

**$R + \alpha R^2$  loop quantum cosmology**Jaume Amorós,<sup>1,\*</sup> Jaume de Haro,<sup>1,†</sup> and Sergei D. Odintsov<sup>2-4,‡</sup><sup>1</sup>*Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain*<sup>2</sup>*Institució Catalana de Recerca i Estudis Avançats (ICREA), Barcelona, Spain*<sup>3</sup>*Institut de Ciències de l'Espai (CSIC-IEEC), Campus UAB, Facultat de Ciències, Torre C5-Par-2a pl, E-08193 Bellaterra (Barcelona), Spain*<sup>4</sup>*Tomsk State Pedagogical University, 634061 Tomsk; National Research Tomsk State University, 634050 Tomsk, Russia; and King Abdulaziz University, Jeddah 21441, Saudi Arabia*

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Working in Einstein frame, we introduce, in order to avoid singularities, holonomy corrections to the  $f(R) = R + \alpha R^2$  model. We perform a detailed analytical and numerical study when holonomy corrections are taken into account in both Jordan and Einstein frames, obtaining, in Jordan frame, a dynamics which differs qualitatively, at early times, from the one of the original model. More precisely, when holonomy corrections are taken into account, the Universe is not singular, starting at early times in the contracting phase and bouncing to enter the expanding one where, as in the original model, it inflates. This dynamics is completely different from the one obtained in the original  $R + \alpha R^2$  model, where the Universe is singular at early times and never bounces. Moreover, we show that these holonomy corrections may lead to better predictions for the inflationary phase as compared with current observations.

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**I. INTRODUCTION**

Two kinds of quantum geometric corrections come from the discrete nature of space-time assumed in loop quantum cosmology (LQC): inverse volume corrections [1] and holonomy corrections (see, for instance, [2]). Dealing with the flat Friedmann-Lemître-Robertson-Walker (FLRW) geometry, which is the case in our paper, inverse volume corrections have problems because of arbitrary rescalings [3]; more precisely, since the scale factor can be arbitrarily rescaled in a flat metric, these inverse volume corrections could appear at any arbitrary scale, losing their physical meaning. They make sense only in a closed Universe, leading to a bounce that avoids the big bang and big crunch singularity [4]. On the other hand, holonomy corrections, which are well introduced for compact and noncompact geometries, provide a big bounce that avoids singularities like the big bang and big rip (see, for example, [5]).

On the other hand, it is well known that, in general,  $f(R)$  gravity does not avoid singularities, except for particular nonsingular cases where the  $R^2$  term plays an important role, as demonstrated in [6]. In order to avoid them, one could introduce holonomy corrections in  $f(R)$  gravity. The extension of LQC to  $f(R)$  gravity has been recently developed in [7,8], where holonomy corrections are introduced in the Einstein frame (EF), because in that frame the gravitational part of the Hamiltonian is linear in the scalar curvature and the matter part is given by a scalar field.

It is important to recall that the idea to introduce holonomy corrections via the EF was performed in [9] by studying the graceful exit problem in the pre-big bang scenario, i.e., studying the regularization of the singularity that divides the pre- and post-big bang branches in pre-big bang models.

This extension simplifies very much when one considers the flat FLRW geometry. In that case, in order to take into account geometric effects, one has to replace the Ashtekar connection by a suitable sinus function (see, for instance, [10]), obtaining the holonomy corrected Friedmann equation in EF. Finally, from the holonomy corrected Friedmann equation in EF and through the relation between the corresponding variables in both frames, one obtains the holonomy corrected  $f(R)$  theory in the Jordan frame (JF).

Our main objective is to apply, for the flat FLRW geometry, holonomy corrections to the  $f(R) = R + \alpha R^2$  model (also called  $R^2$  gravity) and study its dynamics. To do this, first of all we perform a detailed analysis of  $R^2$  gravity without corrections. When holonomy corrections in the model are taken into account, one obtains a very complicated dynamical equation in the JF. Fortunately, dynamical equations simplify very much in EF (in fact, the dynamics is given by the well-known holonomy corrected Friedmann equation in LQC plus the Klein-Gordon equation in flat FLRW geometry), which allows us to perform a very deep analytical and numerical analysis, whose results can be translated to the JF. Our conclusion is that when holonomy corrections are taken into account, the Universe starts at the critical point ( $H = 0, \dot{H} = 0$ ) (the Hubble parameter and its derivative vanish) and makes small

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oscillations around the critical point before entering the contracting phase, which it leaves by bouncing (see [11] for a review of bounce cosmology). Then it enters the expanding phase where, as in the classical model, it reaches an inflationary stage which it leaves at late times and comes back to the critical point, once again, in an oscillating way.

The paper is organized as follows. In Sec. II, we review  $f(R)$  gravity in Jordan and Einstein frames. In Sec. III, we introduce holonomy corrections to  $f(R)$  gravity. The idea is very simple: working in EF,  $f(R)$  gravity is formulated as Einstein gravity plus a scalar field. Then, the idea, as in standard LQC for the flat FLRW geometry, is to replace the Ashtekar connection by a suitable sinus function. Section IV is devoted to the study of  $R^2$  gravity without holonomy corrections. After performing the change of variable  $p^2 = H$ , where  $H$  is the Hubble parameter, the obtained dynamical equation can be understood as the dynamics of a particle under the action of a quadratic potential with dissipation. This system is very simple and the phase portrait can be drawn with all the details. In Sec. V, we analyze the model with holonomy corrections. We start working in EF due to the simplicity of equations and, once we have studied the dynamics in EF, we obtain the dynamics in JF from the formulas that relate both frames. Moreover, we obtain in EF the corrected expressions of the slow-roll parameters, the values of the spectral index for scalar perturbations, and the ratio of tensor to scalar perturbations, showing that holonomy corrections help to match correctly the theoretical results obtained from  $R^2$  gravity with current observations. Section VI is devoted to discussing a possible unification of inflation and current cosmic acceleration in the framework of loop quantum  $f(R)$  theories. We will show that when one considers the current suggested models for such unification, this extension and/or its analytical study is, in general, unworkable. The only model we have been able to deal with is  $R^2$  plus a small cosmological constant. For such a model, we have performed a detailed analytical study and the results are shown at the end of the work.

## II. CLASSICAL DYNAMICAL EQUATIONS IN DIFFERENT FRAMES

In this section, we review the relations between Jordan and Einstein frames in  $f(R)$  gravity for the flat FLRW geometry.

The Lagrangian in JF for the flat FLRW geometry is given by  $\mathcal{L}_{\text{JF}} = \frac{a^3}{2} f(R)$ , where the scalar curvature is  $R = 6\dot{H} + 12H^2$ , with  $H = \frac{\dot{a}}{a}$  as the Hubble parameter, and the corresponding modified Friedmann equation in  $f(R)$  gravity can be obtained from Ostrogradskii's construction [10], giving as a result

$$6f_{RR}(R)\dot{R}H + (6H^2 - R)f_R(R) + f(R) = 0, \quad (2.1)$$

where  $f_R(R) \equiv \frac{\partial f(R)}{\partial R}$ . Taking the derivative of Eq. (2.1) with respect to time and using the relation  $R = 6(\dot{H} + 2H^2)$ , one obtains the equivalent equation

$$f_{RR}(R)(\ddot{R} - \dot{R}H) + f_{RRR}(R)\dot{R}^2 + 2f_R(R)\left(\frac{R}{2} - 2H^2\right) = 0. \quad (2.2)$$

To work in the EF, one has to perform the change of variables [12]

$$\tilde{a} = \sqrt{f_R(R)}a; \quad d\tilde{t} = \sqrt{f_R(R)}dt. \quad (2.3)$$

Then, in that frame the Lagrangian density, for flat FLRW geometries, is

$$\begin{aligned} \mathcal{L}_{\text{EF}} &= \tilde{a}^3 \left( \frac{1}{2} \tilde{R} + \frac{1}{2} (\tilde{\phi}')^2 - V(\tilde{\phi}) \right) \\ \Leftrightarrow \mathcal{L}_{\text{EF}} &= -3(\tilde{a}')^2 \tilde{a} + \tilde{a}^3 \left( \frac{1}{2} (\tilde{\phi}')^2 - V(\tilde{\phi}) \right), \end{aligned} \quad (2.4)$$

where  $'$  means the derivative with respect to time  $\tilde{t}$ . Here,  $\tilde{a}$  and  $\tilde{\phi}$  have to be considered as independent variables, and of course,  $\tilde{R} = 6\tilde{H}' + 12\tilde{H}^2$ .

The relation between both frames is given through the relations

$$\tilde{\phi} = \sqrt{\frac{3}{2}} \ln(f_R(R)); \quad V(\tilde{\phi}) = \frac{Rf_R(R) - f(R)}{2f_R^2(R)}, \quad (2.5)$$

and a simple calculation shows that the Friedmann equation in the EF, i.e.,  $\tilde{H}^2 = \frac{1}{3}\tilde{\rho}$ , obtained from the Hamiltonian constraint

$$\begin{aligned} \mathcal{H}_{\text{EF}} &\equiv \tilde{a}' \frac{\partial \mathcal{L}_{\text{EF}}}{\partial \tilde{a}'} + \tilde{\phi}' \frac{\partial \mathcal{L}_{\text{EF}}}{\partial \tilde{\phi}'} - \mathcal{L}_{\text{EF}} \\ &= -3(\tilde{a}')^2 \tilde{a} + \tilde{a}^3 \left( \frac{1}{2} (\tilde{\phi}')^2 + V(\tilde{\phi}) \right) = 0, \end{aligned} \quad (2.6)$$

where  $\tilde{\rho} \equiv \frac{1}{2} (\tilde{\phi}')^2 + V(\tilde{\phi})$ , is equivalent to Eq. (2.1). However, the Friedmann equation in EF,  $\tilde{H}^2 = \frac{1}{3}\tilde{\rho}$ , is a constraint instead of a dynamical equation. The dynamics is given by the conservation equation  $\tilde{\rho}' = -3\tilde{H}(\tilde{\phi}')^2$  or the Raychauduri one  $\tilde{H}' = -\frac{1}{2}(\tilde{\phi}')^2$ , which are equivalent to Eq. (2.2).

Note that combining, in EF, the conservation and Friedmann equations, one obtains

$$(\tilde{\rho}')^2 = 3\tilde{\rho}(\tilde{\phi}')^2, \quad (2.7)$$

and coming back to the JF this equation is a second order differential equation in  $R$  (it only contains  $R$ ,  $\dot{R}$ , and  $\ddot{R}$ ) which is equivalent to Eqs. (2.1) and (2.2).

Finally, we show the following relations between both frames, which will be important when we extend LQC to  $R^2$  gravity:

$$H = \sqrt{f_R(R)} \left( \tilde{H} - \frac{1}{\sqrt{6}} \tilde{\phi}' \right);$$

$$R = f_R(R) \left( \tilde{R} + \left( \tilde{\phi}' \right)^2 + \sqrt{6} \frac{\partial V(\tilde{\phi})}{\partial \tilde{\phi}} \right). \quad (2.8)$$

### III. $f(R)$ LOOP QUANTUM COSMOLOGY

The idea to extend loop quantum cosmology to  $f(R)$  theories [ $f(R)$  LQC] has been recently developed in [7,8]. For a flat FLRW geometry, the idea is very simple and goes as follows: working in EF, in the same way as in standard LQC, we can see that the classical variable  $\beta \equiv \gamma \tilde{H}$ , where  $\gamma$  is the Barbero-Immirzi parameter, and the volume  $\tilde{V} \equiv \tilde{a}^3$  are canonically conjugated variables with Poisson bracket  $\{\beta, \tilde{V}\} = \frac{\gamma}{2}$  [13]. Then, in order to take into account the discrete nature of the space, one has to choose a Hilbert space where quantum states are represented by almost periodic functions. However, in this space the variable  $\tilde{\beta}$  does not correspond to a well-defined quantum operator in this space, and since it appears in the Hamiltonian (2.6), because it could be written as

$$\mathcal{H}_{\text{EF}} = -3 \frac{\tilde{\beta}^2}{\gamma^2} \tilde{V} + \tilde{V} \left( \frac{1}{2} (\tilde{\phi}')^2 + V(\tilde{\phi}) \right) = 0, \quad (3.1)$$

in order to have a well-defined quantum theory, one needs to use the general holonomy corrected Hamiltonian in loop quantum gravity (see, for instance, [14,15]).

At an effective level, this is equivalent to introducing the square root of the minimum eigenvalue of the area operator in LQG, namely,  $\lambda = \sqrt{\frac{\sqrt{3}}{2}} \gamma$ , and making the replacement (see [16–18] for a detailed discussion about the justification of this replacement)

$$\tilde{\beta} \rightarrow \frac{\sin(\lambda \tilde{\beta})}{\lambda}, \quad (3.2)$$

in the Hamiltonian (3.1), while keeping on the Poisson bracket  $\{\tilde{\beta}, \tilde{V}\} = \frac{\gamma}{2}$ . Here, it is important to notice that, after the introduction of holonomy corrections,  $\tilde{\beta}$  stops being equal to  $\gamma \tilde{H}$ . This can be shown from the Hamilton equation  $\tilde{V}' = \{\tilde{V}, \mathcal{H}_{\text{EF,LQC}}\}$  [where  $\mathcal{H}_{\text{EF,LQC}}$  is the new Hamiltonian obtained from (3.1) after the replacement (3.2)], because this equation can be written as  $\tilde{V}' = -\frac{\gamma}{2} \frac{\partial \mathcal{H}_{\text{EF,LQC}}}{\partial \tilde{\beta}} = 3 \frac{\sin \lambda \tilde{\beta} \cos \lambda \tilde{\beta}}{\lambda \gamma}$  which is equivalent to

$$\tilde{H} = \frac{\sin 2\lambda \tilde{\beta}}{2\lambda \gamma} \Leftrightarrow \tilde{\beta} = \frac{1}{2\lambda} \arcsin 2\lambda \gamma \tilde{H}. \quad (3.3)$$

**Remark III.1.** It is stated in [19] that there are many different inequivalent loop quantizations, i.e., different pairs of canonically conjugated variables leading to

inequivalent quantum realizations. Here we have used the so-called new quantization of LQC (also known as  $\bar{\mu}$  quantization) [14] based in the use of variables  $(\tilde{\beta}, \tilde{V})$ , which is the unique choice consistent with the physical requirements proposed in [19].

**Remark III.2.** It is important to stress that the replacement (3.2) is only valid for spatially flat models, which is our case. When the spatial curvature does not vanish, holonomy corrections have to be introduced in a nontrivial way (see, for instance, [20]).

Finally, from the Hamilton equation  $\tilde{V}' = \{\tilde{V}, \mathcal{H}_{\text{EF,LQC}}\}$  and the Hamiltonian constraint  $\mathcal{H}_{\text{EF,LQC}} = 0$ , one obtains the corresponding holonomy corrected version of the classical Friedmann equation [13], that is,

$$\tilde{H}^2 = \frac{1}{3} \tilde{\rho} \left( 1 - \frac{\tilde{\rho}}{\tilde{\rho}_c} \right), \quad (3.4)$$

where  $\tilde{\rho}_c \equiv \frac{3}{\lambda^2 \gamma^2}$  is the so-called *critical density* in the EF.

As has been discussed in detail in [21], this equation depicts an ellipse in the plane  $(\tilde{H}, \tilde{\rho})$ , and the dynamics along this curve is very simple: for a nonphantom field, the Universe moves clockwise from the contracting to the expanding phase, starting and ending at the critical point  $(0,0)$  and bouncing only once at  $(0, \tilde{\rho}_c)$ .

Finally, note that in the JF, the holonomy corrected Friedmann equation acquires the complicated form

$$6f_{RR}(R)\dot{R}H + (6H^2 - R)f_R(R) + f(R) = -\frac{(\frac{3}{2}f_{RR}^2(R)\dot{R}^2 + (Rf_R(R) - f(R))f_R(R))^2}{2f_R^4(R)\tilde{\rho}_c}. \quad (3.5)$$

### IV. $R^2$ GRAVITY

In this section we study with all the details the classical model  $f(R) = R + \alpha R^2$ , with  $\alpha > 0$ . This model contains a quadratic correction to the scalar curvature and is a modified version of the Starobinsky model [22], where the author considered quantum vacuum effects due to massless fields conformally coupled with gravity. Note that such (eternal) trace-anomaly driven inflation was proposed earlier in Ref. [23].

For this model, the classical equation (2.1) becomes

$$12\alpha H\dot{R} + 6H^2 + 12\alpha RH^2 - \alpha R^2 = 0 \Leftrightarrow H^2 = -12\alpha \left( 3\dot{H}H^2 + H\dot{H} - \frac{1}{2}\dot{H}^2 \right), \quad (4.1)$$

which coincides, when the parameter  $\beta$  vanishes, with the dynamical equation studied in [22]

$$H^2 = -12\alpha \left( 3\dot{H}H^2 + H\ddot{H} - \frac{1}{2}\dot{H}^2 \right) + \beta H^4, \quad \text{where } \beta > 0. \quad (4.2)$$

It is very simple to show that Eq. (4.1) leads to an inflationary epoch [24,25]. Effectively, when the slow-roll initial condition  $|\dot{H}| \ll H^2$  is fulfilled, Eq. (4.1) becomes  $\ddot{H} = -3H\dot{H} - \frac{H}{12\alpha}$ , which has the following particular solution in the expanding phase ( $H > 0$ )

$$\begin{aligned} \dot{H}(t) &= -\frac{1}{36\alpha} \Rightarrow H(t) = \frac{t_1 - t}{36\alpha} \\ \Rightarrow a(t) &= a(t_1)e^{-18\alpha H^2(t)} \quad \text{for } t < t_1. \end{aligned} \quad (4.3)$$

If  $t_i$  and  $t_f$  are the beginning and the end of inflation ( $t_i < t_f < t_1$ ), then one will have

$$a(t_f) = a(t_i)e^{18\alpha(H^2(t_i) - H^2(t_f))} \cong a(t_i)e^{18\alpha H^2(t_i)}, \quad (4.4)$$

and the 60  $e$ -folds needed to solve the flatness and horizon problems will be obtained when  $\alpha H^2(t_i)$  is approximately 3.3.

Unfortunately,  $R^2$  gravity contains singularities at early times; that is, all solutions have divergent scalar curvature at early times. To show that, one has to perform the change of variables  $p^2(t) = H(t) > 0$  [26] (in this model, the Universe does not bounce); then Eq. (4.1), which is not well-defined at singular value  $H = 0$ , becomes the following well-defined equation:

$$\frac{d}{dt} \left( \frac{\dot{p}^2}{2} + W(p) \right) = -3p^2\dot{p}^2, \quad (4.5)$$

where  $W(p) = \frac{p^2}{48\alpha}$ .

We can see that the system (4.5) is dissipative. To understand its dynamics, we can imagine a ‘‘particle’’ rolling down along the parabola  $W(p)$ , losing energy and oscillating, at late times, around  $p = 0$ . As a consequence, when time goes back the particle gains energy and finally  $|p| \rightarrow \infty$  ( $H \rightarrow \infty$ ); i.e., all the solutions are singular at early times. One also can check this fact as follows: we write Eq. (4.5) as

$$\ddot{p} + \frac{p}{24\alpha} = -3p^2\dot{p}, \quad (4.6)$$

and look for, at early times, solutions of the form  $p(t) = \frac{C}{(t-\bar{t})^r}$ , where  $C$  and  $r$  are parameters. Inserting this expression in (4.6) and retaining the leading terms when  $t \gtrsim \bar{t}$ , one obtains the equation

$$\frac{r(r+1)C}{(t-\bar{t})^{r+2}} = \frac{3rC^3}{(t-\bar{t})^{3r+1}}, \quad (4.7)$$

which has singular solutions at  $t = \bar{t}$  of the form  $p(t) = \sqrt{\frac{1}{2(t-\bar{t})}}$ .

**Remark IV.1.** In the contracting phase, we can perform the change of variable  $p^2(t) = -H(t) > 0$ , obtaining the system

$$\frac{d}{dt} \left( \frac{\dot{p}^2}{2} + W(p) \right) = 3p^2\dot{p}^2, \quad (4.8)$$

where  $W(p) = \frac{p^2}{48\alpha}$ . We can see that in the contracting phase the system is antidissipative (the Universe gains energy); in this case the Universe starts oscillating around the bottom of the potential, leaving it gradually, and becomes singular at late times.

Equation (4.6) is also useful to obtain the inflationary period and the dynamics at late times. Effectively, when initially one has  $\dot{p} \cong 0$ , Eq. (4.6) becomes  $p\dot{p} = -\frac{1}{72\alpha}$ , whose inflationary solution is once again

$$H(t) = p^2(t) = \frac{t_1 - t}{36\alpha}. \quad (4.9)$$

On the other hand, to obtain the dynamics at late times we follow the same method used in chaotic inflation for a quadratic potential (see page 240 of [27]). Performing the change of variable

$$\begin{aligned} \dot{p}(t) &= \sqrt{2}f(t) \cos(\theta(t)), \\ p(t) &= \sqrt{48\alpha}f(t) \sin(\theta(t)), \end{aligned} \quad (4.10)$$

and inserting these expressions in Eqs. (4.5) and (4.6), one gets the system

$$\begin{cases} \dot{f} = -18\alpha f^3(1 - \cos(4\theta)) \\ \dot{\theta} = \frac{1}{\sqrt{24\alpha}} + 144\alpha f^2 \sin^3(\theta) \cos(\theta). \end{cases} \quad (4.11)$$

Since  $p$  goes to zero at late times, we can disregard the second term in the right-hand side in the second equation of (4.11), obtaining  $\dot{\theta} = \frac{1}{\sqrt{24\alpha}}$ , whose solution is  $\theta(t) = \frac{t}{\sqrt{24\alpha}} + \omega$ ,  $\omega$  being a constant of integration. Inserting this approximate solution in the first equation of (4.11), we obtain a solvable equation whose solution is given by

$$\begin{aligned} f(t) &= \sqrt{\frac{1}{36\alpha t \left( 1 - \frac{\sin\left(\frac{2t}{\sqrt{6\alpha}} + 4\omega\right)}{\frac{2t}{\sqrt{6\alpha}}} \right)}} \\ &\cong \sqrt{\frac{1}{36\alpha t \left( 1 + \frac{\sin\left(\frac{2t}{\sqrt{6\alpha}} + 4\omega\right)}{\frac{2t}{\sqrt{6\alpha}}} \right)}}, \end{aligned} \quad (4.12)$$

and thus, the Hubble parameter reads

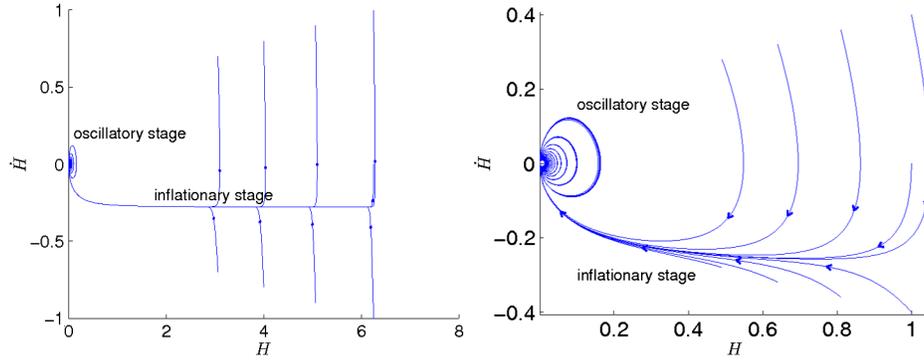


FIG. 1 (color online). Phase portrait for  $\alpha = 0.1$ . The Universe comes from a singularity at early times. When time goes forward, it enters in the attractor inflationary phase, leaving it at early times when the Universe starts to oscillate around  $(0,0)$  without bouncing. In the first part of the figure, we have taken values of  $H$  up to 6 to show clearly the inflationary stage, and in the second one the values of  $H$  are taken up to 1 to show, in more detail, the oscillatory phase. It is clear from the figures that orbits are unbounded, coming from  $\infty$  at early times.

$$H(t) \cong \frac{4}{3t} \left( 1 + \frac{\sin\left(\frac{2t}{\sqrt{6\alpha}} + 4\omega\right)}{\frac{2t}{\sqrt{6\alpha}}} \right) \sin^2\left(\frac{t}{\sqrt{24\alpha}} + \omega\right). \quad (4.13)$$

$$a(t) \cong t^{2/3} \left( 1 + \frac{2}{3} \frac{\sin\left(\frac{t}{\sqrt{6\alpha}}\right)}{\frac{t}{\sqrt{6\alpha}}} \right) \cong t^{2/3}. \quad (4.15)$$

Now, choosing  $\omega = \pi/2$ , one obtains the well-known result [24,28,29]

$$H(t) \cong \frac{4}{3t} \left( 1 + \frac{\sin\left(\frac{2t}{\sqrt{6\alpha}}\right)}{\frac{2t}{\sqrt{6\alpha}}} \right) \cos^2\left(\frac{t}{\sqrt{24\alpha}}\right), \quad (4.14)$$

and after integrating by parts one gets, as Starobinsky in [22],

These analytic results are supported numerically in Fig. 1.

An important remark is in order: note that in the model of [22], one obtains the same equation (4.5) but with the potential  $W(p) = \frac{p^2}{48\alpha} - \frac{\beta p^6}{144\alpha}$ . In this case, the potential has a stable minimum at  $p = 0$  and two unstable maximums at  $p = \pm\beta^{-1/4}$ , which corresponds to the unstable de Sitter solution  $H = \beta^{-1/2}$ . From the shape of this potential, one deduces that there are only two unstable nonsingular solutions (the ones that start at the de Sitter points and end at the bottom of the potential), and two that only are singular at late times (the ones that start at the de Sitter

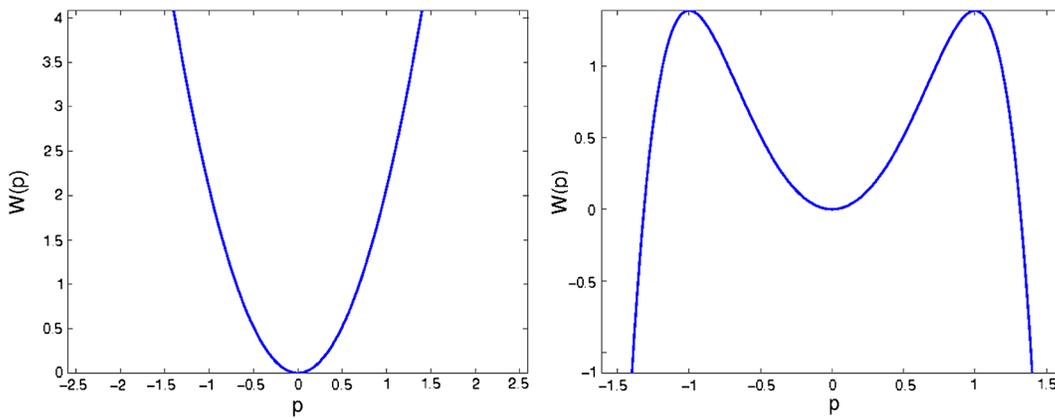


FIG. 2 (color online). Shape of the potentials  $W(p)$  for  $\alpha = 0.01$  and  $\beta = 1$ . The first part of the figure corresponds to the potential given by  $R^2$  gravity and the second one to the potential given by the model suggested in [22]. The dynamics is very easy: one can imagine a particle moving under the action of the potential  $W$  and losing energy. The first potential particles come at early times from  $|p| = \infty$  and end at late times at  $p = 0$  in an oscillating way. For the second potential, there are two unstable de Sitter solutions at  $p = \pm(1/\beta)^{-1/4}$ , so the particle could start at early times at these points and fall down into the wedge of the potential ending, at late times, at  $p = 0$  in an oscillatory way. These are the only nonsingular solutions. All the other orbits are singular at early and/or late times.

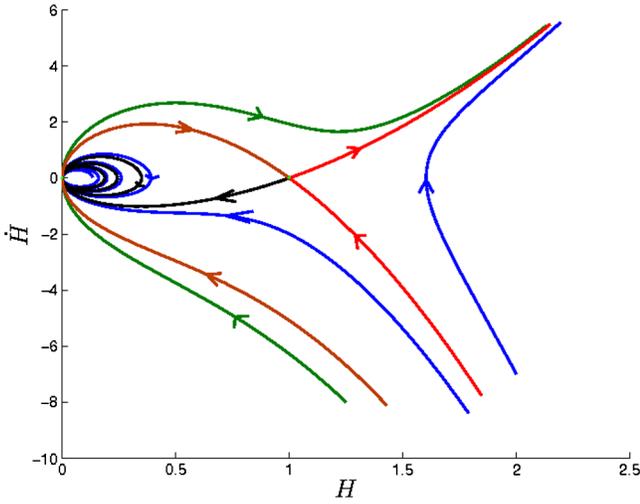


FIG. 3 (color online). Phase portrait for  $\alpha = 0.01$  and  $\beta = 1$  of the Starobinsky model. The unique nonsingular solutions are the de Sitter ones which correspond to the saddle point  $(1,0)$ , which is the unique critical point of the system (painted brown), and the black curve that starts at the critical point and ends oscillating at  $(0,0)$ .

points and end at  $|p| = \infty$ ); all the other solutions are singular at early times. At late times, there are two kinds of solutions: the ones that have enough energy to overpass the wedge of the potential and become singular at late times, and others with less energy that fall down against the wedge of the potential without clearing it due to the dissipation and approach  $p = 0$  with the same oscillatory behavior as in the  $R + \alpha R^2$  model (see Fig. 2 for the shape of potentials and Fig. 3 for the phase portrait of the Starobinsky model).

Note that what is really important in the Starobinsky model at late times is the oscillatory behavior of the scale factor rather than its amplitude, because at late times the period of oscillation of the scale factor is much shorter than the Hubble time, meaning that for a few oscillations the amplitude of the scale factor can be considered constant. This behavior can be thought of as oscillations of a decaying field called *scalaron* [22] that creates light conformally coupled particles, which finally thermalize, yielding a hot Friedmann Universe that matches the Standard Model.

## V. LOOP QUANTUM $R^2$ GRAVITY

We start this section by showing that there exists a wide range of values of  $\alpha$  and  $\tilde{\rho}_c$  for which the  $R^2$  LQC model does not have any singularity. First at all, from the holonomy corrected Friedmann equation (3.5), we deduce that

$$0 \leq \tilde{\rho} \leq \tilde{\rho}_c \quad \text{and} \quad -\sqrt{\frac{\tilde{\rho}_c}{12}} \leq \tilde{H} \leq \sqrt{\frac{\tilde{\rho}_c}{12}}. \quad (5.1)$$

On the other hand, Eq. (2.5), applied to  $R^2$  gravity, leads to the positive potential

$$V(\tilde{\phi}) = \frac{1}{8\alpha} (1 - e^{-\sqrt{\frac{2}{3}}\tilde{\phi}})^2, \quad (5.2)$$

and thus, one also has

$$0 \leq (\tilde{\phi}')^2 \leq 2\tilde{\rho}_c \quad \text{and} \quad 0 \leq V(\tilde{\rho}) \leq \tilde{\rho}_c. \quad (5.3)$$

Using the Raychaudhuri equation in LQC,  $\tilde{H}' = -\frac{1}{2}(\tilde{\phi}')^2(1 - \frac{2\tilde{\rho}}{\tilde{\rho}_c})$ , one deduces that

$$|\tilde{H}'| \leq \frac{1}{2}(\tilde{\phi}')^2 \leq \tilde{\rho}_c \Rightarrow |\tilde{R}| \leq 7\tilde{\rho}_c. \quad (5.4)$$

Moreover, the potential (5.2) satisfies

$$\frac{\partial V(\tilde{\phi})}{\partial \tilde{\phi}} = \frac{1}{f_R(R)} \sqrt{\frac{V(\tilde{\phi})}{3\alpha}}, \quad (5.5)$$

which means [see the second equation of (2.8)]

$$R = f_R(R)(\tilde{R} + (\tilde{\phi}')^2) + \sqrt{\frac{2V(\tilde{\phi})}{\alpha}}, \quad (5.6)$$

and thus,

$$R = \frac{1}{1 - 2\alpha(\tilde{R} + (\tilde{\phi}')^2)} \left( \tilde{R} + (\tilde{\phi}')^2 + \sqrt{\frac{2V(\tilde{\phi})}{\alpha}} \right). \quad (5.7)$$

From the bound  $1 - 2\alpha(\tilde{R} + (\tilde{\phi}')^2) \geq 1 - 18\alpha\tilde{\rho}_c$  one easily deduces

$$|R| \leq \frac{1}{1 - 18\alpha\tilde{\rho}_c} \left( 18\tilde{\rho}_c + \sqrt{\frac{2\tilde{\rho}_c}{\alpha}} \right), \quad (5.8)$$

which is always bounded provided we choose  $\alpha < \frac{1}{18\tilde{\rho}_c}$ .

Finally, since  $|R|$  is bounded, from the first equation of (2.8) one deduces that  $|H|$  is bounded, and consequently  $|\dot{H}| = \frac{1}{6}|R - 12H^2|$  is bounded, meaning that  $R^2$  gravity in LQC has no singularities.

In fact, as we will see, in any case there are singularities when one takes into account holonomy corrections. However, when  $8\alpha\tilde{\rho}_c > 1$  the scalar curvature  $R$  can achieve very large values. To show that, we have to perform a detailed analysis in EF.

### A. $R^2$ LQC in Einstein frame

To perform a deeper analysis of the model we will work in EF, where the dynamical equations are simpler than in the JF one. In fact, when  $f(R) = R + \alpha R^2$ , Eq. (3.5) becomes

$$12\alpha\dot{R}H + 6H^2(1 + 2\alpha R) - \alpha R^2 = -\frac{\alpha^2[2\alpha(R^3 + 3\dot{R}^2) + R^2]^2}{2(1 + 2\alpha R)^4\tilde{\rho}_c}. \quad (5.9)$$

On the other hand, in EF, the field  $\tilde{\phi}$  satisfies the equation

$$\tilde{\phi}'' + 3\tilde{H}\tilde{\phi}' + \frac{\partial V(\tilde{\phi})}{\partial \tilde{\phi}} = 0, \quad (5.10)$$

where the potential  $\tilde{\phi}$  is given by (5.2).

As we have already explained, due to the holonomy effects in EF, the Universe starts in the contracting phase with zero energy, and the energy density increases until it catches up with the critical value  $\tilde{\rho}_c$ , where the Universe bounces and enters in the expanding phase.

Performing the change of variable  $\sqrt{\frac{2}{3}}\tilde{\phi} = \ln \tilde{\psi}$ , i.e.,  $\tilde{\psi} = f_R(R) = 1 + 2\alpha R$  (essentially  $\tilde{\psi}$  is like  $R$ ), one gets

$$\tilde{\psi}''\tilde{\psi} - (\tilde{\psi}')^2 + 3\tilde{H}\tilde{\psi}'\tilde{\psi} + \frac{1}{6\alpha}(\tilde{\psi} - 1) = 0. \quad (5.11)$$

From Eq. (5.11) one can show that the orbits in the plane  $(\tilde{\psi}, \tilde{\psi}')$  are symmetric with respect to the axis  $\tilde{\psi}' = 0$  in the expanding and contracting phase, because Eq. (5.11) remains invariant after performing the replacement  $\tilde{t} \rightarrow -\tilde{t}$  and  $\tilde{H} \rightarrow -\tilde{H}$ . To be more precise, consider in the plane  $(\tilde{\psi}, \tilde{\psi}')$  a trajectory [a solution of (5.11)]  $\sigma_1(t) = (\tilde{\psi}(t), \tilde{\psi}'(t))$  in the contracting  $\tilde{H} < 0$  (respectively, expanding  $\tilde{H} > 0$ ) phase. Then,  $\sigma_2(t) = (\tilde{\psi}(-t), -\tilde{\psi}'(-t))$  is a trajectory in the expanding  $\tilde{H} > 0$  (respectively, contracting  $\tilde{H} < 0$ ) phase.

The energy density, using the new variables, is given by

$$\tilde{\rho} = \frac{3}{4\tilde{\psi}^2} \left( (\tilde{\psi}')^2 + \frac{1}{6\alpha}(\tilde{\psi} - 1)^2 \right), \quad (5.12)$$

which means that  $\tilde{H}$  vanishes at the point  $(\tilde{\psi}, \tilde{\psi}') = (1, 0)$  and over the curve  $\tilde{\rho} = \tilde{\rho}_c$ , with equation

$$\frac{(\tilde{\psi}')^2}{\frac{4\tilde{\rho}_c}{3(1-8\alpha\tilde{\rho}_c)}} + \frac{(\tilde{\psi} - \frac{1}{1-8\alpha\tilde{\rho}_c})^2}{\frac{8\alpha\tilde{\rho}_c}{(1-8\alpha\tilde{\rho}_c)^2}} = 1, \quad (5.13)$$

which produces an ellipse for  $1 - 8\alpha\tilde{\rho}_c > 0$ , a hyperbola for  $1 - 8\alpha\tilde{\rho}_c < 0$ , and a parabola for  $1 - 8\alpha\tilde{\rho}_c = 0$ . Note also that  $(1,0)$  is the unique critical point corresponding to  $\tilde{\rho} = 0$ , which means that all the orbits start and end at this point (the Universe starts and ends at this point), and in the curve (5.13) the Universe in EF bounces, because it corresponds to  $\tilde{\rho} = \tilde{\rho}_c$ .

From the previous analysis, we can conclude that the dynamics, working in EF, goes as follows: the Universe starts in the contracting phase  $\tilde{H} < 0$ , oscillating around the unique critical point  $(1,0)$  and increasing the amplitude of oscillations, and then it reaches the curve  $\tilde{\rho} = \tilde{\rho}_c$ , where it bounces and enters in the expanding phase  $\tilde{H} > 0$ , coming back once again to  $(1,0)$  in an oscillatory way (our analytical study is supported numerically in Fig. 4).

Two important remarks are in order:

- (1) Strictly speaking, the phase portrait in the plane  $(\tilde{\psi}, \tilde{\psi}')$  shows the dynamics of two dynamical systems, because Eq. (5.11) defines two different differential equations, one with  $\tilde{H} > 0$  and the other one with  $\tilde{H} < 0$ . Then, since we have two different autonomous dynamical systems, at each point of the

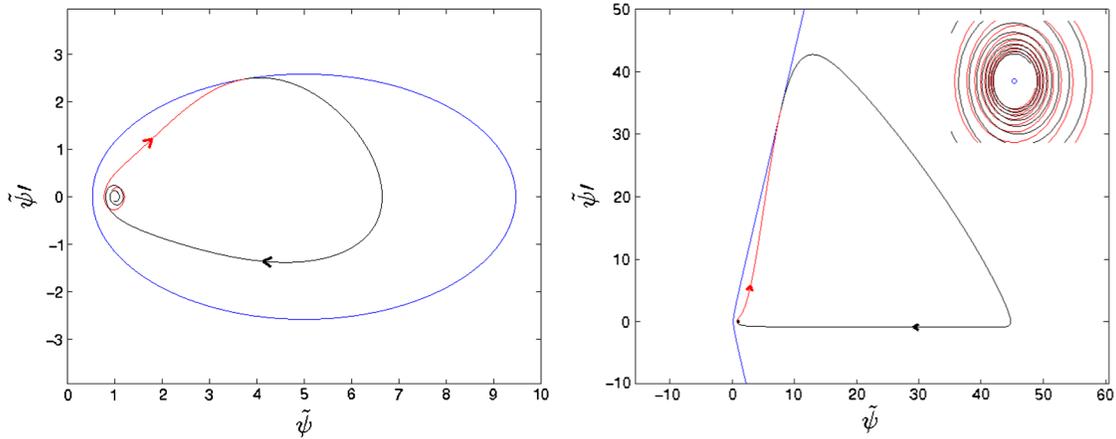


FIG. 4 (color online). In the first part, we have the phase space portrait of an orbit in EF for the case  $1 - 8\alpha\tilde{\rho}_c > 0$  ( $\alpha = 0.1$  and  $\tilde{\rho}_c = 1$ ). The Universe starts, in the contracting phase  $\tilde{H} < 0$ , oscillating around  $(1,0)$  (red curve) and arriving at the ellipse defined by Eq. (5.13) (blue curve), where the Universe bounces, entering the expanding phase  $\tilde{H} > 0$  and coming back to  $(1,0)$  by oscillating (black curve). In the second part, we draw an orbit in EF for the case  $1 - 8\alpha\tilde{\rho}_c < 0$  ( $\alpha = 0.1$  and  $\tilde{\rho}_c = 15$ ). The dynamics is similar; the only difference is that now the blue curve is a hyperbola. At the top of the panel, we have inserted and increased in size the oscillatory behavior around the critical point  $(1,0)$ .

plane  $(\tilde{\psi}, \tilde{\psi}')$ , two different orbits, one with  $\tilde{H} > 0$  and the other one with  $\tilde{H} < 0$ , cross.

- (2) It is important to realize that the system does not contain singularities because all the orbits start and end at the critical point (1,0). In the case  $1 - 8\alpha\tilde{\rho}_c > 0$ , the variables  $\tilde{\psi}$  and  $\tilde{\psi}'$  move inside an ellipse (a compact domain), meaning that, in this case, all the quantities are bounded. Effectively, inside the ellipse the quantities  $\tilde{H}$ ,  $\tilde{R}$ ,  $\tilde{\phi}$ ,  $\tilde{\phi}'$ , and  $\tilde{\psi} = 1 + 2\alpha R$  are bounded. Consequently, it follows from (2.8) that  $H$  is bounded. On the other hand, in the case  $1 - 8\alpha\tilde{\rho}_c < 0$ , the variables  $\tilde{\psi}$  and  $\tilde{\psi}'$  move inside an unbounded region delimited by a hyperbola, meaning that there are orbits where  $\tilde{\psi}$ , and consequently the scalar curvature  $R$ , achieve very large values, which never happens in the other case.

### 1. Inflation in Einstein frame

The slow-roll parameters in EF are given by (see, for example, [30])

$$\tilde{\epsilon} \equiv -\frac{\tilde{H}'}{\tilde{H}^2} \quad \text{and} \quad \tilde{\eta} \equiv \tilde{\epsilon} - \tilde{\delta} = 2\tilde{\epsilon} - \frac{\tilde{\epsilon}'}{2\tilde{H}\tilde{\epsilon}}, \quad (5.14)$$

where  $\tilde{\delta} = \frac{\tilde{\phi}''}{\tilde{H}\tilde{\phi}'}$ .

Slow-roll dynamics requires  $(\tilde{\phi}')^2 \ll V(\tilde{\phi})$  and  $\tilde{\phi}'' \ll \tilde{H}\tilde{\phi}'$ . Then, in the slow-roll phase, the dynamical equations read

$$\tilde{H}^2 = \frac{V(\tilde{\phi})}{3} \left(1 - \frac{V(\tilde{\phi})}{\tilde{\rho}_c}\right) \quad \text{and} \quad 3\tilde{H}\tilde{\phi}' + \frac{\partial V(\tilde{\phi})}{\partial \tilde{\phi}} = 0, \quad (5.15)$$

and thus, in this phase, the slow-roll parameters are approximately

$$\begin{aligned} \tilde{\epsilon} &\cong \frac{1}{2} \left( \frac{1}{V(\tilde{\phi})} \frac{\partial V(\tilde{\phi})}{\partial \tilde{\phi}} \right)^2 \frac{(1 - \frac{2V(\tilde{\phi})}{\tilde{\rho}_c})}{(1 - \frac{V(\tilde{\phi})}{\tilde{\rho}_c})^2} \quad \text{and} \\ \tilde{\eta} &\cong \frac{1}{V(\tilde{\phi})} \frac{\partial^2 V(\tilde{\phi})}{\partial \tilde{\phi}^2} \frac{1}{(1 - \frac{V(\tilde{\phi})}{\tilde{\rho}_c})}. \end{aligned} \quad (5.16)$$

For the potential given by  $R^2$  gravity, i.e., for (5.2), slow-roll conditions ( $|\tilde{\epsilon}| \ll 1$  and  $|\tilde{\eta}| \ll 1$ ) are only satisfied for large positive values of the field. In that case, Eq. (5.16) becomes

$$\tilde{\epsilon} \cong \frac{4}{3} \frac{e^{-\sqrt{\frac{2}{3}}\tilde{\phi}}}{(1 - e^{-\sqrt{\frac{2}{3}}\tilde{\phi}})^4} \frac{\left(1 - \frac{(1 - e^{-\sqrt{\frac{2}{3}}\tilde{\phi}})^2}{4\alpha\tilde{\rho}_c}\right)}{\left(1 - \frac{(1 - e^{-\sqrt{\frac{2}{3}}\tilde{\phi}})^2}{8\alpha\tilde{\rho}_c}\right)^2} \quad (5.17)$$

and

$$\tilde{\eta} \cong \frac{4}{3} \frac{2e^{-\sqrt{\frac{2}{3}}\tilde{\phi}} - e^{-\sqrt{\frac{2}{3}}\tilde{\phi}}}{(1 - e^{-\sqrt{\frac{2}{3}}\tilde{\phi}})^2} \frac{1}{\left(1 - \frac{(1 - e^{-\sqrt{\frac{2}{3}}\tilde{\phi}})^2}{8\alpha\tilde{\rho}_c}\right)}. \quad (5.18)$$

To calculate inflation ends, the values of the slow-roll parameters must be of the order 1, which happens, for positive values of the field  $\tilde{\phi}$ , when it satisfies the equation

$$\frac{e^{-\sqrt{\frac{2}{3}}\tilde{\phi}}}{(1 - e^{-\sqrt{\frac{2}{3}}\tilde{\phi}})^2} \cong \frac{\sqrt{3}}{2}, \quad (5.19)$$

whose solution is

$$\tilde{\phi}_{\text{end}} = -\sqrt{\frac{3}{2}} \ln \left( \frac{1 + \sqrt{3} - \sqrt{2\sqrt{3} + 1}}{\sqrt{3}} \right) > 0. \quad (5.20)$$

And to calculate the number of  $e$ -folds that the scale factor increases during the period of inflation

$$\tilde{N} \equiv \int_{\tilde{t}_{\text{in}}}^{\tilde{t}_{\text{end}}} \tilde{H} d\tilde{t} = \int_{\tilde{\phi}_{\text{in}}}^{\tilde{\phi}_{\text{end}}} \frac{\tilde{H}}{\tilde{\phi}'} d\tilde{\phi}, \quad (5.21)$$

we have to use the slow roll equations (5.15), obtaining

$$\tilde{N} \cong \int_{\tilde{\phi}_{\text{end}}}^{\tilde{\phi}_{\text{in}}} \frac{V(\tilde{\phi})}{\frac{\partial V(\tilde{\phi})}{\partial \tilde{\phi}}} \left(1 - \frac{V(\tilde{\phi})}{\tilde{\rho}_c}\right) d\tilde{\phi}. \quad (5.22)$$

In the case of our potential (5.2), the final number of  $e$ -folds is approximately

$$\tilde{N} \cong \frac{3}{4} e^{\sqrt{\frac{2}{3}}\tilde{\phi}_{\text{in}}}. \quad (5.23)$$

On the other hand, for a given value of  $\tilde{N}$  the slow-roll parameters are

$$\tilde{\epsilon} \cong \frac{3}{4\tilde{N}^2} \frac{(1 - \frac{1}{4\alpha\tilde{\rho}_c})}{(1 - \frac{1}{8\alpha\tilde{\rho}_c})^2} \quad \text{and} \quad \tilde{\eta} \cong -\frac{1}{\tilde{N}} \frac{1}{(1 - \frac{1}{8\alpha\tilde{\rho}_c})}. \quad (5.24)$$

With these values, the spectral index of scalar perturbations, namely,  $\tilde{n}_s$ , and the ratio of tensor to scalar perturbations, namely,  $\tilde{r}$ , are approximately

$$\begin{aligned} \tilde{n}_s &\cong 1 - 6\tilde{\eta} + 2\tilde{\eta} \\ &\cong 1 - \frac{2}{\tilde{N}} \frac{1}{(1 - \frac{1}{8\alpha\tilde{\rho}_c})}, \\ \tilde{r} &\cong 16\tilde{\epsilon} \cong \frac{12}{\tilde{N}^2} \frac{(1 - \frac{1}{4\alpha\tilde{\rho}_c})}{(1 - \frac{1}{8\alpha\tilde{\rho}_c})^2}, \end{aligned} \quad (5.25)$$

which coincide when holonomy corrections are disregarded, i.e., when  $\tilde{\rho}_c \rightarrow \infty$ , with the values obtained in [31].

A very important remark is in order: the latest Planck data give for the spectral index the approximate value  $\tilde{n}_s = 0.9603 \pm 0.0073$ . If one disregards the loop corrections, to achieve the value 0.96 one has to take  $\tilde{N} = 50$   $e$ -folds, which does not give enough inflation to solve the flatness and horizon problems. However, if one takes into account holonomy corrections, for the values  $8\alpha\tilde{\rho}_c \cong 6$  and  $\tilde{N} = 60$  (the minimum number of  $e$ -folds required to solve the horizon and flatness problems), one obtains the desired result. Moreover, for these same values, one obtains  $\tilde{r} = 0.0031$ , which satisfies the current bound  $\tilde{r} < 0.11$ .

To be more precise, if one disregards loop corrections, 60  $e$ -folds are only achieved when  $0.9666 \leq \tilde{n}_s \leq \tilde{n}_{s,\max} = 0.9676$ ; in fact, for  $\tilde{n}_s = 0.9676$ , one obtains 61.72  $e$ -folds, which means that, in this model without corrections, it is impossible for the Universe to inflate more than 61.72  $e$ -folds. However, including loop quantum effects one easily achieves a greater number of  $e$ -folds; for example, for  $\tilde{n}_s = 0.9676$ , one obtains 70  $e$ -folds, choosing  $8\alpha\tilde{\rho}_c \cong 8.46$ . To sum up, we have shown that loop quantum corrections could be essential to match  $R^2$  inflation correctly with the current observational data.

### B. $R^2$ LQC in Jordan frame

To study the dynamics in the JF from the results obtained in the EF, we look for the points in the space  $(\tilde{\psi}, \tilde{\psi}')$  where the Universe could bounce in the JF; i.e., we look for the points where  $H = 0$ . Since  $H = \sqrt{f_R(R)}(\tilde{H} - \frac{1}{\sqrt{6}}\tilde{\phi}')$ , one has to solve the equation  $\tilde{H}^2 = \frac{1}{4}\frac{(\tilde{\psi}')^2}{\tilde{\psi}^2}$ , which gives, for  $\tilde{\psi} > 1$ , the following curve:

$$\frac{(\tilde{\psi}')^2}{\frac{\tilde{\rho}_c}{12(1-\sqrt{8\alpha\tilde{\rho}_c})}} + \frac{\left(\tilde{\psi} - \frac{1-\sqrt{2\alpha\tilde{\rho}_c}}{1-\sqrt{8\alpha\tilde{\rho}_c}}\right)^2}{\frac{2\alpha\tilde{\rho}_c}{(1-\sqrt{8\alpha\tilde{\rho}_c})^2}} = 1, \quad (5.26)$$

which, as in EF, produces an ellipse for  $1 - 8\alpha\tilde{\rho}_c > 0$ , a hyperbola for  $1 - 8\alpha\tilde{\rho}_c < 0$ , and a parabola for  $1 - 8\alpha\tilde{\rho}_c = 0$ . And for  $0 < \tilde{\psi} < 1$ , the curve is

$$\frac{(\tilde{\psi}')^2}{\frac{\tilde{\rho}_c}{12(1+\sqrt{8\alpha\tilde{\rho}_c})}} + \frac{\left(\tilde{\psi} - \frac{1+\sqrt{2\alpha\tilde{\rho}_c}}{1+\sqrt{8\alpha\tilde{\rho}_c}}\right)^2}{\frac{2\alpha\tilde{\rho}_c}{(1+\sqrt{8\alpha\tilde{\rho}_c})^2}} = 1, \quad (5.27)$$

which is always an ellipse. Then, when in EF the orbits in the plane  $(\tilde{\psi}, \tilde{\psi}')$  reach those curves, the Universe in the JF could bounce. To assure that it bounces, the equation  $\tilde{H} = \frac{\tilde{\psi}'}{2\tilde{\psi}}$  must be satisfied.

Now we are ready to explain the dynamics in JF from the results already obtained in EF: in EF the dynamics starts in the contracting phase and ends in the expanding one at the critical point  $(\tilde{\psi}, \tilde{\psi}') = (1, 0)$ . From the relation between both frames

$$\begin{aligned} H &= \sqrt{\tilde{\psi}} \left( \tilde{H} - \frac{\tilde{\psi}'}{2\tilde{\psi}} \right), \\ \dot{H} &= \frac{\tilde{\psi}'}{2} \left( \tilde{H} - \frac{\tilde{\psi}'}{2\tilde{\psi}} \right) + \tilde{\psi} \left( \tilde{H}' - \frac{1}{2} \left( \frac{\tilde{\psi}'}{\tilde{\psi}} \right)' \right), \end{aligned} \quad (5.28)$$

which is obtained from the first equation of (2.8) and its derivative, one deduces that, in JF, the Universe starts and ends at  $(H = 0, \dot{H} = 0)$ . Note that to calculate explicitly  $\dot{H}$  one has to use the Raychaudhuri equation  $\tilde{H}' = -\frac{3}{4} \left( 1 - \frac{2\tilde{\rho}}{\tilde{\rho}_c} \right) \left( \frac{\tilde{\psi}'}{\tilde{\psi}} \right)^2$  and the field equation (5.11). Moreover, since in EF the orbits of the system at early and late times oscillate around the point  $(\tilde{\psi}, \tilde{\psi}') = (1, 0)$ , crossing many times the curves (5.26) and (5.27), one can conclude that in JF the orbits of the system at early times oscillate around the point  $(H, \dot{H}) = (0, 0)$ , meaning that the Universe makes small bounces many times, and when it leaves this oscillatory regime, it enters the contracting phase and bounces [in EF when the orbit reach the curve (5.13)] to enter the expanding phase, where the Universe inflates and finally, at late times, it goes asymptotically to the critical point (0,0) in an oscillating way, that is, bouncing again many times.

Note that this behavior is completely different from the one obtained by disregarding holonomy corrections where, in JF, as we have already seen in Sec. IV, the Universe never bounces and is singular at early times. Moreover, it is important to remark that the holonomy corrected equation (5.9) is not singular at  $H = 0$ , and thus, the orbits can cross the axis  $H = 0$ , which allows the Universe to bounce. Of course, that does not happen in classical  $R^2$  gravity, where the corresponding dynamical equation [Eq. (4.1)] is not defined at  $H = 0$ .

Numerically, the dynamics in the plane  $(H, \dot{H})$  is easily derived via (5.28) from the one in EF, which is very simple as we have already shown. In Fig. 5, we have depicted in the plane  $(H, \dot{H})$  the orbits depicted in Fig. 4.

Note also that Eq. (5.11) defines two different dynamical systems, which means that in the plane  $(H, \dot{H})$ , two different orbits, one with  $\tilde{H} > 0$  and the other one with  $\tilde{H} < 0$ , cross at each point. Moreover, the invariance of Eq. (5.11) with respect to the replacement  $\tilde{t} \rightarrow -\tilde{t}$  and  $\tilde{H} \rightarrow -\tilde{H}$  means that the phase portrait in the plane  $(H, \dot{H})$  has a symmetry with respect to the axis  $\dot{H} = 0$ . More precisely, given a piece of an orbit with  $\tilde{H} > 0$  (respectively,  $\tilde{H} < 0$ ) in EF, there is a symmetric piece, with respect to the axis  $H = 0$ , of an orbit with  $\tilde{H} < 0$  (respectively,  $\tilde{H} > 0$ ) in EF.

A final remark is in order: in JF, the dynamics of our extension of LQC to  $R^2$  gravity in the vacuum (we have not considered any scalar field) is given by Eq. (5.9), which is a second order differential equation on  $H$ , mathematically meaning that it is a first order differential system in variables  $(H, \dot{H})$ . Working in EF, we depict the dynamics with Eq. (5.10), which is also a second order in  $\tilde{\phi}$ , meaning

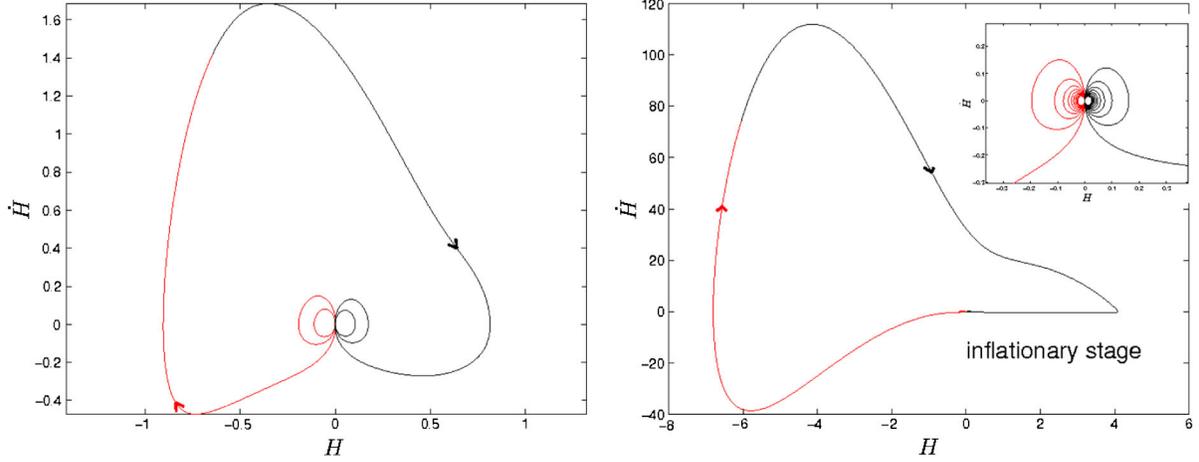


FIG. 5 (color online). In the first part of the figure, we have the phase space portrait of an orbit in JF for the case  $1 - 8\alpha\tilde{\rho}_c > 0$  ( $\alpha = 0.1$  and  $\tilde{\rho}_c = 1$ ). The Universe starts oscillating around  $(0,0)$ . Then it enters the contracting phase ( $H < 0$ ) and bounces, entering the expanding phase  $H > 0$ , coming back to  $(0,0)$  by oscillating. In the second part, we draw an orbit in JF for the case  $1 - 8\alpha\tilde{\rho}_c < 0$  ( $\alpha = 0.1$  and  $\tilde{\rho}_c = 15$ ). The dynamics is similar to that described in the other part, but there is enough inflation here in the expanding phase.

that the dynamics is given by a first order differential system in variables  $(\dot{\phi}, \phi')$ . (Note that from the holonomy corrected Friedmann equation,  $\dot{H}$  is merely a function of  $\phi$  and  $\dot{\phi}$ , in the same way as in standard LQC.)

The same happens in  $f(R)$  gravity “à la Palatini,” where the connection is a free variable (see, for instance, [32]), and in teleparallel  $f(T)$  gravity [21], when the stress tensor is depicted by a scalar field  $\phi$ , because in both cases the corresponding modified Friedmann equation relates the Hubble parameter with the energy density; i.e., the Hubble parameter is a function of  $\phi$  and  $\dot{\phi}$ , meaning that the dynamics is given by the conservation equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0, \quad (5.29)$$

which is a second order differential equation in  $\phi$ .

However, if one considers standard  $f(R)$  gravity or  $f(R)$  LQC, i.e., if the connection is fixed to be the Levi-Civita one, coupled with a scalar field  $\phi$ , the number of degrees of freedom will increase, because apart from the modified Friedmann equation in  $f(R)$  gravity, which is second order in  $H$ , one has to consider the conservation equation, which is second order in  $\phi$ , meaning that one will have a first order differential system in the plane  $(H, \dot{H}, \phi, \dot{\phi})$ .

## VI. INFLATION AND DARK ENERGY IN $R^2$ LQC

Some time ago, the unification of the early time inflation with late time dark energy (DE) in frames of modified gravity was proposed [33]. Later, several improved models containing DE have been suggested to unify inflation with the current acceleration of the Universe. In this work, the idea is to add to  $R^2$  gravity a correction  $g(R)$  given a model of the form  $f(R) = R + \alpha R^2 + g(R)$  that takes into account

the accelerated expansion of the Universe and passes the Solar System tests. Two of the best-regarded examples of these corrections are  $g(R) = \lambda(e^{-bR} - 1)$ , where  $\lambda$  and  $b$  are positive constants [34], and  $g(R) = -m^2 \frac{c_1(R/m^2)^n}{c_2(R/m^2)^n + 1}$ , where  $n > 0$  and  $c_1, c_2$  are dimensionless parameters [35].

The problem with these kinds of models is that they lead to very complicated potentials in EF, complicating considerably their extension to LQC. Moreover, it is nearly impossible to perform a detailed analytical study and it is not evident how to perform numerical computations. For this reason, in order to deal with DE, we will consider the simplest model: we will add a small cosmological constant to our model; i.e., we will consider the  $f(R) = R + \alpha R^2 - 2\Lambda$  model.

When one does not take into account holonomy corrections, the system after the change  $p = H^2$  has the same form as (4.5) but with the potential  $W(p) = \frac{p^2}{48\alpha} + \frac{\Lambda}{144\alpha p^2}$ . This potential satisfies  $V(0) = V(\infty) = \infty$ , meaning that the dynamics can be restricted to positive values of  $p$ . The potential only has a minimum at the point  $p = (\frac{\Lambda}{3})^{1/4}$  (de Sitter solution), and thus at late times all the solutions go asymptotically to this point, oscillating around it. Moreover, the inflationary solution given in (4.9) is also an attractor when the cosmological constant is taken into account. Finally, it is easy to show that the solutions are singular at early times. When a cosmological constant is considered, there are two kind of solutions: the ones that, as in  $R^2$  gravity without cosmological constant, are given by  $p(t) = \sqrt{\frac{1}{2(t-\bar{t})}}$ , and the other ones given by  $p(t) = (\frac{\Lambda}{36\alpha}(t-\bar{t}))^{1/6}$ , which vanish at  $t = \bar{t}$  but have divergent scalar curvature.

Incorporating the cosmological constant into the EF model, we have obtained the following potential

$V(\tilde{\phi}) = \frac{1}{8\alpha}(1 - e^{-\sqrt{\frac{2}{3}}\tilde{\phi}})^2 + \Lambda e^{-\sqrt{\frac{2}{3}}\tilde{\phi}}$ , which has a minimum at  $\tilde{\phi}_{\min} = \sqrt{\frac{3}{2}}\ln(1 + 8\alpha\Lambda)$ . That means that, at late times in the plane  $(\tilde{\phi}, \tilde{\phi}')$  of EF, all the solutions oscillate around  $\tilde{Q}_{\min} \equiv (\tilde{\phi}_{\min}, 0)$ . When we introduce loop quantum effects in EF, the orbits will oscillate initially around  $\tilde{Q}_{\min}$  in the contracting phase, i.e.,  $\tilde{H} < 0$ . In fact,  $\tilde{Q}_{\min}$  in the contracting phase corresponds to the anti-de Sitter solution  $\tilde{H}_- = -\sqrt{\frac{V(\tilde{\phi}_{\min})}{3}}(1 - \frac{V(\tilde{\phi}_{\min})}{\tilde{\rho}_c})$ , where  $V(\tilde{\phi}_{\min}) = \frac{\Lambda}{8\alpha\Lambda+1}$  is the minimum value of the potential. After leaving the anti-de Sitter phase, the orbits move into the contracting phase before bouncing and entering the expanding one where the Universe inflates, and finally they oscillate asymptotically to the de Sitter solution  $\tilde{H}_+ \equiv -\tilde{H}_-$ .

In JF, the dynamics is very similar: the Universe starts oscillating around the anti-de Sitter solution  $H_- = \sqrt{8\alpha\Lambda + 1}\tilde{H}_-$ , and after leaving the anti-de Sitter phase, it moves into the contracting phase  $H < 0$ , which it leaves by bouncing. Then it enters the expanding phase where it inflates and finally, at late times, it oscillates around the de Sitter solution  $H_+ = \sqrt{8\alpha\Lambda + 1}\tilde{H}_+$ . This oscillatory behavior at late times is essential, because it excites the light fields coupled with gravity that will reheat the Universe [22,24,28], yielding a hot Universe that matches the  $\Lambda$ CDM model.

## VII. CONCLUSIONS

We have introduced holonomy corrections to  $R^2$  gravity in order to avoid early time singularities that appear in this

model. We have performed a detailed analytical and numerical analysis which shows that the new model is not singular due to the quantum geometric corrections (holonomy corrections) coming from the discrete nature of space-time assumed in LQC. The new model is more involved than the original one. For this reason, in order to understand the dynamics in JF, a previous analysis must be performed in EF, where the dynamical equations greatly simplify. This allows us to perform a detailed study of its dynamics, which is essential in order to have a global idea of the system in JF. From this analysis we conclude that, when quantum geometric corrections are taken into account, the Universe evolves from the contracting phase to the expanding one through a big bounce, and when it enters the expanding phase, as in the classical model, it inflates in such a way that these holonomy corrections lead to theoretical predictions that match correctly with current observational data. Finally, we remark that it would be interesting to study different versions of  $f(R)$  gravity, for instance, with several power-law type terms, in order to understand how such theories which normally do not support the inflation behave in the LQC approach.

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