

Actuator-fault Detection and Isolation based on Set-theoretic Approaches

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Abstract

In this paper, an actuator-fault detection and isolation (FDI) approach is proposed. The FDI approach is based on a bank of interval observers, each of which is designed to match a healthy or faulty system mode. To ensure reliable FDI for all considered actuator faults, a collection of invariant set-based FDI conditions are established for the proposed technique. Under these guaranteed FDI conditions, all the considered faults can be detected and isolated during the transition induced by fault occurrences. Comparing with the existing set-based FDI approaches, the advantage of the proposed technique consists in that it combines the advantages of interval observers in the transient-state functioning and the advantages of invariant sets in the steady-state functioning. This paper is completed with the study of a continuous stirred-tank reactor (CSTR), which illustrates the effectiveness of the proposed method.

Keywords: Fault detection and isolation, invariant sets, interval observers, zonotopes, linear systems.

1. Introduction

The interval observers, as one of set-theoretic FDI approaches, are well-known for robust fault detection (FD) [1, 2, 3], which consists in propagating the effect of uncertainties through the system models to generate real-time intervals for the real outputs. Provided that the system is healthy, the current outputs should be inside the output intervals estimated by the interval observer based on the healthy system model. When the system is affected by faults, once the current outputs violate their intervals, the FD task will be triggered. In the literature, there exist different types of set-theoretic FD approaches: the set-valued observer [4], the set-membership state estimation [5, 6], the invariant set-based [7, 8] and the interval observer-based approaches [2, 3]. Regarding fault isolation (FI), interval observers (or other related techniques such as the set-membership estimation) generally turn to other FI techniques such as the fault signature matrix approach [9, 10]. So far, few works have been addressed for the FI application of interval observers, especially when considering that one wants to obtain guaranteed FI once the considered actuator faults affect the system.

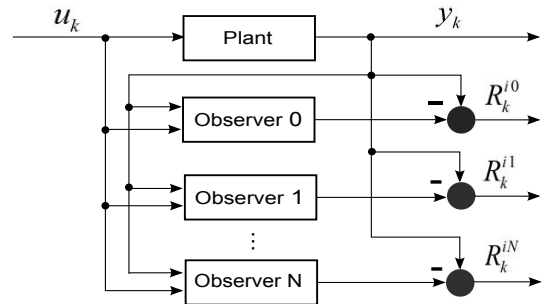


Figure 1: Interval observer-based FDI scheme

The objective of this paper is to propose an interval observer-based FDI approach in which both FD and FI are implemented by means of interval observers and without relying on other FI techniques. To provide FDI guarantees, invariant sets are used to establish FDI conditions [8, 11]. In [12], the relationship of interval observers and invariant sets in FD is investigated and some useful results for the present paper are given there. In a previous work [13], a bank of interval observers, where each observer is designed to match either one healthy or one of the faulty modes, is used to implement FDI. This paper follows the framework proposed in [13] and further proposes new techniques to enhance the performance of the FDI framework

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shown in Figure 1.

In this paper, the design of interval observers is based on the discrete-time Luenberger observer structure, and the uncertainties (disturbances, offsets and noises, etc) and faults with unknown magnitudes but known bounds are considered. In principle, the proposed method can be extended to the plant with parametric uncertainties [2]. Additionally, considering the balance among the expressional concision, computational precision and complexity, this paper uses zonotopes as the containment set to propagate the effect of uncertainties in interval observers (see [5, 6] for zonotopes in state estimation).

The contribution of this paper is threefold. First, it provides a novel perspective to the FI application of interval observers by merging the notions of interval observers and invariant sets. Second, the proposed method extends this FDI framework for unknown faults but with known bounds. Third, this technique can detect and isolate faults during the transition between different modes with less conservative FDI conditions. In this endeavor, this paper builds on the primer results in [13, 14].

The remainder of this paper is organized as follows. Section 2 introduces the plant and interval observers. Section 3 defines residual zonotopes and derives their bounding zonotopes. In Section 4, a collection of invariant set-based FDI conditions are established. The FDI algorithm is presented in Section 5. In Section 6, a CSTR example is used to illustrate the effectiveness of this approach. Finally, Section 7 draws general conclusions.

The notation \oplus represents the Minkowski sum (Minkowski sum of two sets A and B is defined by $A \oplus B = \{a + b : a \in A, b \in B\}$), $|\cdot|$ denotes the elementwise absolute value, and \mathbb{B}^r is a box composed of r unitary intervals. The inequalities are understood elementwise. The bold matrices denote, such as \mathbf{A} , interval matrices, and $mid(\cdot)$ and $diam(\cdot)$ compute the center and diameter of an interval matrix, respectively. The notation I denotes the identity matrix with compatible dimensions. Given a vector $g \in \mathbb{R}^n$ and a matrix $G \in \mathbb{R}^{n \times m}$ ($n \leq m$), a zonotope X with an order m is defined as $X = g \oplus G\mathbb{B}^m$, where g and G are called the center and segment matrix of the zonotope, respectively. For some $\epsilon > 0$, one denotes $\mathbb{B}_{p_0}^n(\epsilon) = \{x \in \mathbb{R}^n : \|x\|_{p_0} \leq \epsilon\}$, where $\|x\|_{p_0}$ is the p_0 -norm of vector x .

2. Plant and interval observers

2.1. Plant dynamics

The linear discrete time-invariant plant under the effect of actuator faults is modelled as

$$x_{k+1} = Ax_k + BF_i u_k + \omega_k, \quad (1a)$$

$$y_k = Cx_k + \eta_k, \quad (1b)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times p}$ and $C \in \mathbb{R}^{q \times n}$ are constant matrices, $x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^p$ and $y_k \in \mathbb{R}^q$ are states, inputs and outputs at the k -th time instant, respectively, ω_k and η_k represent system uncertainties (disturbances, noises, offsets, etc) in states and outputs, respectively, and F_i ($i \in \mathbb{I} = \{0, 1, \dots, N\}$) represents a finite range of actuator-fault modes important to system

performance or safety) is a $p \times p$ diagonal matrix modelling the i -th actuator mode (healthy or faulty)¹.

Note that F_0 is the identity matrix with compatible dimensions and describes the healthy actuator mode, and F_i with $i \in \mathbb{I} \setminus \{0\}$ models the fault-affected system. The diagonal elements of F_i ($i \in \mathbb{I} \setminus \{0\}$) belong to the interval $[0, 1]$, where an element taking the value 0 or 1 represents the complete outage or health of the corresponding actuator, respectively, while taking a value inside $(0, 1)$ denotes the partial performance degradation of the corresponding actuator.

Assumption 2.1. The uncertainties ω_k and η_k are unknown but bounded by known sets W and V with the form

$$W = \{\omega \in \mathbb{R}^n : |\omega - \omega^c| \leq \bar{\omega}, \omega^c \in \mathbb{R}^n, \bar{\omega} \in \mathbb{R}^n\},$$

$$V = \{\eta \in \mathbb{R}^q : |\eta - \eta^c| \leq \bar{\eta}, \eta^c \in \mathbb{R}^q, \bar{\eta} \in \mathbb{R}^q\},$$

respectively, where the vectors ω^c , η^c , $\bar{\omega}$ and $\bar{\eta}$ are constant.

Note that, the sets W and V can be rewritten as zonotopes

$$W = \omega^c \oplus H_{\bar{\omega}} \mathbb{B}^n, \quad (2a)$$

$$V = \eta^c \oplus H_{\bar{\eta}} \mathbb{B}^q, \quad (2b)$$

where $H_{\bar{\omega}} \in \mathbb{R}^{n \times n}$ and $H_{\bar{\eta}} \in \mathbb{R}^{q \times q}$ are diagonal matrices with diagonal entries composed of $\bar{\omega}$ and $\bar{\eta}$, respectively.

Assumption 2.2. The pairs (A, BF_i) for all $i \in \mathbb{I}$ are stabilizable and the given control inputs u_k guarantee the stability of the plant (1) (or the plant (1) is stable) and the pair (A, C) is detectable.

Assumption 2.3. In the i -th actuator-fault mode, F_i is bounded by an interval matrix \mathbf{F}_i , i.e., $F_i \in \mathbf{F}_i$, where F_i (the actual magnitude of the i -th fault) is unknown and \mathbf{F}_i (the bound of the magnitude of the i -th fault) is known.

Note that F_i is used to model the i -th actuator fault, while \mathbf{F}_i is used to describe the range of fault magnitude (an important feature for system performance or safety) of the corresponding actuators in the i -th mode.

Remark 2.1. A fault occurrence indicates a change from F_i to F_j ($i \neq j$) in (1). Since the actuator-fault magnitude is unknown, when the system is in the functioning of the i -th mode, F_i can be any value inside its bound \mathbf{F}_i , which implies that F_i can be constant or time-varying with any profile.

2.2. Interval observers

Interval observers use the plant inputs and outputs to estimate the state and output sets. The interval observer corresponding to the healthy functioning is introduced first to explain the general design procedure, which is further employed for the description of the rest of interval observers.

¹A system mode characterizes the system as being under a certain dynamics, which corresponds to either healthy or faulty behaviors.

2.2.1. Interval observer for healthy mode

Considering the plant (1), in the healthy mode, the healthy interval observer with Luenberger structure

$$\hat{x}_{k+1} = A\hat{x}_k + BF_0u_k + L_0(y_k - \hat{y}_k) + \check{\omega}_k, \quad (3a)$$

$$\hat{y}_k = C\hat{x}_k + \check{\eta}_k, \quad (3b)$$

is designed to monitor the system in the healthy functioning, where the matrix L_0 is the observer gain.

In the Luenberger observer structure (3), the uncertain variables $\check{\omega}_k$ and $\check{\eta}_k$ are used to describe the effect of ω_k and η_k in the plant (1) on the state and output estimations \hat{x}_k and \hat{y}_k (i.e., the effect on the state and output sets estimated by the corresponding interval observer), respectively. The uncertain variables $\check{\omega}_k$ and $\check{\eta}_k$ are different from ω_k and η_k but have the same sets, respectively, i.e., $\check{\omega}_k \in W$ and $\check{\eta}_k \in V$. By substituting (3b) into (3a), (3) can be equivalently transformed into

$$\hat{x}_{k+1} = (A - L_0C)\hat{x}_k + BF_0u_k + L_0y_k - L_0\check{\eta}_k + \check{\omega}_k, \quad (4a)$$

$$\hat{y}_k = C\hat{x}_k + \check{\eta}_k. \quad (4b)$$

Note that $\check{\omega}_k$ and $\check{\eta}_k$ respectively emulate ω_k and η_k from (1) and are used to cover the effect of ω_k and η_k on the state and output estimations from (3), while acknowledging that the actual noises are not measurable but manipulated set-wise.

In the healthy mode, the healthy interval observer able to estimate the state and output sets that bound the states and outputs of the plant (1) can be obtained as

$$\hat{X}_{k+1}^0 = (A - L_0C)\hat{X}_k^0 \oplus \{BF_0u_k\} \oplus \{L_0y_k\} \oplus (-L_0)V \oplus W, \quad (5a)$$

$$\hat{Y}_k^0 = C\hat{X}_k^0 \oplus V \quad (5b)$$

by substituting the sets W and V of $\check{\omega}_k$ and $\check{\eta}_k$ into (4), where \hat{X}_k^0 and \hat{Y}_k^0 are the estimated state and output sets at time instant k , respectively, and L_0 is chosen to ensure that $A - L_0C$ is a Schur matrix, which is always possible under Assumption 2.2.

Remark 2.2. The interval observer is a set-based observer that converges under the Schur-matrix hypothesis for the matrix $A - L_0C$ independent of the topology of the sets in the construction. Naturally, these properties are inherited in the case of zonotopic sets used in this paper.

Since ω_k and η_k are bounded by zonotopes in (2), zonotopes to bound the estimated outputs and states can be constructed by introducing zonotopic description of $\check{\omega}_k$ and $\check{\eta}_k$ into the observer mapping (4) and using zonotope arithmetic at each time instant.

Assumption 2.4. The initial state of the plant is represented as x_0 and all the interval observers are initialized by a common zonotopic set \hat{X}_0 such that $x_0 \in \hat{X}_0$ holds.

Actually, an interval observer estimates the bounds for states and outputs starting from an initial zonotope and the initial zonotope can be arbitrarily given. For convenience, it is assumed that the initial zonotope of each observer can contain the

initial state of the plant and all interval observers use the same initial zonotope as in Assumption 2.4. However, how to give an initial zonotope for each observer is decided by the designer.

Remark 2.3. Under Assumption 2.4, it is guaranteed that the current states and outputs of the plant are always bounded by the state and output zonotopes estimated by an interval observer whose internal model matches the current mode model.

In the proposed technique, since interval observers are based on zonotopes, the discussion is also based on zonotopes in the remaining of the paper and some relevant features of zonotopes are introduced in Appendices.

2.2.2. Interval observers for actuator-fault modes

Similarly, the interval observer, used to monitor the j -th actuator-fault mode ($j \in \mathbb{I} \setminus \{0\}$), is designed as

$$\hat{X}_{k+1}^j = (A - L_jC)\hat{X}_k^j \oplus \{B\mathbf{F}_j u_k\} \oplus \{L_j y_k\} \oplus (-L_j)V \oplus W, \quad (6a)$$

$$\hat{Y}_k^j = C\hat{X}_k^j \oplus V, \quad (6b)$$

where j is the index of the interval observer, \hat{X}_k^j and \hat{Y}_k^j are the estimated state and output zonotopes at time instant k , respectively, and the observer gain L_j is chosen to ensure that $A - L_jC$ is a Schur matrix, which is guaranteed by Assumption 2.2.

Remark 2.4. The gain matrix L_j of the j -th interval observer is independent of those of the other interval observers, i.e., the gain matrices of interval observers are separately designed.

Remark 2.5. In Appendices, if H_k , \mathbf{B} and u_k are zero, Property B.5 will reduce to the computation of a zonotope to bound the multiplication of an interval matrix and a vector. In this case, by using the reduced result of Property B.5, the term $\mathbf{F}_j u_k$ in (6) can be overapproximated by a zonotope whose center and segment matrix are $\text{mid}(\mathbf{F}_j)u_k$ and $\frac{\text{diam}(\mathbf{F}_j)}{2}u_k$, respectively.

As per Remark 2.5, $\mathbf{F}_j u_k$ in (6) can be replaced by its zonotopic overapproximation. In this way, one can obtain a feasible form of (6), i.e., an overapproximation of (6). Moreover, using zonotope manipulations, the obtained feasible form of (6) can be equivalently split into the center-segment matrix description

$$\hat{x}_{k+1}^{j,c} = (A - L_jC)\hat{x}_k^{j,c} + B\text{mid}(\mathbf{F}_j)u_k + L_j y_k - L_j \eta^c + w^c, \quad (7a)$$

$$\hat{H}_{k+1}^{j,x} = [(A - L_jC)\hat{H}_k^{j,x} \quad B\frac{\text{diam}(\mathbf{F}_j)}{2}u_k \quad -L_j H_{\bar{\eta}} \quad H_{\bar{\omega}}], \quad (7b)$$

$$\hat{y}_k^{j,c} = C\hat{x}_k^{j,c} + \eta^c, \quad (7c)$$

$$\hat{H}_k^{j,y} = [C\hat{H}_k^{j,x} \quad H_{\bar{\eta}}], \quad (7d)$$

where $\hat{x}_{k+1}^{j,c}$ and $\hat{y}_k^{j,c}$ are the centers of \hat{X}_{k+1}^j and \hat{Y}_k^j , and $\hat{H}_{k+1}^{j,x}$ and $\hat{H}_k^{j,y}$ are the segment matrices of \hat{X}_{k+1}^j and \hat{Y}_k^j , respectively.

For simplicity, one uses the same notations \hat{X}_{k+1}^j and \hat{Y}_k^j to denote the state and output set estimations of (6) and (7). It can

be observed that \hat{X}_{k+1}^j and \hat{Y}_k^j in (7) are the overapproximations of \hat{X}_k^j and \hat{Y}_k^j in (6), respectively, where the former is the particular implementation of interval observers while the latter is the theoretical expression of interval observers.

Note that, since only (7) is used to estimate state and output sets in the proposed approach, \hat{X}_{k+1}^j and \hat{Y}_k^j in the remaining of the paper denote the state and output sets estimated by (7).

Remark 2.6. Letting $j = 0$ in (7), the center $\hat{x}_{k+1}^{0,c}$ and segment matrix $\hat{H}_{k+1}^{0,x}$ of \hat{X}_{k+1}^0 , and the center $\hat{y}_k^{0,c}$ and segment matrix $\hat{H}_k^{0,y}$ of \hat{Y}_k^0 , corresponding to the healthy interval observer, can be accurately obtained.

3. Residual zonotopes

3.1. Residual zonotopes

For the model-based FDI approach, it is necessary to define the residuals for fault diagnosis. Different from the traditional residual definition (a residual is defined as a vector), the residuals are defined as sets (zonotopes) in the proposed technique. As per (1), (5) and (7), the residual zonotope is defined as

$$\begin{aligned} R_k^{ij} &= \{y_k\} \oplus (-\hat{Y}_k^j) \\ &= \{Cx_k + \eta_k\} \oplus \{(-C\hat{X}_k^j) \oplus (-V)\} \\ &= C\{\{x_k\} \oplus (-\hat{X}_k^j)\} \oplus \{\eta_k\} \oplus (-V), \end{aligned} \quad (8)$$

where R_k^{ij} denotes the residual zonotope estimated by the j -th ($j \in \mathbb{I}$) interval observer when the plant is in the i -th ($i \in \mathbb{I}$) mode at time instant k .

In order to obtain the set values of R_k^{ij} , the zonotope $\tilde{X}_k^{ij} = \{x_k\} \oplus (-\hat{X}_k^j)$ in (8) should be considered, which is derived as

$$\tilde{X}_k^{ij} = \{x_k\} \oplus (-\hat{X}_k^j) = \{(x_k - \hat{x}_k^{j,c})\} \oplus \hat{H}_k^{j,x} \mathbb{B}^{s_k^j}, \quad (9)$$

where s_k^j represents the order of the zonotope \tilde{X}_k^{ij} .

According to (1) and (7), at time instant $k + 1$, using $\hat{x}_{k+1}^{j,c}$ and $\hat{H}_{k+1}^{j,x}$ to denote $x_k - \hat{x}_{k+1}^{j,c}$ and $\hat{H}_{k+1}^{j,x}$ as seen in (9), the center $\tilde{x}_{k+1}^{ij,c}$ and segment matrix $\tilde{H}_{k+1}^{ij,x}$ of \tilde{X}_{k+1}^{ij} are computed as

$$\begin{aligned} \tilde{x}_{k+1}^{ij,c} &= (A - L_j C) \tilde{x}_k^{ij,c} + B(F_i - \text{mid}(\mathbf{F}_j))u_k \\ &\quad - L_j(\eta_k - \eta^c) + (\omega_k - \omega^c), \end{aligned} \quad (10a)$$

$$\begin{aligned} \tilde{H}_{k+1}^{ij,x} &= \hat{H}_{k+1}^{j,x}, \\ \hat{H}_{k+1}^{j,x} &= [(A - L_j C) \hat{H}_k^{j,x} B \frac{\text{diam}(\mathbf{F}_j)}{2} u_k - L_j H_{\bar{\eta}} H_{\bar{\omega}}]. \end{aligned} \quad (10b)$$

By substituting (9) into (8), the expression of the residual zonotope can be restated as

$$R_k^{ij} = C\tilde{X}_k^{ij} \oplus \{\eta_k\} \oplus (-V). \quad (11)$$

3.2. Adaptive bounds for residual zonotopes

Substituting \mathbf{F}_i , W and V into (10a) to respectively replace F_i , ω_k and η_k and using zonotope operations, one can obtain a bounding zonotope \check{X}_{k+1}^{ij} to bound \tilde{X}_{k+1}^{ij} described by (9) and

(10). Moreover, the center $\check{x}_{k+1}^{ij,c}$ and segment matrix $\check{H}_{k+1}^{ij,x}$ of \check{X}_{k+1}^{ij} can be derived as

$$\check{x}_{k+1}^{ij,c} = (A - L_j C) \check{x}_k^{ij,c} + B \text{mid}(\mathbf{F}_i) u_k - B \text{mid}(\mathbf{F}_j) u_k, \quad (12a)$$

$$\begin{aligned} \check{H}_{k+1}^{ij,x} &= [(A - L_j C) \check{H}_k^{ij,x} B \frac{\text{diam}(\mathbf{F}_i)}{2} u_k - B \frac{\text{diam}(\mathbf{F}_j)}{2} u_k \\ &\quad L_j H_{\bar{\eta}} - L_j H_{\bar{\eta}} H_{\bar{\omega}} - H_{\bar{\omega}}]. \end{aligned} \quad (12b)$$

Using Remark 2.5 and zonotope manipulations, an equivalent compact form of (12) can be derived as

$$\begin{aligned} \check{X}_{k+1}^{ij} &= (A - L_j C) \check{X}_k^{ij} \oplus B \check{U}_i \oplus B(-\check{U}_j) \oplus L_j(-V) \\ &\quad \oplus W \oplus L_j V \oplus (-W), \end{aligned} \quad (13)$$

where the sets \check{U}_i and \check{U}_j are zonotopes computed as

$$\check{U}_i = \{\text{mid}(\mathbf{F}_i) u_k\} \oplus \frac{\text{diam}(\mathbf{F}_i)}{2} u_k \mathbb{B}^{s_{\check{u}_i}},$$

and

$$\check{U}_j = \{\text{mid}(\mathbf{F}_j) u_k\} \oplus \frac{\text{diam}(\mathbf{F}_j)}{2} u_k \mathbb{B}^{s_{\check{u}_j}},$$

where $s_{\check{u}_i}$ and $s_{\check{u}_j}$ are the orders of \check{U}_i and \check{U}_j , respectively.

Proposition 3.1. As long as $\check{X}_{k^*}^{ij} \subseteq \tilde{X}_{k^*}^{ij}$ holds, \check{X}_k^{ij} will be always bounded by \tilde{X}_k^{ij} for all time instants $k > k^*$.

Proof : Since (12) is obtained by substituting the bounds of F_i , ω_k and η_k into (10), it follows that if, at time instant k^* , $\check{X}_{k^*}^{ij} \subseteq \tilde{X}_{k^*}^{ij}$ holds, then after k^* , the inclusion will always hold. \square

As per Proposition 3.1, by introducing \check{X}_k^{ij} and V into (11), a computable bound for R_k^{ij} can be obtained as

$$\check{R}_k^{ij} = C\check{X}_k^{ij} \oplus V \oplus (-V). \quad (14)$$

3.3. Static bounds for residual zonotopes

In (13), the adaptive bound \check{X}_k^{ij} of \tilde{X}_k^{ij} always tracks the evolution of control inputs. Since FDI conditions of the proposed approach are established by using fixed steady sets, in order to establish the FDI conditions, it is necessary to obtain a static bound for \check{X}_k^{ij} (not affected by the evolution of inputs). This will be detailed in the following contents.

Assumption 3.1. The inputs u_k of the plant are bounded by

$$U = \{u_k \in \mathbb{R}^p : |u_k - u^c| \leq \bar{u}, u^c \in \mathbb{R}^p, \bar{u} \in \mathbb{R}^p\},$$

where the vectors u^c and \bar{u} are constant. Furthermore, the set U can be rewritten as a zonotope $U = u^c \oplus H_{\bar{u}} \mathbb{B}^p$, where $H_{\bar{u}} \in \mathbb{R}^{p \times p}$ is a diagonal matrix with the main diagonal being \bar{u} .

By replacing u_k in (12) with its bound U , one can obtain a static bound denoted as \check{X}_{k+1}^{ij} for both \check{X}_{k+1}^{ij} and \check{X}_{k+1}^{ij} and the set-based dynamics of \check{X}_{k+1}^{ij} is expressed as

$$\begin{aligned} \check{X}_{k+1}^{ij} &= (A - L_j C) \check{X}_k^{ij} \oplus B \check{U}_i \oplus B(-\check{U}_j) \oplus L_j(-V) \\ &\quad \oplus W \oplus L_j V \oplus (-W), \end{aligned} \quad (15)$$

where, according to Property B.4 and Property B.5 in Appendices, the sets \check{U}_i and \check{U}_j are zonotopes computed as

$$\check{U}_i = \{mid(\mathbf{F}_i)u_c\} \oplus [seg(\diamond(\mathbf{F}_i H_{\bar{u}}))] \frac{diam(\mathbf{F}_i)}{2} u_c] \mathbb{B}^{s_{\bar{u}_i}},$$

and

$$\check{U}_j = \{mid(\mathbf{F}_j)u_c\} \oplus [seg(\diamond(\mathbf{F}_j H_{\bar{u}}))] \frac{diam(\mathbf{F}_j)}{2} u_c] \mathbb{B}^{s_{\bar{u}_j}},$$

where $s_{\bar{u}_i}$ and $s_{\bar{u}_j}$ are the orders of \check{U}_i and \check{U}_j , respectively.

Thus, from set-theoretic point of view, one can define an equivalent dynamics for (15) as

$$\begin{aligned} \check{x}_{k+1}^{ij} = & (A - L_j C) \check{x}_k^{ij} + B \check{u}_{ik} - B \check{u}_{jk} - L_j \eta_k + \check{\omega}_k \\ & + L_j \check{\eta}_k - \omega_k, \end{aligned} \quad (16)$$

where $\check{u}_{ik} \in \check{U}_i$, $\check{u}_{jk} \in \check{U}_j$, $\check{\eta}_k \in V$ and $\check{\omega}_k \in W$.

As per the notions of robust positively invariant (RPI) and minimal robust positively invariant (mRPI) sets in Property A.1, Property A.2 and Theorem A.1 in Appendices, an RPI set for (16) can be computed and further be written in the zonotopic form. Moreover, using Proposition A.1 in Appendices and the RPI set as an initial set for (15), after a finite number of iterations, an RPI approximation (denoted as S_{ij}) with an arbitrarily expected precision to the mRPI set of (16) can be computed. Since the mRPI set is the limit set of (15), as long as the precision of S_{ij} is satisfactory, S_{ij} can reliably replace the use of the mRPI set.

In the proposed approach, for each mode, the corresponding interval observer is designed according to the mode model. By substituting \check{X}_k^{ij} in (15) into (8), a static bound \check{R}_k^{ij} for both R_k^{ij} and \check{R}_k^{ij} can be obtained as

$$\check{R}_k^{ij} = C \check{X}_k^{ij} \oplus V \oplus (-V). \quad (17)$$

Remark 3.1. Two different residual-related sets \check{R}_k^{ij} and \check{R}_k^{ij} are considered in the proposed approach. \check{R}_k^{ij} and \check{R}_k^{ij} as the two different bounds of R_k^{ij} have different uses, i.e., \check{R}_k^{ij} is used for the on-line FI during the transition while \check{R}_k^{ij} is used to establish the FDI conditions. This will be elaborated in the following.

Since the set S_{ij} is an RPI approximation of the mRPI set \check{X}_∞^{ij} , one can obtain a suitable approximation \check{R}_∞^{ij} for \check{R}_k^{ij} , which is expressed as

$$\check{R}_\infty^{ij} = C S_{ij} \oplus V \oplus (-V). \quad (18)$$

Note that, in Section 3, when $i = 0$ and $j = 0$, the relevant conclusions reduce to the case corresponding to the healthy interval observer under the healthy mode.

4. Guaranteed FDI conditions

This section establishes the FDI conditions at steady state by using the static bound of residual zonotopes and the notion of invariant sets.

4.1. Theoretical FDI conditions

In the proposed approach, the theoretical FDI conditions are established by investigating the dynamic behaviors of the system at infinity. As k tends to infinity, a collection of guaranteed FDI conditions can be established using the static residual-bounding zonotopes \check{R}_∞^{ij} as indicated in (17). The general conclusion is summarized in the following theorem.

Theorem 4.1. *Given the plant (1) and a bank of interval observers (5) and (6), for any mode i ($i \in \mathbb{I}$), if the static residual-bounding zonotopes corresponding to interval observers satisfy*

$$\mathbf{0} \in \check{R}_\infty^{ii} \text{ and } \mathbf{0} \notin \check{R}_\infty^{ij}, j \neq i, i, j \in \mathbb{I}, \quad (19)$$

once a considered mode occurs, the detection and isolation of the mode can be guaranteed as the system converges to the steady state of the mode.

Proof : The proof includes three parts. The first one is to prove that (19) is asymptotic FDI conditions for the proposed method. The second one focuses on the dynamic behaviors of the static residual-bounding zonotopes at infinity, i.e., \check{R}_∞^{ij} translates the behaviors of the plant at steady state, which guarantees FDI. The third one is to prove that (19) guarantees FDI during the transition induced by mode switching.

1) The satisfaction of (19) implies that only residual zonotopes estimated by the interval observer matching the current mode contain the origin $\mathbf{0}$ at infinity while residual zonotopes, estimated by the interval observers not matching the current mode, exclude $\mathbf{0}$ at infinity. Thus, (19) guarantees that the considered faults satisfying the theorem are detectable and isolable.

2) Without loss of generality, the following proof is based on the relevant set-based dynamics. Equation (15) shows that the time-variant term is $(A - L_j C) \check{X}_k^{ij}$, which means that the differences of \check{X}_k^{ij} at different time instants, are determined by the shape of \check{X}_0^{ij} , while the contractive factor $A - L_j C$ is determined by the placement of the eigenvalues of the matrix $A - L_j C$ (system matrix of the dynamics of the j -th interval observer). Thus, after a waiting time assessed by the eigenvalues of the interval observer after a mode switching, (15) enters into its steady state. Then, the set values of \check{X}_k^{ij} after entering into the steady state can be sufficiently close² to the set \check{X}_∞^{ij} , which implies that \check{X}_∞^{ij} can be used to approximately describe the dynamic behaviors of the whole process after the waiting time. Thus, as long as Theorem 4.1 is satisfied, FDI of all the considered faults can be guaranteed once they occur.

3) According to 1) and 2), it is known that the considered faults can be detected and isolated at latest when the system enters into a new steady state, which is implemented by finding the interval observer that estimates residual zonotopes that can include $\mathbf{0}$. Regarding the implementation of FI during the transition, it will be detailed in Section 5. \square

² \check{X}_k^{ij} is inside the set described as the Minkowski sum of $\{P_{ij}\} \oplus (1 + \epsilon) \{\check{X}_\infty^{ij} \oplus \{-P_{ij}\}\}$, where P_{ij} denotes the center of \check{X}_∞^{ij} and ϵ is a scalar that satisfies $\epsilon > 0$.

Remark 4.1. According to Section 3, at infinity, one has $R_\infty^{ij} \subseteq \check{R}_\infty^{ij} \subseteq \hat{R}_\infty^{ij}$. Thus, if \check{R}_∞^{ij} satisfies Theorem 4.1, it implies that the same conclusion can be drawn for R_∞^{ij} , which guarantees that all the considered faults are detectable and isolable by a bank of interval observers. Similar with Proposition 3.1, if $R_{k^*}^{ij} \subseteq \check{R}_{k^*}^{ij} \subseteq \hat{R}_{k^*}^{ij}$ holds, $R_k^{ij} \subseteq \check{R}_k^{ij} \subseteq \hat{R}_k^{ij}$ will always hold for all time instants $k \geq k^*$. Since the adaptive bound \check{R}_k^{ij} is less conservative than the static bound \hat{R}_k^{ij} , the FI task of this proposed approach is done by using \check{R}_k^{ij} . This will be detailed in next contents.

4.2. Practical FDI conditions

Theoretically, \check{R}_∞^{ij} should be used to establish and check the FDI conditions as explained in Theorem 4.1. However, since \check{R}_∞^{ij} can not be accurately computed but only approximated, Theorem 4.1 has only theoretical value.

To establish a collection of offline precheckable FDI conditions for practical applications, one has to turn to the approximation of \check{R}_∞^{ij} defined in Section 3. Based on (18) and Theorem 4.1, a collection of practical FDI conditions are given as

$$\mathbf{0} \in \hat{R}_\infty^{ii} \text{ and } \mathbf{0} \notin \hat{R}_\infty^{ij}, j \neq i, i, j \in \mathbb{I}. \quad (20)$$

If all the considered faults satisfy (20), it is assured that all of them are detectable and isolable by the proposed FDI approach. The guaranteed FDI conditions are a collection of sufficient conditions, not necessary conditions due to the series of approximations contained in the approach. Thus, their satisfaction can guarantee FDI, but their violation does not imply that the faults are not detectable or isolable with extra effort.

5. FDI algorithm

Under the satisfaction of the FDI conditions, the proposed FDI algorithm is elaborated in this section.

5.1. Fault detection and isolation

The proposed approach implements FD by testing if residual zonotopes estimated by the interval observer matching the current system mode can include the origin at each time instant. The FD principle is summarized in the following theorem.

Theorem 5.1. *If the system is in the steady-state functioning of the m -th mode, residual zonotopes estimated by the m -th interval observer can always satisfy*

$$\mathbf{0} \in R_k^{mm}, m \in \mathbb{I}, \quad (21)$$

which implies that, whenever a violation of (21) is detected, it is indicated that a fault has occurred in the system.

Theorem 5.1 follows the interval observer-based FD approach in [13]. Please refer to Section IV in [13] for the details. To explain the FI principle, it is assumed that the system is in the m -th mode and that a fault is detected at time instant k_d . Thus, $R_{k_d}^{fj}$ ($f, j \in \mathbb{I} \setminus \{m\}$) can be obtained at time instant k_d , where f denotes the index of a new but unknown mode.

Furthermore, for the j -th interval observer, an initial zonotope at time instant k_d , denoted as $\check{X}_{k_d}^{jj}$ that satisfies $\check{X}_{k_d}^{jj} \supseteq \hat{X}_{k_d}^{fj}$, i.e., $\check{R}_{k_d}^{jj} \supseteq R_{k_d}^{fj}$, is constructed. This initial zonotope $\check{X}_{k_d}^{jj}$ is used to initialize the dynamics \check{X}_{k+1}^{jj} given by (13), which corresponds to the j -th interval observer. After this initialization, one can try to isolate faults during the transition.

Proposition 5.1. *After a fault is detected and the dynamics \check{X}_{k+1}^{jj} are initialized, if the j -th interval observer matches the current and unknown mode, \check{R}_k^{jj} should always fully bound R_k^{fj} after the FD time instant k_d , i.e., $\check{R}_k^{jj} \supseteq R_k^{fj}$ ($k \geq k_d$), while if the j -th interval observer does not match the current mode, \check{R}_k^{jj} can only fully contain R_k^{fj} at the first several steps after FD and will finally diverge.*

Proposition 5.1 states the transient FI principle proposed in this paper, which is guaranteed by Theorem 4.1. The relevant details will be further discussed below.

With respect to each interval observer (excluding the m -th one), the adaptive bound \check{R}_k^{jj} is obtained by initializing the corresponding dynamics of \check{X}_{k+1}^{jj} . Thus, starting from the FD time k_d , the fault can be isolated by real-time testing if

$$R_{l_d}^{fj} \subseteq \check{R}_{l_d}^{jj}, l_d > k_d, f, j \in \mathbb{I} \setminus \{m\} \quad (22)$$

is violated for each interval observer. By iteratively testing (22) till the time instant when one and only one interval observer can satisfy (22), it implies that the current fault is isolated at the time instant and the fault is indexed by the index of the interval observer.

Because of the FDI conditions, one can ensure that the fault can be isolated before the system reaches its new steady state. But the particular time needed for FI is unknown, which depends on the system dynamics and faults. It is proved that the proposed FI method can isolate the faults during the transient state and avoid waiting a period until the complete disappearance of transient behaviors to make FI decisions.

Remark 5.1. For the j -th interval observer, $\check{R}_{k_d}^{jj}$ and $\check{R}_{k_d}^{ij}$ ($i \neq j, i, j \in \mathbb{I} \setminus \{m\}$) may intersect. If the intersections always contain $R_{k_d}^{fj}$ during the transition, even though the j -th interval observer does not match the new mode, it is still possible that $\check{R}_k^{jj} \supseteq R_k^{fj}$ ($k \geq k_d$) persistently holds. Consequently, this fact may disturb the FI accuracy of the proposed criterion (22).

In order to solve the problem in Remark 5.1, one turns to the FI mechanism in 3) of Theorem 4.1, i.e., testing if

$$\mathbf{0} \in R_k^{fj}, \quad (23)$$

once the system enters into a new steady state. If (23) holds after entering into the steady state of a new mode, it implies that j is the index of the new mode. Otherwise, j does not indicate the new mode and should be removed from the candidate modes. To judge if the system has entered into the steady state of a new mode, it is necessary to define a waiting time. The waiting time is used to describe the duration of the transient behaviors after a fault is detected, see [13] for the details.

Definition 1. The waiting time T is defined as, at least, the maximum of the settling time of all the interval observers, such that residual zonotopes estimated by the interval observer matching the current system mode include $\mathbf{0}$ while residual zonotopes estimated by interval observers not matching the current system mode exclude $\mathbf{0}$ after a fault is detected.

Assumption 5.1. All the considered actuator faults are persistent and the persistent time T_s^i ($i \in \mathbb{I}$) of the i -th fault is not shorter than the waiting time T , i.e., $T_s^i \geq T$.

According to the previous discussions, the ultimate FI algorithm proposed in this paper is a combination of the two different FI strategies in (22) and (23). Under the satisfaction of Theorem 4.1, the following theorem is used to summarize the proposed FI algorithm.

Theorem 5.2. *Once a fault is detected, the FI strategy (22) is firstly used to isolate the fault during the transition. If after a waiting time, there are still at least two interval observers that satisfy (22), then the FI algorithm is switched into the FI strategy (23) for the final FI decision.*

Eventually, by combining the FD strategy in Theorem 5.1 and the FI strategy in Theorem 5.2, the effectiveness of the proposed FDI approach can be guaranteed by Theorem 4.1. The FDI procedure of the proposed approach is summarized in Algorithm 1. In Algorithm 1, $\text{length}(\cdot)$ computes the number of the elements of a set. Finally, there will be one and only one element in \mathbb{I}_m that indicates the new mode, for simplicity, the notation $f = \mathbb{I}_m$ is directly used at the end of the algorithm.

5.2. Initial zonotopes

It is assumed that a fault is detected at time instant k_d . As per Section 5.1, at the FD time k_d , all the corresponding bounding zonotope dynamics \check{X}_k^{jj} ($j \in \mathbb{I} \setminus \{m\}$) should be initialized by their corresponding initial zonotope, denoted as $\check{X}_{k_d}^{jj}$, such that

$$\check{X}_{k_d}^{jj} \supseteq \check{X}_{k_d}^{fj} \quad (24)$$

implying that $\check{R}_{k_d}^{jj} \supseteq R_{k_d}^{fj}$ ($f, j \in \mathbb{I} \setminus \{m\}$) holds. This initialization is a key precondition for the proposed approach to implement FDI during the transition. Thus, a key point is to construct $\check{X}_{k_d}^{jj}$ for all the corresponding dynamics of \check{X}_k^{jj} .

Here, the idea is to use the obtainable information $R_{k_d}^{fj}$ at time instant k_d to construct the zonotope $\check{X}_{k_d}^{jj}$ satisfying (24). By defining a zonotope $V_0 = H_{\bar{\eta}} \mathbb{B}^q$, (11) can be transformed into

$$R_k^{fj} = C\check{X}_k^{fj} \oplus \{\eta_k - \eta^c\} \oplus (-V_0). \quad (25)$$

By adding $-(\eta_k - \eta^c)$ to both sides of (25), (25) turns into

$$R_k^{fj} \oplus \{-(\eta_k - \eta^c)\} = C\check{X}_k^{fj} \oplus (-V_0). \quad (26)$$

Considering $-(\eta_k - \eta^c) \in (-V_0)$, one can further obtain

$$C\check{X}_k^{fj} \oplus (-V_0) \subset R_k^{fj} \oplus (-V_0). \quad (27)$$

Algorithm 1 Proposed FDI algorithm

Require: T, \hat{X}_0 , mode index $i \in \mathbb{I}$;

Ensure: Fault index f ;

```

1: Initialization:  $i = m, f = m$  and  $\hat{X}_0^{mj} = \hat{X}_0$  ( $m, j \in \mathbb{I}$ );
2: At time instant  $k$ : Switching  $\leftarrow$  FALSE,  $\mathbf{0} \in R_k^{mm}$  and  $\mathbf{0} \notin R_k^{mj}, j \in \mathbb{I} \setminus \{m\}$ ;
3: while Switching  $\neq$  TRUE do
4:    $k \leftarrow k + 1$ ;
5:   Obtain  $R_k^{mm}$ ;
6:   if  $\mathbf{0} \notin R_k^{mm}$  then
7:     Switching  $\leftarrow$  TRUE;
8:     Construct initial zonotopes  $\check{X}_{k_d}^{jj}, j \in \mathbb{I} \setminus \{m\}$ ;
9:     Initialize all the dynamics  $\check{X}_k^{jj}$  described by (13);
10:  end if
11: end while
12:  $\mathbb{I}_m = \mathbb{I} \setminus \{m\}$ ;
13: Timer =  $T$ ;
14: while Timer  $\neq$  0 do
15:    $k \leftarrow k + 1$ ;
16:   if  $\text{length}(\mathbb{I}_m) \neq 1$  then
17:     Obtain all  $R_k^{fj}$  and  $\check{R}_k^{jj}, f, j \in \mathbb{I}_m$ ;
18:     for  $j \in \mathbb{I}_m$  do
19:       if  $R_k^{fj} \not\subseteq \check{R}_k^{jj}$  then
20:         Remove  $j$  from  $\mathbb{I}_m$ ;
21:       end if
22:     end for
23:   end if
24:   if  $\text{length}(\mathbb{I}_m) = 1$  then
25:      $f = \mathbb{I}_m$ ;
26:     Timer = 0;
27:   else
28:     Timer = Timer - 1;
29:   end if
30: end while
31: if  $f = m$  then
32:   Obtain all  $R_k^{fj}, j \in \mathbb{I}_m$ ;
33:   for  $j \in \mathbb{I}_m$  do
34:     if  $\mathbf{0} \notin R_k^{fj}$  then
35:       Remove  $j$  from  $\mathbb{I}_m$ ;
36:     end if
37:   end for
38:    $f = \mathbb{I}_m$ ;
39: end if
40: return  $f$ ;
```

Eventually, a key expression is obtained from (27) as

$$C\check{X}_k^{fj} \subset R_k^{fj}. \quad (28)$$

Since R_k^{fj} is a zonotope, it can be written in the zonotopic form $R_k^{fj} = r_k^{fj,c} \oplus H_k^{fj,r} \mathbb{B}^{s_k^{fj,r}}$. By using the zonotopic form of R_k^{fj} , (28) can be equivalently expressed as a group of q inequal-

ities and the i -th inequality out of the q inequalities has the form

$$|C(i)\tilde{x}_k^{fj} - r_k^{fj,c}(i)| \leq \|H_k^{fj,r}(i)\|, \quad i = 1, 2, \dots, q, \quad (29)$$

where $C(i)$ denotes the i -th row of C , and $r_k^{fj,c}(i)$ and $H_k^{fj,r}(i)$ denote the i -th component of $r_k^{fj,c}$ and the i -th row of $H_k^{fj,r}$.

According to Property B.3 in Appendices, each inequality out of the q inequalities of (29) determines a strip. This implies that the q strips determined by the q inequalities should form a closed set. This closed set (denoted as \tilde{X}_k^{fj}) can be computed by using Property B.3. Note that \tilde{X}_k^{fj} is able to contain \tilde{X}_k^{fj} , i.e., $\tilde{X}_k^{fj} \supseteq \tilde{X}_k^{fj}$, which can be used as an initial zonotope that satisfies (24) at time instant k_d , i.e.,

$$\tilde{X}_{k_d}^{fj} = \tilde{X}_{k_d}^{fj}.$$

However, since Property B.3 can only produce a zonotope approximation for the intersection of a zonotope and a strip, to construct \tilde{X}_k^{fj} using Property B.3, an initial zonotope has to be given to the approach proposed in Property B.3 as a starting set.

Remark 5.2. The initial zonotope (denoted as \tilde{X}) of the approach in Property B.3 is defined as a zonotope that contains the physical constraint set of \tilde{X}_k^{fj} for any interval observer in any considered mode. Since there always exist the physical constraints on any system, a proper set \tilde{X} can be easily found.

Thus, by using \tilde{X} as an initial zonotope for Property B.3, at the FD time instant k_d , $\tilde{X}_{k_d}^{fj}$ can be computed as the initial zonotope $\tilde{X}_{k_d}^{fj}$ to initialize the dynamics of the corresponding bounding zonotopes \tilde{X}_k^{jj} ($j \in \mathbb{I} \setminus \{m\}$) described by (13). Using the generated residual-bounding zonotope sequences, FDI during the transition can be implemented.

Note that, in the case that C is invertible, (28) can be further transformed into

$$\tilde{X}_k^{fj} \subset C^{-1}R_k^{fj},$$

where C^{-1} represents the inverse of C . In this case, at the FD time k_d , $C^{-1}R_{k_d}^{fj}$ is directly used as the initial zonotope $\tilde{X}_{k_d}^{fj}$.

5.3. Discussions of FDI framework

Indeed, under Theorem 4.1, this proposed FDI framework based on a bank of interval observers implies two different FI mechanisms. The first one, proposed in this paper, can detect and isolate the considered faults during the transition between different modes. The second one, proposed in [13], requires the system to wait a specific period (a waiting time) until it enters into steady state, to perform the FI task. Eventually, the interval observer estimating residual zonotopes that can contain the origin indicates a new mode. Please refer to [13] for the details of the second mechanism.

The advantage of the first mechanism consists in its FI quickness, but it requires more computational resources. Comparatively, the second one isolates faults after the system has already been at steady state, which implies that more time is needed. But the second one does not rely on the information provided

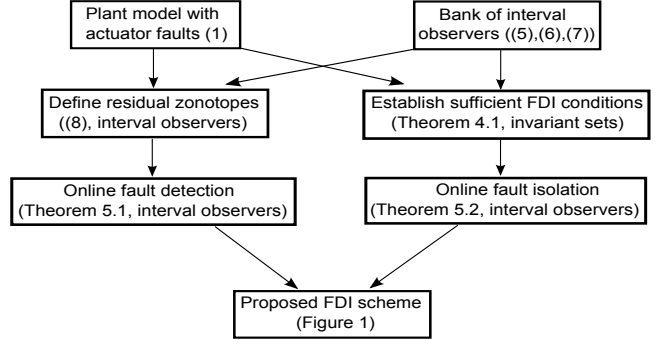


Figure 2: The design procedure of the proposed FDI approach

by residual-bounding zonotopes in real time, which means less computational load.

These two FI strategies inside the same framework are complementary to each other. In applications, the FI strategy is chosen as per the particular requirements and considerations of FI tasks. To summarize, a flow chart is presented in Figure 2 to show the design procedure of the proposed approach.

6. Illustrative example

In this paper, a CSTR presented in [15] is used to illustrate the effectiveness of this approach. The CSTR considers an exothermic irreversible reaction $A \rightarrow B$. Based on the reactant mass balance and energy balance in the reactor, the process is depicted by a non-linear dynamic model given in [15] (please read [15] for all the details about the CSTR case study in the present paper). As per [15], c_A is the concentration of the component A , T is the reactor temperature, q_c is the input and c_A is the output, and the nominal values for the CSTR model parameters are given in Table 1. The operating point of the CSTR is given as

$$c_{A_o} = 8.235 \times 10^{-2} \text{ mol/l}, \quad (30a)$$

$$T_o = 441.81 \text{ K}. \quad (30b)$$

The discrete-time linear model of the system around the operating point defined by (30) is obtained as

$$x_{k+1} = \begin{bmatrix} 0.8976 & -0.0002 \\ -0.4894 & 0.7606 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 0.0024 \end{bmatrix} Fu_k + \omega_k^i, \quad (31a)$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k, \quad (31b)$$

where ω_k^i is a bounded signal to model the linearization and discretization errors³ between the non-linear model and linear discrete-time model for the i -th mode.

The proposed approach only requires the bounds of ω_k^i and does not require their real-time values. In simulation, empirical values are given for the bounds of ω_k^i as $\omega^{0,c} = \begin{bmatrix} 0.001 & 0.001 \end{bmatrix}^T$, $\bar{\omega}^0 = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$,

³In real situations, the errors are possible to be different for different modes, thus, this does not conflict with (1).

$$\omega^{1c} = [0.002 \quad 0.002]^T, \bar{\omega}^1 = [0.015 \quad 0.015]^T, \\ \omega^{2c} = [0.003 \quad 0.003]^T, \bar{\omega}^2 = [0.03 \quad 0.03]^T.$$

The faults affecting the valve position corresponding to the coolant flow are considered, i.e., the flow rate of the coolant is affected. Thus, the faults are modelled as F shown in (31), where 0 and 1 denote the complete jam and healthy functioning of the valve, respectively, and a value inside (0, 1) denotes that the valve loses partial performance. Here, two faults are considered, i.e., F_0 (healthy), F_1 (fault 1) and F_2 (fault 2).

It is known the particular magnitude of faults is unknown in reality. Thus, one considers the bounds of F_1 and F_2 , which are denoted as intervals

$$\mathbf{F}_1 = [0.1, 0.3], \mathbf{F}_2 = [0.5, 0.7]. \quad (32)$$

If \mathbf{F}_1 and \mathbf{F}_2 satisfy the proposed FDI conditions, a fault occurrence with any fault magnitude inside \mathbf{F}_1 or \mathbf{F}_2 is detectable and isolable. These two operating regions \mathbf{F}_1 and \mathbf{F}_1 of the valve are monitored by two interval observers. Whenever the operating situation of the valve drops into either of the two regions, they can be detected and isolated by the proposed approach.

Based on (31) and (32), three interval observers with the form indicated in (5) and (6) are designed to monitor the linearized continuous-time model. The gain matrices and initial conditions for the interval observers and the waiting time for the steady-state FI are given as

- observer gains:

$$L_0 = L_1 = L_2 = \begin{bmatrix} 0.1582 \\ 11.6106 \end{bmatrix},$$

- initial conditions:

$$x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \hat{X}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0.05 & 0 & 0 & 0.05 \\ 0.05 & 0 & 0.05 & 0 \end{bmatrix} \mathbb{B}^4,$$

- waiting time T :

$$T = 20\Delta t.$$

The actual magnitudes of actuator faults are given as

$$F_1 = 0.15, F_2 = 0.55,$$

which are inside the bounds indicated in (32), respectively.

In simulation, the input around the operating point is a sinusoidal signal that oscillates in an interval

$$\Delta q_c \in [-20, 20].$$

According to Theorem A.1 and Proposition A.1 in Appendices and iterating (15) thirty steps, as explained in Section 3.3, the RPI approximations of the limit sets of the static residual-bounding zonotopes for each interval observer are computed. Furthermore, the interval hulls of these RPI approximations are presented as

Table 1: Parameters of the CSTR

Variable	Symbol	Nominal value
Tank volume	V	100 [l]
Feed flow rate	q	100 [l/min]
Feed concentration	c_{Af}	1 [mol/l]
Feed temperature	T_f	350 [K]
Coolant flow rate	q_c	100 [mol/l]
Coolant temperature	T_c	350 [K]
Densities	ρ, ρ_c	1000 [g/l]
Specific heats	C_p, C_{pc}	1 [cal/(g K)]
Pre-exponential factor	k_0	7.2×10^{10} [1/min]
Exponential factor	E/R	9.98×10^3 [K]
Heat of reaction	$-\Delta H$	2.0×10^5 [cal/mol]
Heat transfer charact.	hA	7.0×10^5 [1/(min K)]
Sampling period	Δt	0.1 [min]

- for interval observer 0:

$$\hat{R}_\infty^{10} = [0.0477, 0.0719],$$

$$\hat{R}_\infty^{20} = [0.1035, 0.1356],$$

- for interval observer 1:

$$\hat{R}_\infty^{01} = [-0.0719, -0.0477],$$

$$\hat{R}_\infty^{21} = [0.0396, 0.0799],$$

- for interval observer 2:

$$\hat{R}_\infty^{02} = [-0.1356, -0.1035],$$

$$\hat{R}_\infty^{12} = [-0.0799, -0.0396].$$

As per the discussions in this paper, \hat{R}_∞^{00} , \hat{R}_∞^{11} and \hat{R}_∞^{22} can always include $\mathbf{0}$, they are omitted here. It can be observed that all the RPI approximations corresponding to a bank of interval observers satisfy the FDI conditions as established in (18), which implies that the proposed technique can be used for FDI.

Remark 5.2 suggests an initial zonotope \tilde{X} determined by the plant physical constraints. \tilde{X} is used by Property B.3 to construct initial zonotopes for the initialization of the dynamics of the static residual-bounding zonotopes whenever a fault is detected. By simulation, \tilde{X} is empirically given as

$$\tilde{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0.15 & 0 & 0.15 \\ 0 & 5 & 5 \end{bmatrix} \mathbb{B}^3,$$

which can bound \tilde{X}_k^{ij} in any possible situation.

Besides, the parameter λ in Property B.3 (note that a selection strategy of λ can be found in [5]) is given as

$$\lambda = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

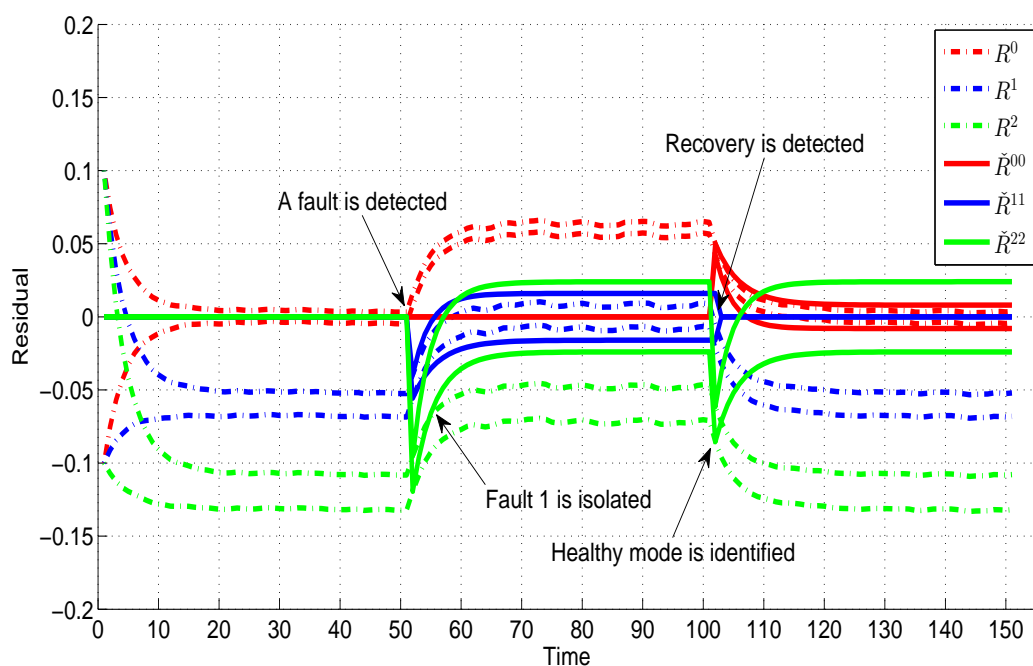


Figure 3: FDI of the fault 1

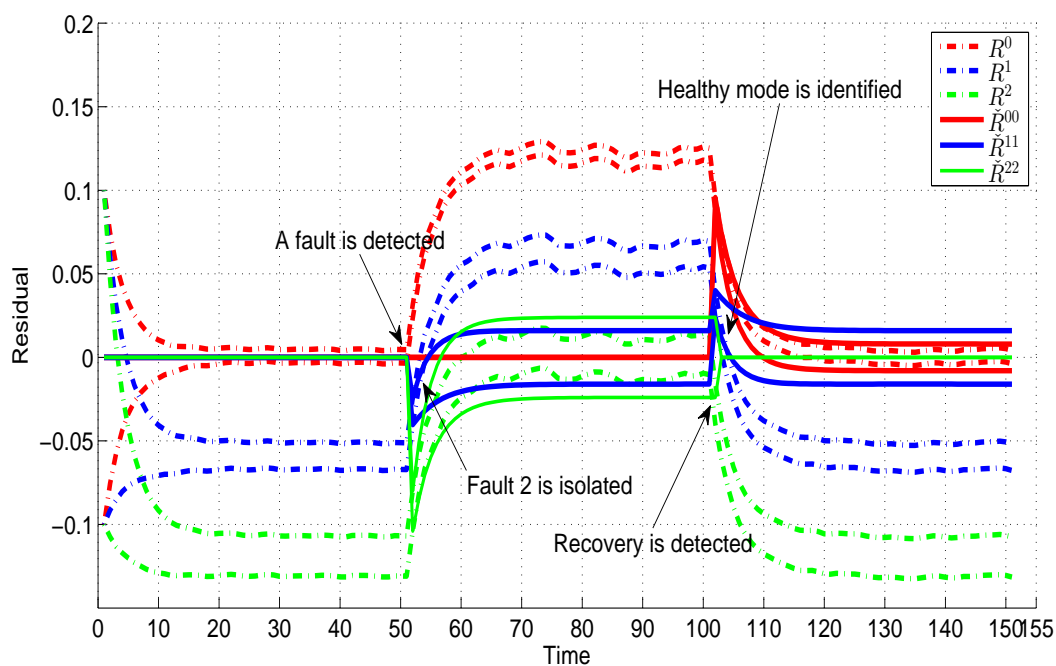


Figure 4: FDI of the fault 2

In this example, the fault modes 1 and 2 are simulated separately. The fault scenarios for both fault modes are set as follows: from time instants 0 to 49, the actuator is healthy, from time instants 50 to 99 an actuator fault occurs and from time instants 100 to 150 the actuator recovers to the healthy mode.

The simulation results of the fault 1 are presented in Figure 3. From time instants 0 to 49, the actuator is healthy, thus, residual zonotopes (R^0 in Figure 3) estimated by the healthy interval observer can always contain the origin. At time instant 50, the fault 1 occurs. Then, R^0 excludes the origin at time instant 52,

which indicates the fault is detected at time instant 52.

At the same time, \check{R}^{11} and \check{R}^{22} corresponding to the interval observers 1 and 2 are initialized to start the transient FI task. At time instant 53, it can be observed that $R^1 \subseteq \check{R}^{11}$ but $R^2 \not\subseteq \check{R}^{22}$, which implies that the fault 2 does not occur while the fault 1 has occurred in the system. The same conclusion can be drawn when one analyzes the steady-state behaviors and it can be observed that $\mathbf{0} \in R^1$, $\mathbf{0} \notin R^0$ and $\mathbf{0} \notin R^2$ after T , which also indicates that the fault 1 has occurred in the system. Besides, from time instants 100 to 150, a recovery process is introduced, which can be understood in the same way. Regarding the fault 2, the results are given in Figure 4, which can be explained similarly as the fault scenario 1. Thus, as per the results, the proposed FDI technique is effective to detect the faults and further isolate the faults during the transition between two different modes.

Remark 6.1. It is known that there are two different FI strategies in the proposed FI approach. One is for the transient-state FI and the other is for the steady-state FI. However, one should notice that the emphasis of the proposed FI approach consists in the transient-state FI strategy (this is analyzed in the two previous paragraphs), while the steady-state FI strategy is used as FI guarantees of the transient-state FI strategy when the transient-state FI strategy loses its effectiveness.

7. Conclusions

In this paper, an FDI approach using a bank of interval observers is proposed, where invariant set-based FDI conditions are established to guarantee FDI. Under the FDI conditions, the approach can provide two different FI mechanisms that can be selected according to the need of actual applications. The first FI mechanism can isolate faults during the transition between different modes while the second one usually needs more time to isolate faults but with less computational load. The future research consists in reducing the FDI conditions and extending the approach into the system with parametric uncertainties.

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Appendix A. Invariant Sets

In this paper, the linear discrete time-invariant dynamics

$$x_{k+1} = A_o x_k + B_o \delta_k \quad (\text{A.1})$$

are used to explain the notions of invariant sets, where A_o and B_o are constant matrices and A_o is a Schur matrix, δ_k belongs to $\Delta = \{\delta : |\delta - \delta^\circ| \leq \bar{\delta}\}$ with δ° and $\bar{\delta}$ being constant vectors.

Definition A.1. A set $X \subset \mathbb{R}^n$ is called an RPI set for Eq.(A.1) if and only if $A_o X \oplus B_o \Delta \subseteq X$.

Definition A.2. The mRPI set of Eq.(A.1) is defined as an RPI set contained in any closed RPI set and the mRPI set is unique and compact.

Definition A.3. Given a scalar $\epsilon > 0$ and a set $\Omega \subset \mathbb{R}^n$, the set $\Phi \subset \mathbb{R}^n$ is an outer ϵ -approximation of Ω if $\Omega \subseteq \Phi \subseteq \Omega \oplus \mathbb{B}_{p_0}^n(\epsilon)$ and it is an inner ϵ -approximation of Ω if $\Phi \subseteq \Omega \subseteq \Phi \oplus \mathbb{B}_{p_0}^n(\epsilon)$.

Theorem A.1. ([8],[16]). Considering Eq.(A.1) and letting $A_o = V \Lambda V^{-1}$ be the Jordan decomposition of A_o , the set

$$\Phi(\theta) = \{x \in \mathbb{R}^n : |V^{-1}x| \leq (I - |\Lambda|)^{-1} |V^{-1}B_o| \bar{\delta} + \theta\} \oplus \xi^{\circ},$$

is RPI and attractive for the trajectories of Eq.(A.1), with θ being any (arbitrarily small) vector with positive components, where $\xi^\circ = (I - A_\circ)^{-1} B_\circ \delta^\circ$.

1. For any θ , the set $\Phi(\theta)$ is (positively) invariant, that is, if $x_0 \in \Phi(\theta)$, then $x_k \in \Phi(\theta)$ for all $k \geq 0$.
2. Given $\theta \in \mathbb{R}^n, \theta > 0$, and $x_0 \in \mathbb{R}^n$, there exists $k^* \geq 0$ such that $x_k \in \Phi(\theta)$ for all $k \geq k^*$.

Proposition A.1. ([8]). Considering Eq.(A.1) and denoting X_0 as an initial set, the set sequence

$$X_{j+1} = A_\circ X_j \oplus B_\circ \Delta, \quad j = 1, 2, \dots,$$

converges to the mRPI set of Eq.(A.1), where if X_0 is an RPI set of Eq.(A.1), each iteration of the set sequence is an RPI approximation of the mRPI set.

Appendix B. Zonotopes

Definition B.4. The interval hull $\square X$ of a zonotope denoted as $X = g \oplus G\mathbb{B}^r \subset \mathbb{R}^n$ is the smallest interval box that contains X , i.e., $\square X = \{x : |x_i - g_i| \leq \|G_i\|_1\}$, where G_i is the i -th row of G , and x_i and g_i are the i -th components of x and g , respectively.

Property B.1. Given a zonotope $X = g \oplus G\mathbb{B}^r \subset \mathbb{R}^n$ and a compatible matrix K , $KX = K \oplus KG\mathbb{B}^r$.

Property B.2. Given a zonotope $X = g \oplus G\mathbb{B}^r \subset \mathbb{R}^n$ and an integer s (with $n < s < r$), denote by \hat{G} the matrix resulting from the recording of the columns of the matrix G in decreasing Euclidean norm. $X \subseteq g \oplus [\hat{G}_T \quad Q]\mathbb{B}^s$ where \hat{G}_T is obtained from the first $s - n$ columns of the matrix \hat{G} and $Q \in \mathbb{R}^{n \times n}$ is a diagonal matrix whose elements satisfy $Q_{ii} = \sum_{j=s-n+1}^r |\hat{G}_{ij}|$, $i = 1, \dots, n$.

Property B.3. ([5]). Given a zonotope $X = g \oplus G\mathbb{B}^r \subset \mathbb{R}^n$, a strip $S = \{x \in \mathbb{R}^n \mid |cx - d| \leq \sigma\}$ and a vector $\lambda \in \mathbb{R}^n$, then $X \cap S \subseteq \hat{X}(\lambda) = \hat{g}(\lambda) \oplus \hat{G}(\lambda)\mathbb{B}^{r+1}$ holds where $\hat{g}(\lambda) = g + \lambda(d - cg)$ and $\hat{G}(\lambda) = [(I - \lambda c)G \quad \sigma\lambda]$.

Property B.4. ([5]). Given a family of zonotopes denoted by $X = g \oplus \mathbf{M}\mathbb{B}^m$, where $g \in \mathbb{R}^n$ is a real vector and $\mathbf{M} \in \mathbb{R}^{n \times m}$ is an interval matrix, a zonotope inclusion $\diamond(X)$ is defined by

$$\diamond(X) = g \oplus [\text{mid}(\mathbf{M}) \quad G]\mathbb{B}^{m+n},$$

where the matrix G is a diagonal matrix with

$$G_{ii} = \sum_{j=1}^m \frac{\text{diam}(\mathbf{M})_{ij}}{2}, \quad i = 1, 2, \dots, n.$$

Property B.5. ([2]). Given $X_{k+1} = \mathbf{A}X_k \oplus \mathbf{B}u_k$, where \mathbf{A} and \mathbf{B} are interval matrices and u_k is the input at time instant k , if X_k is a zonotope with the center g_k and segment matrix H_k , X_{k+1} can be bounded by a zonotope

$$X_{k+1}^e = g_{k+1} \oplus H_{k+1}\mathbb{B}^r,$$

where

$$\begin{aligned} g_{k+1} &= \text{mid}(\mathbf{A})g_k + \text{mid}(\mathbf{B})u_k, \\ H_{k+1} &= [J_1 \quad J_2 \quad J_3], \\ J_1 &= \text{seg}(\diamond(\mathbf{A}H_k)), \\ J_2 &= \frac{\text{diam}(\mathbf{A})}{2}g_k, \\ J_3 &= \frac{\text{diam}(\mathbf{B})}{2}u_k, \end{aligned}$$

and $\text{seg}(\cdot)$ computes the segment matrix of a zonotope.