyset \text{ (Stable and unstable fixed points merge and disappear).}$$

Hence, the fixed point disappears in a border-collision bifurcation. Note that this result is local; thus, the cycle is suppressed locally to the corner collision point. The existence of other cycles might be possible according to the global properties of the map that cannot be studied analytically and must therefore be validated by appropriate simulations.

For example, for $c_2 = 1$ and $c_1 = 0.5, \beta = 1.067$, Fig. 3 shows the analytical piecewise-linear map, the numerically computed Poincaré map, and the line $x_1(k+1) = x_1(k)$.

Fig. 4(b) shows the evolution of the continuous system in the case considered here. We see that the control is successful in suppressing the cycle of the open-loop system [16]. As expected, the strategy is local and a larger limit cycle is detected in Fig. 4(b). This is due to the fact that the map shown in Fig. 3 eventually intersects the identity line for larger values of x_1. Thus, the control is only effective in suppressing the limit cycle in the region of interest. Global control strategies should instead be used to achieve global results.

B. $\mathbf{F}(\mathbf{x})$ *Unknown,* $\mathbf{u} = \mathbf{c}$

Now, each component of the control signal in region S_1 is defined as $u_i = c_i, i = 1, 2$. In this case, the controlled system inside the corner is described by

$$\begin{aligned} \dot{x_1} &= f_1 + c_1 \\ \dot{x_2} &= f_2 + c_2 \end{aligned} \tag{42}$$

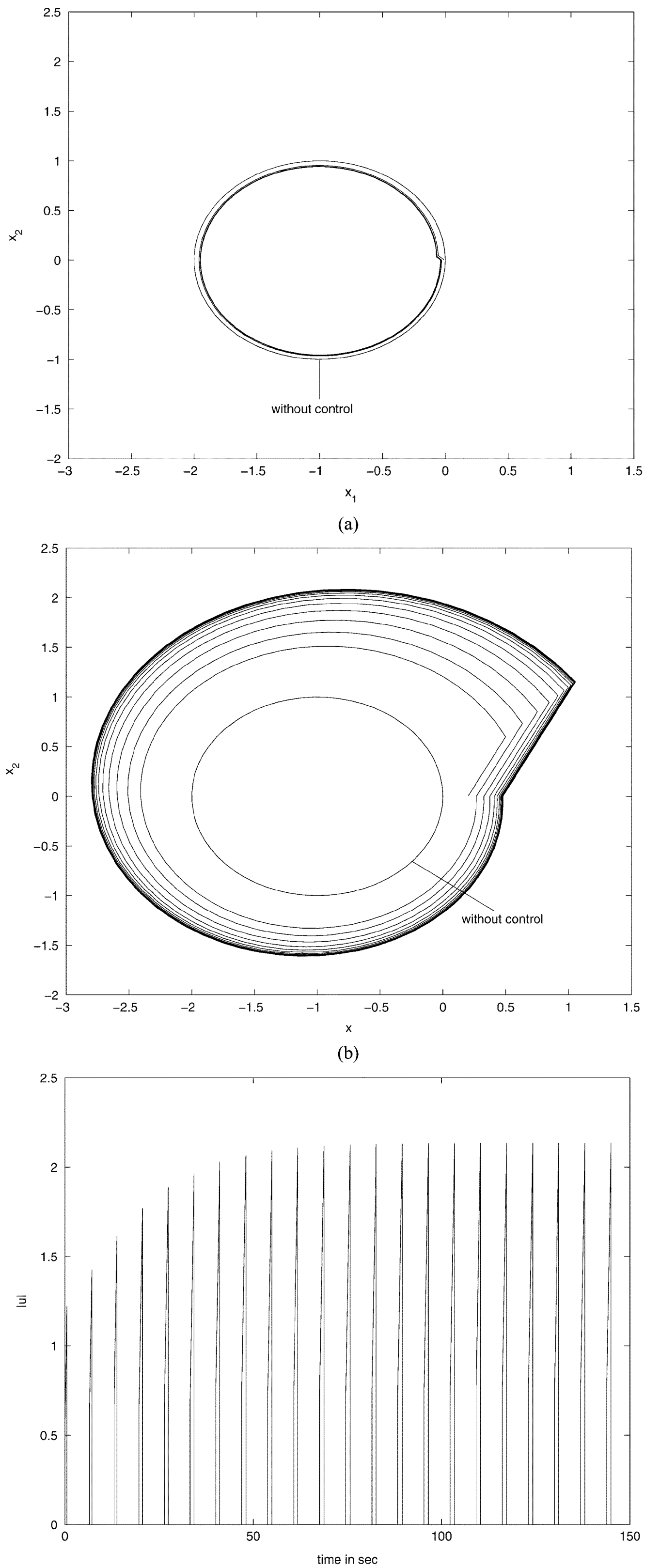
and knowing that $f_1^0 = 0, f_2^0 = 1, g_1^0 = c_1$, and $g_2^0 = 1 + c_2$, the Poincaré map is given by

$$x_1 \rightarrow \begin{cases} Px_1, & \text{if noncrossing} \\ P\left(x_1 - c_1 \frac{x_1 - b}{c_1 - c_2 - 1}\right), & \text{if crossing} \end{cases} \tag{43}$$

with $P = \alpha = 0.5335$.

In this case, the slope of the map when crossing is given by

$$\beta = P\frac{c_2 + 1}{c_2 + 1 - c_1}.$$



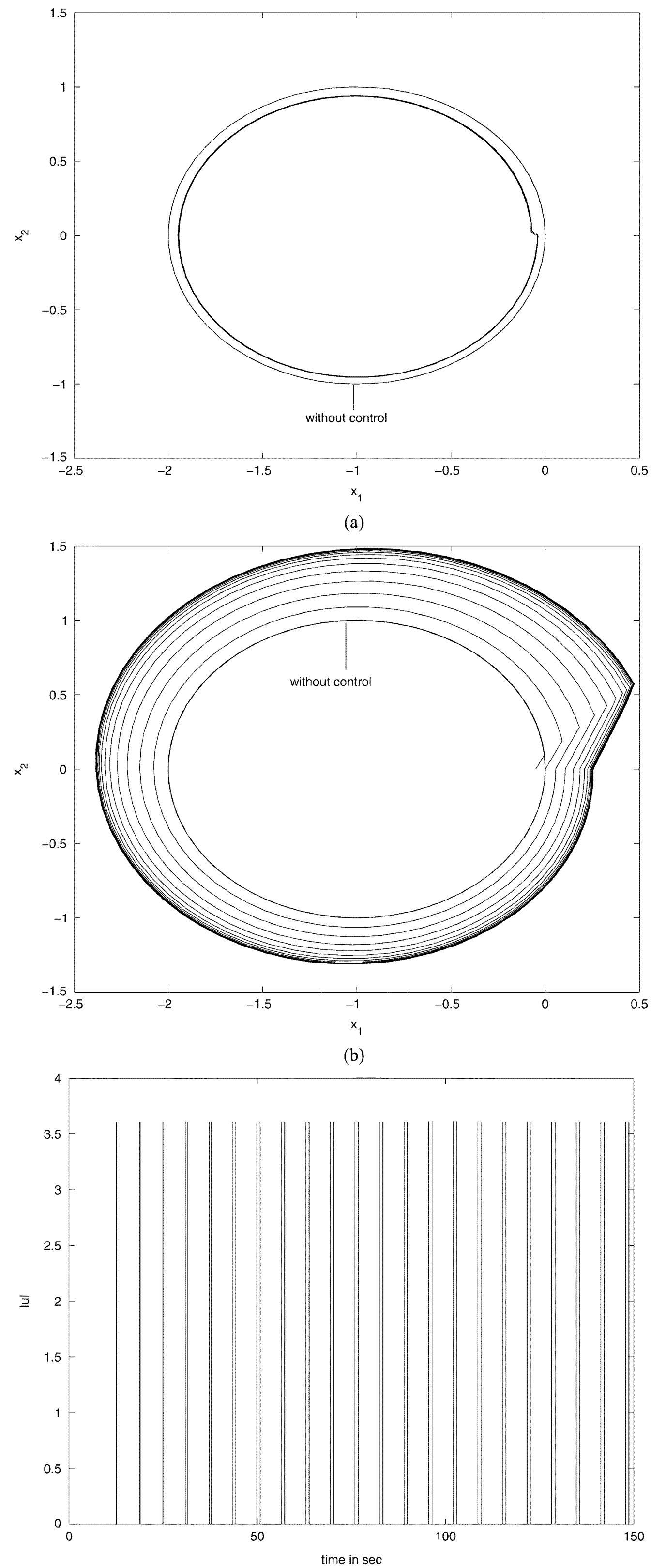


Fig. 4. Evolution of the system controlled with $\mathbf{u} = -\mathbf{F}(\mathbf{x}) + \mathbf{c}$ with different control parameters. Subfigures (a) and (b) are in the same scale. Subfigure (c) shows the behavior of the maximum effort control in case (b). (a) $c_1 = -1$ and $c_2 = 1$. (b) $c_1 = 0.5$ and $c_2 = 1$. (c) Control signal.

Fig. 5. Evolution of the system controlled with $\mathbf{u} = \mathbf{c}$ Subfigures (a) and (b) are in the same scale. Subfigure (c) shows the behavior of the maximum effort control in case (b). (a) $c_1 = -2$ and $c_2 = 1$, (b) $c_1 = 2$ and $c_2 = 3$, (c) Control signal.

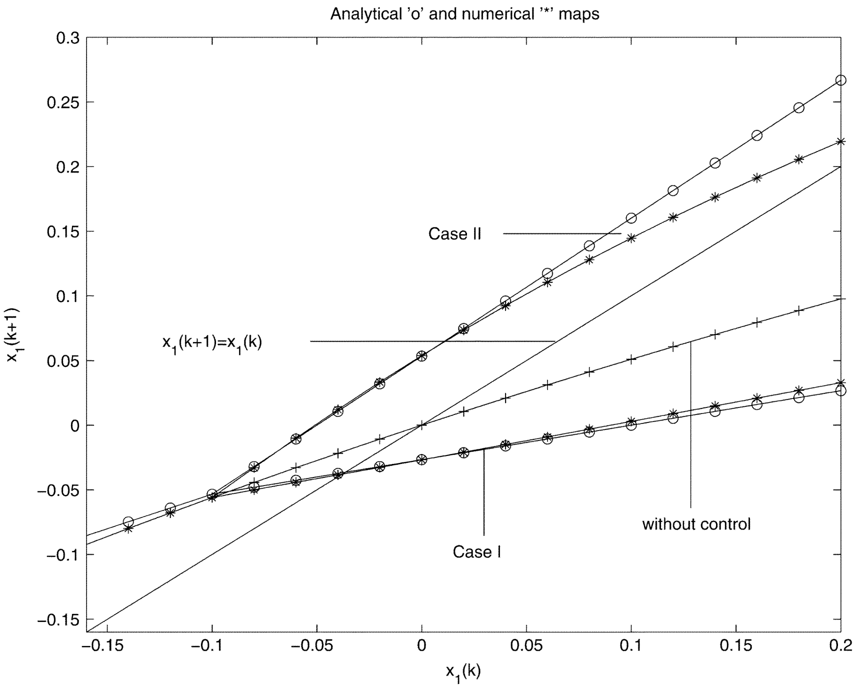


Fig. 6. Case $\mathbf{F}(\mathbf{x})$ unknown. Evolution of the fixed point for several constant values. The corner-collision point is $(-0.1, 0)$. Other bifurcation scenarios are not possible.

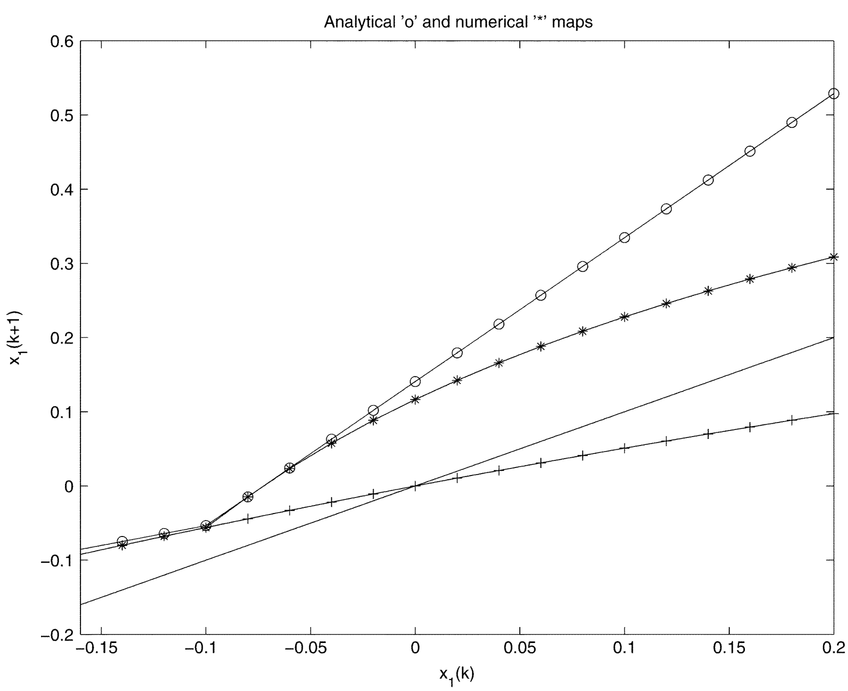


Fig. 7. Piecewise-linear map with a high slope. $c_1 = 2.8$ and $c_2 = 3$. As can be seen, far from the corner, the approximation loses validation.

Again, we need to choose c_1 and c_2 in order to avoid sliding. Following derivations similar to those outlined above, we find that $\beta > 0$ for all c_1 and c_2 satisfying (21). Note that now

$$x_1^* = \frac{Pc_1b}{c_1 - c_2 - 1 + P(1 + c_2)}$$

and so the amplitude of the limit cycle can be controlled with parameters b, c_1, and c_2.

Thus, as before, we can use the controller to achieve two different aims.

1) Case I: $\beta < 1$: Some algebra shows that this case can be possible when

$$c_1 < (1 - P)(c_2 + 1).$$

For example, with $c_2 = 1$ and $c_1 = -2$ the slope is $\beta = 0.2667$. This means that fixed point in a neighborhood continues

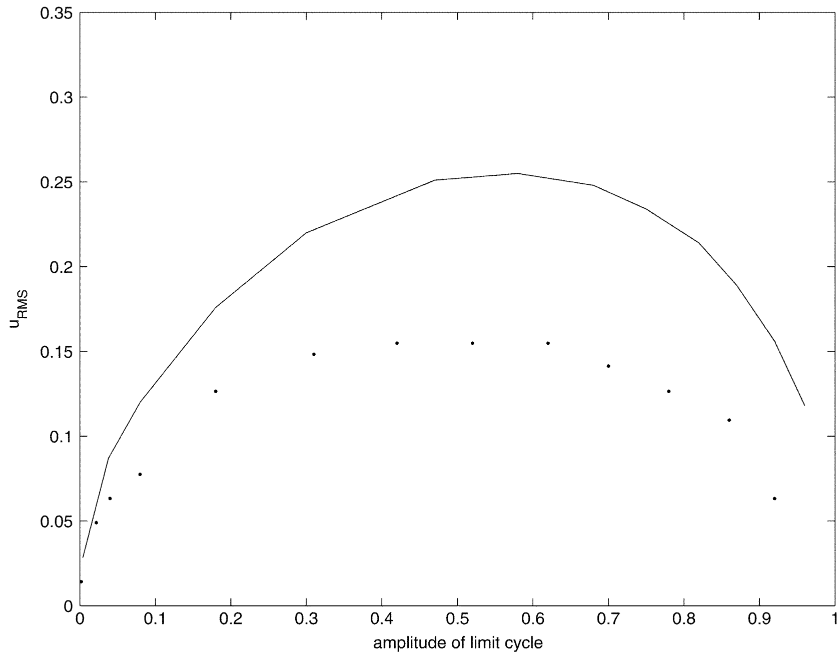


Fig. 8. Relation between amplitude of the limit cycle and rms value when the desired oscillation has an amplitude lower than the original. The solid line corresponds to numerically computed values, while the points are computed with the analytical formula deduced in Section IV.

existing after the bifurcation. Fig. 5(a) shows the behavior of the continuous-time system with and without control. We see that the limit cycle persists as expected with a minor change of its amplitude. This can be emphasised by selecting different values of c_1 and c_2 and hence moving further the fixed point of the map.

2) Case II: $\beta > 1$: Now we have

$$A, b \to \emptyset$$

and the limit cycle disappears locally in a border collision bifurcation. This case is possible too. It can be easily checked that the conditions on c_1 and c_2 are given by

$$(c_2 + 1)(1 - P) < c_1 < c_2 + 1.$$

For example, with $c_2 = 3$ and $c_1 = 2$ the slope is $\beta = 1.067$. Fig. 5(b) shows the behavior of the continuous-time system. We observe that the target cycle has disappeared locally after a border collision bifurcation; the evolution of the map moving toward another fixed point outside the range where the local description is valid. Such a fixed point corresponds to the large-amplitude limit cycle depicted in the figure. Fig. 6 shows the analytical and numerical maps for Cases I and II confirming the excellent agreement between the analysis and the numerics. (Note that, for the sake of brevity, to obtain an analytical approximation of the Poincaré map, we have truncated the discontinuity map to its linear terms.).

It is worth to note that in all the cases, if the slope of the piecewise-linear map is very high, the map based on the linear approximation becomes representative of the system behavior in a relatively small neighborhood of the bifurcation point (see Fig. 7).

3) Control Effort: In this subsection, the analytical formulas derived to estimate the control effort, when $\mathbf{F}(\mathbf{x})$ is unknown, are compared with the numerical values obtained. Constants c_1 and c_2 are varied to obtain different lower or bigger amplitude limit cycles. Once the constants have been fixed, the corner penetration is varied to obtain different amplitudes. There is no unified measure of an amplitude of a limit cycle. The original limit cycle is a circle and a natural measure can be its radius. Since the obtained limit cycles (after the control is applied) are approximately circles also, which are centered at point $(-1, 0)$, we take $x_1 + 1$ (being x_1 the first coordinate when the limit cycle enters S_1) as a measure of the amplitude. We have also computed another measure of the amplitude, given by

$$\sqrt{\frac{1}{T}\int_0^T ((x_1(\tau) + 1)^2 + x_2(\tau)^2) d\tau}$$

with equivalent results.

TABLE I
AMPLITUDES OF LIMIT CYCLES AND RMS VALUES

Amplitude	u_{RMS}
1.47	0.524
1.61	0.612
1.69	0.675
1.79	0.765
1.86	0.837
1.91	0.920

Table I shows the numerically computed rms value for cycles with amplitudes bigger than the amplitude of the original limit cycle (in this case $c_1 = 2$ and $c_2 = 3$). Also, Fig. 8 shows the relation between amplitude of the limit cycle and the numerical (solid line) and analytical (dots) rms value when the desired oscillation has an amplitude lower than the original (in this case $c_1 = -2$ and $c_2 = 1$).

As can be seen from Fig. 8, the limit cycle can disappear (the amplitude is decreased to zero). Thus, the method can also control the appearance and disappearance of limit cycles, though this fact is nonlocal and cannot be explained by the theory in

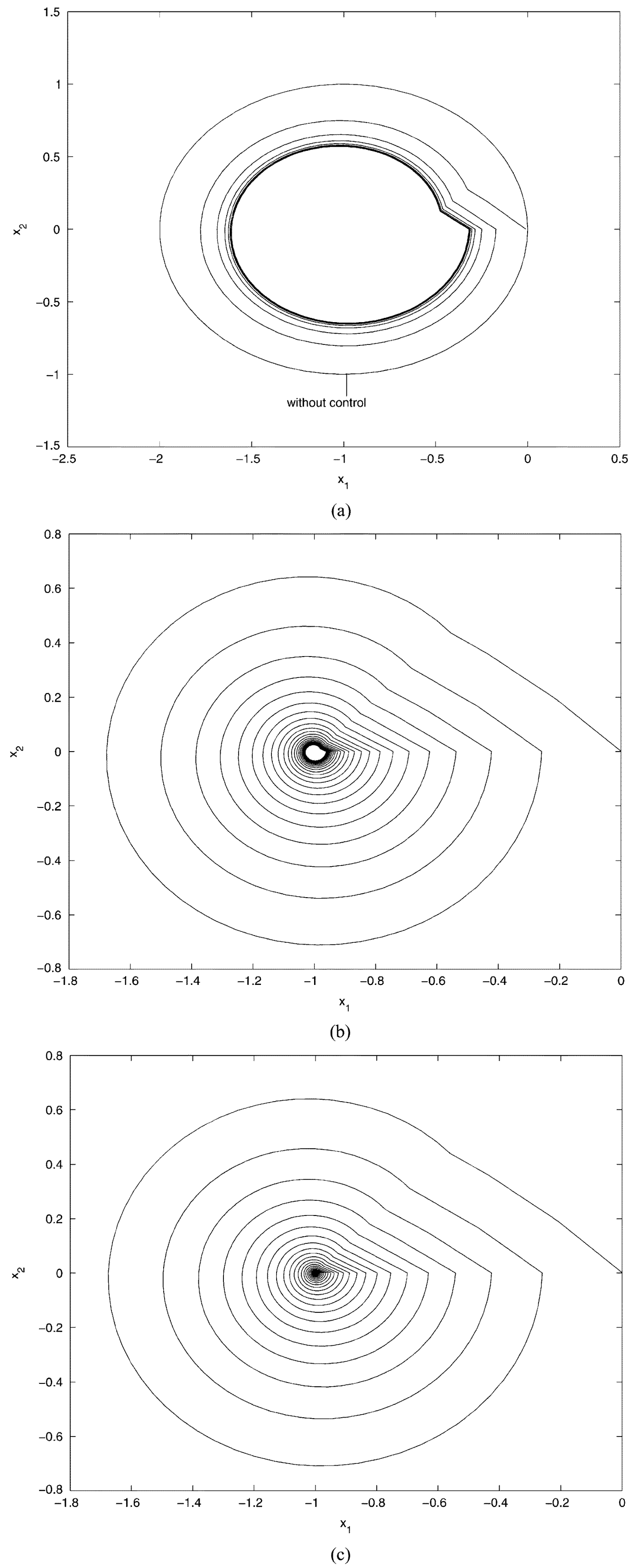


Fig. 9. Different values for the corner penetration give different meaningful amplitude reductions, even suppressing the limit cycle. (a) Corner penetration is set to -0.6. A small reduction in the amplitude of the limit cycle is obtained. (b) Corner penetration is set to -0.99. A considerable reduction of the amplitude is observed. (c) Corner penetration is set to -1, and the limit cycle turns into an equilibrium point. Also, the control effort is reduced to zero at the stationary point.

this paper. Fig. 9 shows different limit cycles when the corner penetration is varied far from the nonsmooth bifurcation.

VI. Conclusion

In this paper, we have shown that it is possible to synthesize a switching control law to suppress or change the main features of a target limit cycle in planar smooth dynamical system. Other authors [5], [6] have studied the bifurcation control problem from a smooth feedback framework. Our approach is different since switching (and thus nonsmooth) control laws are proposed. In so doing, the theory of nonsmooth bifurcations was explicitly used in the design process. Namely, by appropriately selecting the control constants and the switching manifolds, it is possible, to change the properties of the Poincaré map associated to the cycle of interest. The resulting control action is acting on the system in a relatively small neighborhood of the corner-collision point and hence guarantees the achievement of the control goal with a minimal control expenditure. We wish to emphasize that rather than being a technique for the control of bifurcations in nonlinear systems, the strategy presented here aims at exploiting the theory of nonsmooth bifurcations for control system design.

Ongoing research is aimed at further exploring the ideas presented in this paper and establish formal links between the controller gains and the properties of Ω-limit set of the closed-loop system. Also, the extra degrees of freedom corresponding to the control law parameters c_1 and c_2 (chosen here by using the additional constraint of avoiding sliding) can be further exploited to obtain, for example, a given slope β of the map when crossing, or to have solutions satisfying certain performance criteria. Future work will investigate this further and will also be concerned with the experimental validation of this control strategy.

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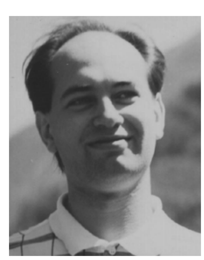

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