

The degree/diameter problem in maximal planar bipartite graphs *

C. Dalfo^a, C. Huemer^a, J. Salas^b

^aUniversitat Politècnica de Catalunya, BarcelonaTech
Dept. de Matemàtica Aplicada IV, Barcelona, Catalonia
{`crystina.dalfo,clemens.huemer`}@upc.edu

^bIIIA-CSIC, Bellaterra, Catalonia
`julian.salas@iiia.csic.es`

Abstract

The (Δ, D) (degree/diameter) problem consists of finding the largest possible number of vertices n among all the graphs with maximum degree Δ and diameter D . We consider the (Δ, D) problem for maximal planar bipartite graphs, that are simple planar graphs in which every face is a quadrangle. We prove that for the $(\Delta, 2)$ problem, the number of vertices is $n = \Delta + 2$; and for the $(\Delta, 3)$ problem, $n = 3\Delta - 1$ if Δ is odd and $n = 3\Delta - 2$ if Δ is even. Then, we study the general case (Δ, D) and obtain that an upper bound on n is approximately $3(2D + 1)(\Delta - 2)^{\lfloor D/2 \rfloor}$, and another one is $C(\Delta - 2)^{\lfloor D/2 \rfloor}$ if $\Delta \geq D$ and C is a sufficiently large constant. Our upper bounds improve for our kind of graphs the one given by Fellows, Hell and Seyffarth for general planar graphs. We also give a lower bound on n for maximal planar bipartite graphs, which is approximately $(\Delta - 2)^k$ if $D = 2k$ ($k \geq 1$), and $3(\Delta - 3)^k$ if $D = 2k + 1$ ($k \geq 4$), for Δ and D sufficiently large in both cases.

1 Introduction

We consider simple graphs $G = G(V, E)$ that are bipartite, planar and with the maximum possible number of edges. In a *bipartite* graph, each cycle has an even length. If a graph can be drawn on the plane without any crossing of its edges, then the graph is called *planar*. A planar bipartite graph is maximal if when we add a new edge, the graph obtained is no longer planar or bipartite. A maximal planar bipartite graph divides the plane only into quadrangles (see Ringel [13]).

The (Δ, D) problem consists of finding the maximum possible number of vertices $n = |V|$ in a graph G with maximum degree Δ and diameter D . This is a prominent topic in graph theory, with results obtained for many cases. Information about this problem for graphs in general can be found in the comprehensive survey by Miller and Širáň [10] and for planar graphs also on the web page by Loz, Pérez-Rosés, and Pineda-Villavicencio [8].

For simple maximal planar graphs (that is, triangulations), the (Δ, D) problem with diameter $D = 2$ and $\Delta \geq 8$ was solved by Seyffarth in [16]. She proved that, in this case, the number of vertices is $n \leq \lfloor \frac{3}{2}\Delta \rfloor + 1$ if $\Delta \geq 8$, and that this bound is best possible. Later, Hell and Seyffarth [9] showed that this result also holds for the larger class of all simple planar graphs. Yang, Lin, and Dai [20] solved the remaining case $\Delta < 8$ for $D = 2$, for both graph classes. Fellows, Hell and Seyffarth [3] found that an upper bound on the number of vertices for planar graphs is $8\Delta + 12$ for diameter $D = 3$, and $3(2D + 1)(2\Delta^{\lfloor D/2 \rfloor} + 1)$ for any diameter.

*Research of C. Dalfo supported by the Ministerio de Educación y Ciencia and the European Regional Development Fund under project MTM2011-28800-C02-01, and by the Catalan Research Council under project 2009SGR1387. Research of C. Huemer partially supported by projects MEC MTM2012-30951 and Gen. Cat. DGR 2009SGR1040 and ESF EUROCORES programme EuroGIGA, CRP ComPoSe: grant EUI-EURC-2011-4306.

Regarding lower bounds on the maximum number of vertices of planar graphs, given a fixed diameter D and a large maximum degree Δ , Fellows, Hell and Seyffarth [4] proved that $n \geq \frac{9}{2}\Delta^{(D-1)/2} - o(\Delta^{(D-1)/2})$ for odd values of D , and $n \geq \frac{3}{2}\Delta^{D/2} - o(\Delta^{D/2})$ for even values of D . Later, Fera-Puron and Pineda-Villavicencio [5] presented a lower bound on the maximum number of vertices for planar graphs of maximum degree $\Delta \geq 6$ and odd diameter $D \geq 5$, which is $(\lfloor \frac{9\Delta}{2} \rfloor - 12) \frac{\Delta(\Delta-1)^{(D-3)/2-2}}{\Delta-2} + 9$. For planar graphs with even diameter $D = 2k$ and maximum degree Δ , Tishchenko [18] obtained the lower bound $\lfloor \frac{3\Delta}{2} \frac{(\Delta-1)^k - 1}{\Delta-2} \rfloor + 1$, and proved that this is also an upper bound for cases with large Δ , concretely for $\Delta \geq 6(12k + 1)$.

Related bounds on the (Δ, D) problem for sparse graph classes and for graphs embedded on surfaces can be found in Nevo, Pineda-Villavicencio and Wood [11], and Pineda-Villavicencio and Wood [12].

In this work we study the (Δ, D) problem for maximal planar bipartite graphs. We show in Section 2, that in the $(\Delta, 2)$ problem $n = \Delta + 2$ and that only the complete bipartite graph $K_{2,\Delta}$ satisfies this equation. Moreover, we solve the $(\Delta, 3)$ problem and prove that $n = 3\Delta - 1$ if Δ is odd, and $n = 3\Delta - 2$ if Δ is even, and we give examples of graphs that satisfy these equations. In Section 3, we study the general case (Δ, D) and obtain that n is bounded from above by approximately $3(2D + 1)(\Delta - 2)^{\lfloor D/2 \rfloor}$. For the case $\Delta \geq D$ we also obtain the upper bound $n \leq C(\Delta - 2)^{\lfloor D/2 \rfloor}$, for some constant C . Our upper bounds improve for our kind of graphs the one given by Fellows, Hell and Seyffarth for general planar graphs. We also give a lower bound on n for maximal planar bipartite graphs, which is approximately $(\Delta - 2)^k$ if $D = 2k$ ($k \geq 1$), and $3(\Delta - 3)^k$ if $D = 2k + 1$ ($k \geq 4$), for Δ and D sufficiently large in both cases. The precise bounds are given in this section.

As a notation, in a bipartite graph we denote the two partition classes with colors black B and white W . Then $n = |B| + |W|$. From the Euler characteristic $|V| - |E| + |F| = 2$, which relates the numbers of vertices $n = |V|$, edges $|E|$ and faces $|F|$ in a planar embedding of G , and the fact that each face is incident to four edges, one obtains the well-known relations $|E| = 2n - 4$ and $|F| = n - 2$.

2 The $(\Delta, 2)$ and $(\Delta, 3)$ problems in maximal planar bipartite graphs

For maximal planar bipartite graphs with diameter $D = 2$, we solve the $(\Delta, 2)$ problem with the following result.

Proposition 2.1. *Consider a maximal planar bipartite graph G with diameter $D = 2$, maximum degree Δ and maximum number of vertices n , then $n = \Delta + 2$. The only graph that satisfies this equation is the complete bipartite graph $K_{2,\Delta}$.*

For maximal planar bipartite graphs with diameter $D = 3$, our main result is the following.

Theorem 2.2. *Consider a maximal planar bipartite graph G with diameter $D = 3$, maximum degree Δ and maximum number of vertices n , then*

$$n = \begin{cases} 3\Delta - 1 & \text{if } \Delta \text{ is odd,} \\ 3\Delta - 2 & \text{if } \Delta \text{ is even.} \end{cases}$$

The proof of this result is implied by some lemmas, which we state below.

The following observation will be used throughout.

Observation 2.3. *In a bipartite graph with diameter $D = 3$, any two vertices from the same bipartition class have at least one common neighbor from the other bipartition class.*

First we study the cases $\Delta = 3$ and $\Delta = 4$.

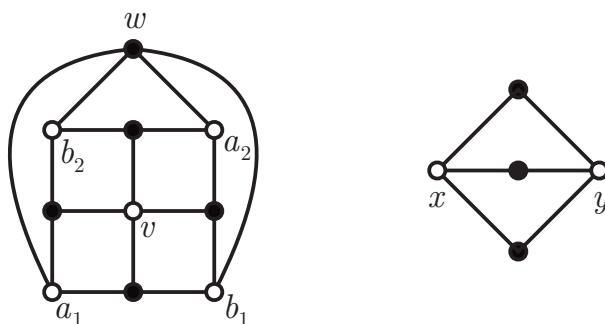


Figure 1: Connecting the white vertices a_1, a_2, b_1 and b_2 on cycle C_v through a black vertex w outside region R_v (left), and a $K_{2,3}$ (right).

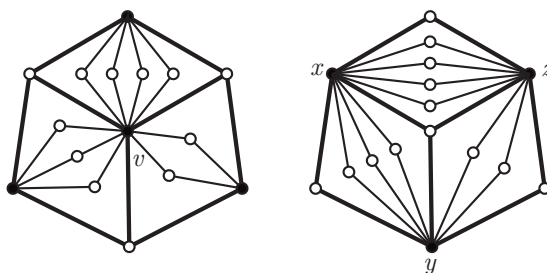


Figure 2: Case (1): $w = 3$

Lemma 2.4. *If G is a maximal planar bipartite graph with maximum degree $\Delta = 3$, then $n \leq 8$. This bound is best possible.*

Lemma 2.5. *Let G be a maximal planar bipartite graph with diameter $D = 3$, maximum degree Δ , and n vertices. If G contains a vertex that has at least four neighbors of degree $d > 2$, then $n \leq 2\Delta + 2$.*

Lemma 2.6. *If G is a maximal planar bipartite graph with diameter $D = 3$, maximum degree $\Delta = 4$ and n vertices, then $n \leq 10$. This bound is best possible.*

Lemma 2.7. *Let G be a maximal planar bipartite graph with diameter $D = 3$ and maximum degree Δ . If G does not contain a vertex that has at least four neighbors of degree $d > 2$, then G has at least $n - 8$ vertices of degree 2.*

We are now ready to finish the proof of Theorem 2.2 with the following lemma, which completes the case $\Delta \geq 5$.

Lemma 2.8. *Let G be a maximal planar bipartite graph with diameter $D = 3$, maximum degree $\Delta \geq 5$ and n vertices. If G contains at least $n - 8$ vertices of degree 2, then $n \leq 3\Delta - 1$ if Δ is odd, and $n \leq 3\Delta - 2$ if Δ is even.*

3 The (Δ, D) problem in maximal planar bipartite graphs

3.1 An upper bound

Fellows, Hell and Seyffarth [3] obtained bounds on the (Δ, D) problem for planar graphs applying the following theorem by Lipton and Tarjan [7].

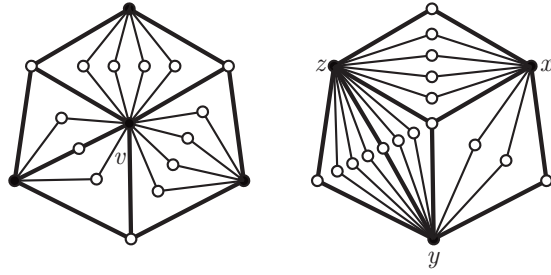


Figure 3: Case (2): $w \geq 4$

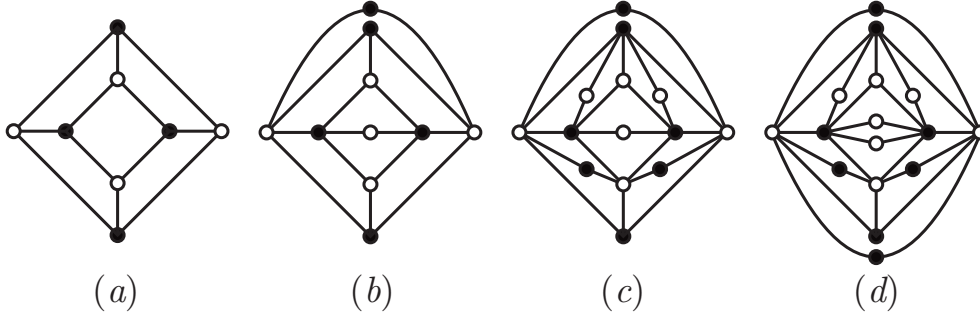


Figure 4: Graphs with $n = 3\Delta - 1$: (a) : $\Delta = 3$ and (c) : $\Delta = 5$; with $n = 3\Delta - 2$: (b) : $\Delta = 4$ and (d) : $\Delta = 6$

Theorem 3.1 ([7]). *Let G be a planar graph on n vertices containing a spanning tree of radius r . Then $V(G)$ can be partitioned into sets A, B and C such that no edges join vertices in A with vertices in B , $|A| \leq \frac{2}{3}n$, $|B| \leq \frac{2}{3}n$, and $|C| \leq 2r + 1$.*

Clearly, this theorem also holds for maximal planar bipartite graphs. We give an upper bound on the number of vertices for this kind of graphs. The cases $D = 2$ and $D = 3$ are studied in Section 2. No maximal planar bipartite graphs with $\Delta = 3$ has more than $n = 8$ vertices. Therefore, we assume that $D \geq 4$ and $\Delta \geq 4$.

Theorem 3.2. *Let G be a maximal planar bipartite graph on n vertices with maximum degree $\Delta \geq 4$ and diameter $D \geq 4$. Then,*

(a) *If $\Delta = 4$: $n \leq 6(2D + 1) \left(\lfloor \frac{D}{2} \rfloor^2 + \lfloor \frac{D}{2} \rfloor + 1 \right)$.*

(b) *If $\Delta > 4$:*

$$n \leq 3(2D + 1) \left[\frac{\sqrt{\Delta(\Delta-4)}}{2(\Delta-4)^2} \left[(\Delta - 4 + \sqrt{\Delta(\Delta-4)}) \left(\frac{\Delta-2-\sqrt{\Delta(\Delta-4)}}{2} \right)^{\lfloor D/2 \rfloor + 1} - 2\sqrt{\Delta(\Delta-4)} + (4 - \Delta + \sqrt{\Delta(\Delta-4)}) \left(\frac{\Delta-2+\sqrt{\Delta(\Delta-4)}}{2} \right)^{\lfloor D/2 \rfloor + 1} \right] + 2 \right], \quad (1)$$

which is approximately $3(2D + 1) \left[(\Delta - 2)^{\lfloor D/2 \rfloor} + 1 \right]$ if Δ is sufficiently large.

The upper bound given by Fellows, Hell and Seyffarth [3] for planar graphs is

$$n \leq 3(2D + 1)(2\Delta^{\lfloor D/2 \rfloor} + 1). \quad (2)$$

As our graphs are planar, this bound also applies to maximal planar bipartite graphs, but our bound is much better for this kind of graphs. See an example for $\Delta = 5$ in Figure 6, with the

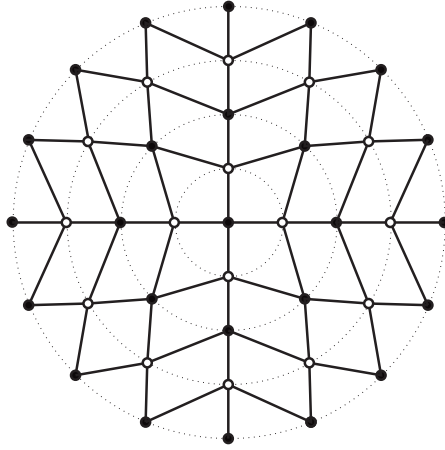


Figure 5: An almost maximal spanning subgraph for $\Delta = 4$

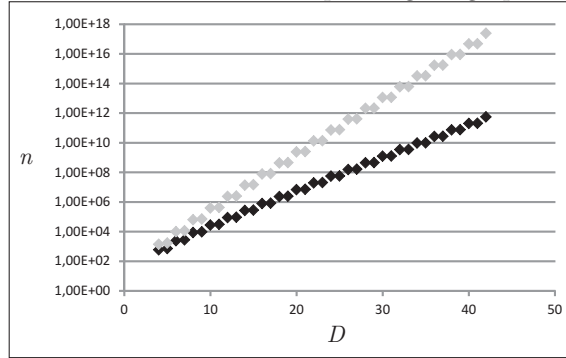


Figure 6: Plot of the log (base 10) of the number of vertices n with respect to the diameter D according to our bound given by Eq. (1) (black points) and the one by Fellows, Hell and Seyffarth given by Eq. (2) (grey points), for $\Delta = 5$ and $4 \leq D \leq 42$

values of our bound given by Theorem 3.2 and the one by Fellows, Hell and Seyffarth.

We also give an alternative upper bound for the (Δ, D) problem of the form $n < C(\Delta - 2)^{\lceil D/2 \rceil}$, for some constant C , which improves the bound of Theorem 3.2 by a factor D , when D is even and sufficiently large. However, it remains for further research to determine the smallest value of C for which this bound holds. It is based on the following theorem of Chepoi, Estellon, and Vaxès [2]. The ball of center $v \in G$ and radius k consists of all vertices of G at distance at most k from v .

Theorem 3.3 ([2]). *There exists a constant C such that any planar graph G of diameter $D \leq 2k$ can be covered with at most C balls of radius k .*

We remark that Gavaille, Peleg, Raspaud, and Sopena in [6] presented a family of planar graphs which requires $C \geq 4$.

Corollary 3.4. *There exists a constant C such that each maximal planar bipartite graph G with maximum degree Δ and diameter D has at most $n \leq C(\Delta - 2)^{\lceil D/2 \rceil}$ vertices.*

We further strengthen the bound for the (Δ, D) problem given in Corollary 3.4 to $C(\Delta - 2)^{\lfloor D/2 \rfloor}$, for the case D odd and $\Delta \geq D$. In the following of this section, we borrow from the works by Tishchenko [18, 19] and use his N -separator theorem. Before we state this theorem, we give some definitions. Let G be a maximal planar graph which is embedded in the plane and let T be a

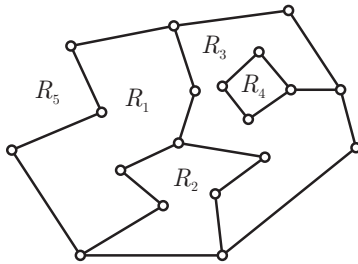


Figure 7: A 5-separator divides the plane into five regions.

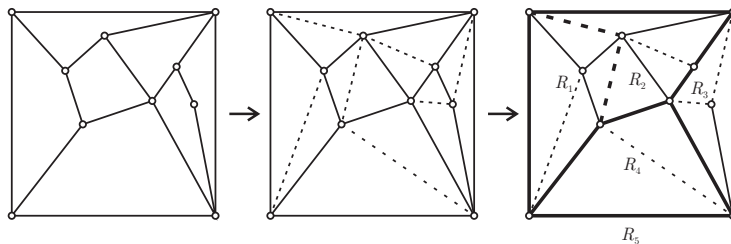


Figure 8: Adding auxiliary edges to a maximal bipartite planar graph and applying the 5-separator theorem.

spanning tree of G . Let C_1, \dots, C_{N-1} be the cycles formed by adding $N-1$ edges of $G \setminus T$ to T . Let S_N be the union of C_i , with $1 \leq i \leq N-1$. S_N partitions the plane into N regions R_1, \dots, R_N . S_N is called an N -separator in G . Figure 7 shows a 5-separator. B_i is the subgraph of G consisting of several (possibly, one) cycles bounding region R_i . A_i is the subgraph of G induced by all the vertices lying either in the interior of R_i or in B_i .

Theorem 3.5 ([19]). *Given a plane triangulation G and its spanning tree T , let $N \geq 2$ be an integer satisfying $|V(G)| > \frac{3N-1}{2}$. Then, an N -separator exists in G such that*

$$\min_{i \in \{1, \dots, N\}} \left\{ |V(A_i)| - \frac{1}{2}|V(B_i)| \right\} \geq \frac{1}{2N-1} \left(|V(G)| + \frac{N-1}{2} \right).$$

Theorem 3.6. *There exists a constant C such that each maximal planar bipartite graph G with maximum degree Δ and diameter D , for $\Delta \geq D$, has at most $n \leq C(\Delta-2)^{\lfloor D/2 \rfloor}$ vertices.*

3.2 A lower bound

In this section we present maximal planar bipartite graphs $G_{\Delta, D}$, with given maximum degree Δ and diameter D , which have a large number $n = n(G_{\Delta, D})$ of vertices.

Theorem 3.7. (a) *For any diameter $D = 2k$ ($k \geq 1$) and maximum degree Δ ($\Delta \geq 5$), there exists a maximal planar bipartite graph $G_{\Delta, D}$ whose number of vertices $n(G_{\Delta, D})$ is*

$$n(G_{\Delta, D}) = \frac{\Delta \left(\Delta - 2 + \sqrt{\Delta(\Delta - 4)} \right)^k + \Delta \left(\Delta - 2 - \sqrt{\Delta(\Delta - 4)} \right)^k}{(\Delta - 4)2^k} - \frac{8}{\Delta - 4},$$

which is approximately $(\Delta - 2)^k$, for Δ and D sufficiently large.

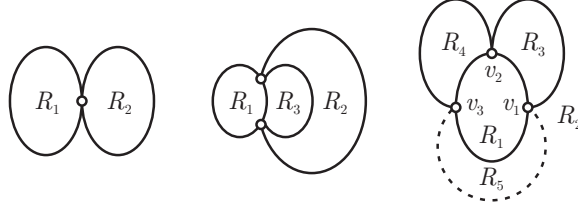


Figure 9: Two regions R_1 and R_2 , which share at most three vertices.

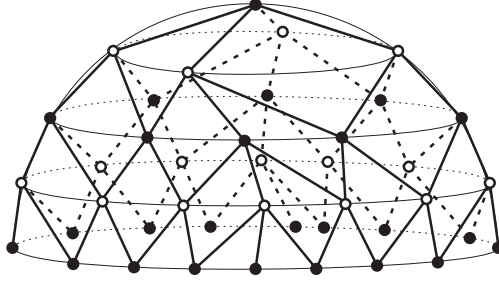


Figure 10: The superior half of a maximal planar bipartite graph drawn on a sphere for $\Delta = 4$

- (b) For any diameter $D = 2k + 1$ ($k \geq 1$) and odd maximum degree Δ ($\Delta \geq 9$), there exists a maximal planar bipartite graph $G_{\Delta,D}$ whose number of vertices $n(G_{\Delta,D})$ is

$$\begin{aligned} n(G_{\Delta,3}) &= 3\Delta - 1 && \text{for } D = 3, \\ n(G_{\Delta,5}) &= 3\Delta^2 - 21\Delta + 26 && \text{for } D = 5, \\ n(G_{\Delta,2k+1}) &= 3\Delta^2 - 21\Delta + 26 + \frac{3(\Delta-7)(\Delta-2)^2((\Delta-3)^{k-2}-1)}{(\Delta-4)} && \text{for } D = 2k + 1 \text{ and } k > 2, \end{aligned}$$

which is approximately $3(\Delta - 3)^k$, for Δ and D sufficiently large.

- (c) For any diameter $D = 2k + 1$ ($k \geq 1$) and even maximum degree Δ ($\Delta \geq 10$), there exists a maximal planar bipartite graph $G_{\Delta,D}$ whose number of vertices $n(G_{\Delta,D})$ is

$$\begin{aligned} n(G_{\Delta,3}) &= 3\Delta - 2 && \text{for } D = 3, \\ n(G_{\Delta,5}) &= 3\Delta^2 - 22\Delta + 26 && \text{for } D = 5, \\ n(G_{\Delta,2k+1}) &= 3\Delta^2 - 22\Delta + 26 + \frac{(3\Delta-22)(\Delta-2)^2((\Delta-3)^{k-2}-1)}{(\Delta-4)} && \text{for } D = 2k + 1 \text{ and } k > 2, \end{aligned}$$

which is approximately $3(\Delta - 3)^k$, for Δ and D sufficiently large.

References

- [1] K. Appel, W. Haken, The solution of the four-color map problem, *Sci. Amer.* **237** (1977) 108 – 121.
- [2] V. Chepoi, B. Estellon, Y. Vaxès, On covering planar graphs with a fixed number of balls, *Discrete Comput. Geom.* **37** (2007) 237 – 244.
- [3] M. Fellows, P. Hell, K. Seyffarth, Large planar graphs with given diameter and maximum degree, *Discrete Appl. Math.* **61** (1995) 133 – 153.
- [4] M. Fellows, P. Hell, K. Seyffarth, Constructions of large planar networks with given degree and diameter, *Networks* **32** (1998) 275 – 281.

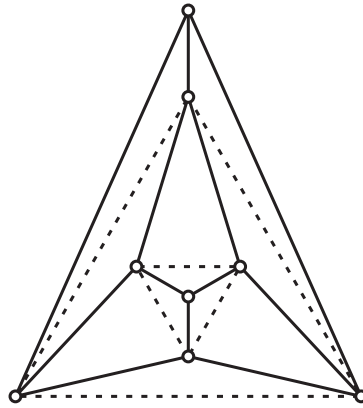


Figure 11: The first step of the iterative construction.

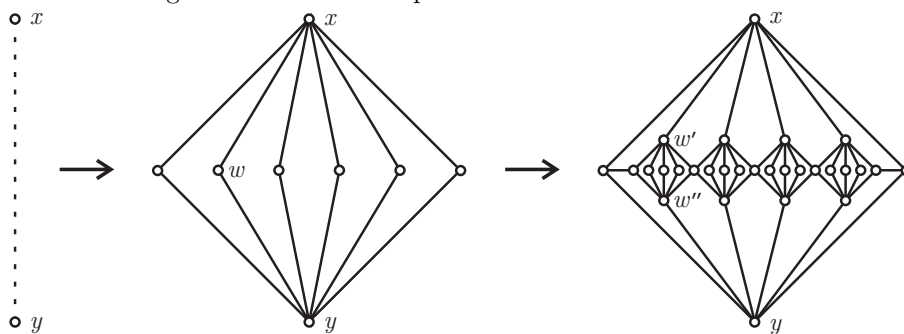


Figure 12: Substitution of an r -diamond. The inductive step.

- [5] R. Fera-Puron, G. Pineda-Villavicencio, Constructions of large graphs on surfaces, *Graph. Combinator.*, DOI 10.1007/s00373-013-1323-y, 2013.
- [6] C. Gavaille, D. Peleg, A. Raspaud, E. Sopena, Small k -dominating sets in planar graphs with applications, 27th International Workshop on Graph-Theoretic Concepts in Computer Science 2001, Lect. Notes Comput. Sc. **2204**, Springer-Verlag, Berlin, (2001) 201 – 216.
- [7] R.J. Lipton, R.E. Tarjan, A separator theorem for planar graphs, *SIAM J. Appl. Math.* **36** (1979) 177 – 189.
- [8] E. Loz, H. Pérez-Rosés, G. Pineda-Villavicencio, The degree/diameter problem for planar graphs,
http://combinatoricswiki.org/wiki/The_Degree_Diameter_Problem_for_Planar_Graphs,
 2008, accessed on 3 March 2014.
- [9] P. Hell, K. Seyffarth, Largest planar graphs of diameter two and fixed maximum degree, *Discrete Math.* **111** (1993) 313 – 322.
- [10] M. Miller, J. Širáň, Moore graphs and beyond: A survey of the degree/diameter problem, *Electron. J. Combin.* **20(2)** (2013) #DS14v2.
- [11] E. Nevo, G. Pineda-Villavicencio, D.R. Wood, On the maximum order of graphs embedded in surfaces, arXiv:1312.1753 (2013).
- [12] G. Pineda-Villavicencio, D.R. Wood, The degree-diameter problem for sparse graph classes, arXiv:1307.4456 (2013).
- [13] G. Ringel, Two trees in maximal planar bipartite graphs, *J. Graph Theory* **17** (1993) 755–758.

- [14] N. Robertson, D.P. Sanders, P.D. Seymour, R. Thomas, A New Proof of the Four Colour Theorem, *Electron. Res. Announc. Amer. Math. Soc.* **2** (1996) 17-25.
- [15] R. Sedgewick, P. Flajolet, *An introduction to the analysis of algorithms*, Addison-Wesley, Reading, MA, 1996.
- [16] K. Seyffarth, Maximal planar graphs of diameter two, *J. Graph Theory* **13** (1989) 619 – 648.
- [17] E. Steinitz, “Polyeder und Raumeinteilungen”, *Encyclopedie der mathematischen Wissenschaften*, **3** (Geometrie) (1922) 1 – 139.
- [18] S.A. Tishchenko, Maximum size of a planar graph with given degree and even diameter, *Eur. J. Combin.* **33** (2012) 380 – 396.
- [19] S.A. Tishchenko, N -separators in planar graphs, *Eur. J. Combin.* **33** (2012) 397 – 407.
- [20] Y. Yang, J. Lin, Y. Dai, Largest planar graphs and largest maximal planar graphs of diameter two, *J. Comput. Appl. Math.* **144** (2002) 349 – 358.