

Power-Factor Compensation of Electrical Circuits

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A FRAMEWORK FOR ANALYSIS AND DESIGN IN THE NONLINEAR NONSINUSOIDAL CASE

Optimizing energy transfer from an ac source to a load is a classical problem in electrical engineering. The design of power apparatus is such that the bulk of the transfer occurs at the fundamental frequency of the source. In practice, the efficiency of this transfer is typically reduced due to the phase shift between voltage and current at the fundamental frequency. The phase shift arises largely due to energy flows characterizing electric motors that dominate the aggregate load. The power factor, defined as the ratio between the real or active power (average of the instantaneous power) and the apparent power (the product of rms values of the voltage and current), then captures the energy-transmission efficiency for a given load.

The standard approach to improving the power factor is to place a compensator between the source and the load. Conceptual design of the compensator typically assumes that the equivalent source consists of an ideal generator having zero Thevenin impedance and producing a fixed, purely sinusoidal voltage [1]. If the load is linear time invariant (LTI), the resulting steady-state current is a shifted sinusoid, and the power factor is the cosine of the phase-shift angle. Power-factor compensation is then achieved by modifying the circuit to reduce the phase shift between the source voltage and the current.

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A fundamental energy-equalization mechanism underlies the phase-shifting action of power-factor compensation. Indeed, it can be shown that the power factor is improved if and only if the difference between the average electric and magnetic energies stored in the circuit is reduced. The optimal power factor is achieved when electric and magnetic energies are equal, which occurs when the impedance seen from the source behaves like a resistor for the source frequency. Unfortunately, standard textbook presentations [1]–[3] do not explain the power-factor compensation in terms of energy equalization but rather rely on an axiomatic definition of reactive power. In the LTI sinusoidal case, reactive power turns out to be proportional to the energy difference mentioned above, and thus reactive-power reduction is tantamount to energy equalization.

POWER-FACTOR COMPENSATION WITH NONSINUSOIDAL SIGNALS

Due to economic and environmental considerations, increasingly stringent efficiency requirements are being placed on electric energy systems [4]. These requirements have resulted, on the one hand, in more widespread use of power semiconductor switching devices that are nearly lossless and thus reduce power dissipation. On the other hand, many electrical devices function over wide operating ranges, where nonlinear phenomena cannot be neglected. These trends have resulted in the widespread presence of nonsinusoidal signals in energy networks at all power levels. An unfortunate consequence of the inclusion of switching devices and the presence of nonlinear loads is additional signal distortion, which has two undesirable effects. First, the introduction of harmonics that are not present in the original waveforms can excite unmodeled dynamics and result in degraded performance. Second, the task of designing power-factor compensators—which, as indicated above, is well understood for sinusoidal signals and relies on fundamental energy-equalization principles—is far from clear in the face of distorted signals. Available compensation technologies include rotating machinery and mechanically or electronically switched capacitors and inductors as well as power electronic converters, such as active filters and flexible ac transmission systems. See [5] for a recent review and [6] for an example of an innovative combination of the two basic classes of compensators.

We can broadly distinguish two approaches to the problem of power-factor compensation in the nonsinusoidal case. The current-tracking approach assumes that we can inject any desired current into the load, which is a reasonable assumption because of the recent availability of polyphase active filters [7], which, through switching action and energy storage, can generate almost arbitrary current profiles. Then, a reference waveform for the source current is defined—typically a scaled version of the source voltage—and the control problem reduces to the selection

of the switching policy to track the reference signal. A vast amount of literature in the power electronics community is devoted to this approach, which is dominant in high-performance applications [8]. We do not pursue the current-tracking approach in this article, but rather refer the interested reader to [9] for a review, from a control theory perspective, of the main existing techniques and pointers for the relevant literature. Typical control schemes for power-factor compensation are linear (PI or predictive controllers) with some excursions into nonlinear control such as hysteresis-based and neuro/fuzzy compensators [8]. In most of these applications, rigorous stability analysis is absent, and discussions center on efficient ways to estimate derivatives of noisy signals.

The second approach to power-factor compensation is based on interconnecting subsystems with energy storage elements. Using this framework, we can define an attribute, namely, additive generalized reactive power, to improve the power factor. Providing the correct definition of generalized reactive power is a longstanding question, with research dating as far back as 1927, when Budeanu suggested an extension to multifrequencies of the classical definition of reactive power for a single harmonic [10]. While deficiencies of this approach are widely documented in the literature [11], [12], Budeanu's approach still dominates influential documents such as IEEE standards. The limitations of Budeanu's approach are illustrated in this article. To the best of our knowledge, all definitions of reactive power reported in the literature are based on orthogonal decompositions of the terminal signals and give various interpretations to the resulting terms. Among the vast literature, we cite here only the comprehensive works [3], [11], which contain an extensive list of references. The detailed discussions at the end of [11] illustrate the degree of controversy and even confusion that exists on this topic. See also [13] for a more recent account of the field and [14] for a detailed description of several common misconceptions.

CONTRIBUTIONS OF THE ARTICLE

The main contribution of this article is the identification of the key role played by cyclodissipativity [15], [16] in power-factor compensation. We prove that a necessary and sufficient condition for a parallel (shunt) lossless compensator to improve the power factor is that the overall system satisfy a cyclodissipativity property. In the spirit of standard passivation [17], this result leads naturally to a formulation of the power-factor-compensation problem as one of rendering the load cyclodissipative. Consequently, we show that cyclodissipativity provides a rigorous mathematical framework for analyzing and designing power-factor compensators for general nonlinear loads operating in nonsinusoidal regimes.

As explained in [18], cyclodissipativity is understood here in terms of the available generalized energy. The idea

is borrowed from thermodynamics [19], where the notion is formulated in a conceptually clearer (though perhaps mathematically less rigorous) manner than in circuits and systems theory. As pointed out in [20], this distinction may be due to the fact that thermodynamics has never made the study of linear systems a central concern, a notable exception being [21]. In contrast, the circuits and systems literature has a tendency to formulate general ideas in terms of their particular manifestation in a linear context, which is perhaps a reason why the generalization of reactive power has proven so elusive.

POWER-FACTOR COMPENSATION

We consider the classical scenario of energy transfer from an n -phase ac generator to a load as depicted in Figure 1. Throughout this article, lower case boldface letters denote column vectors, while upper case boldface letters denote matrices. The voltage and current of the source are denoted by the column vectors $\mathbf{v}_s, \mathbf{i}_s \in \mathbb{R}^n$, while the load is described by a possibly nonlinear, time-varying n -port system Σ . We formulate the power-factor-compensation problem as follows:

C.1) $\mathbf{v}_s \in \mathcal{V}_s \subseteq \mathcal{L}_2^n[0, T] := \{\mathbf{x} : [0, T] \rightarrow \mathbb{R}^n : \|\mathbf{x}\|^2 := (1/T) \int_0^T |\mathbf{x}(\tau)|^2 d\tau < \infty\}$ where $\|\cdot\|$ is the rms value and $|\cdot|$ is the Euclidean norm. Depending on the context, the set \mathcal{V}_s may be equal to $\mathcal{L}_2^n[0, T]$ or it may consist

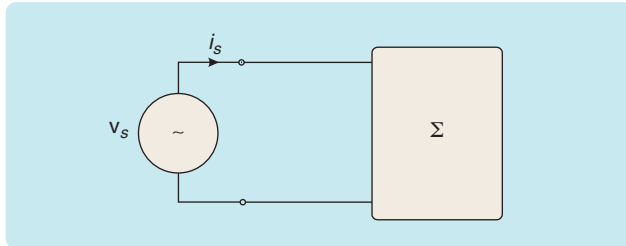


FIGURE 1 Circuit schematic of a polyphase ac system that represents the classical scenario of energy transfer from an n -phase ac generator to a load. The voltage and current of the source are represented by vectors $\mathbf{v}_s, \mathbf{i}_s \in \mathbb{R}^n$, and the load is described by a possibly nonlinear, time-varying n -port system Σ .

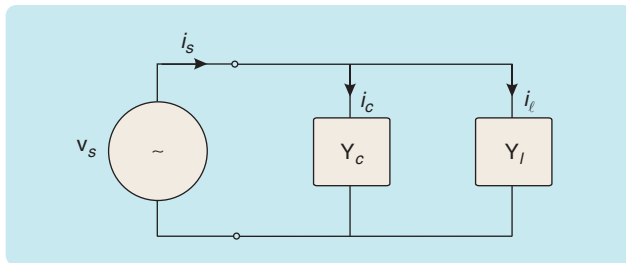


FIGURE 2 Typical compensation configuration in which the compensator, represented by its admittance \mathbf{Y}_c , is placed in shunt. This configuration preserves the rated voltage at the load terminals. The compensator is restricted to be lossless, that is, $\langle \mathbf{v}_s, \mathbf{Y}_c \mathbf{v}_s \rangle = 0$, which means that no power dissipation occurs at the compensator terminals.

of a single periodic signal $\mathbf{v}_s(t) = \mathbf{v}_s(t + T)$ or a set of sinusoids with limited harmonic content, for example, $\mathbf{v}_s(t) = \mathbf{V}_s \sin \omega_0 t$, where $\omega_0 \in [\omega_0^m, \omega_0^M] \subset [0, \infty)$.

C.2) The power-factor-compensation configuration is depicted in Figure 2, where $\mathbf{Y}_c, \mathbf{Y}_l : \mathcal{V}_s \rightarrow \mathcal{L}_2^n[0, T]$ are the admittance operators of the compensator and the load, respectively. That is, $\mathbf{Y}_c : \mathbf{v}_s \mapsto \mathbf{i}_c$ and $\mathbf{Y}_l : \mathbf{v}_s \mapsto \mathbf{i}_l$, where $\mathbf{i}_c, \mathbf{i}_l \in \mathbb{R}^n$ denote the compensator and load currents, respectively. In the simplest LTI case the operators $\mathbf{Y}_c, \mathbf{Y}_l$ can be described by their admittance transfer matrices, which we denote by $\hat{\mathbf{Y}}_c(s), \hat{\mathbf{Y}}_l(s) \in \mathbb{R}^{n \times n}(s)$, where s represents the complex frequency variable $s = j\omega$.

C.3) The power-factor compensator is lossless, that is,

$$\langle \mathbf{v}_s, \mathbf{Y}_c \mathbf{v}_s \rangle = 0 \quad \text{for all } \mathbf{v}_s \in \mathcal{V}_s, \quad (1)$$

where $\langle \mathbf{x}, \mathbf{y} \rangle := \frac{1}{T} \int_0^T \mathbf{x}^\top(t) \mathbf{y}(t) dt$ is the inner product in $\mathcal{L}_2^n[0, T]$.

We make the following fundamental assumption throughout the work.

Assumption A.1

The source is ideal, in the sense that \mathbf{v}_s remains unchanged for all loads Σ .

The standard definition of power factor [2] is given as in Definition 1.

Definition 1

The power factor of the source is defined by

$$PF := \frac{\langle \mathbf{v}_s, \mathbf{i}_s \rangle}{\|\mathbf{v}_s\| \|\mathbf{i}_s\|}, \quad (2)$$

where $P := \langle \mathbf{v}_s, \mathbf{i}_s \rangle$ is the active (real) power and the product $S := \|\mathbf{v}_s\| \|\mathbf{i}_s\|$ is the apparent power.

From (2) and the Cauchy-Schwarz inequality, it follows that $P \leq S$. Hence $PF \in [-1, 1]$ is a dimensionless measure of the energy-transmission efficiency. Indeed, under Assumption A.1, the apparent power S is the highest average power delivered to the load among all loads that have the same rms current $\|\mathbf{i}_s\|$. The apparent power equals the active power if and only if \mathbf{v}_s and \mathbf{i}_s are collinear. If this is not the case, $P < S$ and compensation schemes are introduced to maximize power factor.

Definition 2

Power-factor improvement is achieved with the compensator \mathbf{Y}_c if and only if

$$PF > PF_u := \frac{\langle \mathbf{v}_s, \mathbf{i}_l \rangle}{\|\mathbf{v}_s\| \|\mathbf{i}_l\|}, \quad (3)$$

where PF_u denotes the uncompensated power factor, that is, the value of PF with $\mathbf{Y}_c = 0$.

A consequence of our assumptions is that all signals in the system are periodic, with fundamental period T and belong to the space $\mathcal{L}_2^n[0, T)$. However, as becomes clear below, all derivations remain valid if we replace $\mathcal{L}_2^n[0, T)$ by the set of square-integrable functions $\mathcal{L}_2^n[0, \infty)$. Hence, periodicity is not essential for our developments. Restricting our analysis to $\mathcal{L}_2^n[0, T)$ captures the practically relevant scenario in which, for most power-factor-compensation problems of interest, the system operates in a periodic, though not necessarily sinusoidal, steady state.

In the vast majority of applications, the power-factor compensator is placed in shunt to simplify field installation and to simplify voltage regulation at the load terminals. The compensator is also restricted to be lossless to avoid additional power dissipation or the need to provide an additional source.

Assumption A.1 is tantamount to saying that the source has no impedance, which is justified by the fact that most ac power devices are designed and operated in this manner. For ease of presentation and without loss of generality, we also assume that $\langle v_s, i_s \rangle \geq 0$, which indicates that real (active) power is always delivered from the source to the load.

We bring to the reader's attention the problem of maximum energy transfer, which is related, but fundamentally different, from the power-factor-compensation problem. In the former, the source is not assumed to be ideal, but has an impedance $\mathbf{Z}_s : \mathcal{L}_2^n[0, T) \rightarrow \mathcal{L}_2^n[0, T)$, as shown in Figure 3. The problem is thus to find the load that maximizes the energy transfer for any arbitrary given voltage waveform, as studied in [23]–[25]. Note that the qualifier *any* is important since it distinguishes this problem from the broadband matching problem [26], where a set of voltages is given.

The role of power factor as an indicator of energy-transmission efficiency is usually explained in textbooks as follows [2]. In view of periodicity we can express the q th phase component of the terminal variables in terms of their (exponential) Fourier series as

$$v_{s_q}(t) = \sum_{k=-\infty}^{\infty} \hat{V}_{s_q}(k) \exp(jk\omega_0 t),$$

where $\omega_0 := 2\pi/T$ is the fundamental frequency and, for integers k ,

$$\hat{V}_{s_q}(k) := \frac{1}{T} \int_0^T v_{s_q}(t) \exp(-jk\omega_0 t) dt$$

are the Fourier coefficients of the q th phase element of the voltage, also called spectral lines or harmonics. For details, see "Properties of Periodic Signals." Similar expressions are obtained for the q th phase components of the current vector i_s . Because the product of sinusoidal variables of different frequencies integrated over a com-

mon period is zero, only components of v_s and i_s that are of the same frequency contribute to the average power P . However, if the current is distorted, the rms value of i_s can exceed the rms value of the sum of the current components in phase with the voltage. In this case, the source may not deliver its rated power, although it may deliver its rated rms current.

A CYCLODISSIPATIVITY CHARACTERIZATION OF POWER-FACTOR COMPENSATION

In this section, we prove that power factor is improved if and only if the compensated system satisfies a cyclo-dissipativity property. A corollary of this result is an operator-theoretic characterization of all of the compensators that improve the power factor. Finally, we show that, as in the LTI sinusoidal case, a phase-shifting interpretation of power-factor-compensation action is possible. To formulate our results, we need the Definition 3.

Definition 3

The n -port system of Figure 2 is cyclo-dissipative with respect to the supply rate $w(v_s, i_s)$, where $w : \mathcal{V}_s \times \mathcal{L}_2^n[0, T) \rightarrow \mathbb{R}$, if and only if

$$\int_0^T w(v_s(t), i_s(t)) dt > 0 \quad (4)$$

for all $(v_s, i_s) \in \mathcal{V}_s \times \mathcal{L}_2^n[0, T)$.

Proposition 1

Consider the system of Figure 2 with fixed \mathbf{Y}_ℓ . The compensator \mathbf{Y}_c improves the power factor if and only if the system is cyclo-dissipative with respect to the supply rate

$$w(v_s, i_s) := (\mathbf{Y}_\ell v_s + i_s)^\top (\mathbf{Y}_\ell v_s - i_s). \quad (5)$$

Proof

From Kirchhoff's current law $i_s = i_c + i_\ell$, the relation $i_c = \mathbf{Y}_c v_s$, and the lossless condition (1), it follows that $\langle v_s, i_s \rangle = \langle v_s, i_\ell \rangle$. Consequently, (2) becomes

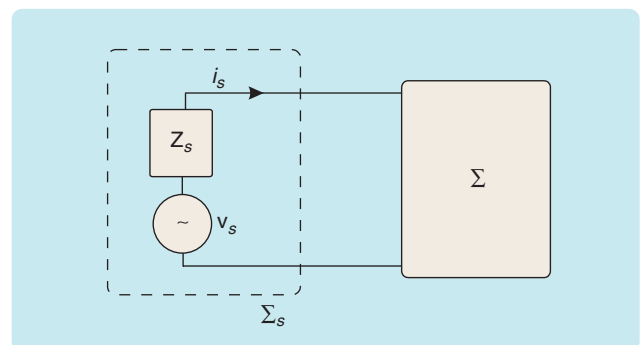


FIGURE 3 Circuit configuration considered for the maximum energy transfer problem. The nonideal source Σ_s has a series impedance \mathbf{Z}_s . The problem here is to find the load Σ that maximizes the energy transfer for an arbitrary voltage waveform.

$$PF = \frac{\langle \mathbf{v}_s, \mathbf{i}_\ell \rangle}{\|\mathbf{v}_s\| \|\mathbf{i}_s\|},$$

and (3) holds if and only if

$$\|\mathbf{i}_s\|^2 < \|\mathbf{Y}_\ell \mathbf{v}_s\|^2, \quad (6)$$

where we use $\mathbf{i}_\ell = \mathbf{Y}_\ell \mathbf{v}_s$. Finally, note that (4) with (5) is equivalent to (6), which yields the desired result.

Corollary 1

Consider the system of Figure 2. Then \mathbf{Y}_c improves the power factor for a given \mathbf{Y}_ℓ if and only if \mathbf{Y}_c satisfies

$$2\langle \mathbf{Y}_\ell \mathbf{v}_s, \mathbf{Y}_c \mathbf{v}_s \rangle + \|\mathbf{Y}_c \mathbf{v}_s\|^2 < 0 \quad \text{for all } \mathbf{v}_s \in \mathcal{V}_s. \quad (7)$$

Dually, given \mathbf{Y}_c , the power factor is improved for all \mathbf{Y}_ℓ that satisfy (7).

Proof

Substituting $\mathbf{i}_s = (\mathbf{Y}_\ell + \mathbf{Y}_c) \mathbf{v}_s$ in (6) yields (7).

To provide a phase-shift interpretation of power-factor compensation, Figure 4 depicts the vector signals \mathbf{v}_s , \mathbf{i}_s , \mathbf{i}_ℓ , and \mathbf{i}_c , where the angles θ and θ_u are understood in the sense of the inner product, as defined below. Note that the lossless condition (1) imposes $\langle \mathbf{i}_c, \mathbf{v}_s \rangle = 0$. Replacing $\mathbf{i}_\ell = \mathbf{Y}_\ell \mathbf{v}_s$ and $\mathbf{i}_c = \mathbf{Y}_c \mathbf{v}_s$ in the power-factor-improvement condition (7) yields

$$\|\mathbf{i}_c\|^2 + 2\langle \mathbf{i}_c, \mathbf{i}_\ell \rangle < 0, \quad (8)$$

which is equivalent to $\|\mathbf{i}_c\| < 2\Delta$, where the distance Δ is defined by

$$\Delta := -\frac{\langle \mathbf{i}_\ell, \mathbf{i}_c \rangle}{\|\mathbf{i}_c\|} > 0.$$

On the other hand, it is clear from Figure 4 that $\|\mathbf{i}_c\| < 2\Delta$ if and only if $\theta < \theta_u$. The equivalence between power-factor improvement and $\theta < \theta_u$ follows directly from the fact that

$$\theta := \cos^{-1} PF, \quad \theta_u := \cos^{-1} PF_u. \quad (9)$$

Properties of Periodic Signals

We briefly review some basic properties of the inner product of periodic signals [39].

DEFINITION S1 (INNER PRODUCT OF FUNCTIONS)

The inner product of two real periodic signals $f(t)$ and $g(t)$ with period T is defined as

$$\langle f, g \rangle := \frac{1}{T} \int_0^T f(t)g(t)dt.$$

PROPERTY S1 (COMPLEX FOURIER SERIES)

The complex Fourier series representation of a periodic signal $f(t)$ with period T is given by

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{F}(n) \exp(jn\omega_0 t),$$

where $\omega_0 := (2\pi/T)$ and $\hat{F}(n)$ are the complex Fourier coefficients given by

$$\hat{F}(n) := \frac{1}{T} \int_0^T f(t) \exp(-jn\omega_0 t) dt.$$

If $f(t)$ is real, then its Fourier coefficients satisfy $\hat{F}(-n) = \hat{F}^*(n)$, where $\hat{F}^*(n)$ denotes the conjugate of $\hat{F}(n)$.

PROPERTY S2 (INNER PRODUCT OF PERIODIC FUNCTIONS IN TERMS OF ITS COMPLEX FOURIER COEFFICIENTS)

Let $f(t)$, $g(t)$ be two real periodic functions with period T and complex Fourier coefficients $\hat{F}(n)$, $\hat{G}(n)$. Then, the inner product of $f(t)$ and $g(t)$ is given by

$$\langle f, g \rangle := \sum_{n=-\infty}^{\infty} \hat{F}(n) \hat{G}^*(n).$$

PROPERTY S3

Let $f(t)$, $g(t)$ be differentiable periodic functions with period T . Then

$$\langle \dot{f}, \dot{g} \rangle = -\langle \dot{f}, g \rangle.$$

PROPERTY S4 (INNER PRODUCT OF SIGNALS WITH DERIVATIVES)

Let $f(t)$, $g(t)$ be real periodic and differentiable functions with period T and complex Fourier coefficients $\hat{F}(n)$, $\hat{G}(n)$. Then,

$$\langle \dot{f}, \dot{g} \rangle := 2\omega_0 \sum_{n=1}^{\infty} n \operatorname{Im}\{\hat{F}(n) \hat{G}^*(n)\},$$

where $\omega_0 := (2\pi/T)$.

PROPERTY S5 (NORM \mathcal{L}_2 OF THE TIME DERIVATIVE OF PERIODIC FUNCTIONS IN TERMS OF ITS COMPLEX FOURIER COEFFICIENTS)

Let $f(t)$ be a real periodic function with period T and complex Fourier coefficients $\hat{F}(n)$. Then,

$$\|\dot{f}\|^2 = \langle \dot{f}, \dot{f} \rangle := \omega_0^2 \sum_{n=-\infty}^{\infty} n^2 |\hat{F}(n)|^2,$$

where $\omega_0 := (2\pi/T)$.

Notice that these functions are well defined and, furthermore, because of the unidirectional energy-transfer assumption, it follows that $\theta \in [-(\pi/2), (\pi/2)]$ and $\theta_u \in [-(\pi/2), (\pi/2)]$.

From (5), we see that the supply rate depends on the load operator. Therefore, the associated cyclodissipativity property (4) cannot be used as a definition of additive generalized reactive power required in a system-interconnection approach. That is, we cannot compensate for a lack of cyclodissipativity in the load by an excess of cyclodissipativity in the compensator as occurs in the LTI case with reactive power, where the compensator is an inductor or a capacitor, depending on whether the reactive power is negative or positive.

Readers familiar with the power-factor-compensation problem may find the statements above to be self-evident. Indeed, under Assumption A.1, power-factor improvement is equivalent to reduction of the rms value of the source current. Now, using $i_s = i_c + i_\ell$ to compute the rms value of i_s yields

$$\|i_s\|^2 = \|i_\ell\|^2 + \|i_c\|^2 + 2\langle i_c, i_\ell \rangle. \quad (10)$$

It is clear from (10) that a necessary and sufficient condition for reducing $\|i_s\|$ from its uncompensated rms value $\|i_\ell\|$ is precisely (8), which, as shown in Proposition 1 is equivalent to power-factor improvement.

Definition 3 of cyclodissipativity is not standard but captures the essence of the property introduced in [16] and [18] for systems with a state realization. In other words, a system is cyclodissipative if it cannot create generalized energy over closed paths. In our case, these paths are defined for port signals, while these paths are typically associated with state trajectories. The system might, however, produce energy along some initial portion of a closed path; if so, the system would not be dissipative. Clearly, every dissipative system is cyclodissipative, stemming from the fact that in the latter case we restrict the set of inputs of interest to those inputs that generate periodic trajectories, a feature that is intrinsic in power-factor-compensation problems.

As in dissipative systems, storage functions and dissipation inequalities can be used to characterize cyclodissipativity provided we eliminate the restriction that the storage functions be nonnegative. This statement corresponds to the following result of [16], where to further emphasize the distinction between storage functions that are bounded and those that are not, the name *virtual storage function* is used.

Theorem 1

Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, $\mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u})$, where $\mathbf{x} \in \mathbf{X} \subset \mathbb{R}^n$ and $\mathbf{u}, \mathbf{y} \in \mathbb{R}^m$, and let \mathbf{X} be the set of reachable and controllable points. Then the system is cyclodissipative if and only if there exists a virtual storage function $V : \mathbf{X} \rightarrow \mathbb{R}$ satisfying

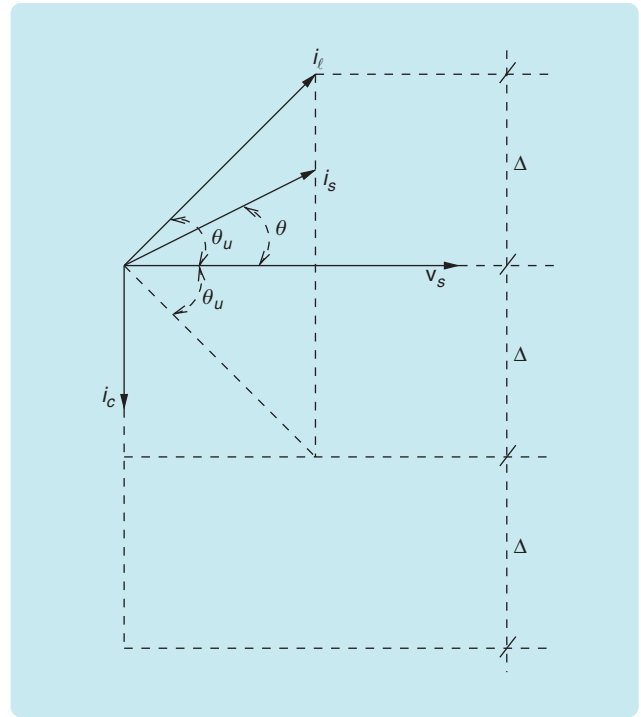


FIGURE 4 Phase-shift interpretation of power-factor compensation. The power factor is improved if and only if $\theta < \theta_u$, which is equivalent to $\|i_c\| < 2\Delta$.

$$V(\mathbf{x}(0)) + \int_0^T w(\mathbf{u}(t), \mathbf{y}(t)) dt \geq V(\mathbf{x}(T))$$

for all $T \geq 0$ and for all $\mathbf{u} \in \mathcal{L}_2^m$.

POWER-FACTOR COMPENSATION IN THE LTI SINUSOIDAL CASE

We now specialize the above derivations to the case in which $n = 1$, $v_s(t) = V_s \sin \omega_0 t$, where $\omega_0 \in [\omega_0^m, \omega_0^M] \subset [0, \infty)$, and the scalar LTI stable operators Y_ℓ, Y_c are described by their admittance transfer functions $\hat{Y}_\ell(j\omega_0)$ and $\hat{Y}_c(j\omega_0)$, respectively. In this case, the steady-state source current is

$$i_s(t) = I_s \sin(\omega_0 t + \theta),$$

where $I_s := V_s |\hat{Y}_\ell(j\omega_0) + \hat{Y}_c(j\omega_0)|$ and $\theta := \angle\{\hat{Y}_\ell(j\omega_0) + \hat{Y}_c(j\omega_0)\}$. Simple calculations confirm that θ and the uncompensated angle $\theta_u := \angle\{\hat{Y}_\ell(j\omega_0)\}$ coincide with (9). We also have the following simple property.

Lemma 1

The scalar LTI operator Y_c is lossless if and only if $\text{Re}\{\hat{Y}_c(j\omega)\} = 0$ for all $\omega \in [0, \infty)$.

Proof

From Parseval's theorem, we have

$$\begin{aligned}\langle v_s, Y_c v_s \rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{V}_s(-j\omega) \hat{Y}_c(j\omega) \hat{V}_s(j\omega) d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re}\{\hat{Y}_c(j\omega)\} |\hat{V}_s(j\omega)|^2 d\omega,\end{aligned}$$

where, to obtain the second identity, we use the fact that $\operatorname{Im}\{\hat{Y}_c(j\omega)\}$ is an odd function of ω .

Proposition 2

In the LTI scalar sinusoidal case, the power factor is improved if and only if

$$\frac{\operatorname{Im}\{\hat{Y}_\ell(j\omega_0)\}}{\operatorname{Im}\{\hat{Y}_c(j\omega_0)\}} < -\frac{1}{2} \quad \text{for all } \omega_0 \in [\omega_0^m, \omega_0^M]. \quad (11)$$

Proof

In this case, the signal space of Figure 4 can be replaced by the complex plane with the admittances' frequency responses taking the place of the signals, as indicated in Figure 5. Notice that because of Lemma 1, $\hat{Y}_c(j\omega_0)$ is purely imaginary. From basic geometry, we see that $\theta < \theta_u$ if and only if (11) holds.

The equivalence between power-factor improvement and $\theta < \theta_u$ is a restatement of the fact that energy-transmission efficiency is improved by reducing the phase shift between the source voltage and current waveforms, a statement that can be found in standard circuits textbooks. However, the explicit characterization (11) does not seem to be widely known.

The action of a power-factor compensator is explained above without resorting to the axiomatic definition of complex power used in textbooks to introduce the notion of reactive power. In contrast with our geometric perspective of power-factor compensation, this mathematical construction cannot easily be extended to the nonlinear nonsin-

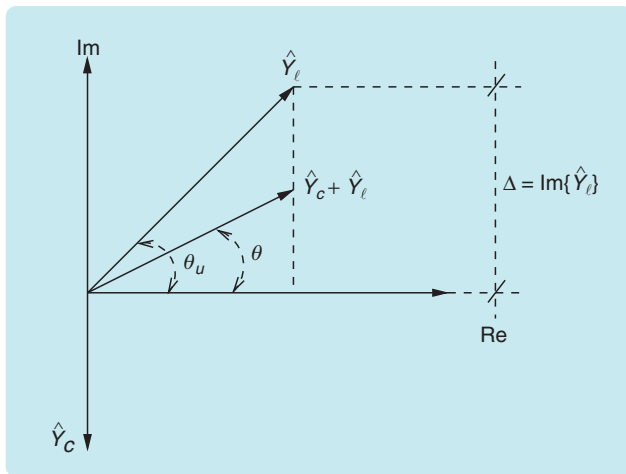


FIGURE 5 Power-factor compensation in the LTI case. Replacing the signals of Figure 4 by their admittances' frequency responses, it follows that $\theta < \theta_u$ if and only if (11) holds.

oidal case. Furthermore, the mathematical background used in the above derivations is elementary.

For clarity, the above analysis is restricted to the scalar case, that is, $n = 1$. Similar derivations can easily be carried out for n -phase systems. For instance, if $\hat{Y}_c(s)$ is diagonal, power-factor improvement is equivalent to

$$[\operatorname{Im}\{\hat{Y}_c(j\omega_0)\}]^{-1} \operatorname{Im}\{\hat{Y}_\ell(j\omega_0)\} < -\frac{1}{2} \mathbf{I}_n \quad \text{for all } \omega_0 \in [\omega_0^m, \omega_0^M].$$

POWER-FACTOR COMPENSATION WITH LTI CAPACITORS AND INDUCTORS

Corollary 1 identifies all of the load admittances for which the source power factor is improved with a given compensator, namely, those load admittances that satisfy inequality (7). In this section, we further explore this condition for LTI capacitive and inductive compensation. For simplicity, we assume throughout this section that the system is single phase, that is, $n = 1$, but the load is possibly nonlinear.

Proposition 3

Consider the system of Figure 2 with $n = 1$ and a fixed LTI capacitor compensator with admittance $\hat{Y}_c(s) = C_c s$, where $C_c > 0$. The following statements are equivalent:

- There exists $C_{\max} > 0$ such that the load is cyclodissipative with supply rate

$$w_C(\dot{v}_s, i_\ell) = -2i_\ell \dot{v}_s - C_{\max} \dot{v}_s^2. \quad (12)$$

- For all C_c satisfying $0 < C_c < C_{\max}$, the power factor is improved.

Proof

Assume i) holds. Integrating $w_C(\dot{v}_s, i_\ell)$ and using Definition 3 yields the cyclodissipation inequality

$$2\langle i_\ell, \dot{v}_s \rangle + C_{\max} \|\dot{v}_s\|^2 \leq 0. \quad (13)$$

Note that (13) implies that $2\langle i_\ell, C_c \dot{v}_s \rangle + \|C_c \dot{v}_s\|^2 \leq 0$ for all $0 < C_c < C_{\max}$. The latter is the condition for power-factor improvement (7) for the case at hand. The converse proof is established by reversing these arguments.

A similar proposition can be established for inductive compensation. In contrast with the upper bound given for C_c , a lower bound on the inductance L_c is imposed. Furthermore, an assumption on v_s is needed to ensure absolute integrability of the supply rate.

Proposition 4

Consider the system of Figure 2 with $n = 1$ and a fixed LTI inductor compensator with admittance $\hat{Y}_c(s) = (1/L_c)s$, where $L_c > 0$. Assume v_s has no dc component. The following statements are equivalent:

- The load is cyclodissipative with supply rate

$$w_L(z, i_\ell) = -2L_{\min} i_\ell z - z^2, \quad (14)$$

for some constant $L_{\min} > 0$ and $\dot{z} = v_s$.

ii) For all $L_c > L_{\min}$, the power factor is improved.

Proposition 3 (respectively, Proposition 4) states that the power factor of a load can be improved with a capacitor (respectively, inductor) if and only if the load is cyclodissipative with supply rate (12) [respectively, (14)]. This result constitutes an extension to the nonlinear, nonsinusoidal case of the definition of inductive (respectively, capacitive) loads. Two questions arise immediately:

Q1 Which loads are cyclodissipative?

Q2 If power-factor improvement is possible for a given signal v_s , what is the optimal value of the capacitance (respectively, inductance)?

An answer to the second question is straightforward and known in the energy-processing community [13], [27]. For illustration, we consider the case of capacitive compensation. In this case, $\|i_s\|^2$ takes the form

$$\|i_s\|^2 = \|i_\ell\|^2 - 2C_c \left\langle \frac{d}{dt} i_\ell, v_s \right\rangle + C_c^2 \|\dot{v}_s\|^2, \quad (15)$$

where Property S3 of the sidebar “Properties of Periodic Signals” is used to obtain the second right hand side term. Equation (15), which is quadratic in the unknown C_c , has the minimizer

$$C_\star = \frac{\left\langle \frac{d}{dt} i_\ell, v_s \right\rangle}{\|\dot{v}_s\|^2}. \quad (16)$$

See also [13] for the polyphase case as well as illustrative examples.

Similar optimization problems for alternative reactive circuit topologies are studied in the circuits literature [3]. However, there seem to be many open problems. For instance, in [27] and [28], it is shown that for RL loads the optimal solution corresponds to a negative inductance, and thus a switched series LC circuit is suggested as an alternative option. A systematic study of this optimization problem may lead to a better understanding of admissible topologies and suboptimal solutions.

Load Cyclodissipativity

Given the strong relationship between cyclodissipativity and energy equalization, we postpone question Q1 to the next section, where we explore the role of energy in the power-factor-compensation problem. However, to illustrate some of the issues involved, we conclude this section with three examples. In each example, we verify cyclodissipativity with supply rates $v_s(d/dt)i_\ell$ and $v_s \int i_\ell$, which are implied by cyclodissipativity with supply rates (12) and (14), respectively. In view of this implication, it is clear that the former properties are necessary (but not sufficient) for capacitor power-factor improvement.

In the first example, taken from [29], we consider arbitrary values of the circuit parameters and prove cyclodissipativity with respect to $v_s(d/dt)i_\ell$ for all $v_s \in \mathcal{L}_2^n[0, T)$, hence establishing a structural property of the circuit. On the other hand, in the remaining examples, we fix the parameters and an element of \mathcal{V}_s . In these examples, we also compare the result of power-factor compensation based on cyclodissipativity with the classical technique of Budeanu, which we summarize as follows. Budeanu [10] defines reactive power as

$$Q_B := \sum_{k=1}^N Q_k = \sum_{k=1}^N 2|\hat{V}_s(k)| |\hat{I}_\ell(k)| \sin \phi(k), \quad (17)$$

where the positive integer N is the number of harmonics of interest and Q_k and $\phi(k)$ are the reactive power and the phase-angle difference of the k th harmonic, respectively. The unit for reactive power is var, as discussed in “Electrical Power Quantities.” The definition (17) is an attempt to generalize, to the case of multiple frequencies, the classical definition [1] of reactive power for a single harmonic $Q := Q_1 = 2|\hat{V}_s(1)| |\hat{I}_\ell(1)| \sin \phi$, where ϕ is the phase-angle difference between voltage and current.

Since the Fourier transform is a linear operator, it follows from generalized Tellegen’s theorem [30] that Q_B in (17) obeys power conservation, and thus we can sum the values of Q_B over the branch elements of a circuit. This property suggests a compensation procedure based on the selection of an inductor or a capacitor depending on whether $Q_B > 0$ or $Q_B < 0$, respectively. Nevertheless, the problem with Budeanu’s reactive power definition (17) essentially boils down to the fact that the reactive powers Q_k at different frequencies may have opposite signs due to

Electrical Power Quantities

In physics, power is the amount of work done or energy transferred per unit of time. The unit of power is the watt (W). However, in electrical systems, the watt is reserved for instantaneous power (the product of the voltage and current as functions of time) and for the active (real) power P consumed by the load. The apparent power S represents the voltage and current delivered to the load. The apparent power S , which is conventionally expressed in volt-amperes (VA), is the product of rms voltage and current. The relationship between the active power and the apparent power is given by the power factor (2). When the load absorbs real power P , electric energy is transformed into other forms of energy, for instance, heat or kinetic energy. In contrast, when the load absorbs reactive power Q , no useful energy is derived [1]. To distinguish reactive power from real power, the unit for reactive power Q , including Budeanu’s reactive power Q_B , is the var, which stands for volt-amperes-reactive [40], [41].

the sine term in (17), in which case the reactive-power terms Q_k cancel each other [12]. Consequently, $Q_B = 0$ can occur for nonzero values of Q_k , that is, despite the presence of reactive power at some frequencies. Thus Q_B is not an effective measure of reactive power. Although this deficiency is widely documented in the literature [11], [12], [14], Budeanu's technique continues to have widespread influence.

Example 1

Consider the nonlinear RLC circuit depicted in Figure 6 with linear elements L , C , and R_2 and a nonlinear current-controlled resistor R_1 with characteristic function $\hat{v}_{R_1} : i_L \mapsto v_{R_1}$, where i_L is the inductor current and v_{R_1} is the voltage across the resistor R_1 . A state-space representation of the circuit is given by

$$\begin{aligned} -L \frac{d}{dt} i_L &= \hat{v}_{R_1}(i_L) + v_C - v_s, \\ C \dot{v}_C &= i_L - \frac{v_C}{R_2}, \end{aligned}$$

where v_C is the capacitor voltage. Motivated by Theorem 1, we prove cyclodissipativity by constructing a virtual storage function. As a candidate, we consider

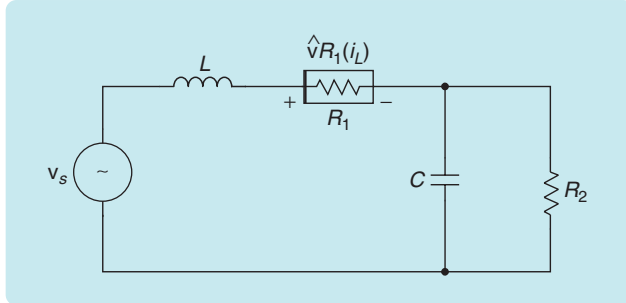


FIGURE 6 Nonlinear RLC circuit of Example 1. This circuit is cyclodissipative with respect to the supply rate $v_s(d/dt)i_L$ if $R_2 \leq \sqrt{L/C}$. Moreover, the circuit is dissipative if, in addition, R_1 is passive, that is, if $\hat{v}_{R_1}(i_L)$ is a first-third quadrant function.

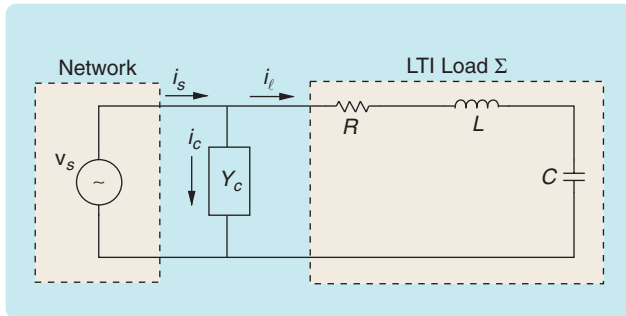


FIGURE 7 LTI series RLC load of Example 2. Since this circuit is cyclodissipative with respect to $v_s(d/dt)i_L$ and $v_s \int i_L$, the power factor can be improved when Y_c is either a shunt capacitor or a shunt inductor.

$$V(i_L, v_C) = \int_0^{i_L} \hat{v}_{R_1}(\tau) d\tau + \frac{R_2}{2} \left[\left(i_L - \frac{1}{R_2} v_C \right)^2 + i_L^2 \right],$$

which is obtained following the constructive procedure of [31]; for details see [29]. The time derivative of $V(i_L, v_C)$ is

$$\dot{V}(i_L, v_C) = \begin{bmatrix} \frac{d}{dt} i_L \\ \frac{d}{dt} v_C \end{bmatrix}^\top \mathbf{A} \begin{bmatrix} \frac{d}{dt} i_L \\ \frac{d}{dt} v_C \end{bmatrix} + v_s \frac{d}{dt} i_L, \quad (18)$$

with

$$\mathbf{A} := \begin{bmatrix} -L & 2R_2C \\ 0 & -C \end{bmatrix}.$$

Notice that the symmetric part of the matrix \mathbf{A} is negative semidefinite if and only if

$$R_2 \leq \sqrt{\frac{L}{C}}. \quad (19)$$

Integrating (18), using the negative semidefiniteness of \mathbf{A} , and invoking Theorem 1, we conclude that the circuit is cyclodissipative with supply rate $v_s(d/dt)i_L$ provided (19) holds. Consequently, the power factor can be improved with a capacitor. Note that, if R_1 is passive, that is, if $\hat{v}_{R_1}(i_L)$ is a first-third quadrant function, then $V(i_L, v_C) \geq 0$ and qualifies as a storage function, proving that the circuit is, in addition, dissipative with supply rate $v_s(d/dt)i_L$.

Example 2

Consider the LTI series RLC circuit of Figure 7 supplied with a periodic voltage source

$$\begin{aligned} v_s(t) = & 360\sqrt{2} \sin(\omega_0 t) + 144\sqrt{2} \sin(3\omega_0 t) \\ & + 42\sqrt{2} \sin(5\omega_0 t) \quad \text{V}, \end{aligned}$$

where $\omega_0 = 100\pi$ rad/s, $R = 15\Omega$, $L = 0.0796$ H, and $C = 0.0212$ mF. The uncompensated circuit has power factor $PF_u = 0.2202$.

Since $\langle v_s, (d/dt)i_l \rangle = 28.28 \times 10^4$ V-A/s and $\|\dot{v}_s\| = 18.85 \times 10^4$ V/s, the power factor can be compensated using a shunt capacitor with capacitance C_c satisfying $0 < C_c < 15.90$ μF . The optimal capacitor given by (16) is $C_c^* = 7.95$ μF , yielding an improved power factor $PF = 0.2281$. Interestingly, the cyclodissipativity condition $\langle v_s, \int i_l \rangle \geq 0$ is also satisfied, in fact, $\langle v_s, \int i_l \rangle = 2.59$ V-A-s. Hence, the compensator system can be a shunt inductor with $0.2580 \leq L_c$ with optimal value

$$L_c^* = \frac{-\|\int v_s\|^2}{\langle \int v_s, i_l \rangle} = 0.5161 \text{ H},$$

yielding an improved power factor $PF = 0.2393$. Budeanu's reactive power (17) is $Q_B = -392.66$ var, a negative value suggesting that the load is predominantly capacitive.

However, as shown above, the power factor can be improved by using either a shunt capacitor or a shunt inductor.

Example 3

For the circuit depicted in Figure 8 with $v_s(t) = 220\sqrt{2}\sin(\omega_0 t) + 70\sqrt{2}\sin(3\omega_0 t)$ V, $\omega_0 = 100\pi$ rad/s, $R = 10\ \Omega$, $L = 0.2$ H, and $C = 0.04$ mF, we have $\langle v_s, (d/dt)i_L \rangle = -8.45 \times 10^4$ V-A/s and $\langle v_s, \int i_L \rangle = -0.3629$ V-A-s. Therefore, the power factor cannot be increased using a capacitor or an inductor. On the other hand, Budeanu's reactive power is $Q_B = 18.26$ var, suggesting that the power factor can be compensated with a capacitor having capacitance $C_B = 0.9211\ \mu\text{F}$. However, comparing $PF = 0.0972$ with $PF_u = 0.0987$, we see that power factor is degraded with the shunt capacitor, as predicted by the cyclodissipativity analysis.

ENERGY EQUALIZATION AND POWER-FACTOR COMPENSATION

We now explore connections between LTI LC power-factor compensation and energy equalization, where the latter is understood in the sense of reducing the difference between the stored magnetic and electrical energies of the circuit. We study conditions for load cyclodissipativity, which is established in Propositions 3 and 4 as equivalent to power-factor improvement. Results on cyclodissipativity of nonlinear RLC circuits are summarized in [29]. It is shown in [32] that general n -port nonlinear RL (respectively, RC) circuits with convex energy functions are cyclo dissipative with supply rate $i_L \dot{v}_s$ (respectively, $v_s(d/dt)i_L$). In [31], a similar property is established for RLC circuits, which is a slight variation of the result given below.

In this section we also prove a one-to-one correspondence between cyclodissipativity and energy equalization for scalar circuits with linear inductors and capacitors and nonlinear resistors. Then, we identify a class of nonlinear RLC circuits for which a large (quantifiable) difference between the average electrical and magnetic energies implies power-factor compensation. Finally, we show by example that this relation may not hold for time-varying linear circuits.

Energy-Equalization Equivalence for Circuits with Linear Inductors and Capacitors

The class of RLC circuits that we consider as load models consists of interconnections of possibly nonlinear lumped dynamic elements (n_L inductors, n_C capacitors) and static elements (n_R resistors). Capacitors and inductors are defined by the physical laws and constitutive relations [22]

$$i_C = \dot{q}_C, \quad v_C = \nabla H_C(q_C), \quad (20)$$

$$v_L = \dot{\phi}_L, \quad i_L = \nabla H_L(\phi_L), \quad (21)$$

respectively, where $i_C, v_C, q_C \in \mathbb{R}^{n_C}$ are the capacitor currents, voltages, and charges, $i_L, v_L, \phi_L \in \mathbb{R}^{n_L}$ are the inductor currents, voltages, and flux-linkages, $H_L: \mathbb{R}^{n_L} \rightarrow \mathbb{R}$ is

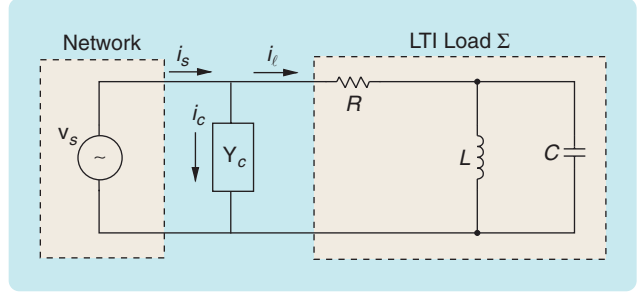


FIGURE 8 LTI series-parallel RLC load of Example 3. Since this circuit is not cyclodissipative with respect to either $v_s(d/dt)i_L$ or $v_s \int i_L$, its power factor cannot be compensated with shunt capacitors or inductors.

the magnetic energy stored in the inductors, $H_C: \mathbb{R}^{n_C} \rightarrow \mathbb{R}$ is the electric energy stored in the capacitors, and ∇ is the gradient operator. We assume that the energy functions H_L and H_C are twice differentiable. For linear capacitors and inductors, H_L and H_C are given by $H_C(q_C) = (1/2)q_C^T C^{-1} q_C$ and $H_L(\phi_L) = (1/2)\phi_L^T L^{-1} \phi_L$, respectively, where $L \in \mathbb{R}^{n_L \times n_L}$ and $C \in \mathbb{R}^{n_C \times n_C}$ are positive definite. For simplicity, we assume that L and C are diagonal. Finally, the circuit has n_{R_L} current-controlled resistors, which are described by their characteristic functions $\hat{v}_{R_i}(i_{R_i})$, $i = 1, \dots, n_{R_L}$, while the n_{R_C} voltage-controlled resistors are described by $\hat{i}_{R_i}(v_{R_i})$, $i = 1, \dots, n_{R_C}$.

Proposition 5

Consider the system of Figure 2 with $n = 1$, $v_s \in \mathcal{L}_2[0, T]$, a (possibly nonlinear) RLC load with time-invariant resistors, and fixed LTI capacitor compensator with admittance $\hat{Y}_c(s) = C_c s$, where $0 < C_c < C_{\max}$. Then the following statements hold:

- i) The power factor is improved if and only if

$$\langle v_L, \nabla^2 H_L v_L \rangle - \langle i_C, \nabla^2 H_C i_C \rangle \geq C_{\max} \omega_0^2 \sum_{k=1}^{\infty} k^2 |\hat{V}_s(k)|^2, \quad (22)$$

where $\hat{V}_s(k)$ is the k th spectral line of $v_s(t)$.

- ii) If the inductors and capacitors are linear, (22) reduces to

$$\begin{aligned} & \sum_{k=1}^{\infty} k^2 \left[\sum_{q=1}^{n_L} L_q |\hat{I}_{L_q}(k)|^2 - \sum_{q=1}^{n_C} C_q |\hat{V}_{C_q}(k)|^2 \right] \\ & \geq \frac{C_{\max}}{2} \sum_{k=1}^{\infty} k^2 |\hat{V}_s(k)|^2, \end{aligned} \quad (23)$$

where C_q and L_q are the q th capacitance and inductance, and $\hat{V}_{C_q}(k)$ and $\hat{I}_{L_q}(k)$ are the spectral lines of the corresponding capacitor voltage and inductor current.

- iii) If, in addition, $v_s(t) = V_s \sin \omega_0 t$, then (22) becomes

$$H_{L_{av}}(\omega_0) - H_{C_{av}}(\omega_0) \geq \frac{C_{\max}}{8} V_s^2,$$

A fundamental energy-equalization mechanism underlies the phase-shifting action of power-factor compensation.

where $H_{C_{av}}(\omega_0) := \sum_{q=1}^{n_C} (1/4) C_q |\hat{V}_{C_q}(1)|^2$ and $H_{L_{av}}(\omega_0) := \sum_{q=1}^{n_L} (1/4) L_q |\hat{I}_{L_q}(1)|^2$ are, respectively, the average electric and magnetic energy stored in the load.

Proof

Applying Tellegen's theorem [22] to the RLC load yields $i_\ell \dot{v}_s = \mathbf{i}_R^\top \dot{\mathbf{v}}_R + \mathbf{i}_L^\top \dot{\mathbf{v}}_L + \mathbf{i}_C^\top \dot{\mathbf{v}}_C$, which upon integration yields

$$\begin{aligned} \langle i_\ell, \dot{v}_s \rangle &= \langle \mathbf{i}_R, \dot{\mathbf{v}}_R \rangle + \langle \mathbf{i}_L, \dot{\mathbf{v}}_L \rangle + \langle \mathbf{i}_C, \dot{\mathbf{v}}_C \rangle \\ &= - \left\langle \frac{d}{dt} \mathbf{i}_L, \mathbf{v}_L \right\rangle + \langle \mathbf{i}_C, \dot{\mathbf{v}}_C \rangle \\ &= - \langle \nabla^2 H_L \mathbf{v}_L, \mathbf{v}_L \rangle + \langle \mathbf{i}_C, \nabla^2 H_C \mathbf{i}_C \rangle, \end{aligned}$$

where the second identity uses the fact that, along periodic trajectories, $\langle \mathbf{i}_R, \dot{\mathbf{v}}_R \rangle = 0$ for time-invariant resistors. The last identity follows from the constitutive relations (20) and (21). The proof of the first claim is completed by replacing the expression above in (13) and computing $\|\dot{v}_s\|^2$ with Property S5 of the sidebar "Properties of Periodic Signals."

The second and third claims are established as follows. From linearity of capacitors and inductors we have

$$\begin{aligned} \langle i_\ell, \dot{v}_s \rangle &= - \langle \mathbf{L}^{-1} \mathbf{v}_L, \mathbf{v}_L \rangle + \langle \mathbf{i}_C, \mathbf{C}^{-1} \mathbf{i}_C \rangle \\ &= - \langle \mathbf{L}^{-1} \dot{\phi}_L, \dot{\phi}_L \rangle + \langle \dot{\mathbf{q}}_C, \mathbf{C}^{-1} \dot{\mathbf{q}}_C \rangle \\ &= 2\omega_0^2 \sum_{k=1}^{\infty} k^2 \left[\sum_{q=1}^{n_C} C_q |\hat{V}_{C_q}(k)|^2 - \sum_{q=1}^{n_L} L_q |\hat{I}_{L_q}(k)|^2 \right], \end{aligned}$$

where (20) and (21) are used for the second identity and Property S5 of the sidebar "Properties of Periodic Signals" to compute the last line. Claim 3 follows by taking one spectral line and using the classical definition of averaged energy stored in linear inductors and capacitors [22].

Results analogous to Proposition 5 can be established for inductive compensation by checking the key cyclodissipation inequality

$$\langle i_\ell, z \rangle + \frac{1}{2L_m} \|z\|^2 \leq 0,$$

which stems from (14). Simple calculations show that the latter is equivalent to

$$\langle \mathbf{q}_C, \nabla H_C \rangle - \langle \phi_L, \nabla H_L \rangle \geq \frac{1}{2L_m} \|z\|^2, \quad (24)$$

which in the LTI sinusoidal case becomes

$$H_{C_{av}}(\omega_0) - H_{L_{av}}(\omega_0) \geq \frac{V_s^2}{8\omega_0^2 L_{\min}}. \quad (25)$$

Inequalities (23) and (25) reveal the energy-equalization mechanism of power-factor compensation in the LTI scalar sinusoidal case, that is, power-factor improvement with a capacitor (respectively, inductor) is possible if and only if the average magnetic (respectively, electrical) energy dominates the average electrical (respectively, magnetic) energy. Claim 2 shows that this interpretation of power-factor compensation remains valid when the source is an arbitrary periodic signal and the resistors are nonlinear, by viewing, in a natural way, $L_q |\hat{I}_{L_q}(k)|^2$ and $C_q |\hat{V}_{C_q}(k)|^2$ as the magnetic and electric energies of the k th harmonic for the q th inductive and capacitive element, respectively. On the other hand, we do not have a natural energy interpretation for (22).

Claim 3 of the proposition is established in [33] using the relation between the impedance $\hat{Z}_\ell(s) = (\hat{V}_s(s)/\hat{I}_\ell(s))$ of an LTI RLC circuit and the averaged stored energies

$$\hat{Z}_\ell(j\omega) = \frac{1}{|\hat{I}_\ell(j\omega)|^2} \{2P_{av}(\omega) + 4j\omega[H_{L_{av}}(\omega) - H_{C_{av}}(\omega)]\}, \quad (26)$$

where $P_{av}(\omega) := \frac{1}{2} \sum_{q=1}^{n_R} R_q |\hat{I}_q(j\omega)|^2$ is the power dissipated in the resistors. The expression (26) appears in equation (5.6) of Chapter 9 of [22]. Indeed, applying Parseval's theorem to the cyclodissipation inequality (13), we obtain the equivalences

$$\langle i_\ell, \dot{v}_s \rangle + \frac{C_{\max}}{2} \|\dot{v}_s\|^2 \leq 0$$

if and only if

$$\operatorname{Re}\{j\omega \hat{Z}_\ell(j\omega)\} |\hat{I}_\ell(j\omega)|^2 + \frac{C_{\max} \omega^2}{2} V_s^2 \leq 0, \quad (27)$$

if and only if

$$4\omega^2 [H_{C_{av}}(\omega) - H_{L_{av}}(\omega)] + \frac{C_{\max} \omega^2}{2} V_s^2 \leq 0.$$

Simple calculations show that (11) of Proposition 2 with $\hat{Y}_\ell(s) = C_{\max} s$ is equivalent to (27). Indeed, it is easy to prove that

$$\operatorname{Re}\{j\omega \hat{Z}_\ell(j\omega)\} = \omega |\hat{Z}_\ell(j\omega)|^2 \operatorname{Im}\{\hat{Y}_\ell(j\omega)\}.$$

Replacing the latter, together with $|\hat{V}_s(j\omega)|^2 = |\hat{Z}_\ell(j\omega)|^2 |\hat{I}_\ell(j\omega)|^2$, in (27) yields $\operatorname{Im}\{\hat{Y}_\ell(j\omega)\} < -C_{\max}(\omega/2)$, which is obtained in (11) for capacitive compensation (see Figure 5).

Necessity of Energy Equalization for Nonlinear RLC Loads

The presence of the energy functions in (22) and (24), which hold for nonlinear RLC loads, suggests that energy equalization is related to power-factor compensation for more general loads. Indeed, Proposition 6 below establishes that a sufficiently large difference between magnetic and electrical energies is necessary for capacitive power-factor compensation. The proof of this result, which is technical and thus is outside the scope of this article, follows from the arguments used in [31]. The dual result for inductive power-factor compensation is also true, but is omitted for brevity.

Proposition 6

Consider a nonlinear topologically complete RLC circuit with a voltage source $v_s \in \mathcal{L}_2^n[0, T]$ in series with inductors and satisfying the following assumptions.

- B.1) The energy functions of the inductors and capacitors are strictly convex.
- B.2) The voltage-controlled resistors are linear and passive.
- B.3) All capacitors have a (voltage-controlled) resistor in parallel, and the value of the resistance is sufficiently small.

Then, the circuit is cyclodissipative with supply rate $(d/dt)i_\ell^T v_s$. Furthermore, if the current-controlled resistors are passive, then the circuit is dissipative.

Assumptions B.1 and B.2 are technical conditions needed to construct the virtual storage function. Assumption B.3 ensures that the electrical energy stored in the capacitors is smaller than the magnetic energy stored in the inductors. As shown in [31], the qualifier *sufficiently small* in Assumption B.3 can be explicitly quantified using an upper bound on the resistances. Indeed, since all capacitors have linear resistors in parallel, it follows that, as the value of the resistances decreases, the currents tend to flow through the resistors, and the energy stored in the capacitors becomes small. The stored energy tends to zero as the resistances go to zero, which is the limiting case when all of the capacitors are short-circuited.

Limits of Energy-Equalization Equivalence

Unfortunately, the energy-equalization interpretation of power-factor compensation breaks down even for simple time-varying linear circuits, as shown in the following example taken from [14].

Example 4

Consider the linear time-varying circuit of Figure 9 with a TRIAC-controlled purely resistive load $R = 10 \Omega$. The TRIAC can be modeled as a switched resistor with characteristic

$$i_\ell(t) = \begin{cases} 0, & \text{if } t \in \left[\frac{kT}{2}, \frac{kT}{2} + \alpha\right), k = 0, 1, \dots, \\ \frac{v_s(t)}{R}, & \text{otherwise,} \end{cases}$$

where $T = 2\pi/\omega_0$ is the fundamental period and $0 \leq \alpha < T/2$ is the TRIAC's firing angle. The uncompensated voltage $v_s(t)$ and current $i_s(t)$ are depicted in Figure 10 for $v_s(t) = 220\sqrt{2}\sin(\omega_0 t)$ V and $v_s(t) = 220\sqrt{2}\sin(\omega_0 t) + 50\sqrt{2}\sin(3\omega_0 t)$ V, with $\omega_0 = 100\pi$ rad/s and $\alpha = T/4 = 0.005$ s. It is important to emphasize that this switched resistor circuit does not contain energy-storage elements. Furthermore, the TRIAC does not satisfy condition $\langle i_R, \dot{v}_R \rangle = \langle i_\ell, \dot{v}_s \rangle = 0$, which is used to establish the proof of Proposition 5.

For the sinusoidal source, we obtain $\langle \dot{v}_s, i_\ell \rangle = -48.4 \times 10^4$ V-A/s and $\|\dot{v}_s\| = 6.91 \times 10^4$ V/s, and thus a shunt capacitor with $0 < C_c < 0.202$ mF improves the power factor. From (16), we obtain that the optimal capacitor is $C_\star = 0.101$ mF, which increases the uncompensated power factor $PF_u = 0.7071$ to $PF = 0.7919$. In this sinusoidal case, Budeanu's analysis is consistent with cyclodissipativity, and both yield the same optimal capacitor compensator.

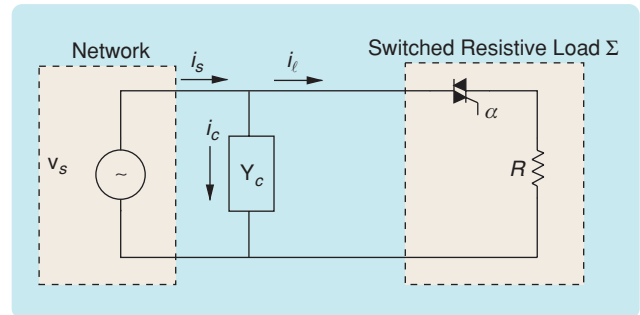


FIGURE 9 Circuit with the TRIAC-controlled resistive load of Example 4. The power factor of this circuit, which does not contain energy-storing elements, can be improved with a capacitor. This example thus shows that power-factor improvement does not imply an order relation between stored energies.

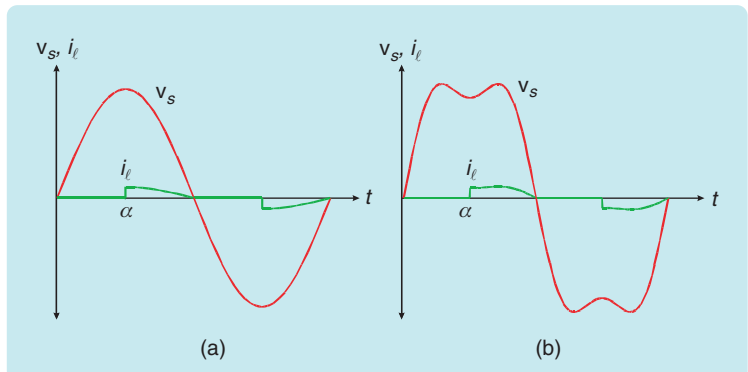


FIGURE 10 Voltage and current waveforms in the (uncompensated) circuit with the TRIAC-controlled resistive load of Example 4. (a) illustrates the corresponding voltage v_s and i_s for the sinusoidal case with $v_s(t) = 220\sqrt{2}\sin(\omega_0 t)$ V, while (b) illustrates the nonsinusoidal case with $v_s(t) = 220\sqrt{2}\sin(\omega_0 t) + 50\sqrt{2}\sin(3\omega_0 t)$ V. In both cases, $\omega_0 = 100\pi$ rad/s, and the TRIAC firing angle is $\alpha = T/4 = 0.005$ s.

If $v_s(t)$ is the two-harmonic periodic signal above, we obtain $\langle v_s, (d/dt)i_L \rangle = 28.9 \times 10^4$ V-A/s. Hence the load can be compensated with a capacitor whose optimal value is $C_* = 0.0413$ mF, yielding $PF = 0.7258$. By using Budeanu's reactive power $Q_B = 1.5 \times 10^3$ var, the resulting capacitor is $C_B = 0.08923$ mF and $PF = 0.7014$. Although the capacitor doubles its value, the power factor is worse than in the uncompensated case.

CONCLUDING REMARKS AND FUTURE WORK

This article advances an analysis and compensator design framework for power-factor compensation based on cyclodissipativity. Although the framework applies to general polyphase unbalanced circuits, we have focused on the problem of power-factor compensation with LTI capacitors or inductors of single-phase loads. We expect the full power of the approach to become evident for polyphase unbalanced loads with possibly nonlinear lossless compensators, where the existing solutions are far from satisfactory [13]. The main obstacle appears to be the lack of knowledge about the load, a piece of information that is essential for a successful design. In this respect, we plan to study simple circuit topologies that capture the essence of the problem, for instance, basic diode and transistor rectifiers. Preliminary calculations reported in [34] and [35] are encouraging.

While we concentrated here on passive shunt compensation, we are aware that current-source-based control is an attractive option in some cases. For these actuators or active filters, which can be modeled by discontinuous differential equations, the control objective is current tracking. See [2] for an introduction and [36] for a modeling procedure consistent with the energy-based approach advocated here. Although nonlinear control strategies have been used for basic topologies [34], [35], [37] many questions remain unanswered [38].

Another important problem in energy-processing systems with distorted signals is the regulation of harmonic content. Although we have not explicitly addressed this issue here, it is clear that improving the power factor reduces the harmonic distortion; a quantification of this effect is a subject of current research. It would also be highly desirable to formulate a natural optimization problem for the interconnection approach to power-factor compensation, especially for general circuit topologies and polyphase loads. Issues including compensator-circuit complexity, existence of the optimal solution, and suboptimality need to be addressed. Once again, we believe the analytically skilled control community, working in collaboration with circuit and energy specialists, can contribute along these lines.

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