

Transient Analysis of Markov Models of Fault-Tolerant Systems with Deferred Repair using Split Regenerative Randomization

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Abstract

The (standard) randomization method is an attractive alternative for the transient analysis of continuous time Markov models. The main advantages of the method are numerical stability, well-controlled computation error and ability to specify the computation error in advance. However, the fact that the method can be computationally very expensive limits its applicability. In this paper, we develop a new method called *split regenerative randomization* which, having the same good properties as standard randomization, can be significantly more efficient. The method covers reliability-like models with a particular but quite general structure and requires the selection of a subset of states and a *regenerative* state satisfying some conditions. For a class of continuous time Markov models, model class C_2 , including typical failure/repair reliability-like models with exponential failure and repair time distributions and deferred repair, natural selections are available for both the subset of states and the regenerative state and, for those natural selections, theoretical results are available assessing the efficiency of the method in terms of “visible” model characteristics. Those results can be used to anticipate when the method can be expected to be competitive. We illustrate the application of the method using a large class C_2 model and show that for models in that class the method can indeed be significantly more efficient than previously available randomization-based methods.

1 Introduction

Homogeneous continuous time Markov chains (CTMCs) are frequently used for performance, dependability and performability modeling. The transient analysis of those models can be significantly more costly than the steady-state analysis, and very costly in absolute terms when the CTMC is large. This makes the development of efficient transient analysis techniques for CTMCs a research topic of great interest. Commonly used methods are ODE solvers and randomization. Good reviews of these methods can be found in [25, 26, 33]. The randomization method (also called uniformization) is attractive because it is numerically stable and the computation error is well-controlled and can be specified in advance. It was first proposed by Grassman [19] and has been further developed by Gross and Miller [20]. The method is offered by well-known performance, dependability and performability modeling packages [2, 13, 16, 18]. The randomization method is based on the following result [22, Theorem 4.19]. Let $X = \{X(t); t \geq 0\}$ be a CTMC with finite state space Ω , let $\lambda_{i,j}$, $i, j \in \Omega$, $i \neq j$ be the transition rate of X from state i to state j , and let $\lambda_i = \sum_{j \in \Omega - \{i\}} \lambda_{i,j}$, $i \in \Omega$ be the output rate of X from state i . Consider any $\Lambda \geq \max_{i \in \Omega} \lambda_i$ and define the homogeneous discrete time Markov chain (DTMC) $\hat{X} = \{\hat{X}_n; n = 0, 1, 2, \dots\}$ with same state space and initial probability distribution as X and transition probabilities $P[\hat{X}_{n+1} = j | \hat{X}_n = i] = P_{i,j} = \lambda_{i,j}/\Lambda$, $i \in \Omega$, $j \neq i$, $P[\hat{X}_{n+1} = i | \hat{X}_n = i] = P_{i,i} = 1 - \lambda_i/\Lambda$, $i \in \Omega$. Let $Q = \{Q(t); t \geq 0\}$ be a Poisson process with arrival rate Λ independent of \hat{X} . We have $P[Q(t) = n] = e^{-\Lambda t} (\Lambda t)^n / n!$. Then, $X = \{X(t); t \geq 0\}$ is probabilistically identical to $\{\hat{X}_{Q(t)}; t \geq 0\}$. We call this the *randomization result*.

Assume that a reward rate structure $r_i \geq 0$, $i \in \Omega$ is defined over the state space of X . The quantity r_i has the meaning of ‘‘rate’’ at which reward is earned while X is in state i . Then, a useful measure to consider is the expected transient reward rate $ETRR(t) = E[r_{X(t)}]$. We will review next a typical implementation of the randomization method for the computation of the measure $ETRR(t)$. Using the randomization result, we can express $ETRR(t)$ in terms of the transient regime of \hat{X} as

$$\begin{aligned} ETRR(t) &= \sum_{i \in \Omega} r_i P[X(t) = i] = \sum_{i \in \Omega} r_i P[\hat{X}_{Q(t)} = i] \\ &= \sum_{i \in \Omega} r_i \sum_{n=0}^{\infty} P[\hat{X}_n = i | Q(t) = n] P[Q(t) = n] = \sum_{i \in \Omega} r_i \sum_{n=0}^{\infty} P[\hat{X}_n = i] P[Q(t) = n] \\ &= \sum_{n=0}^{\infty} \sum_{i \in \Omega} r_i P[\hat{X}_n = i] P[Q(t) = n] = \sum_{n=0}^{\infty} d(n) e^{-\Lambda t} \frac{(\Lambda t)^n}{n!}, \end{aligned}$$

with $d(n) = \sum_{i \in \Omega} r_i P[\hat{X}_n = i]$. In the randomization method, an approximate value for $ETRR(t)$, $ETRR_N^a(t)$, is obtained by truncating the summatory:

$$ETRR_N^a(t) = \sum_{n=0}^N d(n) e^{-\Lambda t} \frac{(\Lambda t)^n}{n!}.$$

Using $0 \leq d(n) \leq r_{\max} = \max_{i \in \Omega} r_i$, we have

$$0 \leq ETRR(t) - ETRR_N^a(t) \leq r_{\max} \sum_{n=N+1}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!}.$$

A usual accuracy requirement is to limit the absolute truncation error to a value $\leq \varepsilon$. Then, N is chosen as

$$N = \min \left\{ m \geq 0 : r_{\max} \sum_{n=m+1}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \leq \varepsilon \right\}.$$

Let $\mathbf{q}(n)$ be the row vector $(P[\widehat{X}_n = i])_{i \in \Omega}$ and let $\mathbf{P} = (P_{i,j})_{i,j \in \Omega}$ be the transition probability matrix of \widehat{X} . Computation of $ETRR_N^a(t)$ requires the knowledge of $\mathbf{q}(n)$, $0 \leq n \leq N$. Vector $\mathbf{q}(0)$ is known, since it is the initial probability row vector of X . Vectors $\mathbf{q}(n)$, $0 < n \leq N$ can be computed from $\mathbf{q}(0)$ using

$$\mathbf{q}(n+1) = \mathbf{q}(n)\mathbf{P}. \quad (1)$$

Stable and efficient computation of the Poisson probabilities $e^{-\Lambda t}(\Lambda t)^n/n!$ avoiding overflows and intermediate underflows is a delicate issue and several alternatives have been proposed [8, 17, 23, 31]. Our implementation of all randomization-based methods will use the approach described in [23, pp. 1028–1029] (see also [1]), which has good numerical stability.

For large models, the computational cost of the randomization method is roughly due to the N vector-matrix multiplications (1). The truncation parameter N increases with Λt and, for that reason, Λ is usually taken equal to $\max_{i \in \Omega} \lambda_i$. Using the well-known result [35, Theorem 3.3.5] that $Q(t)$ has for $\Lambda t \rightarrow \infty$ an asymptotic normal distribution with mean and variance Λt , it is easy to realize that for large Λt and $\varepsilon \ll 1$ the required N will be $\approx \Lambda t$. Then, if the model is large and has to be solved for values of t for which Λt is large, the randomization method will be expensive.

Several variants of the (standard) randomization method have been proposed to improve its efficiency. Miller has used selective randomization to solve reliability models with detailed representation of error handling activities [29]. The idea behind selective randomization [27] is to randomize the model only in a subset of the state space. Reibman and Trivedi [33] have proposed an approach based on the multistep concept. The idea is to compute \mathbf{P}^M explicitly, where M is the length of the multistep, and use the recurrence $\mathbf{q}(n+M) = \mathbf{q}(n)\mathbf{P}^M$ to advance \widehat{X} faster for steps which have negligible contributions to the transient solution of X at time t . Since, for large Λt , the number of $\mathbf{q}(n)$'s with significant contributions is of the order of $\sqrt{\Lambda t}$, the multistep concept allows a significant reduction of the required number of vector-matrix multiplications when Λt is large. However, when \mathbf{P} is sparse, significant fill-in can occur when computing \mathbf{P}^M . Adaptive uniformization [30] is a method in which the randomization rate is adapted depending on the states in which the randomized DTMC can be at a given step. Numerical experiments have shown that adaptive uniformization can be faster than standard randomization for short to medium mission times. In addition, it can be used to solve models with infinite state spaces and not uniformly bounded output rates. Recently, it has been proposed to combine adaptive uniformization and standard randomization to obtain a method which outperforms both for most models [31]. Steady-state detection [25] is another proposal to speed up the randomization method. A method based on steady-state detection which gives error bounds has been developed [38]. Steady-state detection is useful for models which reach their steady-state before the largest time at which the measure must be computed. Another recently proposed randomization-based method is regenerative randomization [9, 10]. That method covers rewarded CTMC models X with finite state space $\Omega = S \cup \{f_1, f_2, \dots, f_A\}$, $A \geq 0$, satisfying some

conditions. In the method, a truncated transformed model is obtained having the same measure as the original model with some arbitrarily small error and the truncated transformed model is, then, solved by standard randomization. The method requires the selection of a regenerative state $r \in S$ and its performance depends on that selection. The truncated transformed model is constructed by characterizing with enough accuracy the behavior of the original model from $S' = S - \{r\}$ up to state r or a state f_i and from r until next hit of r or a state f_i , and its size depends on how fast the randomized DTMC \widehat{X} of X with a randomization rate slightly larger than $\max_{i \in \Omega} \lambda_i$ hits with high probability r or a state f_i starting at a state in S' . For large enough models and large enough t , regenerative randomization will be significantly more efficient than standard randomization. Furthermore, for a class of models, class C', including typical failure/repair models with exponential failure and repair time distributions and repair in every state with failed components, a natural selection for the regenerative state exists and theoretical results are available assessing approximately the performance of the method for that natural selection in terms of "visible" model characteristics. The bounding regenerative randomization method [11] allows to compute inexpensively tight bounds for a certain class of models, class C'', including typical failure/repair reliability-like models with exponential failure and repair time distributions and repair in every state with failed components. Randomization with quasistationarity detection [12] is another recent randomization-based method. The method is applicable to CTMC models with state space $S \cup \{f_1, \dots, f_A\}$, $A \geq 1$, where the states f_i , $1 \leq i \leq A$, are absorbing and all states in S are transient and reachable from each other. The method exploits the existence of a quasistationary distribution in the subset of transient states of DTMCs with a certain structure, and, for large t , can be significantly more efficient than the standard randomization method.

In this paper, we will consider CTMCs X with finite state space Ω such that (C1) $\Omega = S \cup \{f_1, f_2, \dots, f_A\}$, $|S| \geq 3$, $A \geq 1$, where f_i , $1 \leq i \leq A$, are absorbing states and, either all states in S are transient, or X has a single recurrent class $C \subset S$, and (C2) all states in Ω are reachable (from some state with nonnull initial probability), and will develop a method, called *split regenerative randomization*, for computing the measure $m(t) = \sum_{i=1}^A r_{f_i} P[X(t) = f_i]$, where $r_{f_i} \geq 0$, $1 \leq i \leq A$, and all r_{f_i} are different. The method requires the specification of a subset $E \subset S$ and a regenerative state $r \in E$. It will be assumed that E and r are selected so that, being $E' = E - \{r\}$ and $\overline{E} = S - E$, the following conditions hold: (C3) if X has a single recurrent class $C \subset S$, r belongs to that class, (C4) $|E| \geq 2$, (C5) $|\overline{E}| \geq 1$, (C6) r can only be entered from \overline{E} ($\lambda_{i,r} = 0$, $i \in E'$), (C7) r is the only entry point in E ($\lambda_{i,j} = 0$, $i \in \overline{E}$, $j \in E'$), (C8) $\lambda_{r,j} > 0$ for some $j \in E'$. Let Z , Z' and Z'' be the DTMCs to be defined next. Fulfillment of conditions C1 and C3 is sufficient for all states in S of Z to be transient (Proposition 5 in the Appendix) and all states in $S' = S - \{r\}$ of Z' to be transient (Proposition 6 in the Appendix); fulfillment of conditions C1, C3, and C7 is sufficient for all states in \overline{E} of Z'' to be transient (Proposition 7 in the Appendix); that all states in S of Z , all states in S' of Z' and all states in \overline{E} of Z'' are transient guarantee the "benign" behavior of the method (see Section 3). Condition C8 can be easily circumvented in practice by adding, in case $\lambda_{r,j} = 0$ for all $j \in E'$, a tiny transition rate $\lambda \leq 10^{-10} \varepsilon / (2r_{\max} t_{\max})$ from r to some state in E' , where ε is the allowed error, $r_{\max} = \max_{1 \leq i \leq A} r_{f_i}$, and t_{\max} is the largest time at which $m(t)$ must be computed, introducing an error $\leq 10^{-10} \varepsilon$ in $m(t)$, $t \leq t_{\max}$ (see [9]).

Also, if X has a single recurrent class $C \subset S$, by conditions C3 and C8, $|C| \geq 2$, since $|C| = 1$ would imply through condition C3 that r would be absorbing, in contradiction with condition C8. Therefore, when the method is applicable, f_1, f_2, \dots, f_A have to be the only absorbing states. This makes it easy to check whether the method is applicable to an arbitrary finite CTMC with a given subset of states E and a given regenerative state $r \in E$.

The assumed class of models and the measure $m(t)$ have important applications. Thus, S could include the operational states of a fault-tolerant system and entry into a single absorbing state f_1 could model the failure of the system. Then, with $r_{f_1} = 1$ and $P[X(0) = f_1]$ equal to the probability that initially the system is failed, the $m(t)$ measure would be the unreliability of the system at time t . “Bounding” models are also covered. Thus, S could include a proper subset of the set of operational states of a fault-tolerant system, entry into an absorbing state f_1 could model system failure, and entry into another absorbing state f_2 could model entry into an operational state not in S . Then, with $r_{f_1} = 1, r_{f_2} = 0, P[X(0) = f_1]$ equal to the probability that initially the system is failed, and $P[X(0) = f_2]$ equal to the probability that initially the system is in an operational state outside S , $m(t)$ would be a lower bound for the unreliability of the system at time t . Also, S could include a proper subset of the set of operational states of a fault-tolerant system and entry into a single absorbing state f_1 could model either entry into an operational state outside S or system failure. Then, with $r_{f_1} = 1$ and $P[X(0) = f_1]$ equal to the probability that initially the system is in an operational state outside S or failed, $m(t)$ would be an upper bound for the unreliability of the system at time t .

Later, we will consider a class of CTMC models, model class C_2 , including typical failure/repair reliability-like models of fault tolerant systems with exponential failure and repair time distributions and deferred repair, for which the split regenerative randomization method is applicable with natural selections for E and r and, for those natural selections, will obtain theoretical results assessing the performance of the method in terms of “visible” model characteristics. Repair deferment is an interesting approach in fault-tolerant systems in which actions of replacement of failed components are expensive, for instance, because the system is located at a remote site. To illustrate, Figure 1 gives a small failure/repair reliability model of a fault-tolerant system with deferred repair using the pair-and-spare technique [21] which belongs to class C_2 , in which active modules have failure rate λ_M , the spare module does not fail, the failure of an active module is “soft” with probability S_M and “hard” with probability $1 - S_M$, and whether soft or hard, the failure of an active module is covered with probability C_M . Modules in soft failure are independently recovered at rate μ_S and modules in hard failure are repaired by a single repairman at rate μ_H . Repair is deferred till two modules are failed and, when that condition is reached, repair proceeds till reaching the state 1 without failed components. In the pair-and-spare technique, three redundant copies (channels) of the same module are available and when no channel is failed, two of them are active and the third one is in the spare state. Upon failure of an active channel, the spare channel is activated and the system continues working with two active channels. Failure of a second channel leaves the system with a single active channel. Failure of that channel causes system failure. System failure can also occur due to a coverage fault. The states with deferred repair are the states 2 and 3.

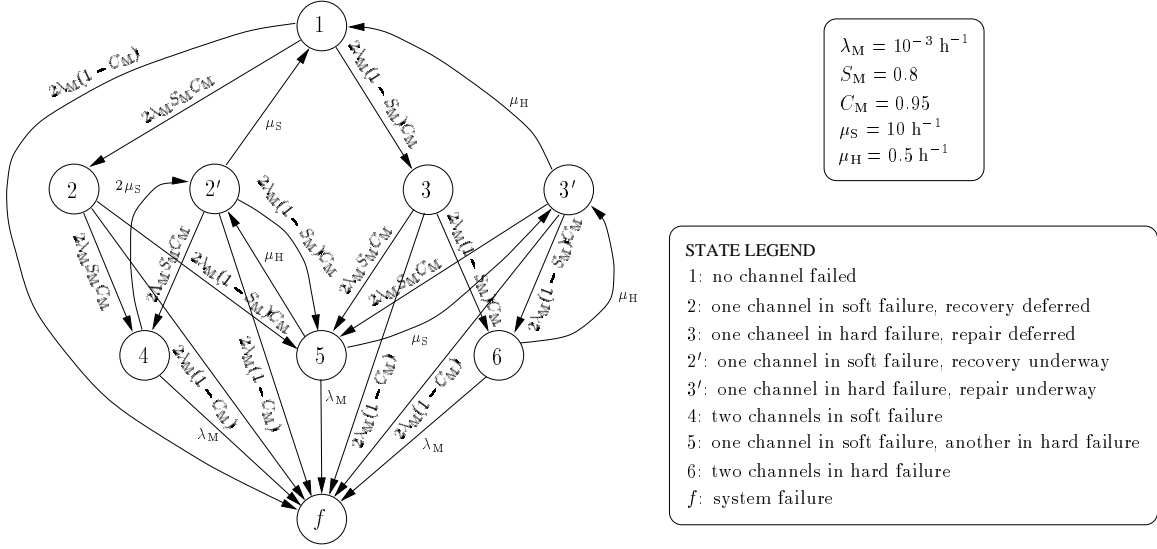


Figure 1: CTMC reliability model of a repairable fault-tolerant system with deferred repair using the pair-and-spare technique.

The poor performance of both the standard randomization and the regenerative randomization methods for typical failure/repair reliability-like models with deferred repair (illustrated by the small reliability model described in Figure 1) and large t is what has motivated the development of the split regenerative randomization method. The poor performance of standard randomization is due to the fact that, for large t , $\max_{i \in \Omega} \lambda_i t$ will be large. The poor performance of regenerative randomization is due to the fact that all transition rates from states with deferred repair (states 2 and 3 in the example of Figure 1) are much smaller than the output rates from states with repair, causing the transition probabilities $P_{i,j}$, $j \neq i$ of the DTMC \hat{X} from states i with deferred repair to be very small and, then, causing \hat{X} to go very slowly from those states to either the regenerative state (a sensible selection for the regenerative state in regenerative randomization would be the state without failed components—state 1 in the example) or an absorbing state (state f in the example). To overcome the problem, the new split regenerative randomization method uses different randomization rates for the states in E and the states in $\Omega - E$, where, for failure/repair reliability-like models with deferred repair, E should be selected so that it includes the states without repair (states 1, 2, and 3 in the example) and r should be selected as the state without failed components (state 1 in the example). This will cause the probabilities $P_{i,j}$, $j \neq i$ from states i with deferred repair to be significantly larger and the new randomized DTMC to go faster from the states with deferred repair to either the regenerative state or an absorbing state.

As in regenerative randomization, the basic idea in split regenerative randomization is to characterize, using a truncated transformed model, the behavior of X from S' up to state r or a state f_i and from r until next hit of either r or a state f_i and, then, solve the truncated transformed model by standard randomization. The truncated transformed model has, however, a more complex structure than the truncated transformed model used in regenerative randomization. That structure includes, in the more general case, besides the absorbing states f_i and an absorbing state a capturing the trun-

cated behavior, a comb of states having as a back a string of states characterizing the non-truncated behavior of X from r till the time X exits E , a comb of states having as a back a string of states characterizing the non-truncated behavior of X from E' till the time X exits E' , and a string of states characterizing the non-truncated behavior of X from \overline{E} till the time X exits \overline{E} . The teeth of the combs characterize non-truncated behavior of X in \overline{E} . The new method has the same good properties as standard randomization: numerical stability, well-controlled computation error, and ability to specify the computation error in advance, and for large t can be significantly faster than other randomization-based methods.

The rest of the paper is organized as follows. Section 2 develops the method. Section 3 proves the so-called benign behavior of the method, discusses qualitatively the efficiency of the method compared with that of standard randomization, and, for class C_2 models, for which the method is applicable with natural selections for E and r , obtains theoretical results assessing the efficiency of the method with those natural selections in terms of “visible” model characteristics. Using a large class C_2 model, Section 4 analyzes the performance of the method and compares it with that of standard randomization, regenerative randomization and adaptive uniformization (which has been shown [30] to improve significantly the performance of standard randomization for failure/repair models with deferred repair for short to medium mission times) and randomization with quasistationarity detection. Finally, Section 5 concludes the paper. The Appendix includes some proofs and some technical results.

2 Split Regenerative Randomization

The split regenerative randomization method combines a model transformation step with the solution of the transformed model by standard randomization. The model transformation step can, conceptually, be further decomposed into two steps. The first step obtains a model, V , with infinite state space such that the $m(t)$ measure can be expressed in terms of the transient regime of V . The second model transformation step obtains a model with finite state space by truncating the state space of V so that the truncated transformed model gives with some upper bounded, arbitrarily small error the $m(t)$ measure. The method uses computationally inexpensive upper bounds for the model truncation error.

In this section, we will develop and describe the split regenerative randomization method. Theoretical properties of the method will be investigated in Section 3. The rest of the section is organized as follows. Section 2.1 will derive and describe the transformed model V with infinite state space. Section 2.2 will deal with the truncation of V . Finally, Section 2.3 will give an algorithmic description of the method, including some implementation details.

2.1 Transformed model with infinite state space

Let $\alpha_i = P[X(0) = i]$, $\alpha_B = \sum_{i \in B} \alpha_i$, $B \subset \Omega$, and $\lambda_{i,B} = \sum_{j \in B} \lambda_{i,j}$, $B \subset \Omega - \{i\}$. Let $\Lambda_E \geq \max_{i \in E} \lambda_i > 0$ and $\Lambda_{\overline{E}} \geq \max_{i \in \overline{E}} \lambda_i > 0$, where the inequalities > 0 come from the fact that the only absorbing states of X are f_1, f_2, \dots, f_A , and let \widehat{X} be the DTMC obtained by randomizing X with rate Λ_E in E and rate $\Lambda_{\overline{E}}$ in $\Omega - E = \overline{E} \cup \{f_1, f_2, \dots, f_A\}$. \widehat{X} will have the same state space and initial probability distribution as X and transition probabilities $P_{i,j} = \lambda_{i,j}/\Lambda_E$, $i \in E$, $j \neq i$, $P_{i,i} = 1 - \lambda_i/\Lambda_E$, $i \in E$, $P_{i,j} = \lambda_{i,j}/\Lambda_{\overline{E}}$, $i \in \overline{E} \cup \{f_1, f_2, \dots, f_A\}$, $j \neq i$, $P_{i,i} = 1 - \lambda_i/\Lambda_{\overline{E}}$, $i \in \overline{E} \cup \{f_1, f_2, \dots, f_A\}$. Let $P_{i,B} = \sum_{j \in B} P_{i,j}$, $B \subset \Omega$. Given a DTMC $Y = \{Y_n; n = 0, 1, 2, \dots\}$, we will use the notation $Y_{m:n}c$ to indicate the predicate that is true when Y_k satisfies condition c for all k , $m \leq k \leq n$ (by convention, the predicate will be true for $m > n$) and by $\#(Y_{m:n}c)$ the number of indices k , $m \leq k \leq n$, for which Y_k satisfies condition c . We can interpret X as the composition of a state visiting process \widehat{X} with independent visit durations with exponential distributions with parameter Λ_E in the states in E and parameter $\Lambda_{\overline{E}}$ in the states in $\overline{E} \cup \{f_1, f_2, \dots, f_A\}$. The correctness of that interpretation can be checked by noting that states are absorbing in \widehat{X} if and only if they are absorbing in X and, for non-absorbing states $i \in S$, computing the kernel of the composite process seen as a semi-Markov process [14] and checking that the kernel probabilities are $q_{i,j}(t) = \lambda_{i,j}/\lambda_i(1 - e^{-\lambda_i t})$, $i \in S$, $j \neq i$, which coincide with the kernel probabilities of X , seen as a semi-Markov process. The check is done in Lemma 1 of the Appendix. We will say that \widehat{X} is the randomized DTMC of X with randomization rate Λ_E in E and $\Lambda_{\overline{E}}$ in $\overline{E} \cup \{f_1, f_2, \dots, f_A\}$ and that X is the derandomized CTMC of \widehat{X} with randomization rate Λ_E in E and $\Lambda_{\overline{E}}$ in $\overline{E} \cup \{f_1, f_2, \dots, f_A\}$.

In order to simplify the description and the implementation of the method, we will assume that the randomization rates Λ_E and $\Lambda_{\overline{E}}$ are taken slightly larger than, respectively, $\max_{i \in E} \lambda_i$ and $\max_{i \in \overline{E}} \lambda_i$ (i.e., $\Lambda_E = (1 + \theta) \max_{i \in E} \lambda_i$, $\Lambda_{\overline{E}} = (1 + \theta) \max_{i \in \overline{E}} \lambda_i$, where θ is a small quantity, say, 10^{-4}). Note that this implies $P_{i,i} > 0$, $i \in \Omega$.

To derive the transformed model V and to justify the theoretical properties of the method, we will introduce three DTMCs, Z , Z' , and Z'' .

The first DTMC, $Z = \{Z_n; n = 0, 1, 2, \dots\}$, follows \widehat{X} from r till re-entry in r . Formally, Z can be defined from a version, \widehat{X}' , of \widehat{X} with initial state r as

$$\begin{aligned} Z_0 &= r, \\ Z_n &= \begin{cases} i & \text{if } \widehat{X}'_{1:n} \neq r \wedge \widehat{X}'_n = i, \quad i \in S' \cup \{f_1, f_2, \dots, f_A\} \\ a & \text{if } \#(\widehat{X}'_{1:n} = r) > 0 \end{cases}, \quad n > 0. \end{aligned} \quad (2)$$

Z has state space $S \cup \{f_1, f_2, \dots, f_A, a\}$, where f_i , $1 \leq i \leq A$, and a are absorbing states and all states in S are transient (see Proposition 5 in the Appendix), and its (possibly) nonnull transition probabilities are

$$P[Z_{n+1} = j \mid Z_n = i] = P_{i,j}, \quad i \in S, j \in S' \cup \{f_1, f_2, \dots, f_A\},$$

$$P[Z_{n+1} = a \mid Z_n = i] = P_{i,r}, \quad i \in S,$$

$$P[Z_{n+1} = f_i | Z_n = f_i] = P[Z_{n+1} = a | Z_n = a] = 1, \quad 1 \leq i \leq A.$$

The second DTMC, $Z' = \{Z'_n; n = 0, 1, 2, \dots\}$, follows \widehat{X} from E' till its first visit to r . Formally Z' can be defined from \widehat{X} as

$$Z'_n = \begin{cases} i & \text{if } \widehat{X}_0 \in E' \wedge \widehat{X}_{1:n} \neq r \wedge \widehat{X}_n = i, \quad i \in S' \cup \{f_1, f_2, \dots, f_A\} \\ a & \text{otherwise} \end{cases}. \quad (3)$$

Z' has state space $S' \cup \{f_1, f_2, \dots, f_A, a\}$, where $f_i, 1 \leq i \leq A$, and a are absorbing states and all states in S' are transient (see Proposition 6 in the Appendix). The initial probability distribution of Z' is $P[Z'_0 = i] = \alpha_i, i \in E', P[Z'_0 = i] = 0, i \in \overline{E} \cup \{f_1, f_2, \dots, f_A\}, P[Z'_0 = a] = \alpha_{\{r\} \cup \overline{E} \cup \{f_1, f_2, \dots, f_A\}}$, and its (possibly) nonnull transition probabilities are

$$P[Z'_{n+1} = j | Z'_n = i] = P_{i,j}, \quad i \in S', j \in S' \cup \{f_1, f_2, \dots, f_A\},$$

$$P[Z'_{n+1} = a | Z'_n = i] = P_{i,r}, \quad i \in S',$$

$$P[Z'_{n+1} = f_i | Z'_n = f_i] = P[Z'_{n+1} = a | Z'_n = a] = 1, \quad 1 \leq i \leq A.$$

The third DTMC, $Z'' = \{Z''_n; n = 0, 1, 2, \dots\}$, follows \widehat{X} from \overline{E} till its first visit to state r . Formally, Z'' can be defined from \widehat{X} as (note that, by condition C7, the only entry point of \widehat{X} in E is state r and, therefore, Z'' cannot visit E)

$$Z''_n = \begin{cases} i & \text{if } \widehat{X}_0 \in \overline{E} \wedge \widehat{X}_{1:n} \neq r \wedge \widehat{X}_n = i, \quad i \in \overline{E} \cup \{f_1, f_2, \dots, f_A\} \\ a & \text{otherwise} \end{cases}. \quad (4)$$

Z'' has state space $\overline{E} \cup \{f_1, f_2, \dots, f_A, a\}$, where $f_i, 1 \leq i \leq A$, and a are absorbing states and all states in \overline{E} are transient (see Proposition 7 in the Appendix). The initial probability distribution of Z'' is $P[Z''_0 = i] = \alpha_i, i \in \overline{E}, P[Z''_0 = f_i] = 0, 1 \leq i \leq A, P[Z''_0 = a] = \alpha_{E \cup \{f_1, f_2, \dots, f_A\}}$, and its (possibly) nonnull transition probabilities are

$$P[Z''_{n+1} = j | Z''_n = i] = P_{i,j}, \quad i \in \overline{E}, j \in \overline{E} \cup \{f_1, f_2, \dots, f_A\},$$

$$P[Z''_{n+1} = a | Z''_n = i] = P_{i,r}, \quad i \in \overline{E},$$

$$P[Z''_{n+1} = f_i | Z''_n = f_i] = P[Z''_{n+1} = a | Z''_n = a] = 1, \quad 1 \leq i \leq A.$$

Let $\mathbf{P} = (P_{i,j})_{i,j \in \Omega}$ be the transition probability matrix of \widehat{X} . Denoting by $\mathbf{P}_{B',B''}, B', B'' \subset \Omega$, the submatrix of \mathbf{P} collecting the transition probabilities from states in B' to states in B'' and letting $\mathbf{P}'_{E,E}$ the matrix identical to $\mathbf{P}_{E,E}$ except that the elements of the column corresponding to state r are 0, the transition probability matrix of Z restricted to its subset of transient states, S , has, with the ordering of states E, \overline{E} , the form

$$\mathbf{P}_Z = \begin{pmatrix} \mathbf{P}'_{E,E} & \mathbf{P}_{E,\overline{E}} \\ \mathbf{0} & \mathbf{P}_{\overline{E},\overline{E}} \end{pmatrix}, \quad (5)$$

where $\mathbf{0}$ is a matrix of all zeroes of appropriate dimensions. The restriction of the transition probability matrix of Z' to its subset of transient states, S' , has with the ordering of states E', \overline{E} the form

$$\mathbf{P}_{Z'} = \begin{pmatrix} \mathbf{P}_{E',E'} & \mathbf{P}_{E',\overline{E}} \\ \mathbf{0} & \mathbf{P}_{\overline{E},\overline{E}} \end{pmatrix}, \quad (6)$$

The transition probability matrix of Z'' restricted to its subset of transient states \overline{E} , $\mathbf{P}_{Z''}$, is

$$\mathbf{P}_{Z''} = \mathbf{P}_{\overline{E}, \overline{E}}, \quad (7)$$

Let $\pi_i(n) = P[Z_n = i]$, $i \in E$, $\pi_i(n, l) = P[Z_n \in E \wedge Z_{n+1:n+l} \in \overline{E} \wedge Z_{n+l} = i]$, $i \in \overline{E}$, $\pi'_i(n) = P[Z'_n = i]$, $i \in E'$, $\pi'_i(n, l) = P[Z'_n \in E' \wedge Z'_{n+1:n+l} \in \overline{E} \wedge Z'_{n+l} = i]$, $i \in \overline{E}$, and $\pi''_i(n) = P[Z''_n = i]$, $i \in \overline{E}$, and consider the row vectors $\boldsymbol{\pi}(n) = (\pi_i(n))_{i \in E}$, $\boldsymbol{\pi}(n, l) = (\pi_i(n, l))_{i \in \overline{E}}$, $\boldsymbol{\pi}'(n) = (\pi'_i(n))_{i \in E'}$, $\boldsymbol{\pi}'(n, l) = (\pi'_i(n, l))_{i \in \overline{E}}$, and $\boldsymbol{\pi}''(n) = (\pi''_i(n))_{i \in \overline{E}}$. Assuming that, within E , state r is numbered first, those vectors, can be computed for $n \geq 0$, $l \geq 1$ using

$$\boldsymbol{\pi}(0) = (1 \ 0 \ 0 \ \cdots \ 0), \quad (8)$$

$$\boldsymbol{\pi}(n+1) = \boldsymbol{\pi}(n)\mathbf{P}'_{E, E}, \quad n \geq 0, \quad (9)$$

$$\boldsymbol{\pi}(n, 1) = \boldsymbol{\pi}(n)\mathbf{P}_{E, \overline{E}}, \quad n \geq 0, \quad (10)$$

$$\boldsymbol{\pi}(n, l+1) = \boldsymbol{\pi}(n, l)\mathbf{P}_{\overline{E}, \overline{E}}, \quad l \geq 1, \quad (11)$$

$$\boldsymbol{\pi}'(0) = (\alpha_i)_{i \in E'}, \quad (12)$$

$$\boldsymbol{\pi}'(n+1) = \boldsymbol{\pi}'(n)\mathbf{P}_{E', E'}, \quad n \geq 0, \quad (13)$$

$$\boldsymbol{\pi}'(n, 1) = \boldsymbol{\pi}'(n)\mathbf{P}_{E', \overline{E}}, \quad n \geq 0, \quad (14)$$

$$\boldsymbol{\pi}'(n, l+1) = \boldsymbol{\pi}'(n, l)\mathbf{P}_{\overline{E}, \overline{E}}, \quad l \geq 1, \quad (15)$$

$$\boldsymbol{\pi}''(0) = (\alpha_i)_{i \in \overline{E}}, \quad (16)$$

$$\boldsymbol{\pi}''(n+1) = \boldsymbol{\pi}''(n)\mathbf{P}_{\overline{E}, \overline{E}}, \quad n \geq 0. \quad (17)$$

To derive the transformed model V , we will consider first the discrete-time stochastic process $\widehat{V} = \{\widehat{V}_n; n = 0, 1, 2, \dots\}$ defined from \widehat{X} as

$$\widehat{V}_n = \begin{cases} s_k & \text{if } 0 \leq k \leq n \wedge \widehat{X}_{n-k} = r \wedge \widehat{X}_{n-k+1:n} \in E' \\ s_{k,l} & \text{if } 0 \leq k \leq n-1 \wedge 1 \leq l \leq n-k \wedge \widehat{X}_{n-k-l} = r \\ & \wedge \widehat{X}_{n-k-l+1:n-l} \in E' \wedge \widehat{X}_{n-l+1:n} \in \overline{E} \\ s'_n & \text{if } \widehat{X}_{0:n} \in E' \\ s'_{k,n-k} & \text{if } 0 \leq k \leq n-1 \wedge \widehat{X}_{0:k} \in E' \wedge \widehat{X}_{k+1:n} \in \overline{E} \\ s''_n & \text{if } \widehat{X}_{0:n} \in \overline{E} \\ f_i & \text{if } \widehat{X}_n = f_i \end{cases}. \quad (18)$$

In words, $\widehat{V}_n = s_k$ if, by step n , \widehat{X} has visited r , the last time it visited r was k steps before and has not left E since then; $\widehat{V}_n = s_{k,l}$ if \widehat{X} has visited r , the last time it visited r was $k+l$ steps before and, since then, has been first $k+1$ steps in E and, after that, l steps in \overline{E} ; $\widehat{V}_n = s'_n$ if, by step n , \widehat{X} has not left E' ; $\widehat{V}_n = s'_{k,n-k}$ if, by step n , \widehat{X} has been in E' the first $k+1$ steps and, after that, has been in \overline{E} $n-k$ steps; $\widehat{V}_n = s''_n$ if, by step n , \widehat{X} has not left \overline{E} ; and $\widehat{V}_n = f_i$ if, by step n , \widehat{X} has been absorbed into f_i . Using the fact that r is the only entry point of \widehat{X} in E , it is possible to check

that the definition of \widehat{V}_n is correct, i.e. that it assigns a single state to every possible trajectory of \widehat{X} , $\widehat{X}_0, \widehat{X}_1, \dots, \widehat{X}_n$. Note that $\widehat{V}_n = s_0$ if and only if $\widehat{X}_n = r$. Let

$$a(k) = \sum_{i \in E} \pi_i(k), \quad (19)$$

$$a(k, l) = \sum_{i \in \overline{E}} \pi_i(k, l), \quad (20)$$

$$a'(k) = \sum_{i \in E'} \pi'_i(k), \quad (21)$$

$$a'(k, l) = \sum_{i \in \overline{E}} \pi'_i(k, l), \quad (22)$$

$$a''(k) = \sum_{i \in \overline{E}} \pi''_i(k), \quad (23)$$

$$v_k^i = \frac{\sum_{j \in E} \pi_j(k) P_{j, f_i}}{a(k)}, \quad (24)$$

$$w_k = \frac{\sum_{i \in E} \pi_i(k) P_{i, E'}}{a(k)}, \quad (25)$$

$$h_k = \frac{\sum_{i \in E} \pi_i(k) P_{i, \overline{E}}}{a(k)}, \quad (26)$$

$$v_{k, l}^i = \frac{\sum_{j \in \overline{E}} \pi_j(k, l) P_{j, f_i}}{a(k, l)}, \quad (27)$$

$$q_{k, l} = \frac{\sum_{i \in \overline{E}} \pi_i(k, l) P_{i, r}}{a(k, l)}, \quad (28)$$

$$w_{k, l} = \frac{\sum_{i \in \overline{E}} \pi_i(k, l) P_{i, \overline{E}}}{a(k, l)}, \quad (29)$$

$$v_k'^i = \frac{\sum_{j \in E'} \pi'_j(k) P_{j, f_i}}{a'(k)}, \quad (30)$$

$$w'_k = \frac{\sum_{i \in E'} \pi'_i(k) P_{i, E'}}{a'(k)}, \quad (31)$$

$$h'_k = \frac{\sum_{i \in E'} \pi'_i(k) P_{i, \overline{E}}}{a'(k)}, \quad (32)$$

$$v_{k, l}'^i = \frac{\sum_{j \in \overline{E}} \pi'_j(k, l) P_{j, f_i}}{a'(k, l)}, \quad (33)$$

$$q'_{k, l} = \frac{\sum_{i \in \overline{E}} \pi'_i(k, l) P_{i, r}}{a'(k, l)}, \quad (34)$$

$$w'_{k, l} = \frac{\sum_{i \in \overline{E}} \pi'_i(k, l) P_{i, \overline{E}}}{a'(k, l)}, \quad (35)$$

$$v_k''^i = \frac{\sum_{j \in \overline{E}} \pi''_j(k) P_{j, f_i}}{a''(k)}, \quad (36)$$

$$q''_k = \frac{\sum_{i \in \overline{E}} \pi''_i(k) P_{i, r}}{a''(k)}, \quad (37)$$

$$w_k'' = \frac{\sum_{i \in \overline{E}} \pi_i''(k) P_{i, \overline{E}}}{a''(k)}. \quad (38)$$

The following proposition shows that \widehat{V} is a DTMC and gives its state space (including only reachable states), initial probability distribution and transition probabilities for the case $\alpha_{E'} > 0, \alpha_{\overline{E}} > 0$. Note that, being $P_{r, E'} > 0$ (by condition C8) and $P_{i, i} > 0, i \in E', \pi_i(k) > 0$, for all $k \geq 0$, for some $i \in E$ and $a(k) > 0$ for all $k \geq 0$. Also, for k such that $a(k, 1) > 0, \pi_i(k, 1) > 0$ for some $i \in \overline{E}$ and, since $P_{i, i} > 0, i \in \overline{E}, \pi_i(k, l) > 0$, for all $l \geq 1$, for some $i \in \overline{E}$ and $a(k, l) > 0$ for all $l \geq 1$. In addition, if $\alpha_{E'} > 0, \pi_i'(0) > 0$ for some $i \in E'$ and, since $P_{i, i} > 0, i \in E', \pi_i'(k) > 0$, for all $k \geq 0$, for some $i \in E'$ and $a'(k) > 0$ for all $k \geq 0$. Assuming $\alpha_{E'} > 0$, for k such that $a'(k, 1) > 0, \pi_i'(k, 1) > 0$ for some $i \in \overline{E}$ and, since $P_{i, i} > 0, i \in \overline{E}, \pi_i'(k, l) > 0$, for all $l \geq 1$, for some $i \in \overline{E}$ and $a'(k, l) > 0$ for all $l \geq 1$. Finally, if $\alpha_{\overline{E}} > 0, \pi_i''(0) > 0$ for some $i \in \overline{E}$ and, since $P_{i, i} > 0, i \in \overline{E}, \pi_i''(k) > 0$, for all $k \geq 0$, for some $i \in \overline{E}$ and $a''(k) > 0$ for all $k \geq 0$. All this guarantees that all transition probabilities of \widehat{V} are well-defined, i.e. that there are not divisions by 0.

Proposition 1. *Assume $\alpha_{E'} > 0$ and $\alpha_{\overline{E}} > 0$. Then, \widehat{V} is a DTMC with reachable state space $E_V \cup \overline{E}_V \cup \{f_1, f_2, \dots, f_A\}$, $E_V = \{s_k, k \geq 0\} \cup \{s'_k, k \geq 0\}$, $\overline{E}_V = \{s_{k,l} : k \geq 0 \wedge a(k, 1) > 0 \wedge l \geq 1\} \cup \{s'_{k,l} : k \geq 0 \wedge a'(k, 1) > 0 \wedge l \geq 1\} \cup \{s''_k, k \geq 0\}$, initial probability distribution $P[\widehat{V}_0 = s_0] = \alpha_r, P[\widehat{V}_0 = s'_0] = \alpha_{E'}, P[\widehat{V}_0 = s''_0] = \alpha_{\overline{E}}, P[\widehat{V}_0 = f_i] = \alpha_{f_i}, 1 \leq i \leq A$, $P[\widehat{V}_0 = i] = 0, i \notin \{s_0, s'_0, s''_0, f_1, f_2, \dots, f_A\}$, and (possibly) non-null transition probabilities*

$$\begin{aligned} P[\widehat{V}_{n+1} = s_0 \mid \widehat{V}_n = s_0] &= P_{r,r}, \\ P[\widehat{V}_{n+1} = f_i \mid \widehat{V}_n = s_k] &= v_k^i, \\ P[\widehat{V}_{n+1} = s_{k+1} \mid \widehat{V}_n = s_k] &= w_k, \\ P[\widehat{V}_{n+1} = s_{k,1} \mid \widehat{V}_n = s_k] &= h_k, \\ P[\widehat{V}_{n+1} = f_i \mid \widehat{V}_n = s_{k,l}] &= v_{k,l}^i, \\ P[\widehat{V}_{n+1} = s_0 \mid \widehat{V}_n = s_{k,l}] &= q_{k,l}, \\ P[\widehat{V}_{n+1} = s_{k,l+1} \mid \widehat{V}_n = s_{k,l}] &= w_{k,l}, \\ P[\widehat{V}_{n+1} = f_i \mid \widehat{V}_n = s'_k] &= v_k'^i, \\ P[\widehat{V}_{n+1} = s'_{k+1} \mid \widehat{V}_n = s'_k] &= w'_k, \\ P[\widehat{V}_{n+1} = s'_{k,1} \mid \widehat{V}_n = s'_k] &= h'_k, \\ P[\widehat{V}_{n+1} = f_i \mid \widehat{V}_n = s'_{k,l}] &= v_{k,l}'^i, \\ P[\widehat{V}_{n+1} = s_0 \mid \widehat{V}_n = s'_{k,l}] &= q'_{k,l}, \\ P[\widehat{V}_{n+1} = s'_{k,l+1} \mid \widehat{V}_n = s'_{k,l}] &= w'_{k,l}, \\ P[\widehat{V}_{n+1} = f_i \mid \widehat{V}_n = s''_k] &= v_k''^i, \\ P[\widehat{V}_{n+1} = s_0 \mid \widehat{V}_n = s''_k] &= q''_k, \\ P[\widehat{V}_{n+1} = s''_{k+1} \mid \widehat{V}_n = s''_k] &= w''_k, \\ P[\widehat{V}_{n+1} = f_i \mid \widehat{V}_n = f_i] &= 1, \end{aligned}$$

where $a(k), a(k, l), a'(k), a'(k, l), a''(k), v_k^i, w_k, h_k, v_{k,l}^i, q_{k,l}, w_{k,l}, v_k'^i, w'_k, h'_k, v_{k,l}'^i, q'_{k,l}, w'_{k,l}, v_k''^i, w''_k$,

q_k'' , and w_k'' are given by (19)-(38) (Figure 2 illustrates the state transition diagram of \widehat{V} for the case $A = 1$).

Proof. See the Appendix. □

The state transition diagram of \widehat{V} has, for the case $\alpha_{E'} > 0$, $\alpha_{\overline{E}} > 0$ considered in Proposition 1, two combs and a string of states. The first comb has as a back the states s_k and as teeth the strings of states $s_{k,l}$ with k fixed. The second comb has as a back the states s'_k and as teeth the strings of states $s'_{k,l}$ with k fixed. The string includes the states s_k'' . When $\alpha_{E'} = 0$, \widehat{V} loses the second comb. When $\alpha_{\overline{E}} = 0$, \widehat{V} loses the string of states s_k'' . Formally, the state space of \widehat{V} can be defined in the general case as $E_V \cup \overline{E}_V \cup \{f_1, f_2, \dots, f_A\}$, where, when $\alpha_{E'} = 0$, E_V does not include the states s'_k and \overline{E}_V does not include the states $s'_{k,l}$ and, when $\alpha_{\overline{E}} = 0$, \overline{E}_V does not include the states s_k'' .

The transformed model with infinite state space is the CTMC $V = \{V(t); t \geq 0\}$ obtained by derandomizing \widehat{V} with rate Λ_E in E_V and rate $\Lambda_{\overline{E}}$ in $\overline{E}_V \cup \{f_1, f_2, \dots, f_A\}$. The CTMC V has the same state space and initial probability distribution as \widehat{V} . Figure 3 illustrates the state transition diagram of V for the case $\alpha_{E'} > 0$, $\alpha_{\overline{E}} > 0$ and $A = 1$. The following theorem establishes that $m(t)$ can be expressed in terms of the transient regime of V .

Theorem 1. $m(t) = \sum_{i=1}^A r_{f_i} P[V(t) = f_i]$.

Proof. It is enough to prove $P[V(t) = f_i] = P[X(t) = f_i]$, $1 \leq i \leq A$. Let $F_{k,l,\Lambda_E,\Lambda_{\overline{E}}}(t)$ be the probability that the sum of k exponential random variables with parameter Λ_E and l exponential random variables with parameter $\Lambda_{\overline{E}}$, being all random variables independent, is $\leq t$. Since \widehat{X} is the randomized DTMC of X with randomization rate Λ_E in E and $\Lambda_{\overline{E}}$ in $\overline{E} \cup \{f_1, f_2, \dots, f_A\}$ and the states f_i are absorbing in X , it is clear that, grouping paths to f_i according to the number of visits that \widehat{X} makes to states in E and \overline{E} before entering f_i ,

$$P[X(t) = f_i] = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P[\#(\widehat{X}_{0:k+l-1} \in E) = k \wedge \#(\widehat{X}_{0:k+l-1} \in \overline{E}) = l \wedge \widehat{X}_{k+l} = f_i] F_{k,l,\Lambda_E,\Lambda_{\overline{E}}}(t).$$

Similarly, since \widehat{V} is the randomized DTMC of V with randomization rate Λ_E in E_V and $\Lambda_{\overline{E}}$ in $\overline{E}_V \cup \{f_1, f_2, \dots, f_A\}$ and the states f_i are absorbing in V , we have

$$P[V(t) = f_i] = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P[\#(\widehat{V}_{0:k+l-1} \in E_V) = k \wedge \#(\widehat{V}_{0:k+l-1} \in \overline{E}_V) = l \wedge \widehat{V}_{k+l} = f_i] F_{k,l,\Lambda_E,\Lambda_{\overline{E}}}(t).$$

But, since (18) $\widehat{V}_n \in E_V$ if only if $\widehat{X}_n \in E$, $\widehat{V}_n \in \overline{E}_V$ if and only if $\widehat{X}_n \in \overline{E}$ and $\widehat{V}_n = f_i$ if and only if $\widehat{X}_n = f_i$,

$$\begin{aligned} P[\#(\widehat{V}_{0:k+l-1} \in E_V) = k \wedge \#(\widehat{V}_{0:k+l-1} \in \overline{E}_V) = l \wedge \widehat{V}_{k+l} = f_i] \\ = P[\#(\widehat{X}_{0:k+l-1} \in E) = k \wedge \#(\widehat{X}_{0:k+l-1} \in \overline{E}) = l \wedge \widehat{X}_{k+l} = f_i], \end{aligned}$$

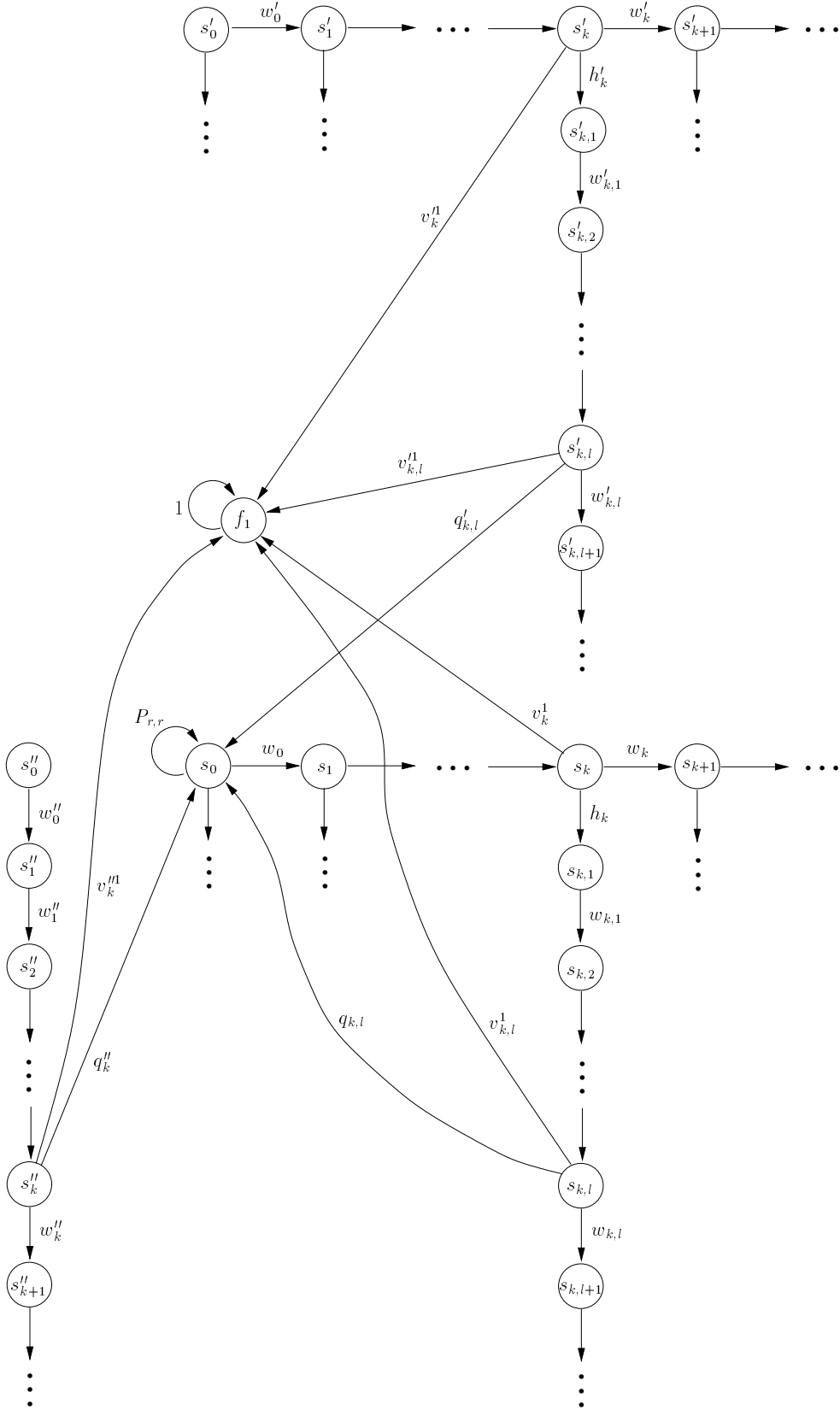


Figure 2: State transition diagram of the DTMC \widehat{V} for the case $\alpha_{E'} > 0$, $\alpha_{\bar{E}} > 0$, and $A = 1$ (there can exist transitions to f_1 from any state and transitions to s_0 from any state $s_{k,l}$, $s_{k,l}'$, and $s_{k,l}''$).

and the result follows. \square

2.2 Truncation of the transformed model

Using Theorem 1, we could compute $m(t)$ by solving V . However, deriving an exact solution for $P[V(t) = f_i]$ does not seem feasible. In this section, we will construct a truncated transformed model, V_T , from which $m(t)$ can be computed with upper bounded, arbitrarily small error. The CTMC V_T is obtained from V by introducing an absorbing state a capturing the truncated behavior and: (1) keeping the states s_k up to s_K , $K \geq 1$, and directing to a the transition rates from s_K ; (2) for each k , $0 \leq k \leq K - 1$, for which $a(k, 1) > 0$, keeping the states $s_{k,l}$ up to $l = K_k \geq 1$ and directing to a the transition rates from s_{k,K_k} ; if $\alpha_{E'} > 0$, (3) keeping the states s'_k up to s'_L , $L \geq 1$, and directing to a the transition rates from s'_L , and (4) for each k , $0 \leq k \leq L - 1$, for which $a'(k, 1) > 0$, keeping the states $s'_{k,l}$ up to $l = L_k \geq 1$ and directing to a the transitions rates from s_{k,L_k} ; and, if $\alpha_{\overline{E}} > 0$, (5) keeping the states s''_k up to s''_M , $M \geq 1$, and directing to a the transition rates from s''_M . V_T can be defined formally from V as

$$V_T(t) = \begin{cases} V(t) & \text{if, by time } t, V \text{ has not exited state } s_K, \text{ a state } s_{k,K_k}, \\ & \text{state } s'_L, \text{ a state } s'_{k,L_k}, \text{ or state } s''_M \\ a & \text{otherwise} \end{cases}. \quad (39)$$

The initial probability distribution of V_T is the same as that of V , i.e. $P[V_T(0) = s_0] = \alpha_r$, $P[V_T(0) = s'_0] = \alpha_{E'}$, $P[V_T(0) = s''_0] = \alpha_{\overline{E}}$, $P[V_T(0) = f_i] = \alpha_{f_i}$, $1 \leq i \leq A$, $P[V_T(0) = i] = 0$, $i \notin \{s_0, s'_0, s''_0, f_1, f_2, \dots, f_A\}$. Let E_V^T denote the set of states in E_V kept in V_T and let \overline{E}_V^T denote the set of states in \overline{E}_V kept in V_T . Note that the state space of V_T is $E_V^T \cup \overline{E}_V^T \cup \{f_1, f_2, \dots, f_A, a\}$. Figure 4 illustrates the state transition diagram of V_T for the case $\alpha_{E'} > 0$, $\alpha_{\overline{E}} > 0$ and $A = 1$.

Using V_T , we can obtain an approximate value for $m(t)$ as

$$m^a(t) = \sum_{i=1}^A r_{f_i} P[V_T(t) = f_i].$$

The following theorem upper bounds the model truncation error.

Theorem 2. $0 \leq m(t) - m^a(t) \leq r_{\max} P[V_T(t) = a] = m^e(t)$, where $r_{\max} = \max_{1 \leq i \leq A} r_{f_i}$.

Proof. Noting that $\sum_{i \in E_V^T \cup \overline{E}_V^T} P[V_T(t) = i] + \sum_{i=1}^A P[V_T(t) = f_i] + P[V_T(t) = a] = 1$, that, according to (39), $P[V_T(t) = i] \leq P[V(t) = i]$, $i \in E_V^T \cup \overline{E}_V^T$, that $E_V^T \subset E_V$ and $\overline{E}_V^T \subset \overline{E}_V$, and

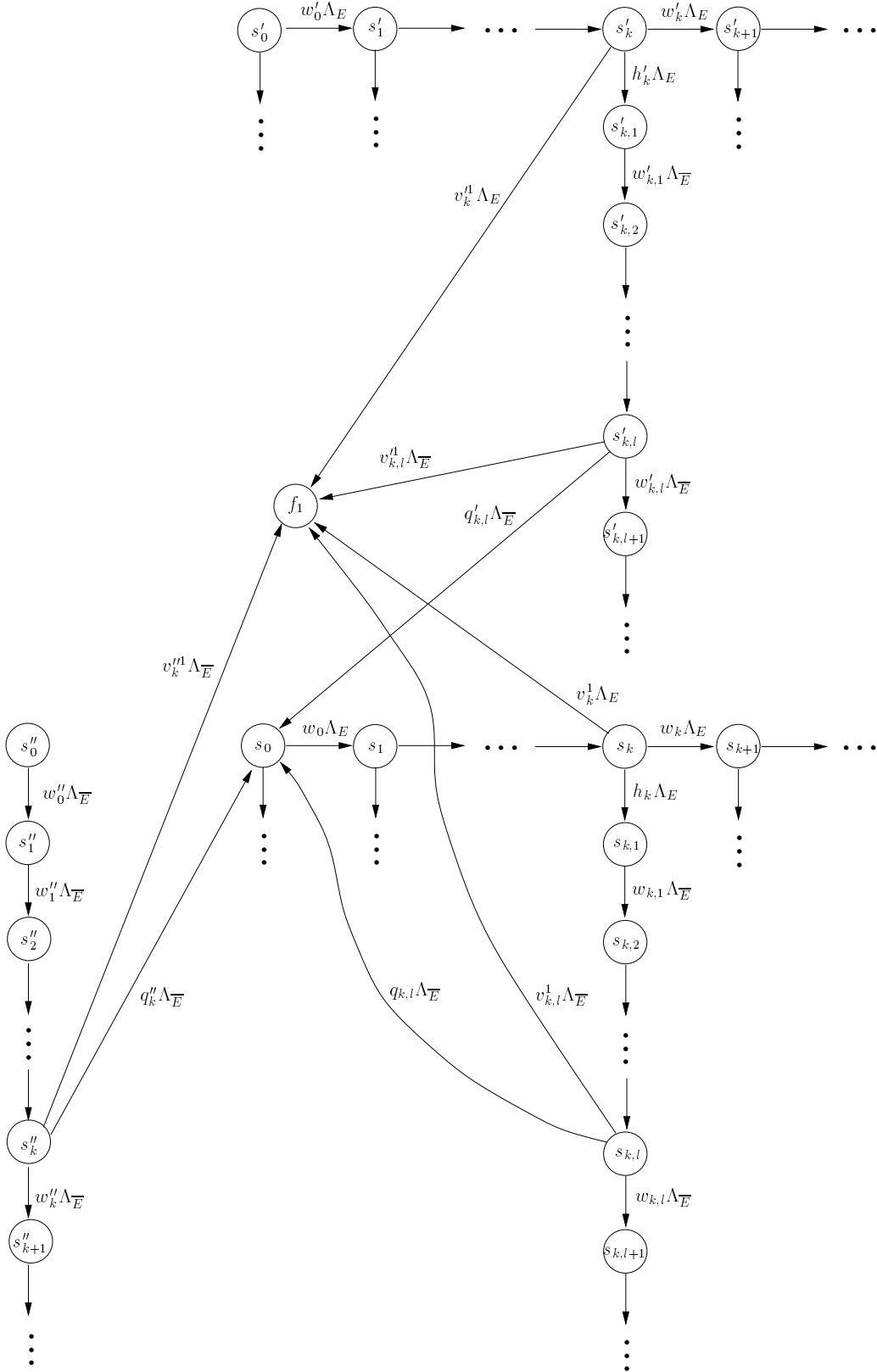


Figure 3: State transition diagram of V for the case $\alpha_{E'} > 0$, $\alpha_{\bar{E}} > 0$, and $A = 1$ (there can exist transitions to f_1 from any state and transitions to s_0 from any state $s_{k,l}$, $s'_{k,l}$, and s''_k).

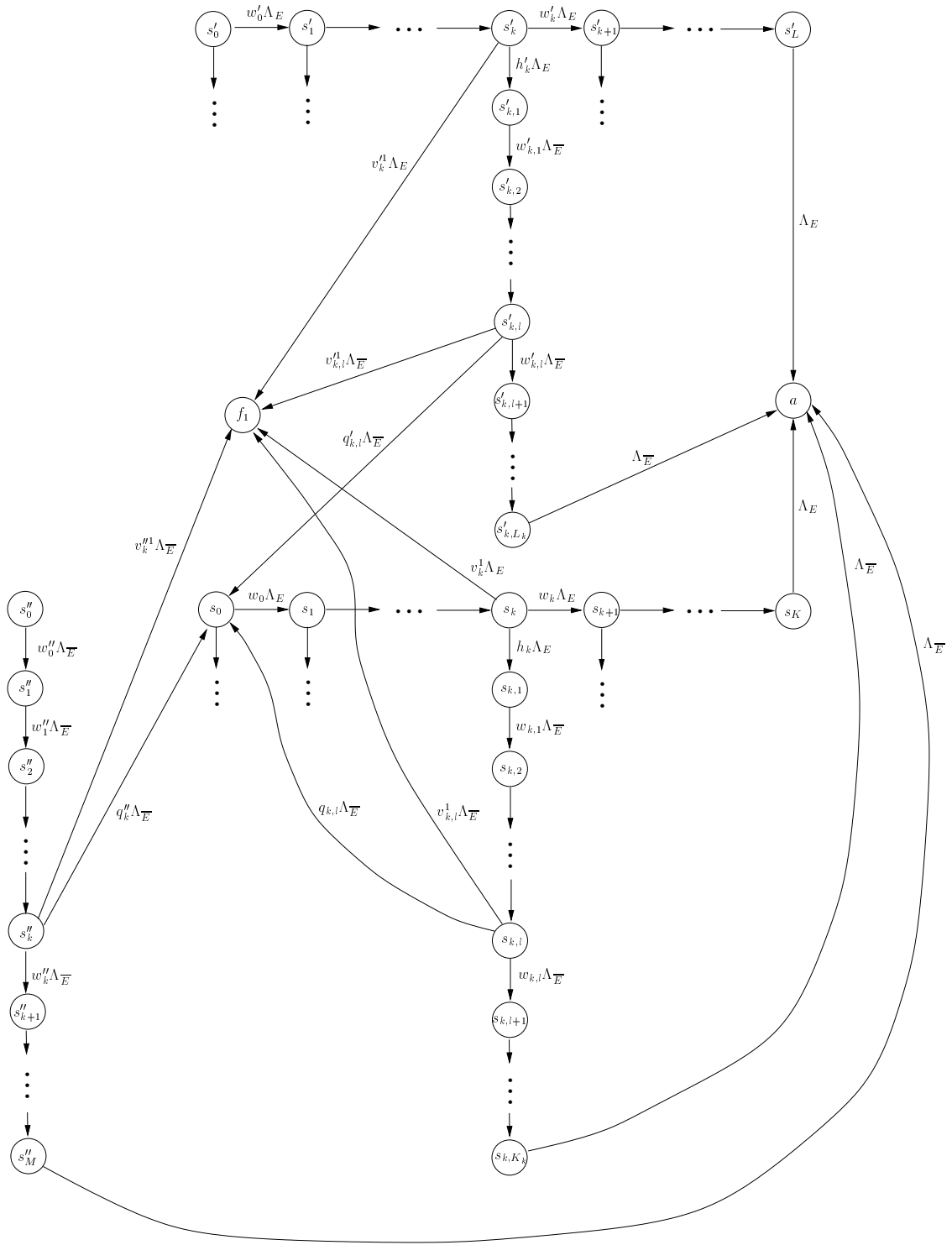


Figure 4: State transition diagram of V_T for the case $\alpha_{E'} > 0$, $\alpha_{\bar{E}} > 0$, and $A = 1$ (there can exist transitions to f_1 from any state s_k , $0 \leq k \leq K - 1$, any state $s_{k,l}$, $1 \leq l \leq K_k - 1$, any state s'_k , $0 \leq k \leq L_1$, any state $s'_{k,l}$, $1 \leq l \leq L_k - 1$, and any state s''_k , $0 \leq k \leq M - 1$; there can exist transitions to s_0 from any state $s_{k,l}$, $1 \leq l \leq K_k - 1$, any state $s'_{k,l}$, $1 \leq l \leq L_k - 1$, and any state s''_k , $0 \leq k \leq M - 1$).

that $\sum_{i \in E_V \cup \bar{E}_V} P[V(t) = i] + \sum_{i=1}^A P[V(t) = f_i] = 1$, we have

$$\begin{aligned}
P[V_T(t) = a] &= 1 - \sum_{i \in E_V^T \cup \bar{E}_V^T} P[V_T(t) = i] - \sum_{i=1}^A P[V_T(t) = f_i] \\
&\geq 1 - \sum_{i \in E_V^T \cup \bar{E}_V^T} P[V(t) = i] - \sum_{i=1}^A P[V_T(t) = f_i] \\
&\geq 1 - \sum_{i \in E_V \cup \bar{E}_V} P[V(t) = i] - \sum_{i=1}^A P[V_T(t) = f_i] \\
&= \sum_{i=1}^A P[V(t) = f_i] - \sum_{i=1}^A P[V_T(t) = f_i]
\end{aligned}$$

and, then, since according to (39), $P[V(t) = f_i] \geq P[V_T(t) = f_i]$ and $r_{f_i} \geq 0$, $1 \leq i \leq A$,

$$\begin{aligned}
0 \leq m(t) - m^a(t) &= \sum_{i=1}^A r_{f_i} \left(P[V(t) = f_i] - P[V_T(t) = f_i] \right) \\
&\leq r_{\max} \sum_{i=1}^A \left(P[V(t) = f_i] - P[V_T(t) = f_i] \right) \\
&\leq r_{\max} P[V_T(t) = a] = m^e(t). \quad \square
\end{aligned}$$

Exact computation of the model truncation error upper bound $m^e(t)$ given by Theorem 2 seems to be expensive. In this section, we will derive an upper bound for $m^e(t)$ made up of several terms which can be computed inexpensively. Split regenerative randomization will use that inexpensive upper bound to control the model truncation error. It is important that the upper bound be inexpensive, since the method may have to compute the terms of the upper bound many times. We will also discuss how the truncation parameters K , K_k , L , L_k and M are selected. To derive the upper bound, we will use the set of results collected by the following proposition.

Proposition 2. (a) For $k \geq 1$,

$$\prod_{i=0}^{k-1} w_i = a(k).$$

(b) For $k \geq 0$,

$$h_k = \frac{a(k, 1)}{a(k)}.$$

(c) For $k \geq 0$ such that $a(k, 1) > 0$,

$$\prod_{i=1}^{l-1} w_{k,i} = \frac{a(k, l)}{a(k, 1)}.$$

(d) Assuming $\alpha_{E'} > 0$, for $k \geq 1$,

$$\prod_{i=0}^{k-1} w'_i = \frac{a'(k)}{\alpha_{E'}}.$$

(e) Assuming $\alpha_{E'} > 0$, for $k \geq 0$,

$$h'_k = \frac{a'(k, 1)}{a'(k)}.$$

(f) Assuming $\alpha_{E'} > 0$, for $k \geq 0$ such that $a'(k, 1) > 0$,

$$\prod_{i=1}^{l-1} w'_{k,i} = \frac{a'(k, l)}{a'(k, 1)}.$$

(g) Assuming $\alpha_{\bar{E}} > 0$, for $k \geq 1$,

$$\prod_{i=0}^{k-1} w''_i = \frac{a''(k)}{\alpha_{\bar{E}}}.$$

Proof. See the Appendix. □

To obtain the inexpensive upper bound for $m^e(t)$, we will decompose $m^e(t)$ into several terms. Let $\gamma_K = \{k : 0 \leq k \leq K - 1 \wedge a(k, 1) > 0\}$, and, assuming $\alpha_{E'} > 0$, let $\gamma'_L = \{k : 0 \leq k \leq L - 1 \wedge a'(k, 1) > 0\}$. The set γ_K collects the indices k for which the comb of V_T having as a back the states s_l has a tooth hanging from state s_k and, assuming $\alpha_{E'} > 0$, γ'_L collects the indices k for which the comb of V_T having as a back the states s'_l has a tooth hanging from state s'_k . The first term, $m_M^{e_l}(t)$, only exists when $\alpha_{\bar{E}} > 0$ and is r_{\max} times the probability that, by time t , V_T will have entered a through s''_M . The second term, $m_L^{e''}(t)$, only exists when $\alpha_{E'} > 0$ and is r_{\max} times the probability that, by time t , V_T will have entered a through s'_L . The third terms, $m_{L,k}^{e''}(t)$, $k \in \gamma'_L$, only exist when $\alpha_{E'} > 0$ and are r_{\max} times the probabilities that, by time t , V_T will have entered a through s'_{k,L_k} . The fourth term, $m_{K,L,M}^{e'''}(t)$, is r_{\max} times the probability that, by time t , V_T will have entered a through s_K . Finally, the fifth terms, $m_{K,L,M,k}^{e'''}(t)$, $k \in \gamma_K$, are r_{\max} times the probabilities that, by time t , V_T will have entered a through s_{k,K_k} .

Assuming $\alpha_{\bar{E}} > 0$, an expression for $m_M^{e_l}(t)$ can be easily obtained by considering that there is only one path through which V_T can enter a through s''_M . That path is $p''_M = (s''_0, s''_1, \dots, s''_M, a)$. V_T is initially in s''_0 with probability $\alpha_{\bar{E}}$ and, then, the probability that V_T will follow that path is (note that all states s''_k have output rate $\Lambda_{\bar{E}}$) $\alpha_{\bar{E}} \prod_{k=0}^{M-1} w''_k$. In addition, since all states of the path have output rate $\Lambda_{\bar{E}}$, the duration of the path (time till V_T enters a) is Erlang with $M + 1$ stages and parameter $\Lambda_{\bar{E}}$. Summarizing all this, we have, using Proposition 2g,

$$m_M^{e_l}(t) = r_{\max} \alpha_{\bar{E}} \left(\prod_{k=0}^{M-1} w''_k \right) P[\text{duration of } p''_M \leq t] = r_{\max} a''(M) \sum_{k=M+1}^{\infty} e^{-\Lambda_{\bar{E}} t} \frac{(\Lambda_{\bar{E}} t)^k}{k!}. \quad (40)$$

Assuming $\alpha_{E'} > 0$, a similar derivation applies to $m_L^{e''}(t)$. Denoting by p'_L the path $(s'_0, s'_1, \dots, s'_L, a)$ and using Proposition 2d,

$$m_L^{e''}(t) = r_{\max} \alpha_{E'} \left(\prod_{k=0}^{L-1} w'_k \right) P[\text{duration of } p'_L \leq t] = r_{\max} a'(L) \sum_{k=L+1}^{\infty} e^{-\Lambda_{E'} t} \frac{(\Lambda_{E'} t)^k}{k!}. \quad (41)$$

Assume $\alpha_{E'} > 0$. Let $k \in \gamma'_L$. The only path through which V_T can enter a through s'_{k,L_k} is $p'_{k,L_k} = (s'_0, s'_1, \dots, s'_k, s'_{k,1}, \dots, s'_{k,L_k}, a)$. The probability that V_T will follow p'_{k,L_k} is $\alpha_{E'} (\prod_{i=0}^{k-1} w'_i) h'_k (\prod_{i=1}^{L_k-1} w'_{k,i})$. Then, using Proposition 2d, e, and f,

$$\begin{aligned} m_{L,k}^{e''}(t) &= r_{\max} \alpha_{E'} \left(\prod_{i=0}^{k-1} w'_i \right) h'_k \left(\prod_{i=1}^{L_k-1} w'_{k,i} \right) P[\text{duration of } p'_{k,L_k} \leq t] \\ &= r_{\max} a'(k) \frac{a'(k,1)}{a'(k)} \frac{a'(k,L_k)}{a'(k,1)} P[\text{duration of } p'_{k,L_k} \leq t] \\ &= r_{\max} a'(k, L_k) P[\text{duration of } p'_{k,L_k} \leq t]. \end{aligned}$$

An exact expression for $P[\text{duration of } p'_{k,L_k} \leq t]$ is complex due to the fact that the states in p'_{k,L_k} before a have different output rates. An easy-to-compute upper bound for $P[\text{duration of } p'_{k,L_k} \leq t]$ can, however, be obtained by neglecting the holding times in the states $s'_{k,1}, \dots, s'_{k,L_k}$. Since all states s'_0, s'_1, \dots, s'_k have output rate Λ_E , this gives the upper bound

$$P[\text{duration of } p'_{k,L_k} \leq t] < \sum_{l=k+1}^{\infty} e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!},$$

yielding

$$m_{L,k}^{e''}(t) < r_{\max} a'(k, L_k) \sum_{l=k+1}^{\infty} e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!}, \quad k \in \gamma'_L. \quad (42)$$

Deriving easy-to-compute upper bounds for $m_{K,L,M}^{e'''}(t)$ and $m_{K,L,M,k}^{e''''}(t)$, $k \in \gamma_K$, is more complex. The following propositions give such upper bounds.

Proposition 3.

$$m_{K,L,M}^{e'''}(t) \leq r_{\max} (\alpha_S - a''(M)) a(K) \sum_{k=K+1}^{\infty} (k - K) e^{-\Lambda_E t} \frac{(\Lambda_E t)^k}{k!}.$$

Proof. See the Appendix. □

Proposition 4.

$$m_{K,L,M,k}^{e''''}(t) \leq r_{\max} (\alpha_S - a''(M)) a(k, K_k) \sum_{l=k+1}^{\infty} (l - k) e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!}, \quad k \in \gamma_K.$$

Proof. See the Appendix. □

Denoting by I_c the indicator function returning the value 1 if condition c is satisfied and the value 0 otherwise, we can summarize the results obtained so far in the following theorem.

Theorem 3.

$$m^e(t) = I_{\alpha_{\bar{E}} > 0} m_M^{e'}(t) + I_{\alpha_{E'} > 0} \left(m_L^{e''}(t) + \sum_{k \in \gamma'_L} m_{L,k}^{e''}(t) \right) + m_{K,L,M}^{e'''}(t) + \sum_{k \in \gamma_K} m_{K,L,M,k}^{e'''}(t),$$

where $m_M^{e'}(t)$ is given by (40), $m_L^{e''}(t)$ is given by (41), upper bounds for $m_{L,k}^{e''}(t)$, $k \in \gamma'_L$, are given by (42), an upper bound for $m_{K,L,M}^{e'''}(t)$ is given by Proposition 3, and upper bounds for $m_{K,L,M,k}^{e'''}(t)$, $k \in \gamma_K$, are given by Proposition 4.

In the split regenerative randomization method, $m(t)$ is computed with error $\leq \varepsilon$. The truncation parameters K , L , M , K_k , $k \in \gamma_K$ and L_k , $k \in \gamma'_L$ are selected so that $m^e(t) \leq \varepsilon/2$ and an error portion $\varepsilon/2$ is still available to account for the truncation error associated with the solution of the truncated transformed model by standard randomization. A sensible approach is to distribute the allowable model truncation error, $\varepsilon/2$, equally among the applicable components, depending on whether $\alpha_{E'} > 0$ and $\alpha_{\bar{E}} > 0$, of $m_M^{e'}(t)$, the sum of $m_L^{e''}(t)$ and the upper bounds for $m_{L,k}^{e''}(t)$, $k \in \gamma'_L$, and the sum of the upper bound for $m_{K,L,M}^{e'''}(t)$ and the upper bounds for $m_{K,L,M,k}^{e'''}(t)$, $k \in \gamma_K$; to distribute equally the allocated error portion for the sum of the upper bound for $m_{K,L,M}^{e'''}(t)$ and the upper bounds for $m_{K,L,M,k}^{e'''}(t)$, $k \in \gamma_K$, between the upper bound for $m_{K,L,M}^{e'''}(t)$ and the sum of the upper bounds for $m_{K,L,M,k}^{e'''}(t)$, $k \in \gamma_K$; in case $\alpha_{E'} > 0$, to distribute equally the allocated error portion for the sum of $m_L^{e''}(t)$ and the upper bounds for $m_{L,k}^{e''}(t)$, $k \in \gamma'_L$, between $m_L^{e''}(t)$ and the sum of the upper bounds for $m_{L,k}^{e''}(t)$, $k \in \gamma'_L$; to distribute equally the allocated error portion for the sum of the upper bounds for $m_{K,L,M,k}^{e'''}(t)$, $k \in \gamma_K$, among the upper bounds, and; finally, if $\alpha_{E'} > 0$, to distribute equally the allocated error portion for the sum of the upper bounds for $m_{L,k}^{e''}(t)$, $k \in \gamma'_L$, among the upper bounds. This leads to a value of M , for the case $\alpha_{\bar{E}} > 0$,

$$M = \min \left\{ m \geq 1 : r_{\max} a''(m) \sum_{k=m+1}^{\infty} e^{-\Lambda_{\bar{E}} t} \frac{(\Lambda_{\bar{E}} t)^k}{k!} \leq \varepsilon_1 \right\}, \quad (43)$$

where $\varepsilon_1 = \varepsilon/6$ if $\alpha_{E'} > 0$ and $\varepsilon_1 = \varepsilon/4$ if $\alpha_{E'} = 0$. The truncation parameter K can, then, be chosen using

$$K = \min \left\{ m \geq 1 : r_{\max} (\alpha_S - a''(M)) a(m) \sum_{k=m+1}^{\infty} (k-m) e^{-\Lambda_E t} \frac{(\Lambda_E t)^k}{k!} \leq \varepsilon_2 \right\}, \quad (44)$$

where $\varepsilon_2 = \varepsilon/12$ if $\alpha_{E'} > 0$ and $\alpha_{\bar{E}} > 0$, $\varepsilon_2 = \varepsilon/8$ if $\alpha_{E'} > 0$ and $\alpha_{\bar{E}} = 0$ or $\alpha_{E'} = 0$ and $\alpha_{\bar{E}} > 0$, and $\varepsilon_2 = \varepsilon/4$ if $\alpha_{E'} = 0$ and $\alpha_{\bar{E}} = 0$. The truncation parameters K_k , $k \in \gamma_K$, can be chosen using

$$K_k = \min \left\{ m \geq 1 : r_{\max} (\alpha_S - a''(M)) a(k, m) \sum_{l=k+1}^{\infty} (l-k) e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!} \leq \frac{\varepsilon_2}{|\gamma_K|} \right\}. \quad (45)$$

Finally, for the case $\alpha_{E'} > 0$, the truncation parameter L can be chosen using

$$L = \min \left\{ m \geq 1 : r_{\max} a'(m) \sum_{k=m+1}^{\infty} e^{-\Lambda_E t} \frac{(\Lambda_E t)^k}{k!} \leq \varepsilon_3 \right\}, \quad (46)$$

where $\varepsilon_3 = \varepsilon/12$ if $\alpha_{\bar{E}} > 0$ and $\varepsilon_3 = \varepsilon/8$ if $\alpha_{\bar{E}} = 0$, and, also for the case $\alpha_{E'} > 0$, the truncation parameters $L_k, k \in \gamma'_L$, can be chosen using

$$L_k = \min \left\{ m \geq 1 : r_{\max} a'(k, m) \sum_{l=k+1}^{\infty} e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!} \leq \frac{\varepsilon_3}{|\gamma'_L|} \right\}. \quad (47)$$

The quantity $\sum_{k=m+1}^{\infty} e^{-\Lambda t} (\Lambda t)^k / k!$ is the probability that in the time interval $[0, t]$ there have been more than m arrivals in a Poisson process with arrival rate Λ and, therefore, $\sum_{k=m+1}^{\infty} e^{-\Lambda t} (\Lambda t)^k / k!$ is increasing with t . Regarding $\sum_{k=m+1}^{\infty} (k - m) e^{-\Lambda t} (\Lambda t)^k / k!$, we can write

$$\sum_{k=m+1}^{\infty} (k - m) e^{-\Lambda t} \frac{(\Lambda t)^k}{k!} = \sum_{k=m+1}^{\infty} \sum_{i=m+1}^k e^{-\Lambda t} \frac{(\Lambda t)^k}{k!} = \sum_{i=m+1}^{\infty} \sum_{k=i}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^k}{k!},$$

and, since each term $\sum_{k=i}^{\infty} e^{-\Lambda t} (\Lambda t)^k / k!$ is increasing with t , $\sum_{k=m+1}^{\infty} (k - m) e^{-\Lambda t} (\Lambda t)^k / k!$ is also increasing with t . Then, in case $m(t)$ has to be computed for several values of t, t_1, t_2, \dots, t_n , suitable truncation parameters for all values of t can be chosen using (43)–(47) with $t = t_{\max} = \max\{t_1, t_2, \dots, t_n\}$.

2.3 Algorithmic Description and Implementation Details

A C-like algorithmic description for the split regenerative randomization method is given in Figures 5 and 6. The algorithm has as inputs the CTMC X , the number of absorbing states A , the reward rates $r_{f_1}, r_{f_2}, \dots, r_{f_A}$, an initial probability distribution row vector $\alpha = (\alpha_i)_{i \in \Omega}$, the subset $E \subset S$, the regenerative state $r \in E$, the allowed error ε , the number of time points n at which $m(t)$ must be computed, and the time points t_1, t_2, \dots, t_n . The algorithm has as outputs the estimates for $m(t), \tilde{m}(t_1), \tilde{m}(t_2), \dots, \tilde{m}(t_n)$, at the given time points. It is assumed that conditions C1–C8 regarding the structure of X and the selections of the subset E and the regenerative state $r \in E$ are satisfied. The truncated transformed model is called V in the algorithmic description. That model is built using the procedures $add_state(V, s, p)$ and $add_transition(V, s, s', \lambda)$. The first procedure adds to V state s with initial probability p ; the second procedure adds to V a transition rate λ from state s to state s' . The algorithm makes two traversals of the backs of the combs: the first one to determine the truncation parameters K and L , and the second one to build the teeth. During the first traversal of the comb having as a back the states s_k , the algorithm determines, through condition $h_k > 0$, which, by Proposition 2b, is equivalent to $a(k, 1) > 0$ the subset γ_K and records in the variable n_k the number of teeth of the comb. Similarly, if $\alpha_{E'} > 0$, during the first traversal of the comb having as a back the states s'_k , the algorithm determines, through condition $h'_k > 0$, which, by Proposition 2e, is equivalent to $a'(k, 1) > 0$ the subset γ'_L and records in the variable n_k the number of teeth of the comb. The algorithm requires the computation, for increasing m , of summatories of the forms

$$S(m, \lambda) = \sum_{k=m+1}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!},$$

$$S'(m, \lambda) = \sum_{k=m+1}^{\infty} (k-m)e^{-\lambda} \frac{\lambda^k}{k!}.$$

Efficient and numerically stable procedures to perform those computations which can be embedded in the algorithmic description are described in [9].

We note that, once \mathbf{P} has been computed, the transition rates of the truncated transformed model are obtained without subtractions. Thus, the split regenerative randomization method has the same excellent numerical stability as the standard randomization method. In addition, in the method, the computation error is well-controlled and can be specified in advance.

The storage overhead of split regenerative randomization with respect to standard randomization includes basically the storage required to hold $P_{i,E'}, i \in E$, $P_{i,\bar{E}}, i \in S$ and the storage required to hold V_T .

3 Theoretical properties

Note that, using (9), $\pi(n) = \pi(0)\mathbf{P}_{E,E}^n$. Also (5), $\mathbf{P}'_{E,E}$ is the restriction of the transition probability matrix of Z to E and all states in E are transient in Z . Then [37, Theorem 4.3], $\pi_i(n)$, $i \in E$ decrease geometrically fast. That $\pi'_i(n)$, $i \in E'$ also decrease geometrically fast can be proved similarly using (13) $\pi'(n) = \pi'(0)\mathbf{P}'_{E',E'}$, the fact that (6) $\mathbf{P}'_{E',E'}$ is the restriction to E' of the transition probability matrix of Z' and that all states in E' are transient in Z' . Finally, that $\pi''_i(k)$, $i \in \bar{E}$ decrease geometrically fast can be proved using (17) $\pi''(n) = \pi''(0)\mathbf{P}''_{\bar{E},\bar{E}}$, the fact that (7) $\mathbf{P}''_{\bar{E},\bar{E}}$ is the restriction to \bar{E} of the transition probability matrix of Z'' and that all states in \bar{E} are transient in Z'' . Then, since the upper bound for $m_{K,L,M}^{e''}(t)$ given by Proposition 3 is upper bounded by $r_{\max} \alpha_{Sa}(K) \sum_{k=K+1}^{\infty} (k-K)e^{-\Lambda_E t} (\Lambda_E t)^k / k!$, which is analogous to the upper bound for the model truncation error term $m_{K,L}^{e''}(t)$ used in regenerative randomization [9] and (41) $m_L^{e''}(t)$ and (40) $m_M^{e'}(t)$ are analogous to the model truncation error term $m_L^{e'}(t)$ used in regenerative randomization [9], we can extend Theorem 3 of [9] and state $K = O(\log(\Lambda_E t / \varepsilon))$, $L = O(\log(1/\varepsilon))$, and $M = O(\log(1/\varepsilon))$. Regarding the lengths of the teeth of the combs, letting $K_{\bar{E}} = \sum_{k \in \gamma_K} K_k$ and $L_{\bar{E}} = \sum_{k \in \gamma'_L} L_k$, we have the following result.

Theorem 4. $K_{\bar{E}} = O((\log(\Lambda_E t / \varepsilon))^2)$ and $L_{\bar{E}} = O((\log(1/\varepsilon))^2)$.

Proof. See the Appendix. □

In summary, $K, L, M, K_{\bar{E}}$ and $L_{\bar{E}}$ are all smooth functions of both t and ε . That property is called *benign behavior* and implies that, for large enough X and large enough t , split regenerative randomization will be significantly less costly than standard randomization. This is because: (1) the cost of the first phase of split regenerative randomization (generation of the truncated transformed model) is made up of components approximately proportional to, respectively, $K, L, M, K_{\bar{E}}$ and $L_{\bar{E}}$, while the cost of standard randomization is for large t approximately proportional to $\max_{i \in \Omega} \lambda_i t$, and (2) being the maximum output rate of the truncated transformed model at most $(1 + \theta)$ times

Inputs: $X, A, r_{f_1}, r_{f_2}, \dots, r_{f_A}, \alpha, E, r, \varepsilon, n, t_1, t_2, \dots, t_n$
Outputs: $\tilde{m}(t_1), \tilde{m}(t_2), \dots, \tilde{m}(t_n)$
 $r_{\max} = \max\{r_{f_1}, r_{f_2}, \dots, r_{f_A}\};$
 $t_{\max} = \max\{t_1, t_2, \dots, t_n\};$
 $\Lambda_E = (1 + 10^{-4}) \max_{i \in E} \lambda_i; \Lambda_{\bar{E}} = (1 + 10^{-4}) \max_{i \in \bar{E}} \lambda_i;$
Obtain \mathbf{P} :
 $\alpha_{E'} = \sum_{i \in E'} \alpha_i; \alpha_{\bar{E}} = \sum_{i \in \bar{E}} \alpha_i; \alpha_S = \alpha_r + \alpha_{E'} + \alpha_{\bar{E}};$
for $(i \in E) P_{i,E'} = \sum_{j \in E', P_{i,j} > 0} P_{i,j};$ for $(i \in S) P_{i,\bar{E}} = \sum_{j \in \bar{E}, P_{i,j} > 0} P_{i,j};$
Build CTMC V including state s_0 with initial probability α_r , state a with initial probability 0 and states $f_i, 1 \leq i \leq A$, with initial probabilities α_{f_i} ;
if $(\alpha_{\bar{E}} > 0)$ {
 if $(\alpha_{E'} == 0)$ $tol = \varepsilon/4$; else $tol = \varepsilon/6$;
 add_state $(V, s''_0, \alpha_{\bar{E}}); \pi'' = (\alpha_i)_{i \in \bar{E}}; a'' = \alpha_{\bar{E}}; M = 0;$
 do{
 for $(i = 1; i \leq A; i++)$ {
 $v''^i = \sum_{j \in \bar{E}, P_{j,i} > 0} \pi''_j P_{j,i} / a''; \text{if } (v''^i > 0) \text{ add_transition}(V, s''_M, f_i, v''^i \Lambda_{\bar{E}});$
 }
 $w'' = \sum_{i \in \bar{E}} \pi''_i P_{i,\bar{E}} / a''; q'' = \sum_{i \in \bar{E}} \pi''_i P_{i,r} / a'';$
 add_state $(V, s''_{M+1}, 0); \text{add_transition}(V, s''_M, s''_{M+1}, w'' \Lambda_{\bar{E}});$
 if $(q'' > 0)$ add_transition $(V, s''_M, s_0, q'' \Lambda_{\bar{E}});$
 $n\pi'' = \pi'' \mathbf{P}'_{\bar{E}, \bar{E}}; \pi'' = n\pi''; M++; a'' = \sum_{i \in \bar{E}} \pi''_i;$
 }
 until $(r_{\max} a'' \sum_{k=M+1}^{\infty} e^{-\Lambda_{\bar{E}} t_{\max}} (\Lambda_{\bar{E}} t_{\max})^k / k! \leq tol)$;
 add_transition $(V, s''_M, a, \Lambda_{\bar{E}});$
} else $a'' = 0;$
if $(\alpha_{E'} > 0 \ \&\& \ \alpha_{\bar{E}} > 0)$ $tol = \varepsilon/12$; else if $(\alpha_{E'} > 0 \ || \ \alpha_{\bar{E}} > 0)$ $tol = \varepsilon/8$; else $tol = \varepsilon/4$;
 $\pi = (I_{i=r})_{i \in E}; a = 1; K = 0; n\mathbf{k} = 0;$
do{
 for $(i = 1; i \leq A; i++)$ {
 $v^i = \sum_{j \in E, P_{j,i} > 0} \pi_j P_{j,i} / a; \text{if } (v^i > 0) \text{ add_transition}(V, s_K, f_i, v^i \Lambda_E);$
 }
 $h_K = \sum_{i \in E} \pi_i P_{i,\bar{E}} / a; w = \sum_{i \in E} \pi_i P_{i,E'} / a;$
 add_state $(V, s_{K+1}, 0); \text{add_transition}(V, s_K, s_{K+1}, w \Lambda_E);$
 if $(h_K > 0)$ $n\mathbf{k}++$;
 $n\pi = \pi \mathbf{P}'_{E, E}; \pi = n\pi; K++; a = \sum_{i \in E} \pi_i;$
}
until $(r_{\max} (\alpha_S - a'') a \sum_{k=K+1}^{\infty} (k - K) e^{-\Lambda_E t_{\max}} (\Lambda_E t_{\max})^k / k! \leq tol)$;
add_transition $(V, s_K, a, \Lambda_E);$
 $\pi = (I_{i=r})_{i \in E};$
for $(k = 0; k \leq K - 1; k++)$ {
 if $(h_k > 0)$ {
 $\pi^{\bar{E}} = \pi \mathbf{P}'_{E, \bar{E}}; K' = 1; a = \sum_{i \in \bar{E}} \pi_i^{\bar{E}}; \text{add_state}(V, s_{k,1}, 0); \text{add_transition}(V, s_k, s_{k,1}, h_k \Lambda_E);$
 while $(r_{\max} (\alpha_S - a'') a \sum_{l=k+1}^{\infty} (l - k) e^{-\Lambda_E t_{\max}} (\Lambda_E t_{\max})^l / l! > tol / n\mathbf{k})$ {
 for $(i = 1; i \leq A; i++)$ {
 $v^i = \sum_{j \in \bar{E}, P_{j,i} > 0} \pi_j^{\bar{E}} P_{j,i} / a; \text{if } (v^i > 0) \text{ add_transition}(V, s_{k,K'}, f_i, v^i \Lambda_{\bar{E}});$
 }
 $w = \sum_{i \in \bar{E}} \pi_i^{\bar{E}} P_{i,\bar{E}} / a; q = \sum_{i \in \bar{E}} \pi_i^{\bar{E}} P_{i,r} / a;$
 add_state $(V, s_{k,K'+1}, 0); \text{add_transition}(V, s_{k,K'}, s_{k,K'+1}, w \Lambda_{\bar{E}});$
 if $(q > 0)$ add_transition $(V, s_{k,K'}, s_0, q \Lambda_{\bar{E}});$
 $n\pi^{\bar{E}} = \pi^{\bar{E}} \mathbf{P}'_{\bar{E}, \bar{E}}; \pi^{\bar{E}} = n\pi^{\bar{E}}; K'++; a = \sum_{i \in \bar{E}} \pi_i^{\bar{E}};$
 }
 add_transition $(V, s_{k,K'}, a, \Lambda_{\bar{E}});$
 }
 **if $(k < K - 1)$ $n\pi = \pi \mathbf{P}'_{E, E}; \pi = n\pi;$ }
}
}**

Figure 5: Algorithmic description of split regenerative randomization.


```

if ( $\alpha_{E'} > 0$ ) {
  if ( $\alpha_{\overline{E}} > 0$ )  $tol = \varepsilon/12$ ; else  $tol = \varepsilon/8$ ;
   $add\_state(V, s'_0, \alpha_{E'})$ ;  $\boldsymbol{\pi}' = (\alpha_i)_{i \in E'}$ ;  $a' = \sum_{i \in E'} \alpha_i$ ;  $L = 0$ ;  $n \_k = 0$ ;
  do {
    for ( $i = 1$ ;  $i \leq A$ ;  $i++$ ) {
       $v^i = \sum_{j \in E', P_{j, f_i} > 0} \pi'_j P_{j, f_i} / a'$ ; if ( $v^i > 0$ )  $add\_transition(V, s'_L, f_i, v^i \Lambda_E)$ ;
    }
     $w' = \sum_{i \in E'} \pi'_i P_{i, E'} / a'$ ;  $h'_L = \sum_{i \in E'} \pi'_i P_{i, \overline{E}} / a'$ ;
     $add\_state(V, s'_{L+1}, 0)$ ;  $add\_transition(V, s'_L, s'_{L+1}, w' \Lambda_E)$ ;
    if ( $h'_L > 0$ )  $n \_k++$ ;
     $n \boldsymbol{\pi}' = \boldsymbol{\pi}' \mathbf{P}_{E', E'}$ ;  $\boldsymbol{\pi}' = n \boldsymbol{\pi}'$ ;  $L++$ ;  $a' = \sum_{i \in E'} \pi'_i$ ;
  }
  until ( $r_{\max} a' \sum_{k=L+1}^{\infty} e^{-\Lambda_E t_{\max}} (\Lambda_E t_{\max})^k / k! \leq tol$ );
   $add\_transition(V, s'_L, a, \Lambda_E)$ ;
   $\boldsymbol{\pi}' = (\alpha_i)_{i \in E'}$ ;
  for ( $k = 0$ ;  $k \leq L - 1$ ;  $k++$ ) {
    if ( $h'_k > 0$ ) {
       $\boldsymbol{\pi}^{\overline{E}} = \boldsymbol{\pi}' \mathbf{P}_{E', \overline{E}}$ ;  $L' = 1$ ;  $a' = \sum_{i \in \overline{E}} \pi_i^{\overline{E}}$ ;  $add\_state(V, s'_{k,1}, 0)$ ;  $add\_transition(V, s'_k, s'_{k,1}, h'_k \Lambda_E)$ ;
      while ( $r_{\max} a' \sum_{l=k+1}^{\infty} e^{-\Lambda_E t_{\max}} (\Lambda_E t_{\max})^l / l! > tol / n \_k$ ) {
        for ( $i = 1$ ;  $i \leq A$ ;  $i++$ ) {
           $v^i = \sum_{j \in \overline{E}, P_{j, f_i} > 0} \pi_j^{\overline{E}} P_{j, f_i} / a'$ ; if ( $v^i > 0$ )  $add\_transition(V, s'_{k, L'}, f_i, v^i \Lambda_{\overline{E}})$ ;
        }
         $w' = \sum_{i \in \overline{E}} \pi_i^{\overline{E}} P_{i, \overline{E}} / a'$ ;  $q' = \sum_{i \in \overline{E}} \pi_i^{\overline{E}} P_{i, r} / a'$ ;
         $add\_state(V, s'_{k, L'+1}, 0)$ ;  $add\_transition(V, s'_{k, L'}, s'_{k, L'+1}, w' \Lambda_{\overline{E}})$ ;
        if ( $q' > 0$ )  $add\_transition(V, s'_{k, L'}, s_0, q' \Lambda_{\overline{E}})$ ;
         $n \boldsymbol{\pi}^{\overline{E}} = \boldsymbol{\pi}^{\overline{E}} \mathbf{P}_{\overline{E}, \overline{E}}$ ;  $\boldsymbol{\pi}^{\overline{E}} = n \boldsymbol{\pi}^{\overline{E}}$ ;  $L'++$ ;  $a' = \sum_{i \in \overline{E}} \pi_i^{\overline{E}}$ ;
      }
       $add\_transition(V, s'_{k, L'}, a, \Lambda_{\overline{E}})$ ;
    }
    if ( $k < L - 1$ ) {  $n \boldsymbol{\pi}' = \boldsymbol{\pi}' \mathbf{P}_{E', E'}$ ;  $\boldsymbol{\pi}' = n \boldsymbol{\pi}'$ ; }
  }
}
 $\Lambda = \max\{\Lambda_E, \Lambda_{\overline{E}}\}$ ;  $N = \min\{m \geq 0 : r_{\max} \sum_{k=m+1}^{\infty} e^{-\Lambda t_{\max}} (\Lambda t_{\max})^k / k! \leq \varepsilon/2\}$ ;
Let  $\widehat{V}$  be the randomized DTMC of  $V$  with randomization rate  $\Lambda$ ;
Give  $N$  steps to  $\widehat{V}$  and compute  $d(k) = \sum_{i=1}^A r_{f_i} P[\widehat{V}_k = f_i]$ ,  $k = 0, 1, \dots, N$ ;
for ( $i = 1$ ;  $i \leq n$ ;  $i++$ ) for ( $k = 0$ ,  $\tilde{m}(t_i) = 0$ ;  $k \leq N$ ;  $k++$ )  $\tilde{m}(t_i) += d(k) e^{-\Lambda t_i} (\Lambda t_i)^k / k!$ .

```

Figure 6: Algorithmic description of split regenerative randomization (continuation).

the maximum output rate of the original model, the cost of the second phase of split regenerative randomization (solution of the truncated transformed model by standard randomization) will scale with the cost of standard randomization at most as the size of the truncated transformed model scales with the size of the original model, X .

The performance of split regenerative randomization depends, of course, on the selections for E and the regenerative state r , since those selections influence the behavior of $a(k)$, $a'(k)$, $a''(k)$, $a(k, l)$, and $a'(k, l)$ and the required values for the truncation parameters $K, L, M, K_k, k \in \gamma_K$ and $L_k, k \in \gamma'_L$. Ideally, E and r should be chosen so that $a(k)$, $a'(k)$, $a''(k)$, $a(k, l)$, and $a'(k, l)$ decrease as fast as possible. For general models, automatic selection of E and r does not seem to be easy in general, and, then, the method relies on the user's intuition to make appropriate selections for E and r . We will consider next a class of models, class C_2 , for which natural selections for E and r exist, and, for models in that class and those natural selections, will obtain stronger theoretical results than the “benign” behavior.

The model class C_2 includes all CTMCs X with finite state space $\Omega = S \cup \{f_1, f_2, \dots, f_A\}$, $|S| \geq 3, A \geq 1$, where $f_i, 1 \leq i \leq A$, are absorbing states and, either all states in S are transient, or X has a single recurrent class $C \subset S$, in which all states are reachable (from some state with non-null initial probability), and in which there exists a partition $S_0 \cup S_1 \cup \dots \cup S_{N_C} \cup \bar{S}_1 \cup \bar{S}_2 \cup \dots \cup \bar{S}_{\bar{N}_C}$ for S satisfying the following properties:

$$(P1) \ S_0 = \{o\} \text{ (i.e. } |S_0| = 1).$$

$$(P2) \ \text{if } X \text{ has a single recurrent class } C \subset S, o \in C.$$

$$(P3) \ |S_0 \cup S_1 \cup \dots \cup S_{N_C}| \geq 2, \text{ and } |\bar{S}_1 \cup \bar{S}_2 \cup \dots \cup \bar{S}_{\bar{N}_C}| \geq 1.$$

$$(P4) \ \lambda_{o, S_1 \cup \dots \cup S_{N_C}} > 0.$$

$$(P5) \ \text{for each } i \in S_k, 0 < k \leq N_C, \lambda_{i, S_0 \cup \dots \cup S_{k-1}} = 0.$$

$$(P6) \ \text{for each } i \in \bar{S}_k, 1 \leq k \leq \bar{N}_C, \lambda_{i, S_1 \cup \dots \cup S_{N_C}} = 0.$$

$$(P7) \ \max_{1 \leq k \leq \bar{N}_C} \max_{i \in \bar{S}_k} \lambda_{i, \bar{S}_k - \{i\} \cup \bar{S}_{k+1} \cup \dots \cup \bar{S}_{\bar{N}_C}} \text{ is significantly smaller than} \\ \min_{1 \leq k \leq \bar{N}_C} \min_{i \in \bar{S}_k} \lambda_{i, S_0 \cup \bar{S}_1 \cup \dots \cup \bar{S}_{k-1} \cup \{f_1, f_2, \dots, f_A\}} > 0.$$

The class includes typical reliability-like failure/repair models with exponential failure and repair time distributions in which repair is deferred until some condition on the subset of failed components is fulfilled and, then, proceeds till the state in which no component is failed is reached, when failure rates are significantly smaller than repair rates. For those models, a partition for S for which properties P1—P7 are satisfied is the partition in which S_k includes the operational states without repair and the same number of failed components, with the subsets S_k sorted by increasing number of failed components, and \bar{S}_k includes the operational states with repair and the same number of failed components, with the subsets \bar{S}_k similarly sorted by increasing number of failed components. As an illustration, a partition for $S = \{1, 2, 2', 3, 3', 4, 5, 6\}$ showing that the CTMC reliability

model depicted in Figure 1 belongs to class C_2 would be $S_0 = \{1\}$, $S_1 = \{2, 3\}$, $\bar{S}_1 = \{2', 3'\}$, $\bar{S}_2 = \{4, 5, 6\}$. Similar models with exponential failure time distributions and repair times with acyclic phase-type distributions [32] (which can be used to fit distributions of non-exponential positive random variables [7]), are also covered by model class C_2 , provided that failure rates are significantly smaller than the rates of the CTMCs defining the phase-type distributions. With the selections $E = S_0 \cup S_1 \cup \dots \cup S_{N_C}$ and $r = o$ models in class C_2 satisfy the conditions C1–C8 making the split regenerative randomization method applicable. Furthermore, with those selections, because of property P7, the models move “fast” from states in \bar{E} to either state o or a state f_i , making those selections natural ones. Consider a class C_2 model and a partition $S_0 \cup S_1 \cup \dots \cup S_{N_C} \cup \bar{S}_1 \cup \bar{S}_2 \cup \dots \cup S_{\bar{N}_C}$ for S satisfying properties P1–P7. Then, we have the following result¹:

Theorem 5. *For class C_2 models and the selections $E = S_0 \cup S_1 \cup \dots \cup S_{N_C}$ and $r = o$, $a(k) \leq h_E(k)$ and $a'(k) \leq \alpha_{E'} h_{E'}(k)$, where, for $k \rightarrow \infty$, $h_E(k) \sim C \binom{k}{p-1} q_E^k$, $C > 0$, p integer ≥ 1 , $h_{E'}(k) \sim C' \binom{k}{p'-1} q_{E'}^k$, $C' > 0$, p' integer ≥ 1 , and*

$$q_E = 1 - \frac{\min_{0 < k \leq N_C} \min_{i \in S_k} \lambda_i}{(1 + \theta) \max_{0 \leq k \leq N_C} \max_{i \in S_k} \lambda_i}.$$

Proof. In the proof, we will denote by \mathbf{A}^T the transpose of a vector or matrix \mathbf{A} and by $\rho(\mathbf{A})$ the spectral radius of a square matrix \mathbf{A} . From (13), for $k \geq 0$, $\boldsymbol{\pi}'(k)^T = (\mathbf{P}_{E', E'}^T)^k \boldsymbol{\pi}'(0)^T$. Using, then, (21), $\pi'_i(k) \geq 0$, $i \in E'$ and (12) $\sum_{i \in E'} \pi'_i(0) = \alpha_{E'}$,

$$\begin{aligned} a'(k) &= \sum_{i \in E'} \pi'_i(k) = \|\boldsymbol{\pi}'(k)^T\|_1 = \|(\mathbf{P}_{E', E'}^T)^k \boldsymbol{\pi}'(0)^T\|_1 \leq e q \|\mathbf{P}_{E', E'}^T\|_1 \|\boldsymbol{\pi}'(0)^T\|_1 \\ &= \|(\mathbf{P}_{E', E'}^T)^k\|_1 \sum_{i \in E'} \pi'_i(0) = \alpha_{E'} \|\mathbf{P}_{E', E'}^T\|_1. \end{aligned}$$

With the ordering of states S_1, \dots, S_{N_C} , the elements in the upper triangular portion of $\mathbf{P}_{E', E'}^T$ are 0 and the diagonal elements of $\mathbf{P}_{E', E'}^T$ have values

$$1 - \frac{\lambda_i}{(1 + \theta) \max_{0 \leq k \leq N_C} \max_{i \in S_k} \lambda_i}, \quad i \in S_k, 0 < k \leq N_C,$$

and $\rho(\mathbf{P}_{E', E'}^T) = q_E$. Then [41, Theorem 3.1]², there exist $C' > 0$ and p' integer ≥ 1 such that, for $k \rightarrow \infty$, $\|(\mathbf{P}_{E', E'}^T)^k\|_1 \sim C' \binom{k}{p'-1} q_E^k$.

From (9), for $k \geq 0$, $\boldsymbol{\pi}(k)^T = (\mathbf{P}_{E, E}^T)^k \boldsymbol{\pi}(0)^T$. Using, then, (19), $\pi_i(k) \geq 0$ and (8) $\sum_{i \in E} \pi_i(0) = 1$,

$$\begin{aligned} a(k) &= \sum_{i \in E} \pi_i(k) = \|\boldsymbol{\pi}(k)^T\|_1 = \|(\mathbf{P}_{E, E}^T)^k \boldsymbol{\pi}(0)^T\|_1 \\ &\leq \|(\mathbf{P}_{E, E}^T)^k\|_1 \|\boldsymbol{\pi}(0)^T\|_1 = \|(\mathbf{P}_{E, E}^T)^k\|_1 \sum_{i \in E} \pi_i(0) = \|(\mathbf{P}_{E, E}^T)^k\|_1. \end{aligned}$$

¹ $x(k) \sim y(k)$ for $k \rightarrow \infty$ denotes $\lim_{k \rightarrow \infty} x(k)/y(k) = 1$.

²Strictly speaking, Theorem 3.1 of [41] asserts the result for the Euclidean norm, but the results easily extends to the considered 1-norm.

With the ordering of states S_0, S_1, \dots, S_{N_C} , $\mathbf{P}'_{E,E}$ has the form:

$$\mathbf{P}'_{E,E} = \begin{pmatrix} 0 & \mathbf{0} \\ \boldsymbol{\beta} & \mathbf{P}'_{E',E'} \end{pmatrix},$$

where $\mathbf{0}$ denotes a row vector with all its components equal to 0 and $\boldsymbol{\beta}$ denotes a column vector. According to Lemma 2 in the Appendix, $\rho(\mathbf{P}'_{E,E}) = \rho(\mathbf{P}'_{E',E'})$ and, therefore, $\rho(\mathbf{P}'_{E,E}) = q_E$. Then, applying again [41, Theorem 3.1], there exist $C > 0$ and p integer ≥ 1 such that, for $k \rightarrow \infty$, $\|(\mathbf{P}'_{E,E})^k\|_1 \sim C \binom{k}{p-1} q_E^k$. \square

Theorem 5 tells that, for class C_2 models with the selections $E = S_0 \cup S_1 \cup \dots \cup S_{N_C}$ and $r = o$, both $a(k)$ and $a'(k)$ are upper bounded by something which decays asymptotically by a factor with value q_E . For small enough ε , the K and L required by split regenerative randomization will be mainly determined by how fast $a(k)$ and $a'(k)$ decay and, then, Theorem 5 suggests that the truncation parameters K and L for class C_2 models with the natural selections for E and r can be roughly upper bounded using q_E^m instead of, respectively, $r_{\max}(\alpha_S - a''(M))a(m) \sum_{k=m+1}^{\infty} (k-m)e^{-\Lambda_E t} (\Lambda_E t)^k / k!$ in (44) and $r_{\max} a'(m) \sum_{k=m+1}^{\infty} e^{-\Lambda_E t} (\Lambda_E t)^k / k!$ in (46). Let $R_E = \max_{0 \leq k \leq N_C} \max_{i \in S_k} \lambda_i / \min_{0 < k \leq N_C} \min_{i \in S_k} \lambda_i \geq 1$. We have $q_E = 1 - 1/((1 + \theta)R_E) \approx 1 - 1/R_E$. For $R_E \geq 1$ close to 1, q_E will be small and the required K and L will be small and, as R_E gets apart from 1, q_E will get closer to 1 and the required K and L will increase.

Let

$$\delta = \frac{\max_{1 \leq k \leq \overline{N}_C} \max_{i \in \overline{S}_k} \lambda_{i, \overline{S}_k - \{i\} \cup \overline{S}_{k+1} \cup \dots \cup \overline{S}_{\overline{N}_C}}}{\min_{1 \leq k \leq \overline{N}_C} \min_{i \in \overline{S}_k} \lambda_{i, S_0 \cup \overline{S}_1 \cup \dots \cup \overline{S}_{k-1} \cup \{f_1, \dots, f_A\}}}.$$

The parameter δ can be seen as a ‘‘rarity’’ parameter measuring how strongly property P7 is satisfied. In terms of the rarity parameter δ , we can model the transition rates from states $i \in \overline{S}_k$ to states $j \in \overline{S}_k - \{i\} \cup \overline{S}_{k+1} \cup \dots \cup \overline{S}_{\overline{N}_C}$, $1 \leq k \leq \overline{N}_C$ as $\lambda_{i,j} = \Lambda_{i,j} \delta$, where $\Lambda_{i,j}$ are constants ≤ 1 , and the remaining transition rates as constants and study the behavior of split regenerative randomization for class C_2 models with the selections $E = S_0 \cup S_1 \cup \dots \cup S_{N_C}$ and $r = o$ as $\delta \rightarrow 0$. Let $P_{i,j}(\delta)$ denote the transition probabilities of \widehat{X} as a function of the rarity parameter δ and let $\mathbf{P}_{\overline{E},\overline{E}}(\delta)$ denote the transition probability matrix $\mathbf{P}_{\overline{E},\overline{E}}$ as a function of δ . Note that, for $i \in \overline{S}_k$, $j \in \overline{S}_k - \{i\} \cup \overline{S}_{k+1} \cup \dots \cup \overline{S}_{\overline{N}_C}$, $1 \leq k \leq \overline{N}_C$, $\lim_{\delta \rightarrow 0} P_{i,j}(\delta) = 0$ and that, using property P6,

$$\lim_{\delta \rightarrow 0} P_{i,i}(\delta) = 1 - \frac{\lambda_{i, S_0 \cup \overline{S}_1 \cup \dots \cup \overline{S}_{k-1} \cup \{f_1, \dots, f_A\}}}{(1 + \theta) \max_{1 \leq k \leq \overline{N}_C} \max_{i \in \overline{S}_k} \lambda_{i, S_0 \cup \overline{S}_1 \cup \dots \cup \overline{S}_{k-1} \cup \{f_1, \dots, f_A\}}}, i \in \overline{S}_k, 1 \leq k \leq \overline{N}_C.$$

Denoting by \mathbf{A}^T the transpose of a vector or matrix \mathbf{A} , we have the following result:

Theorem 6. *For class C_2 models and the selections $E = S_0 \cup S_1 \cup \dots \cup S_{N_C}$ and $r = o$, $a(k, l) \leq h_{\overline{E}}(l-1)$, $a'(k, l) \leq \alpha_{\overline{E}} h_{\overline{E}}(l-1)$ and $a''(l) \leq \alpha_{\overline{E}} h_{\overline{E}}(l)$, where, for $l \rightarrow \infty$, $h_{\overline{E}}(l) \sim C(\delta) \binom{l}{p(\delta)-1} \rho(\mathbf{P}_{\overline{E},\overline{E}}(\delta))^l$, $C(\delta) > 0$, $p(\delta)$ integer ≥ 1 , with $\lim_{\delta \rightarrow 0} \rho(\mathbf{P}_{\overline{E},\overline{E}}(\delta))^l = q_{\overline{E}}$,*

$$q_{\overline{E}} = 1 - \frac{\min_{1 \leq k \leq \overline{N}_C} \min_{i \in \overline{S}_k} \lambda_{i, S_0 \cup \overline{S}_1 \cup \dots \cup \overline{S}_{k-1} \cup \{f_1, \dots, f_A\}}}{(1 + \theta) \max_{1 \leq k \leq \overline{N}_C} \max_{i \in \overline{S}_k} \lambda_{i, S_0 \cup \overline{S}_1 \cup \dots \cup \overline{S}_{k-1} \cup \{f_1, \dots, f_A\}}}.$$

Proof. Note that, using (10) and (11), for $k \geq 0$ and $l \geq 1$, $\boldsymbol{\pi}(k, l)^T = (\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T)^{l-1} \mathbf{P}_{\overline{E}, \overline{E}}^T \boldsymbol{\pi}(k)^T$. Then, using (20), $\pi_i(k, l) \geq 0$, $i \in \overline{E}$, $\|\mathbf{P}_{\overline{E}, \overline{E}}^T\|_1 \leq 1$, and $\|\boldsymbol{\pi}(k)^T\|_1 \leq 1$, for $k \geq 0$ and $l \geq 1$,

$$\begin{aligned} a(k, l) &= \sum_{i \in \overline{E}} \pi_i(k, l) = \|\boldsymbol{\pi}(k, l)^T\|_1 = \|(\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T)^{l-1} \mathbf{P}_{\overline{E}, \overline{E}}^T \boldsymbol{\pi}(k)^T\|_1 \\ &\leq \|(\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T)^{l-1}\|_1 \|\mathbf{P}_{\overline{E}, \overline{E}}^T\|_1 \|\boldsymbol{\pi}(k)^T\|_1 \leq \|(\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T)^{l-1}\|_1. \end{aligned}$$

Similarly, using (14) and (15), for $k \geq 0$ and $l \geq 1$, $\boldsymbol{\pi}'(k, l)^T = (\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T)^{l-1} \mathbf{P}_{\overline{E}, \overline{E}}^T \boldsymbol{\pi}'(k)^T$, and using (22), $\pi'_i(k, l) \geq 0$, $i \in \overline{E}$, $\|\mathbf{P}_{\overline{E}, \overline{E}}^T\|_1 \leq 1$ and, from (12) and (13), taking into account $\|\mathbf{P}_{\overline{E}', \overline{E}'}^T\|_1 \leq 1$, $\|\boldsymbol{\pi}'(k)^T\|_1 = \|(\mathbf{P}_{\overline{E}', \overline{E}'}^T)^k \boldsymbol{\pi}'(0)^T\|_1 \leq \|\mathbf{P}_{\overline{E}', \overline{E}'}^T\|_1^k \|\boldsymbol{\pi}'(0)^T\|_1 \leq \|\boldsymbol{\pi}'(0)^T\|_1 = \alpha_{\overline{E}'}$,

$$\begin{aligned} a'(k, l) &= \sum_{i \in \overline{E}} \pi'_i(k, l) = \|\boldsymbol{\pi}'(k, l)^T\|_1 = \|(\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T)^{l-1} \mathbf{P}_{\overline{E}, \overline{E}}^T \boldsymbol{\pi}'(k)^T\|_1 \\ &\leq \|(\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T)^{l-1}\|_1 \|\mathbf{P}_{\overline{E}, \overline{E}}^T\|_1 \|\boldsymbol{\pi}'(k)^T\|_1 \leq \alpha_{\overline{E}'} \|(\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T)^{l-1}\|_1. \end{aligned}$$

Finally, noting that (17) $\boldsymbol{\pi}''(l)^T = (\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T)^l \boldsymbol{\pi}''(0)^T$ and using (23), $\pi''_i(l) \geq 0$, $i \in \overline{E}$ and (16) $\|\boldsymbol{\pi}''(0)^T\|_1 = \alpha_{\overline{E}}$,

$$\begin{aligned} a''(l) &= \sum_{i \in \overline{E}} \pi''_i(l) = \|\boldsymbol{\pi}''(l)^T\|_1 = \|(\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T)^l \boldsymbol{\pi}''(0)^T\|_1 \leq \|(\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T)^l\|_1 \|\boldsymbol{\pi}''(0)^T\|_1 \\ &= \alpha_{\overline{E}} \|(\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T)^l\|_1. \end{aligned}$$

Let $\rho(\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T)$ denote the spectral radius of $\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T$. We have [41, Theorem 3.1] that there exist $C(\delta) > 0$, $p(\delta)$ integer ≥ 1 such that, for $l \rightarrow \infty$, $\|(\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T)^l\|_1 \sim C(\delta) \binom{l}{p(\delta)-1} \rho(\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T)^l$. Also, since the eigenvalues of a matrix are continuous functions of the elements of the matrix [36, Theorem 3.13], $\lim_{\delta \rightarrow 0} \rho(\mathbf{P}_{\overline{E}, \overline{E}}(\delta)^T) = \rho(\mathbf{P}_{\overline{E}, \overline{E}}(0)^T)$. But, with the ordering of states $\overline{S}_1, \dots, \overline{S}_{\overline{N}_C}$, the elements in the lower triangular portion of $\mathbf{P}_{\overline{E}, \overline{E}}(0)^T$ are 0 and the diagonal elements have values

$$1 - \frac{\lambda_{i, S_0 \cup \overline{S}_1 \cup \dots \cup \overline{S}_{k-1} \cup \{f_1, \dots, f_A\}}}{(1 + \theta) \max_{1 \leq k \leq \overline{N}_C} \max_{i \in \overline{S}_k} \lambda_{i, S_0 \cup \overline{S}_1 \cup \dots \cup \overline{S}_{k-1} \cup \{f_1, \dots, f_A\}}}, \quad i \in \overline{S}_k, 1 \leq k \leq \overline{N}_C$$

and, therefore, $\rho(\mathbf{P}_{\overline{E}, \overline{E}}(0)^T) = q_{\overline{E}}$. \square

Theorem 6 tells that, for class C_2 models with the selections $E = S_0 \cup S_1 \cup \dots \cup S_{N_C}$ and $r = o$, $a''(l)$, $a(k, l)$ and $a'(k, l)$ are upper bounded by something which decays asymptotically by a factor with value $q_{\overline{E}}$ for $\delta \rightarrow 0$. For small enough ε , the M , K_k , $k \in \gamma_K$, and L_k , $k \in \gamma'_L$, required by split regenerative randomization will be mainly determined by how fast $a''(l)$, $a(k, l)$ and $a'(k, l)$ decay with l and, then, Theorem 6 suggests that the truncation parameters M , K_k , $k \in \gamma_K$, and L_k , $k \in \gamma'_L$, for class C_2 models with the natural selections for E and r can be roughly upper bounded using $q_{\overline{E}}^m$ instead of, respectively, $r_{\max} a''(m) \sum_{k=m+1}^{\infty} e^{-\Lambda_{\overline{E}} t} (\Lambda_{\overline{E}} t)^k / k!$ in (43), $r_{\max} (\alpha_S - a''(M)) a(k, m) \sum_{l=k+1}^{\infty} (l-k) e^{-\Lambda_{\overline{E}} t} (\Lambda_{\overline{E}} t)^l / l!$ in (45), and $r_{\max} a'(k, m) \sum_{l=k+1}^{\infty} e^{-\Lambda_{\overline{E}} t} (\Lambda_{\overline{E}} t)^l / l!$ in (47). Let $R_{\overline{E}} = \max_{1 \leq k \leq \overline{N}_C} \max_{i \in \overline{S}_k} \lambda_i / \min_{1 \leq k \leq \overline{N}_C} \min_{i \in \overline{S}_k} \lambda_i \geq 1$. We have $q_{\overline{E}} \approx 1 - 1/((1 + \theta) R_{\overline{E}}) \approx 1 - 1/R_{\overline{E}}$. As $R_{\overline{E}}$ gets larger, $q_{\overline{E}}$ gets closer to 1 and the required M , K_k , $k \in \gamma_K$, and L_k , $k \in \gamma'_L$, should get larger. According to the discussion and the numerical

experiments in [9], for small enough ε and $R_{\overline{E}} \gg 1$, M , K_k , $k \in \gamma_K$ and L_k , $k \in \gamma'_L$ can be roughly upper bounded by $30R_{\overline{E}}$.

Note that both R_E and $R_{\overline{E}}$ are “visible” model characteristics (they can be easily estimated) and, then, the truncation parameters K , L , M , K_k , $k \in \gamma_K$, and L_k , $k \in \gamma'_L$ can be, in practice, roughly upper bounded, and those rough upper bounds can be used to predict the performance of split regenerative randomization and anticipate when the method can be expected to be significantly faster than standard randomization for class C_2 models with the natural selections for E and r .

4 Numerical Analysis

In this section, using a class C_2 model of a 16×16 ASEN-MAX interconnection network [24] under a non-backtracking routing protocol, we will analyze the performance of split regenerative randomization and will compare it with that of standard randomization, regenerative randomization [9], adaptive uniformization [30], and randomization with quasistationarity detection [12]. The dependability analysis of interconnection networks of that kind is challenging and has been the subject of intensive research (see, for instance, [3, 4, 5, 15, 28, 39, 40]). All that research is concerned with reliability analysis in a non-repairable context or steady-state availability analysis using (quite sophisticated) combinatorial techniques.

The architecture of the 16×16 ASEN-MAX interconnection network is depicted in Figure 7. The interconnection network includes a stage of multiplexers, three stages of switches, and a stage of demultiplexers. Links are assumed perfect (they do not fail). The network is assumed to be operational if the set of unfailed components is such that, in the presence of a single request from any source to any destination, the non-backtracking routing protocol used by the interconnection network succeeds in routing the request. That assumption is guaranteed to be pessimistic [34]. In addition, we will consider that faults in multiplexers, switches and demultiplexers may not be covered and that an uncovered fault leads to system failure. The structure of the interconnection network is such that, assuming perfect fault coverage, the interconnection network tolerates the failure of any single component. Multiplexers and demultiplexers fail with rate λ_M ; switches fail with rate λ_S . Multiplexer and demultiplexer faults are covered with probability C_M ; switch faults are covered with probability C_S . There is a single repairman. Repair starts when the number of failed components gets ≥ 2 and, then, proceeds till all failed components have been repaired. Multiplexers and demultiplexers are repaired at rate μ_M ; switches are repaired at rate μ_S . The repair of switches has priority over the repair of multiplexers and demultiplexers. Failed components with same repair priority are chosen at random by the repairman. There is repair preemption. Thus, if a switch fails when a multiplexer or demultiplexer is being repaired, the repair of the multiplexer or demultiplexer is interrupted and the repairman starts immediately the repair of the failed switch. We will use the model parameter values $\lambda_M = 4 \times 10^{-6} \text{ h}^{-1}$, $\lambda_S = 1.2 \times 10^{-5} \text{ h}^{-1}$, $C_M = 0.995$, $C_S = 0.99$, $\mu_M = 2 \text{ h}^{-1}$, and for μ_S two values: 0.4 h^{-1} and 0.08 h^{-1} . Regarding the initial state of the system we will consider two cases: case (1) the initial state is the state without failed components, and case

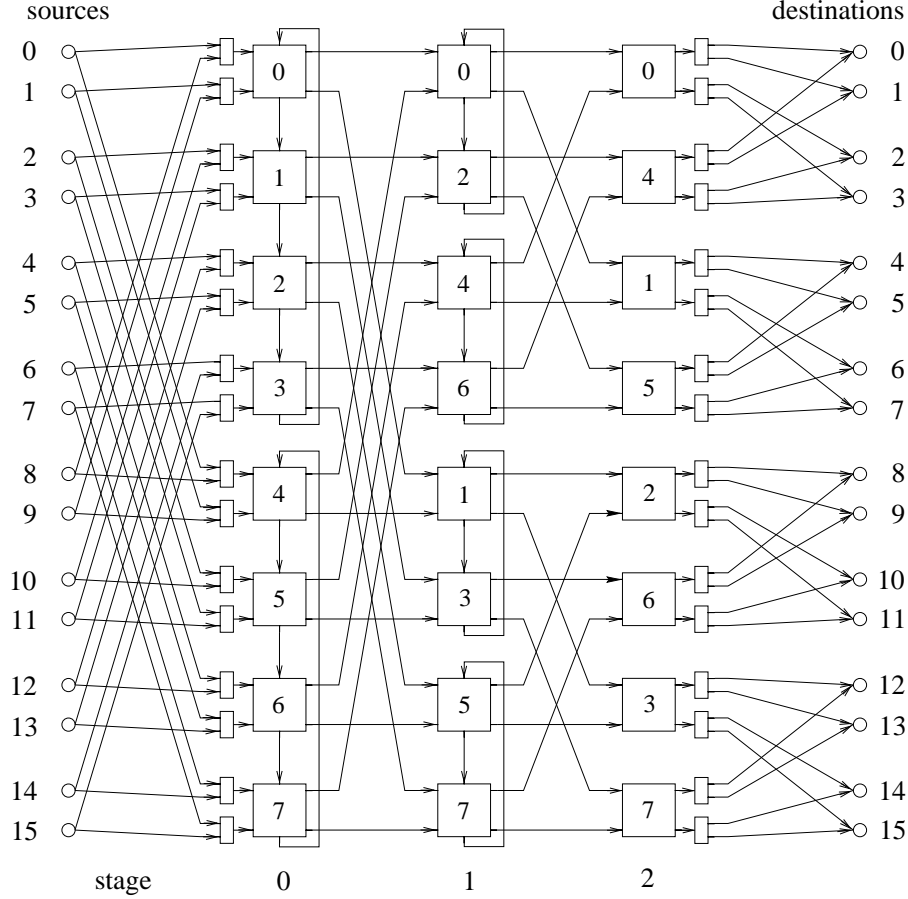


Figure 7: Architecture of the 16×16 ASEN-MAX interconnection network.

(2) with probability 0.5, the initial state is the state in which the only failed component is the switch 0 of stage 0, and, with probability 0.5, the initial state is the state in which the failed components are switch 0 of stage 0 and the top multiplexer.

The measure of interest is the unreliability at time t , $ur(t)$. That measure could be computed using a CTMC with an absorbing state representing that the system has failed and a subset of transient states including all states in which the system is operational. However, the interconnection network has a large number of components (56) and such CTMC is extremely large, making an exact numerical computation of $ur(t)$ unfeasible. The problem can be circumvented by using “bounding” models such as those described in Section 1. A model yielding a lower bound for $ur(t)$ would have a state space $S \cup \{f_1, f_2\}$, where S is a subset of operational states and f_1 and f_2 are absorbing states and a model yielding an upper bound for $ur(t)$ would have a state space $S \cup \{f_1\}$, where S is a subset of operational states and f_1 is an absorbing state. Including in S the operational states with up to 4 failed components gives very tight bounds. Thus, for $\mu_S = 0.4 \text{ h}^{-1}$, case 1 and $t = 100,000 \text{ h}$, we obtain a lower bound for $ur(t)$, $ur_{lb}(t) = 0.652058783$, and an upper bound for $ur(t)$, $ur_{ub}(t) = 0.652058789$. With that selection, $|S| = 315,045$ and both bounding models have manageable sizes. All states in S are transient and both bounding models belong to model class C_2 . We will compare the performance of the methods using the bounding model yielding $ur_{ub}(t)$. For

split regenerative randomization we will take for \bar{E} and r the natural selections, i.e. \bar{E} will include the states in S without repair and r will be the single state o without failed components. Case 1 will illustrate the case $\alpha_{E'} = 0, \alpha_{\bar{E}} = 0$. Case 2 will illustrate the more general case ($\alpha_{E'} > 0, \alpha_{\bar{E}} > 0$). The model also belongs to the model class C considered in [9] and, for regeneration randomization, we will take the selection $r = o$. For all methods we set an absolute error requirement $\varepsilon = 10^{-9}$. All CPU times are measured running the methods with a single target time t on a workstation with a Sun-Blade 1000 processor and 4 GB of memory .

We will start by illustrating the performance of split regenerative randomization. In particular, we will illustrate the behavior of the truncation parameters of the transformed model as a function of t . Table 1 gives K and $K_{\bar{E}}$ for case 1 and both values for μ_S as a function of t . Table 2 gives $K, L, M, K_{\bar{E}},$ and $L_{\bar{E}}$ for case 2 and both values for μ_S as a function of t . We can note that all truncation parameters grow smoothly with t . The truncation parameters K and L have very small values in all cases. This is due to the fact that, having the system many components with quite similar failure rates, the output rates from states in \bar{E} are very similar and, then, R_E is only slightly larger than 1 and q_E is very small ($q_E = 0.0289$). The rough upper bounds for K and L suggested in Section 3 are quite good. Thus, for case 2, $\mu_S = 0.4 \text{ h}^{-1}$ and $t = 100,000 \text{ h}$ the rough upper bound for K is 7, quite close to $K = 9$ and the rough upper bound for L is 7, identical to the actual value of L . The truncation parameters $M, K_k, k \in \gamma_K,$ and $L_k, k \in \gamma'_L,$ have small values for $\mu_S = 0.4 \text{ h}^{-1}$, since in that case the output rates from states in \bar{E} are very similar and $R_{\bar{E}}$ is not large ($R_{\bar{E}} \approx 5$). For $\mu_S = 0.08 \text{ h}^{-1}$, those truncation parameters have higher values, since the output rates from states in \bar{E} are not so similar and $R_{\bar{E}}$ is larger ($R_{\bar{E}} \approx 25$). The truncation parameters $K_k, k \in \gamma_K,$ and $L_k, k \in \gamma'_L,$ decrease as k increases. Thus, for case 2, $\mu_S = 0.4 \text{ h}^{-1}$ and $t = 100,000 \text{ h}$ we have $K_1 = 145, K_2 = 128, K_3 = 112, K_4 = 95, K_5 = 78, K_6 = 60, K_7 = 43,$ and $K_8 = 25$ ($a(0, 1) = 0$ and, then, the comb having as a back the states s_0, s_1, \dots, s_K does not have any tooth hanging from state s_0), and $L_0 = 126, L_1 = 109, L_2 = 92, L_3 = 75, L_4 = 58, L_5 = 41,$ and $L_6 = 22$. We can also note that the rough upper bound $30R_{\bar{E}}$ for the truncation parameters $M, K_k, k \in \gamma_K,$ and $L_k, k \in \gamma'_L,$ is satisfied and is a little loose. Thus, for $\mu_S = 0.4 \text{ h}^{-1}$ the rough upper bound would have a value ≈ 150 , which is a bit larger than the values of $K_k, k \in \gamma_K, L_k, k \in \gamma'_L,$ and M for case 2, $\mu_S = 0.4 \text{ h}^{-1}$ and $t = 100,000 \text{ h}$. This seems to confirm the applicability of that rough upper bound.

We will compare next the performance of split regenerative randomization (SRR), standard randomization (SR), regenerative randomization (RR), adaptive uniformization (AU), and randomization with quasistationarity detection (RQD). To be fair in the comparison, for AU we use the AU layered uniformization variant for AU processes with converged rate described in [30]. This ensures for AU the same numerical stability as all other three methods have. Figure 8 gives the CPU times for all four methods as a function of t for case 1, $\mu_S = 0.4 \text{ h}^{-1}$ and $\mu_S = 0.08 \text{ h}^{-1}$. Figure 9 gives the CPU times for case 2. We can note that for case 1, i.e. when the initial probability distribution is concentrated in the state o without failed components, AU is less expensive than SR for small and medium times and slightly more expensive for large times. This corroborates the known behavior for AU [30]. However, for case 2, i.e. when the initial probability distribution is not concentrated in state o , AU is about as expensive as SR for small and medium times. This is due to the fact that

Table 1: K and $K_{\overline{E}}$ as a function of t for case 1 and both values for μ_S .

t (h)	$\mu_S = 0.4 \text{ h}^{-1}$		$\mu_S = 0.08 \text{ h}^{-1}$	
	K	$K_{\overline{E}}$	K	$K_{\overline{E}}$
10	3	69	3	359
20	3	87	3	463
50	3	110	3	592
100	3	128	3	686
200	4	167	4	896
500	4	208	4	1,120
1,000	5	262	5	1,414
2,000	5	306	5	1,658
5,000	6	395	6	2,148
10,000	7	474	7	2,572
20,000	8	544	8	2,955
50,000	8	592	8	3,226
100,000	9	641	9	3,499

Table 2: K , L , M , $K_{\overline{E}}$, and $L_{\overline{E}}$ as a function of t for case 2 and both values for μ_S .

t (h)	$\mu_S = 0.4 \text{ h}^{-1}$					$\mu_S = 0.08 \text{ h}^{-1}$				
	K	L	M	$K_{\overline{E}}$	$L_{\overline{E}}$	K	L	M	$K_{\overline{E}}$	$L_{\overline{E}}$
10	3	2	43	81	141	3	2	51	421	760
20	3	2	63	99	151	3	2	81	524	816
50	3	3	98	121	189	3	3	155	651	1,023
100	3	3	100	138	210	3	3	263	745	1,137
200	4	3	100	183	231	4	3	444	988	1,249
500	4	4	100	223	284	4	4	554	1,209	1,541
1,000	5	4	100	283	316	5	4	554	1,535	1,717
2,000	6	5	100	347	372	6	5	554	1,877	2,024
5,000	7	6	100	441	446	7	6	554	2,398	2,415
10,000	7	6	100	505	481	7	6	554	2,750	2,609
20,000	8	7	100	583	521	8	7	554	3,166	2,817
50,000	9	7	100	664	523	9	7	554	3,556	2,835
100,000	9	7	100	686	523	9	7	554	3,738	2,835

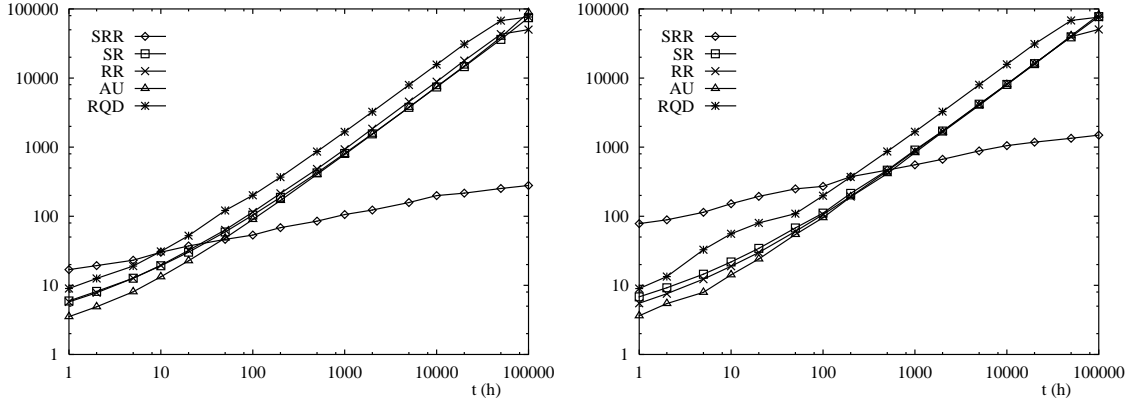


Figure 8: CPU times in seconds under split regenerative randomization (SRR), standard randomization (SR), regenerative randomization (RR), adaptive uniformization (AU), and randomization with quasistationarity detection (RQD) for case 1 and $\mu_S = 0.4 \text{ h}^{-1}$ (left), $\mu_S = 0.08 \text{ h}^{-1}$ (right), as a function of t .

the adapted randomization rate in AU is large since the initial step. Regarding the behavior of RR with respect to SR, for case 1, RR is about as expensive as SR for almost all t . As t increases, RR becomes less expensive than SR, but the time at which RR becomes less expensive is very large (about 100,000 h). For case 2, RR is significantly more expensive than SR except for very large t (about 100,000 h). RQD is more expensive than RR for case 1 and less expensive than RR for case 2. For large t , SRR is by far the less expensive method. For case 1, $\mu_S = 0.4 \text{ h}^{-1}$ and $t = 100,000 \text{ h}$ SRR is 180 times less expensive than the less expensive of the other methods (RR). For case 1, $\mu_S = 0.08 \text{ h}^{-1}$ and $t = 100,000 \text{ h}$ SRR is 33.8 times less expensive than the less expensive of the other methods (RR). For case 2, $\mu_S = 0.4 \text{ h}^{-1}$ and $t = 100,000 \text{ h}$ SRR is 128 times less expensive than the less expensive of the other methods (SR). For case 2, $\mu_S = 0.08 \text{ h}^{-1}$ and $t = 100,000 \text{ h}$ SRR is 23.0 times less expensive than the less expensive of the other methods (SR). These speedups make the CPU times required by SRR for large t ($t = 100,000 \text{ h}$) reasonable, ranging from 279 s for case 1 and $\mu_S = 0.4 \text{ h}^{-1}$ to 3,145 s for case 2 and $\mu_S = 0.08 \text{ h}^{-1}$. The time t beyond which SRR becomes attractive seems to increase as repair rates become more different ($R_{\bar{E}}$ increases) and is larger when the initial probability distribution is not concentrated in the state o without failed components. Both tendencies are due to the increase of the size of the truncated transformed model and, thus, the increase of the computational cost of both phases of the method (generation of the truncated transformed model and solution of that model by standard randomization).

5 Conclusions

Motivated by the poor performance of both standard randomization and regenerative randomization for typical failure/repair reliability-like models with deferred repair, we have developed a new numerical method, called split regenerative randomization, for the transient analysis of a class of CTMC models with a particular but quite general structure. The method has the same good proper-

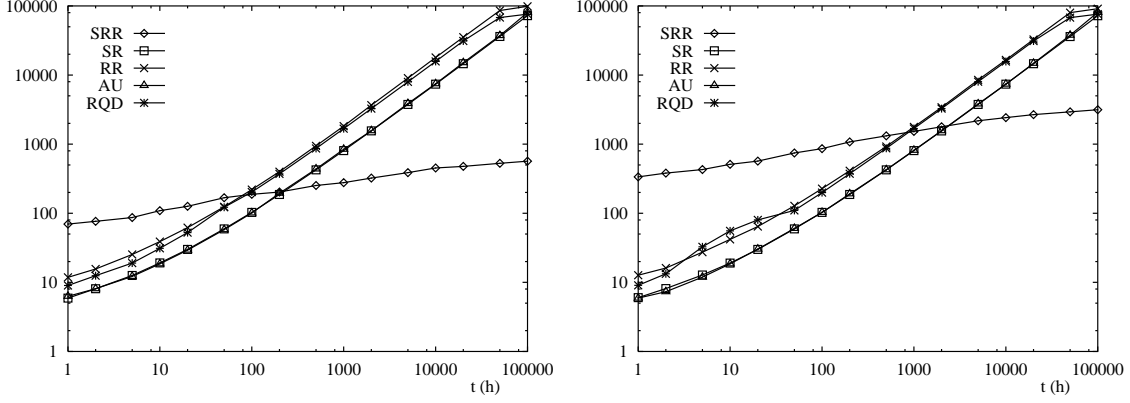


Figure 9: CPU times in seconds under split regenerative randomization (SRR), standard randomization (SR), regenerative randomization (RR), adaptive uniformization (AU), and randomization with quasistationarity detection (RQD) for case 2 and $\mu_S = 0.4 \text{ h}^{-1}$ (left), $\mu_S = 0.08 \text{ h}^{-1}$ (right), as a function of t .

ties as the well-known standard randomization method: numerical stability, well-controlled computation error, and ability to specify the computation error in advance, and, for large enough models and large enough times, will be significantly faster than that method. The method requires the selection of a subset of states E and a regenerative state $r \in E$ and its performance depends on those selections. For a class of models, class C_2 , including typical failure/repair reliability-like models with exponential failure and repair time distribution and deferred repair, natural selections for E and r exist and, for those natural selections, theoretical results have been obtained assessing approximately the performance of the method in terms of “visible” model characteristics. We have shown that, for class C_2 models, the method can indeed be significantly faster than other randomization-based methods and allows the transient analysis of some large models of that type in affordable CPU times.

Appendix

Proof of Proposition 1. The initial probability distribution of \widehat{V} follows from its definition (18). It is also clear from the definition of \widehat{V} that \widehat{V}_n can only be a state s_k , $0 \leq k \leq n$, a state $s_{k,l}$, $0 \leq k \leq n-1$, $1 \leq l \leq n-k$, state s'_n , a state $s'_{k,n-k}$, $0 \leq k \leq n-1$, state s''_n , or a state f_i , $1 \leq i \leq A$. Furthermore, \widehat{V}_n can be state $s_{k,l}$ if and only if $\widehat{X}_{n-k-l} = r$, $\widehat{X}_{n-k-l+1:n-l} \in E'$ and $\widehat{X}_{n-l+1:n} \in \overline{E}$ with nonnull probability and this requires that, for the version of \widehat{X} , \widehat{X}' , with initial state r , $\widehat{X}'_{1:k} \in E'$ and $\widehat{X}'_{k+1:k+l} \in \overline{E}$ with nonnull probability, which requires $\widehat{X}'_{1:k} \in E'$ and $\widehat{X}'_{k+1} \in \overline{E}$ with nonnull probability, and, taking into account the definition of Z (2), this is equivalent to $Z_{0:k} \in E$ and $Z_{k+1} \in \overline{E}$ with nonnull probability, which, taking into account the structure (5) of the transition probability matrix of Z restricted to its transient states, this is equivalent to $Z_k \in E$ and $Z_{k+1} \in \overline{E}$ with nonnull probability, i.e. $a(k,1) = \sum_{i \in \overline{E}} \pi_i(k,1) > 0$. In addition, \widehat{V}_n can be state $s'_{k,n-k}$ if and only if $\widehat{X}_{0:k} \in E'$ and $\widehat{X}_{k+1:n} \in \overline{E}$ with nonnull probability, which requires

$\widehat{X}_{0:k} \in E'$ and $\widehat{X}_{k+1} \in \overline{E}$ with nonnull probability, and, taking into account the definition of Z' (3), this is equivalent to $Z'_{0:k} \in E'$ and $Z'_{k+1} \in \overline{E}$ with nonnull probability, which, taking into account the structure (6) of the restriction of the transition probability matrix of Z' restricted to its transient states, this is equivalent to $Z'_k \in E'$ and $Z'_{k+1} \in \overline{E}$ with nonnull probability, i.e. $a'(k, 1) = \sum_{i \in \overline{E}} \pi'_i(k, 1) > 0$. In summary, we have $\widehat{V}_n \in \{s_k : 0 \leq k \leq n\} \cup \{s_{k,l} : 0 \leq k \leq n-1 \wedge 1 \leq l \leq n-k \wedge a(k, 1) > 0\} \cup \{s'_n\} \cup \{s'_{k,n-k} : 0 \leq k \leq n-1 \wedge a'(k, 1) > 0\} \cup \{s''_n\} \cup \{f_1, f_2, \dots, f_A\}$. This implies that the reachable state space of \widehat{V} is a subset of that stated by the proposition.

It is also clear from the definition of \widehat{V} (18) that: (1) from state s_0 , \widehat{V} can only jump to states f_i , $1 \leq i \leq A$, s_0 , s_1 and $s_{0,1}$, (2) from state s_k , $k \geq 1$, by condition C6, which implies $P_{i,r} = 0$, $i \in E'$, \widehat{V} can only jump to states f_i , $1 \leq i \leq A$, s_{k+1} and $s_{k,1}$, (3) from state $s_{k,l}$, by condition C7, which implies $P_{i,j} = 0$, $i \in \overline{E}$, $j \in E'$, \widehat{V} can only jump to states f_i , $1 \leq i \leq A$, s_0 and $s_{k,l+1}$, (4) from state s'_k , by condition C6, \widehat{V} can only jump to states f_i , $1 \leq i \leq A$, s'_{k+1} and $s'_{k,1}$, (5) from state $s'_{k,l}$, by condition C7, \widehat{V} can only jump to states f_i , $1 \leq i \leq A$, s_0 and $s'_{k,l+1}$, and (6) from state s''_k , by condition C7, \widehat{V} can only jump to states f_i , $1 \leq i \leq A$, s_0 and s''_{k+1} .

Since $\widehat{V}_n = f_i$ if and only if $\widehat{X}_n = f_i$ and f_i is absorbing in \widehat{X} , from state f_i , \widehat{V} can only jump to f_i and $P[\widehat{V}_{n+1} = f_i | \widehat{V}_n = f_i \wedge \widehat{V}_{n-1} = x_{n-1} \wedge \dots \wedge \widehat{V}_0 = x_0] = P[\widehat{V}_{n+1} = f_i | \widehat{V}_n = f_i] = 1$. To completely verify that \widehat{V} is a DTMC we must show that: (a) $P[\widehat{V}_{n+1} = y | \widehat{V}_n = s_k \wedge \widehat{V}_{n-1} = x_{n-1} \wedge \dots \wedge \widehat{V}_0 = x_0] = P[\widehat{V}_{n+1} = y | \widehat{V}_n = s_k]$ for all feasible paths $(x_0, x_1, \dots, x_{n-1}, s_k)$ of \widehat{V} , (b) $P[\widehat{V}_{n+1} = y | \widehat{V}_n = s_{k,l} \wedge \widehat{V}_{n-1} = x_{n-1} \wedge \dots \wedge \widehat{V}_0 = x_0] = P[\widehat{V}_{n+1} = y | \widehat{V}_n = s_{k,l}]$ for all feasible paths $(x_0, x_1, \dots, x_{n-1}, s_{k,l})$ of \widehat{V} , (c) $P[\widehat{V}_{n+1} = y | \widehat{V}_n = s'_n \wedge \widehat{V}_{n-1} = x_{n-1} \wedge \dots \wedge \widehat{V}_0 = x_0] = P[\widehat{V}_{n+1} = y | \widehat{V}_n = s'_n]$ for all feasible paths $(x_0, x_1, \dots, x_{n-1}, s'_n)$ of \widehat{V} , (d) $P[\widehat{V}_{n+1} = y | \widehat{V}_n = s'_{k,n-k} \wedge \widehat{V}_{n-1} = x_{n-1} \wedge \dots \wedge \widehat{V}_0 = x_0] = P[\widehat{V}_{n+1} = y | \widehat{V}_n = s'_{k,n-k}]$ for all feasible paths $(x_0, x_1, \dots, x_{n-1}, s'_{k,n-k})$ of \widehat{V} , and (e) $P[\widehat{V}_{n+1} = y | \widehat{V}_n = s''_n \wedge \widehat{V}_{n-1} = x_{n-1} \wedge \dots \wedge \widehat{V}_0 = x_0] = P[\widehat{V}_{n+1} = y | \widehat{V}_n = s''_n]$ for all feasible paths $(x_0, x_1, \dots, x_{n-1}, s''_n)$ of \widehat{V} . To prove (a) note that $\widehat{V}_n = s_k$ implies $\widehat{V}_{n-k} = s_0$, $\widehat{V}_{n-k+1} = s_1, \dots, \widehat{V}_{n-1} = s_{k-1}$ and, given the definition of \widehat{V} (18), the behavior of \widehat{V} after the steps m at which \widehat{V} hits state s_0 is independent of $\widehat{X}_0, \widehat{X}_1, \dots, \widehat{X}_{m-1}$ and, therefore, of $\widehat{V}_0, \widehat{V}_1, \dots, \widehat{V}_{m-1}$, and, then, $P[\widehat{V}_{n+1} = y | \widehat{V}_n = s_k \wedge \dots \wedge \widehat{V}_{n-k} = s_0 \wedge \widehat{V}_{n-k-1} = x_{n-k-1} \wedge \dots \wedge \widehat{V}_0 = x_0] = P[\widehat{V}_{n+1} = y | \widehat{V}_n = s_k \wedge \dots \wedge \widehat{V}_{n-k} = s_0] = P[\widehat{V}_{n+1} = y | \widehat{V}_n = s_k]$. To prove (b) note that $\widehat{V}_n = s_{k,l}$ implies $\widehat{V}_{n-k-l} = s_0, \dots, \widehat{V}_{n-l} = s_k, \widehat{V}_{n-l+1} = s_{k,1}, \dots, \widehat{V}_{n-1} = s_{k,l-1}$, and, since, as previously shown, \widehat{V} loses its memory at the steps at which it hits state s_0 , $P[\widehat{V}_{n+1} = y | \widehat{V}_n = s_{k,l} \wedge \dots \wedge \widehat{V}_{n-l+1} = s_{k,1} \wedge \widehat{V}_{n-l} = s_k \wedge \dots \wedge \widehat{V}_{n-k-l} = s_0 \wedge \widehat{V}_{n-k-l-1} = x_{n-k-l-1} \wedge \dots \wedge \widehat{V}_0 = x_0] = P[\widehat{V}_{n+1} = y | \widehat{V}_n = s_{k,l} \wedge \dots \wedge \widehat{V}_{n-l+1} = s_{k,1} \wedge \widehat{V}_{n-l} = s_k \wedge \dots \wedge \widehat{V}_{n-k-l} = s_0] = P[\widehat{V}_{n+1} = y | \widehat{V}_n = s_{k,l}]$. To prove (c) note that $\widehat{V}_n = s'_n$ implies $\widehat{V}_0 = s'_0, \widehat{V}_1 = s'_1, \dots, \widehat{V}_{n-1} = s'_{n-1}$ and, then, $P[\widehat{V}_{n+1} = y | \widehat{V}_n = s'_n \wedge \dots \wedge \widehat{V}_0 = s'_0] = P[\widehat{V}_{n+1} = y | \widehat{V}_n = s'_n]$. To prove (d) note that $\widehat{V}_n = s'_{k,n-k}$ implies $\widehat{V}_0 = s'_0, \dots, \widehat{V}_k = s'_k, \widehat{V}_{k+1} = s'_{k,1}, \dots, \widehat{V}_{n-1} = s'_{k,n-k-1}$ and, then, $P[\widehat{V}_{n+1} = y | \widehat{V}_n = s'_{k,n-k} \wedge \dots \wedge \widehat{V}_{k+1} = s'_{k,1} \wedge \widehat{V}_k = s'_k \wedge \dots \wedge \widehat{V}_0 = s'_0] = P[\widehat{V}_{n+1} = y | \widehat{V}_n = s'_{k,n-k}]$. Finally, to prove (e) note that $\widehat{V}_n = s''_n$ implies $\widehat{V}_0 = s''_0, \dots, \widehat{V}_{n-1} = s''_{n-1}$ and, then, $P[\widehat{V}_{n+1} = y | \widehat{V}_n = s''_n \wedge \dots \wedge \widehat{V}_0 = s''_0] = P[\widehat{V}_{n+1} = y | \widehat{V}_n = s''_n]$.

We will verify next the values of the transition probabilities from states $s_k, s_{k,l}, s'_k, s'_{k,l}$, and s''_k . We will use $\widehat{V}_n \in \{s_k : 0 \leq k \leq n\} \cup \{s_{k,l} : 0 \leq k \leq n-1 \wedge 1 \leq l \leq n-k \wedge a(k, 1) >$

$0\} \cup \{s'_n\} \cup \{s'_{k,n-k} : 0 \leq k \leq n-1 \wedge a'(k,1) > 0\} \cup \{s''_n\} \cup \{f_1, f_2, \dots, f_A\}$.

Case a ($P[\widehat{V}_n = s_0] > 0$, which implies (18) $P[\widehat{X}_n = r] > 0$): Taking into account that $\widehat{V}_n = s_0$ if and only if $\widehat{X}_n = r$ and $\widehat{V}_n = f_i$ if and only if $\widehat{X}_n = f_i$, (8), and (19) $a(0) = \sum_{i \in E} \pi_i(0) = 1$:

$$\begin{aligned}
P[\widehat{V}_{n+1} = f_i \mid \widehat{V}_n = s_0] &= \frac{P[\widehat{V}_n = s_0 \wedge \widehat{V}_{n+1} = f_i]}{P[\widehat{V}_n = s_0]} \\
&= \frac{P[\widehat{X}_n = r \wedge \widehat{X}_{n+1} = f_i]}{P[\widehat{X}_n = r]} \\
&= P[\widehat{X}_{n+1} = f_i \mid \widehat{X}_n = r] = P_{r,f_i} = \sum_{j \in E} \pi_j(0) P_{j,f_i} \\
&= \frac{\sum_{j \in E} \pi_j(0) P_{j,f_i}}{a(0)} = v_0^i.
\end{aligned}$$

Similarly,

$$\begin{aligned}
P[\widehat{V}_{n+1} = s_0 \mid \widehat{V}_n = s_0] &= \frac{P[\widehat{V}_n = s_0 \wedge \widehat{V}_{n+1} = s_0]}{P[\widehat{V}_n = s_0]} \\
&= \frac{P[\widehat{X}_n = r \wedge \widehat{X}_{n+1} = r]}{P[\widehat{X}_n = r]} = P[\widehat{X}_{n+1} = r \mid \widehat{X}_n = r] = P_{r,r},
\end{aligned}$$

$$\begin{aligned}
P[\widehat{V}_{n+1} = s_1 \mid \widehat{V}_n = s_0] &= \frac{P[\widehat{V}_n = s_0 \wedge \widehat{V}_{n+1} = s_1]}{P[\widehat{V}_n = s_0]} \\
&= \frac{P[\widehat{X}_n = r \wedge \widehat{X}_{n+1} \in E']}{P[\widehat{X}_n = r]} = P[\widehat{X}_{n+1} \in E' \mid \widehat{X}_n = r] \\
&= P_{r,E'} = \sum_{i \in E} \pi_i(0) P_{i,E'} = \frac{\sum_{i \in E} \pi_i(0) P_{i,E'}}{a(0)} = w_0,
\end{aligned}$$

$$\begin{aligned}
P[\widehat{V}_{n+1} = s_{0,1} \mid \widehat{V}_n = s_0] &= \frac{P[\widehat{V}_n = s_0 \wedge \widehat{V}_{n+1} = s_{0,1}]}{P[\widehat{V}_n = s_0]} \\
&= \frac{P[\widehat{X}_n = r \wedge \widehat{X}_{n+1} \in \overline{E}]}{P[\widehat{X}_n = r]} = P[\widehat{X}_{n+1} \in \overline{E} \mid \widehat{X}_n = r] \\
&= P_{r,\overline{E}} = \sum_{i \in E} \pi_i(0) P_{i,\overline{E}} = \frac{\sum_{i \in E} \pi_i(0) P_{i,\overline{E}}}{a(0)} = h_0.
\end{aligned}$$

Case b ($P[\widehat{V}_n = s_k] > 0, 1 \leq k \leq n$, which implies (18) $P[\widehat{X}_{n-k} = r] > 0$): Taking into account the definition of \widehat{V} (18), that the DTMC $\{\widehat{X}_{m+k}; k = 0, 1, 2, \dots\}$ conditioned to $\widehat{X}_m = r$ is probabilistically identical to $\widehat{X}' = \{\widehat{X}'_k; k = 0, 1, 2, \dots\}$, the definition of Z (2), that, because of the structure of the restriction of the transition probability matrix of Z to its non-absorbing states (5) and the fact that $Z_0 = r, Z_{1:k} \in E'$ if and only if $Z_k \in E$, that $P[Z_{k+1} = f_i \mid Z_k = j] = P_{j,f_i}$,

$j \in E$, and (19):

$$\begin{aligned}
P[\widehat{V}_{n+1} = f_i \mid \widehat{V}_n = s_k] &= \frac{P[\widehat{V}_n = s_k \wedge \widehat{V}_{n+1} = f_i]}{P[\widehat{V}_n = s_k]} \\
&= \frac{P[\widehat{X}_{n-k} = r \wedge \widehat{X}_{n-k+1:n} \in E' \wedge \widehat{X}_{n+1} = f_i]}{P[\widehat{X}_{n-k} = r \wedge \widehat{X}_{n-k+1:n} \in E']} \\
&= \frac{P[\widehat{X}_{n-k+1:n} \in E' \wedge \widehat{X}_{n+1} = f_i \mid \widehat{X}_{n-k} = r]}{P[\widehat{X}_{n-k+1:n} \in E' \mid \widehat{X}_{n-k} = r]} \\
&= \frac{P[\widehat{X}'_{1:k} \in E' \wedge \widehat{X}'_{k+1} = f_i]}{P[\widehat{X}'_{1:k} \in E']} = \frac{P[Z_{1:k} \in E' \wedge Z_{k+1} = f_i]}{P[Z_{1:k} \in E']} \\
&= \frac{P[Z_k \in E \wedge Z_{k+1} = f_i]}{P[Z_k \in E]} = \frac{\sum_{j \in E} P[Z_k = j] P[Z_{k+1} = f_i \mid Z_k = j]}{\sum_{j \in E} P[Z_k = j]} \\
&= \frac{\sum_{j \in E} \pi_j(k) P_{j, f_i}}{\sum_{j \in E} \pi_j(k)} = \frac{\sum_{j \in E} \pi_j(k) P_{j, f_i}}{a(k)} = v_k^i.
\end{aligned}$$

Similarly,

$$\begin{aligned}
P[\widehat{V}_{n+1} = s_{k+1} \mid \widehat{V}_n = s_k] &= \frac{P[\widehat{V}_n = s_k \wedge \widehat{V}_{n+1} = s_{k+1}]}{P[\widehat{V}_n = s_k]} \\
&= \frac{P[\widehat{X}_{n-k} = r \wedge \widehat{X}_{n-k+1:n+1} \in E']}{P[\widehat{X}_{n-k} = r \wedge \widehat{X}_{n-k+1:n} \in E']} = \frac{P[\widehat{X}_{n-k+1:n+1} \in E' \mid \widehat{X}_{n-k} = r]}{P[\widehat{X}_{n-k+1:n} \in E' \mid \widehat{X}_{n-k} = r]} \\
&= \frac{P[\widehat{X}'_{1:k+1} \in E']}{P[\widehat{X}'_{1:k} \in E']} = \frac{P[Z_{1:k+1} \in E']}{P[Z_{1:k} \in E']} = \frac{P[Z_k \in E \wedge Z_{k+1} \in E']}{P[Z_k \in E]} \\
&= \frac{\sum_{i \in E} P[Z_k = i] P[Z_{k+1} \in E' \mid Z_k = i]}{\sum_{i \in E} P[Z_k = i]} = \frac{\sum_{i \in E} \pi_i(k) P_{i, E'}}{\sum_{i \in E} \pi_i(k)} = \frac{\sum_{i \in E} \pi_i(k) P_{i, E'}}{a(k)} = w_k,
\end{aligned}$$

$$\begin{aligned}
P[\widehat{V}_{n+1} = s_{k,1} \mid \widehat{V}_n = s_k] &= \frac{P[\widehat{V}_n = s_k \wedge \widehat{V}_{n+1} = s_{k,1}]}{P[\widehat{V}_n = s_k]} \\
&= \frac{P[\widehat{X}_{n-k} = r \wedge \widehat{X}_{n-k+1:n} \in E' \wedge \widehat{X}_{n+1} \in \overline{E}]}{P[\widehat{X}_{n-k} = r \wedge \widehat{X}_{n-k+1:n} \in E']} \\
&= \frac{P[\widehat{X}_{n-k+1:n} \in E' \wedge \widehat{X}_{n+1} \in \overline{E} \mid \widehat{X}_{n-k} = r]}{P[\widehat{X}_{n-k+1:n} \in E' \mid \widehat{X}_{n-k} = r]} \\
&= \frac{P[\widehat{X}'_{1:k} \in E' \wedge \widehat{X}'_{k+1} \in \overline{E}]}{P[\widehat{X}'_{1:k} \in E']} = \frac{P[Z_{1:k} \in E' \wedge Z_{k+1} \in \overline{E}]}{P[Z_{1:k} \in E']} \\
&= \frac{P[Z_k \in E \wedge Z_{k+1} \in \overline{E}]}{P[Z_k \in E]} = \frac{\sum_{i \in E} P[Z_k = i] P[Z_{k+1} \in \overline{E} \mid Z_k = i]}{\sum_{i \in E} P[Z_k = i]} \\
&= \frac{\sum_{i \in E} \pi_i(k) P_{i, \overline{E}}}{\sum_{i \in E} \pi_i(k)} = \frac{\sum_{i \in E} \pi_i(k) P_{i, \overline{E}}}{a(k)} = h_k.
\end{aligned}$$

Case c ($P[\widehat{V}_n = s_{k,l}] > 0, 0 \leq k \leq n-1, 1 \leq l \leq n-k, a(k,1) > 0$, which implies (18) $P[\widehat{X}_{n-k-l} = r] > 0$): Taking into account the definition of \widehat{V} (18), that the DTMC $\{\widehat{X}_{m+k}; k = 0, 1, 2, \dots\}$ conditioned to $\widehat{X}_m = r$ is probabilistically identical to $\widehat{X}' = \{\widehat{X}'_k; k = 0, 1, 2, \dots\}$, the definition of Z (2), that, because of the structure of the transition probability matrix of Z restricted

to its non-absorbing states (5) and the fact that $Z_0 = r$, $Z_{1:k} \in E'$ if and only if $Z_k \in E$, that $P[Z_{n+1} = f_i | Z_n = j] = P_{j,f_i}$, $j \in \overline{E}$, and (20):

$$\begin{aligned}
P[\widehat{V}_{n+1} = f_i | \widehat{V}_n = s_{k,l}] &= \frac{P[\widehat{V}_n = s_{k,l} \wedge \widehat{V}_{n+1} = f_i]}{P[\widehat{V}_n = s_{k,l}]} \\
&= \frac{P[\widehat{X}_{n-k-l} = r \wedge \widehat{X}_{n-k-l+1:n-l} \in E' \wedge \widehat{X}_{n-l+1:n} \in \overline{E} \wedge \widehat{X}_{n+1} = f_i]}{P[\widehat{X}_{n-k-l} = r \wedge \widehat{X}_{n-k-l+1:n-l} \in E' \wedge \widehat{X}_{n-l+1:n} \in \overline{E}]} \\
&= \frac{P[\widehat{X}_{n-k-l+1:n-l} \in E' \wedge \widehat{X}_{n-l+1:n} \in \overline{E} \wedge \widehat{X}_{n+1} = f_i | \widehat{X}_{n-k-l} = r]}{P[\widehat{X}_{n-k-l+1:n-l} \in E' \wedge \widehat{X}_{n-l+1:n} \in \overline{E} | \widehat{X}_{n-k-l} = r]} \\
&= \frac{P[\widehat{X}'_{1:k} \in E' \wedge \widehat{X}'_{k+1:k+l} \in \overline{E} \wedge \widehat{X}'_{k+l+1} = f_i]}{P[\widehat{X}'_{1:k} \in E' \wedge \widehat{X}'_{k+1:k+l} \in \overline{E}]} \\
&= \frac{P[Z_{1:k} \in E' \wedge Z_{k+1:k+l} \in \overline{E} \wedge Z_{k+l+1} = f_i]}{P[Z_{1:k} \in E' \wedge Z_{k+1:k+l} \in \overline{E}]} \\
&= \frac{P[Z_k \in E \wedge Z_{k+1:k+l} \in \overline{E} \wedge Z_{k+l+1} = f_i]}{P[Z_k \in E \wedge Z_{k+1:k+l} \in \overline{E}]} \\
&= \frac{\sum_{j \in \overline{E}} P[Z_k \in E \wedge Z_{k+1:k+l} \in \overline{E} \wedge Z_{k+l} = j] P[Z_{k+l+1} = f_i | Z_{k+l} = j]}{\sum_{j \in \overline{E}} P[Z_k \in E \wedge Z_{k+1:k+l} \in \overline{E} \wedge Z_{k+l} = j]} \\
&= \frac{\sum_{j \in \overline{E}} \pi_j(k, l) P_{j,f_i}}{\sum_{j \in \overline{E}} \pi_j(k, l)} = \frac{\sum_{j \in \overline{E}} \pi_j(k, l) P_{j,f_i}}{a(k, l)} = v_{k,l}^i.
\end{aligned}$$

Similarly,

$$\begin{aligned}
P[\widehat{V}_{n+1} = s_0 | \widehat{V}_n = s_{k,l}] &= \frac{P[\widehat{V}_n = s_{k,l} \wedge \widehat{V}_{n+1} = s_0]}{P[\widehat{V}_n = s_{k,l}]} \\
&= \frac{P[\widehat{X}_{n-k-l} = r \wedge \widehat{X}_{n-k-l+1:n-l} \in E' \wedge \widehat{X}_{n-l+1:n} \in \overline{E} \wedge \widehat{X}_{n+1} = r]}{P[\widehat{X}_{n-k-l} = r \wedge \widehat{X}_{n-k-l+1:n-l} \in E' \wedge \widehat{X}_{n-l+1:n} \in \overline{E}]} \\
&= \frac{P[\widehat{X}_{n-k-l+1:n-l} \in E' \wedge \widehat{X}_{n-l+1:n} \in \overline{E} \wedge \widehat{X}_{n+1} = r | \widehat{X}_{n-k-l} = r]}{P[\widehat{X}_{n-k-l+1:n-l} \in E' \wedge \widehat{X}_{n-l+1:n} \in \overline{E} | \widehat{X}_{n-k-l} = r]} \\
&= \frac{P[\widehat{X}'_{1:k} \in E' \wedge \widehat{X}'_{k+1:k+l} \in \overline{E} \wedge \widehat{X}'_{k+l+1} = r]}{P[\widehat{X}'_{1:k} \in E' \wedge \widehat{X}'_{k+1:k+l} \in \overline{E}]} \\
&= \frac{P[Z_{1:k} \in E' \wedge Z_{k+1:k+l} \in \overline{E} \wedge Z_{k+l+1} = a]}{P[Z_{1:k} \in E' \wedge Z_{k+1:k+l} \in \overline{E}]} \\
&= \frac{P[Z_k \in E \wedge Z_{k+1:k+l} \in \overline{E} \wedge Z_{k+l+1} = a]}{P[Z_k \in E \wedge Z_{k+1:k+l} \in \overline{E}]} \\
&= \frac{\sum_{i \in \overline{E}} P[Z_k \in E \wedge Z_{k+1:k+l} \in \overline{E} \wedge Z_{k+l} = i] P[Z_{k+l+1} = a | Z_{k+l} = i]}{\sum_{i \in \overline{E}} P[Z_k \in E \wedge Z_{k+1:k+l} \in \overline{E} \wedge Z_{k+l} = i]} \\
&= \frac{\sum_{i \in \overline{E}} \pi_i(k, l) P_{i,r}}{\sum_{i \in \overline{E}} \pi_i(k, l)} = \frac{\sum_{i \in \overline{E}} \pi_i(k, l) P_{i,r}}{a(k, l)} = q_{k,l},
\end{aligned}$$

$$\begin{aligned}
P[\widehat{V}_{n+1} = s_{k,l+1} | \widehat{V}_n = s_{k,l}] &= \frac{P[\widehat{V}_n = s_{k,l} \wedge \widehat{V}_{n+1} = s_{k,l+1}]}{P[\widehat{V}_n = s_{k,l}]} \\
&= \frac{P[\widehat{X}_{n-k-l} = r \wedge \widehat{X}_{n-k-l+1:n-l} \in E' \wedge \widehat{X}_{n-l+1:n+1} \in \overline{E}]}{P[\widehat{X}_{n-k-l} = r \wedge \widehat{X}_{n-k-l+1:n-l} \in E' \wedge \widehat{X}_{n-l+1:n} \in \overline{E}]}
\end{aligned}$$

$$\begin{aligned}
&= \frac{P[\widehat{X}_{n-k-l+1:n-l} \in E' \wedge \widehat{X}_{n-l+1:n+1} \in \overline{E} \mid \widehat{X}_{n-k-l} = r]}{P[\widehat{X}_{n-k-l+1:n-l} \in E' \wedge \widehat{X}_{n-l+1:n} \in \overline{E} \mid \widehat{X}_{n-k-l} = r]} \\
&= \frac{P[\widehat{X}'_{1:k} \in E' \wedge \widehat{X}'_{k+1:k+l+1} \in \overline{E}]}{P[\widehat{X}'_{1:k} \in E' \wedge \widehat{X}'_{k+1:k+l} \in \overline{E}]} = \frac{P[Z_{1:k} \in E' \wedge Z_{k+1:k+l+1} \in \overline{E}]}{P[Z_{1:k} \in E' \wedge Z_{k+1:k+l} \in \overline{E}]} \\
&= \frac{P[Z_k \in E \wedge Z_{k+1:k+l+1} \in \overline{E}]}{P[Z_k \in E \wedge Z_{k+1:k+l} \in \overline{E}]} = \frac{P[Z_k \in E \wedge Z_{k+1:k+l} \in \overline{E} \wedge Z_{k+l+1} \in \overline{E}]}{P[Z_k \in E \wedge Z_{k+1:k+l} \in \overline{E}]} \\
&= \frac{\sum_{i \in \overline{E}} P[Z_k \in E \wedge Z_{k+1:k+l} \in \overline{E} \wedge Z_{k+l} = i] P[Z_{k+l+1} \in \overline{E} \mid Z_{k+l} = i]}{\sum_{i \in \overline{E}} P[Z_k \in E \wedge Z_{k+1:k+l} \in \overline{E} \wedge Z_{k+l} = i]} \\
&= \frac{\sum_{i \in \overline{E}} \pi_i(k, l) P_{i, \overline{E}}}{\sum_{i \in \overline{E}} \pi_i(k, l)} = \frac{\sum_{i \in \overline{E}} \pi_i(k, l) P_{i, \overline{E}}}{a(k, l)} = w_{k, l}.
\end{aligned}$$

Case d ($P[\widehat{V}_n = s'_n] > 0$): Taking into account the definition of \widehat{V} (18), the definition of Z' (3), that because of the structure of the restriction of the transition probability matrix of Z' to its non-absorbing states (6) $Z'_{0:n} \in E'$ if and only if $Z'_n \in E'$, that $P[Z'_{n+1} = f_i \mid Z'_n = j] = P_{j, f_i}$, $j \in E'$, and (21):

$$\begin{aligned}
P[\widehat{V}_{n+1} = f_i \mid \widehat{V}_n = s'_n] &= \frac{P[\widehat{V}_n = s'_n \wedge \widehat{V}_{n+1} = f_i]}{P[\widehat{V}_n = s'_n]} = \frac{P[\widehat{X}_{0:n} \in E' \wedge \widehat{X}_{n+1} = f_i]}{P[\widehat{X}_{0:n} \in E']} \\
&= \frac{P[Z'_{0:n} \in E' \wedge Z'_{n+1} = f_i]}{P[Z'_{0:n} \in E']} = \frac{P[Z'_n \in E' \wedge Z'_{n+1} = f_i]}{P[Z'_n \in E']} \\
&= \frac{\sum_{j \in E'} P[Z'_n = j] P[Z'_{n+1} = f_i \mid Z'_n = j]}{\sum_{j \in E'} P[Z'_n = j]} \\
&= \frac{\sum_{j \in E'} \pi'_j(n) P_{j, f_i}}{\sum_{j \in E'} \pi'_j(n)} = \frac{\sum_{j \in E'} \pi'_j(n) P_{j, f_i}}{a'(n)} = v_n^i.
\end{aligned}$$

Similarly,

$$\begin{aligned}
P[\widehat{V}_{n+1} = s'_{n+1} \mid \widehat{V}_n = s'_n] &= \frac{P[\widehat{V}_n = s'_n \wedge \widehat{V}_{n+1} = s'_{n+1}]}{P[\widehat{V}_n = s'_n]} = \frac{P[\widehat{X}_{0:n+1} \in E']}{P[\widehat{X}_{0:n} \in E']} \\
&= \frac{P[Z'_{0:n+1} \in E']}{P[Z'_{0:n} \in E']} = \frac{P[Z'_n \in E' \wedge Z'_{n+1} \in E']}{P[Z'_n \in E']} \\
&= \frac{\sum_{i \in E'} P[Z'_n = i] P[Z'_{n+1} \in E' \mid Z'_n = i]}{\sum_{i \in E'} P[Z'_n = i]} \\
&= \frac{\sum_{i \in E'} \pi'_i(n) P_{i, E'}}{\sum_{i \in E'} \pi'_i(n)} = \frac{\sum_{i \in E'} \pi'_i(n) P_{i, E'}}{a'(n)} = w'_n.
\end{aligned}$$

$$\begin{aligned}
P[\widehat{V}_{n+1} = s'_{n,1} \mid \widehat{V}_n = s'_n] &= \frac{P[\widehat{V}_n = s'_n \wedge \widehat{V}_{n+1} = s'_{n,1}]}{P[\widehat{V}_n = s'_n]} = \frac{P[\widehat{X}_{0:n} \in E' \wedge \widehat{X}_{n+1} \in \overline{E}]}{P[\widehat{X}_{0:n} \in E']} \\
&= \frac{P[Z'_{0:n} \in E' \wedge Z'_{n+1} \in \overline{E}]}{P[Z'_{0:n} \in E']} = \frac{P[Z'_n \in E' \wedge Z'_{n+1} \in \overline{E}]}{P[Z'_n \in E']} \\
&= \frac{\sum_{i \in E'} P[Z'_n = i] P[Z'_{n+1} \in \overline{E} \mid Z'_n = i]}{\sum_{i \in E'} P[Z'_n = i]} \\
&= \frac{\sum_{i \in E'} \pi'_i(n) P_{i, \overline{E}}}{\sum_{i \in E'} \pi'_i(n)} = \frac{\sum_{i \in E'} \pi'_i(n) P_{i, \overline{E}}}{a'(n)} = h'_n.
\end{aligned}$$

Case e ($P[\widehat{V}_n = s'_{k,n-k}] > 0$, $0 \leq k \leq n-1$, $a'(k, 1) > 0$): Taking into account the definition of \widehat{V} (18), the definition of Z' (3), that because of the structure of the restriction of the transition probability matrix of Z' to its non-absorbing states (6) $Z'_{0:n} \in E'$ if and only if $Z'_n \in E'$, that $P[Z'_{n+1} = f_i | Z'_n = j] = P_{j,f_i}$, $j \in \overline{E}$, and (22):

$$\begin{aligned}
P[\widehat{V}_{n+1} = f_i | \widehat{V}_n = s'_{k,n-k}] &= \frac{P[\widehat{V}_n = s'_{k,n-k} \wedge \widehat{V}_{n+1} = f_i]}{P[\widehat{V}_n = s'_{k,n-k}]} \\
&= \frac{P[\widehat{X}_{0:k} \in E' \wedge \widehat{X}_{k+1:n} \in \overline{E} \wedge \widehat{X}_{n+1} = f_i]}{P[\widehat{X}_{0:k} \in E' \wedge \widehat{X}_{k+1:n} \in \overline{E}]} \\
&= \frac{P[Z'_{0:k} \in E' \wedge Z'_{k+1:n} \in \overline{E} \wedge Z'_{n+1} = f_i]}{P[Z'_{0:k} \in E' \wedge Z'_{k+1:n} \in \overline{E}]} = \frac{P[Z'_k \in E' \wedge Z'_{k+1:n} \in \overline{E} \wedge Z'_{n+1} = f_i]}{P[Z'_k \in E' \wedge Z'_{k+1:n} \in \overline{E}]} \\
&= \frac{\sum_{j \in \overline{E}} P[Z'_k \in E' \wedge Z'_{k+1:n} \in \overline{E} \wedge Z'_n = j] P[Z'_{n+1} = f_i | Z'_n = j]}{\sum_{j \in \overline{E}} P[Z'_k \in E' \wedge Z'_{k+1:n} \in \overline{E} \wedge Z'_n = j]} \\
&= \frac{\sum_{j \in \overline{E}} \pi'_j(k, n-k) P_{j,f_i}}{\sum_{j \in \overline{E}} \pi'_j(k, n-k)} = \frac{\sum_{j \in \overline{E}} \pi'_j(k, n-k) P_{j,f_i}}{a'(k, n-k)} = v_{k,n-k}^i.
\end{aligned}$$

Similarly,

$$\begin{aligned}
P[\widehat{V}_{n+1} = s_0 | \widehat{V}_n = s'_{k,n-k}] &= \frac{P[\widehat{V}_n = s'_{k,n-k} \wedge \widehat{V}_{n+1} = s_0]}{P[\widehat{V}_n = s'_{k,n-k}]} \\
&= \frac{P[\widehat{X}_{0:k} \in E' \wedge \widehat{X}_{k+1:n} \in \overline{E} \wedge \widehat{X}_{n+1} = r]}{P[\widehat{X}_{0:k} \in E' \wedge \widehat{X}_{k+1:n} \in \overline{E}]} \\
&= \frac{P[Z'_{0:k} \in E' \wedge Z'_{k+1:n} \in \overline{E} \wedge Z'_{n+1} = a]}{P[Z'_{0:k} \in E' \wedge Z'_{k+1:n} \in \overline{E}]} = \frac{P[Z'_k \in E' \wedge Z'_{k+1:n} \in \overline{E} \wedge Z'_{n+1} = a]}{[Z'_k \in E' \wedge Z'_{k+1:n} \in \overline{E}]} \\
&= \frac{\sum_{i \in \overline{E}} P[Z'_k \in E' \wedge Z'_{k+1:n} \in \overline{E} \wedge Z'_n = i] P[Z'_{n+1} = a | Z'_n = i]}{\sum_{i \in \overline{E}} P[Z'_k \in E' \wedge Z'_{k+1:n} \in \overline{E} \wedge Z'_n = i]} \\
&= \frac{\sum_{i \in \overline{E}} \pi'_i(k, n-k) P_{i,r}}{\sum_{i \in \overline{E}} \pi'_i(k, n-k)} = \frac{\sum_{i \in \overline{E}} \pi'_i(k, n-k) P_{i,r}}{a'(k, n-k)} = q'_{k,n-k},
\end{aligned}$$

$$\begin{aligned}
P[\widehat{V}_{n+1} = s'_{k,n-k+1} | \widehat{V}_n = s'_{k,n-k}] &= \frac{P[\widehat{V}_n = s'_{k,n-k} \wedge \widehat{V}_{n+1} = s'_{k,n-k+1}]}{P[\widehat{V}_n = s'_{k,n-k}]} \\
&= \frac{P[\widehat{X}_{0:k} \in E' \wedge \widehat{X}_{k+1:n+1} \in \overline{E}]}{P[\widehat{X}_{0:k} \in E' \wedge \widehat{X}_{k+1:n} \in \overline{E}]} = \frac{P[Z'_{0:k} \in E' \wedge Z'_{k+1:n+1} \in \overline{E}]}{P[Z'_{0:k} \in E' \wedge Z'_{k+1:n} \in \overline{E}]} \\
&= \frac{P[Z'_k \in E' \wedge Z'_{k+1:n} \in \overline{E} \wedge Z'_{n+1} \in \overline{E}]}{P[Z'_k \in E' \wedge Z'_{k+1:n} \in \overline{E}]} \\
&= \frac{\sum_{i \in \overline{E}} P[Z'_k \in E' \wedge Z'_{k+1:n} \in \overline{E} \wedge Z'_n = i] P[Z'_{n+1} \in \overline{E} | Z'_n = i]}{\sum_{i \in \overline{E}} P[Z'_k \in E' \wedge Z'_{k+1:n} \in \overline{E} \wedge Z'_n = i]} \\
&= \frac{\sum_{i \in \overline{E}} \pi'_i(k, n-k) P_{i,\overline{E}}}{\sum_{i \in \overline{E}} \pi'_i(k, n-k)} = \frac{\sum_{i \in \overline{E}} \pi'_i(k, n-k) P_{i,\overline{E}}}{a'(k, n-k)} = w'_{k,n-k}.
\end{aligned}$$

Case f ($P[\widehat{V}_n = s''_n] > 0$): Taking into account the definition of \widehat{V} (18), the definition of Z'' (4), that, because (7) the restriction of the transition probability matrix of Z'' to its non-absorbing states is $\mathbf{P}_{\overline{E}, \overline{E}}$, $Z''_{0:n} \in \overline{E}$ if and only if $Z''_n \in \overline{E}$, that $P[Z''_{n+1} = f_i | Z''_n = j] = P_{j, f_i}$, $j \in \overline{E}$, and (23):

$$\begin{aligned}
P[\widehat{V}_{n+1} = f_i | \widehat{V}_n = s''_n] &= \frac{P[\widehat{V}_n = s''_n \wedge \widehat{V}_{n+1} = f_i]}{P[\widehat{V}_n = s''_n]} = \frac{P[\widehat{X}_{0:n} \in \overline{E} \wedge \widehat{X}_{n+1} = f_i]}{P[\widehat{X}_{0:n} \in \overline{E}]} \\
&= \frac{P[Z''_{0:n} \in \overline{E} \wedge Z''_{n+1} = f_i]}{P[Z''_{0:n} \in \overline{E}]} = \frac{P[Z''_n \in \overline{E} \wedge Z''_{n+1} = f_i]}{P[Z''_n \in \overline{E}]} \\
&= \frac{\sum_{j \in \overline{E}} P[Z''_n = j] P[Z''_{n+1} = f_i | Z''_n = j]}{\sum_{j \in \overline{E}} P[Z''_n = j]} \\
&= \frac{\sum_{j \in \overline{E}} \pi''_j(n) P_{j, f_i}}{\sum_{j \in \overline{E}} \pi''_j(n)} = \frac{\sum_{j \in \overline{E}} \pi''_j(n) P_{j, f_i}}{a''(n)} = v''_n{}^i.
\end{aligned}$$

Similarly,

$$\begin{aligned}
P[\widehat{V}_{n+1} = s_0 | \widehat{V}_n = s''_n] &= \frac{P[\widehat{V}_n = s''_n \wedge \widehat{V}_{n+1} = s_0]}{P[\widehat{V}_n = s''_n]} = \frac{P[\widehat{X}_{0:n} \in \overline{E} \wedge \widehat{X}_{n+1} = r]}{P[\widehat{X}_{0:n} \in \overline{E}]} \\
&= \frac{P[Z''_{0:n} \in \overline{E} \wedge Z''_{n+1} = a]}{P[Z''_{0:n} \in \overline{E}]} = \frac{P[Z''_n \in \overline{E} \wedge Z''_{n+1} = a]}{P[Z''_n \in \overline{E}]} \\
&= \frac{\sum_{i \in \overline{E}} P[Z''_n = i] P[Z''_{n+1} = a | Z''_n = i]}{\sum_{i \in \overline{E}} P[Z''_n = i]} \\
&= \frac{\sum_{i \in \overline{E}} \pi''_i(n) P_{i, r}}{\sum_{i \in \overline{E}} \pi''_i(n)} = \frac{\sum_{i \in \overline{E}} \pi''_i(n) P_{i, r}}{a''(n)} = q''_n,
\end{aligned}$$

$$\begin{aligned}
P[\widehat{V}_{n+1} = s''_{n+1} | \widehat{V}_n = s''_n] &= \frac{P[\widehat{V}_n = s''_n \wedge \widehat{V}_{n+1} = s''_{n+1}]}{P[\widehat{V}_n = s''_n]} = \frac{P[\widehat{X}_{0:n+1} \in \overline{E}]}{P[\widehat{X}_{0:n} \in \overline{E}]} \\
&= \frac{P[Z''_{0:n+1} \in \overline{E}]}{P[Z''_{0:n} \in \overline{E}]} = \frac{P[Z''_n \in \overline{E} \wedge Z''_{n+1} \in \overline{E}]}{P[Z''_n \in \overline{E}]} = \frac{\sum_{i \in \overline{E}} P[Z''_n = i] P[Z''_{n+1} \in \overline{E} | Z''_n = i]}{\sum_{i \in \overline{E}} P[Z''_n = i]} \\
&= \frac{\sum_{i \in \overline{E}} \pi''_i(n) P_{i, \overline{E}}}{\sum_{i \in \overline{E}} \pi''_i(n)} = \frac{\sum_{i \in \overline{E}} \pi''_i(n) P_{i, \overline{E}}}{a''(n)} = w''_n.
\end{aligned}$$

It remains to show that all states s_k , $k \geq 0$ are reachable, all states $s_{k,l}$, $k \geq 0$, $l \geq 1$ such that $a(k, 1) > 0$ are reachable, all states s'_k , $k \geq 0$ are reachable, all states $s'_{k,l}$, $k \geq 0$, $l \geq 1$ such that $a'(k, 1) > 0$ are reachable, all states s''_k , $k \geq 0$ are reachable, and all states f_i , $1 \leq i \leq A$ are reachable. That all states f_i , $1 \leq i \leq A$ are reachable follows from $\widehat{V}_n = f_i$ if and only if $\widehat{X}_n = f_i$ and the states f_i being reachable in \widehat{X} (because they are reachable in X). That s'_0 is reachable follows from $P[\widehat{V}_0 = s'_0] = \alpha_{E'} > 0$. That s''_0 is reachable follows from $P[\widehat{V}_0 = s''_0] = \alpha_{\overline{E}} > 0$. That s_0 is reachable follows from $\widehat{V}_n = s_0$ if and only if $\widehat{X}_n = r$ and the state r being reachable in \widehat{X} (because it is reachable in X). The reachability of the remaining states follows, then, from $w_k > 0$, $k \geq 0$, $w'_k > 0$, $k \geq 0$, $w''_k > 0$, $k \geq 0$, $h_k > 0$ for $k \geq 0$ if $a(k, 1) > 0$, $h'_k > 0$ for $k \geq 0$ if $a'(k, 1) > 0$, $w_{k,l} > 0$ for $k \geq 0$ such that $a(k, 1) > 0$ and $l \geq 1$, and $w'_{k,l} > 0$ for $k \geq 0$ such that $a'(k, 1) > 0$ and $l \geq 1$. All this follows from Proposition 2 and $a(k) > 0$, $k \geq 0$; $a'(k) > 0$,

$k \geq 0$; $a''(k) > 0$, $k \geq 0$; $a(k, l) > 0$, $l \geq 1$ for k such that $a(k, 1) > 0$; and $a'(k, l) > 0$, $l \geq 1$ for k such that $a'(k, 1) > 0$. \square

Proof of Proposition 2. Part a) Remember that $a(k) > 0$, $k \geq 0$. Using (9), (19) and (25),

$$w_k = \frac{\sum_{i \in E} \pi_i(k) P_{i, E'}}{a(k)} = \frac{\boldsymbol{\pi}(k) \mathbf{P}'_{E, E} \mathbf{1}}{a(k)} = \frac{\boldsymbol{\pi}(k+1) \mathbf{1}}{a(k)} = \frac{a(k+1)}{a(k)},$$

where $\mathbf{1}$ is a column vector with all its elements equal to 1 of appropriate dimension. Considering, then, that (8) $a(0) = 1$,

$$\prod_{i=0}^{k-1} w_i = \prod_{i=0}^{k-1} \frac{a(i+1)}{a(i)} = \frac{a(k)}{a(0)} = a(k).$$

Part b) Using (10), (20) and (26),

$$h_k = \frac{\sum_{i \in E} \pi_i(k) P_{i, \bar{E}}}{a(k)} = \frac{\boldsymbol{\pi}(k) \mathbf{P}_{E, \bar{E}} \mathbf{1}}{a(k)} = \frac{\boldsymbol{\pi}(k, 1) \mathbf{1}}{a(k)} = \frac{a(k, 1)}{a(k)}.$$

Part c) Remember that $a(k, 1) > 0$ implies $a(k, l) > 0$, $l \geq 1$. Using (11), (20) and (29),

$$w_{k, l} = \frac{\sum_{i \in \bar{E}} \pi_i(k, l) P_{i, \bar{E}}}{a(k, l)} = \frac{\boldsymbol{\pi}(k, l) \mathbf{P}_{\bar{E}, \bar{E}} \mathbf{1}}{a(k, l)} = \frac{\boldsymbol{\pi}(k, l+1) \mathbf{1}}{a(k, l)} = \frac{a(k, l+1)}{a(k, l)},$$

which implies the result.

Part d) Remember that, assuming $\alpha_{E'} > 0$, $a'(k) > 0$, $k \geq 0$. Using (13), (21) and (31),

$$w'_k = \frac{\sum_{i \in E'} \pi'_i(k) P_{i, E'}}{a'(k)} = \frac{\boldsymbol{\pi}'(k) \mathbf{P}_{E', E'} \mathbf{1}}{a'(k)} = \frac{\boldsymbol{\pi}'(k+1) \mathbf{1}}{a'(k)} = \frac{a'(k+1)}{a'(k)}.$$

Considering, then, that (12) $a'(0) = \alpha_{E'}$,

$$\prod_{i=0}^{k-1} w'_i = \prod_{i=0}^{k-1} \frac{a'(i+1)}{a'(i)} = \frac{a'(k)}{a'(0)} = \frac{a'(k)}{\alpha_{E'}}.$$

Part e) Using (14), (22) and (32),

$$h'_k = \frac{\sum_{i \in E'} \pi'_i(k) P_{i, \bar{E}}}{a'(k)} = \frac{\boldsymbol{\pi}'(k) \mathbf{P}_{E', \bar{E}} \mathbf{1}}{a'(k)} = \frac{\boldsymbol{\pi}'(k, 1) \mathbf{1}}{a'(k)} = \frac{a'(k, 1)}{a'(k)}.$$

Part f) Remember that, assuming $\alpha_{E'} > 0$, $a'(k, l) > 0$, $l \geq 1$ for k such that $a'(k, 1) > 0$. Using (15), (22) and (35),

$$w'_{k, l} = \frac{\sum_{i \in \bar{E}} \pi'_i(k, l) P_{i, \bar{E}}}{a'(k, l)} = \frac{\boldsymbol{\pi}'(k, l) \mathbf{P}_{\bar{E}, \bar{E}} \mathbf{1}}{a'(k, l)} = \frac{\boldsymbol{\pi}'(k, l+1) \mathbf{1}}{a'(k, l)} = \frac{a'(k, l+1)}{a'(k, l)},$$

which implies the result.

Part g) Remember that, assuming $\alpha_{\bar{E}} > 0$, $a''(k) > 0$, $k \geq 0$. Using (17), (23) and (38),

$$w''_k = \frac{\sum_{i \in \bar{E}} \pi''_i(k) P_{i, \bar{E}}}{a''(k)} = \frac{\boldsymbol{\pi}''(k) \mathbf{P}_{\bar{E}, \bar{E}} \mathbf{1}}{a''(k)} = \frac{\boldsymbol{\pi}''(k+1) \mathbf{1}}{a''(k)} = \frac{a''(k+1)}{a''(k)}.$$

Considering, then, that (16) $a''(0) = \alpha_{\overline{E}}$,

$$\prod_{i=0}^{k-1} w_i'' = \prod_{i=0}^{k-1} \frac{a''(i+1)}{a''(i)} = \frac{a''(k)}{a''(0)} = \frac{a''(k)}{\alpha_{\overline{E}}}. \quad \square$$

Proof of Proposition 3. We will start by considering the CTMC V_T' differing from V_T in that: (1) the teeth of the comb with the states s_k in the back have not been truncated (they are as in the CTMC V), (2) the transitions to the absorbing states f_i , $1 \leq i \leq A$, have been redirected to s_0 , and (3) the absorbing states f_i , $1 \leq i \leq A$, have been merged into state a . Let \mathcal{P} be the set of paths through which V_T can enter state a through s_K . Every $p \in \mathcal{P}$ is a sequence of states $(x_0, x_1, \dots, x_{L(p)})$, $L(p) \geq 1$, with $P[V_T(0) = x_0] > 0$, nonnull transition rate from every state x_l to x_{l+1} , $0 \leq l < L(p)$, $x_{L(p)-1} = s_K$, and $x_{L(p)} = a$. Let \mathcal{P}' be the set of paths through which V_T' can enter state a through s_K . By construction, $\mathcal{P} \subset \mathcal{P}'$. Also, denoting by $P(p)$, $p \in \mathcal{P}$ the probability that V_T will follow path p and by $P'(p)$, $p \in \mathcal{P}'$ the probability that V_T' will follow path p , we have $P(p) = P'(p)$, $p \in \mathcal{P}$. Further, since the states of V_T different from f_i , $1 \leq i \leq A$, have the same output rates in V_T' as in V_T , denoting by $D(p)$, $p \in \mathcal{P}$ the duration of path p in V_T (time till V_T hits a) and by $D'(p)$, $p \in \mathcal{P}'$ the duration of path p in V_T' , $D(p)$ and $D'(p)$, $p \in \mathcal{P}$ have the same distribution. Then,

$$m_{K,L,M}^{e'''}(t) = r_{\max} \sum_{p \in \mathcal{P}} P(p)P[D(p) \leq t] \leq r_{\max} \sum_{p \in \mathcal{P}'} P'(p)P[D'(p) \leq t]. \quad (48)$$

Consider now the CTMC V_T'' obtained from V_T' by substituting the states with output rate $\Lambda_{\overline{E}}$ (states $s_{k,l}$, $k \in \gamma_K$, $l \geq 1$; if $\alpha_{E'} > 0$, states $s'_{k,l}$, $k \in \gamma'_L$, $1 \leq l \leq L_k$; and, if $\alpha_{\overline{E}} > 0$, states s''_k , $0 \leq k \leq M$) by instantaneous switches with jump probabilities to their successors equal to the jump probabilities of those states in V_T' . Regarding the paths $p \in \mathcal{P}'$, this is equivalent to neglect in every path the holding times in the states with output rate $\Lambda_{\overline{E}}$. Then, denoting by $D''(p)$, $p \in \mathcal{P}'$ the duration of the path p in V_T'' neglecting the holding times in the states with output rate $\Lambda_{\overline{E}}$, we have $P[D''(p) \leq t] \geq P[D'(p) \leq t]$, $p \in \mathcal{P}'$ and

$$P[\text{by time } t, V_T'' \text{ has entered } a \text{ through } s_K] = \sum_{p \in \mathcal{P}'} P'(p)P[D''(p) \leq t] \geq \sum_{p \in \mathcal{P}'} P'(p)P[D'(p) \leq t],$$

which combined with (48) gives

$$m_{K,L,M}^{e'''}(t) \leq r_{\max} P[\text{by time } t, V_T'' \text{ has entered } a \text{ through } s_K]. \quad (49)$$

We will derive next the initial probability distribution and the state transition diagram of V_T'' . By construction, the initial probability distribution of V_T' is

$$P[V_T'(0) = s_0] = P[V_T(0) = s_0] = \alpha_r,$$

$$\text{for } \alpha_{E'} > 0, \quad P[V_T'(0) = s'_0] = P[V_T(0) = s'_0] = \alpha_{E'},$$

$$\text{for } \alpha_{\overline{E}} > 0, \quad P[V_T'(0) = s''_0] = P[V_T(0) = s''_0] = \alpha_{\overline{E}},$$

$$P[V_T'(0) = a] = P[V_T(0) = a] + \sum_{i=1}^A P[V_T(0) = f_i] = \sum_{i=1}^A \alpha_{f_i},$$

$$P[V_T'(0) = i] = 0, \quad i \notin \{s_0, s_0', s_0'', a\}.$$

The initial probability distribution and state transition diagram of V_T'' can, then, be obtained from the initial probability distribution and state transition diagram of V_T' by looking at the states with output rate $\Lambda_{\overline{E}}$ as instantaneous switches and eliminating those switches in a way similar as vanishing markings are eliminated in GSPNs [6]. Figure 10 depicts the state transition diagram of V_T' for the case $\alpha_{E'} > 0$, $\alpha_{\overline{E}} > 0$ and $A = 1$ with states with output rate $\Lambda_{\overline{E}}$ replaced by instantaneous switches (drawn as squares), indicating the switching probabilities.

Assume $\alpha_{\overline{E}} > 0$. The only instantaneous switch with nonnull initial probability is state s_0'' . That switch has initial probability $\alpha_{\overline{E}}$. Using Proposition 2g, the probability that V_T' (with states with output rate $\Lambda_{\overline{E}}$ replaced by instantaneous switches) will follow path $(s_0'', s_1'', \dots, s_M'', a)$ is $\alpha_{\overline{E}} \prod_{k=0}^{M-1} w_k'' = a''(M)$. With probability $\alpha_{\overline{E}}(1 - \prod_{k=0}^{M-1} w_k'') = \alpha_{\overline{E}} - a''(M)$, V_T' will start at s_0'' and will exit the subset of states $\{s_0'', s_1'', \dots, s_M''\}$ through state s_0 . Then, the initial probability distribution of V_T'' will be

$$P[V_T''(0) = s_0] = P[V_T'(0) = s_0] + \alpha_{\overline{E}} - a''(M) = \alpha_r + \alpha_{\overline{E}} - a''(M),$$

$$\text{for } \alpha_{E'} > 0, \quad P[V_T''(0) = s_0'] = P[V_T'(0) = s_0'] = \alpha_{E'},$$

$$P[V_T''(0) = a] = P[V_T'(0) = a] + a''(M) = \sum_{i=1}^A \alpha_{f_i} + a''(M),$$

$$P[V_T''(0) = i] = 0, \quad i \notin \{s_0, s_0', a\}.$$

If $\alpha_{E'} > 0$, elimination of the switches $s_{k,l}', k \in \gamma_L', 1 \leq l \leq L_k$, introduces transitions in V_T'' from states $s_k', k \in \gamma_L'$, to states a and s_0 . Taking into account that the probability that, starting at $s_{k,1}', V_T'$ will exit the subset of states $\{s_{k,1}', s_{k,2}', \dots, s_{k,L_k}'\}$ through state a is $\prod_{i=1}^{L_k-1} w_{k,i}'$, for each $k \in \gamma_L'$, elimination of the switches $s_{k,1}', s_{k,2}', \dots, s_{k,L_k}'$ will introduce a transition rate from s_k' to a equal to $\phi_k' \Lambda_E$, with $\phi_k' = h_k' \prod_{i=1}^{L_k-1} w_{k,i}'$. Assuming conventionally that 0 by a non-defined quantity is 0, V_T'' will have that transition rate from every state $s_k', 0 \leq k \leq L-1$, to a . Since the output rate of every state s_k' is Λ_E , elimination of the switches $s_{k,1}', s_{k,2}', \dots, s_{k,L_k}'$ will introduce a transition rate from every state $s_k', 0 \leq k \leq L-1$, to s_0 equal to $\Lambda_E - w_k' \Lambda_E - \phi_k' \Lambda_E = (1 - w_k' - \phi_k') \Lambda_E$. Elimination of the states $s_{k,l}, k \in \gamma_K$, will introduce transition rates from $s_k, k \in \gamma_K, k > 0$, to state s_0 . The probability that, starting at a state $s_{k,1}, k \in \gamma_K, V_T'$ will exit the subset of states $\{s_{k,1}, s_{k,2}, \dots\}$ through state s_0 is 1. This is a consequence of the fact that the limit for $m \rightarrow \infty$ of $\prod_{i=1}^m w_{k,i} = a(k, m+1)/a(k, 1)$, by Proposition 2c, is 0 because, as shown in the proof of Theorem 4, $a(k, l)$ decreases geometrically fast with l . Then, since the output rate of the states $s_k, 1 \leq k \leq K-1$, is Λ_E , elimination of the switches $s_{k,l}, k \in \gamma_K, l \geq 1$, will result in a transition rate to state s_0 from every state $s_k, 1 \leq k \leq K-1$, equal to $\Lambda_E - w_k \Lambda_E = (1 - w_k) \Lambda_E$. Figure 11 depicts the state transition diagram of V_T'' for the case $\alpha_{E'} > 0$. For the case $\alpha_{E'} = 0$, the states s_k' and the transitions from those states disappear.

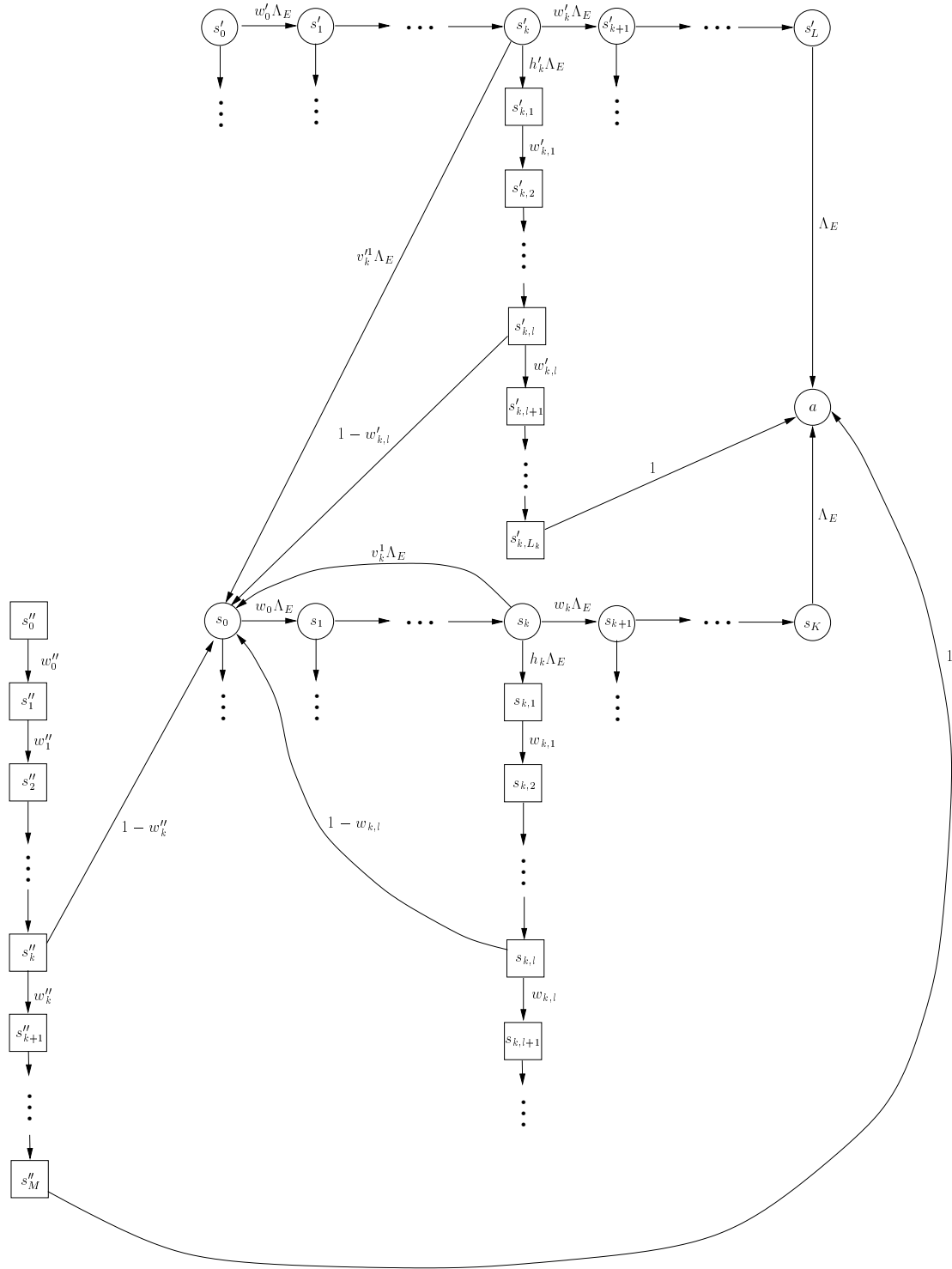


Figure 10: State transition diagram of V'_T with states with outpurate $\Lambda_{\bar{E}}$ replaced by instantaneous switches for the case $\alpha_{E'} > 0$, $\alpha_{\bar{E}} > 0$, and $A = 1$ (there can exist transitions to s_0 from any state s_k , $0 < k \leq K - 1$, any state $s_{k,l}$, $l \geq 1$, any state s'_k , $0 \leq k \leq L - 1$, any state $s'_{k,l}$, $1 \leq k \leq L_k - 1$, and any state s''_k , $0 \leq k \leq M - 1$).

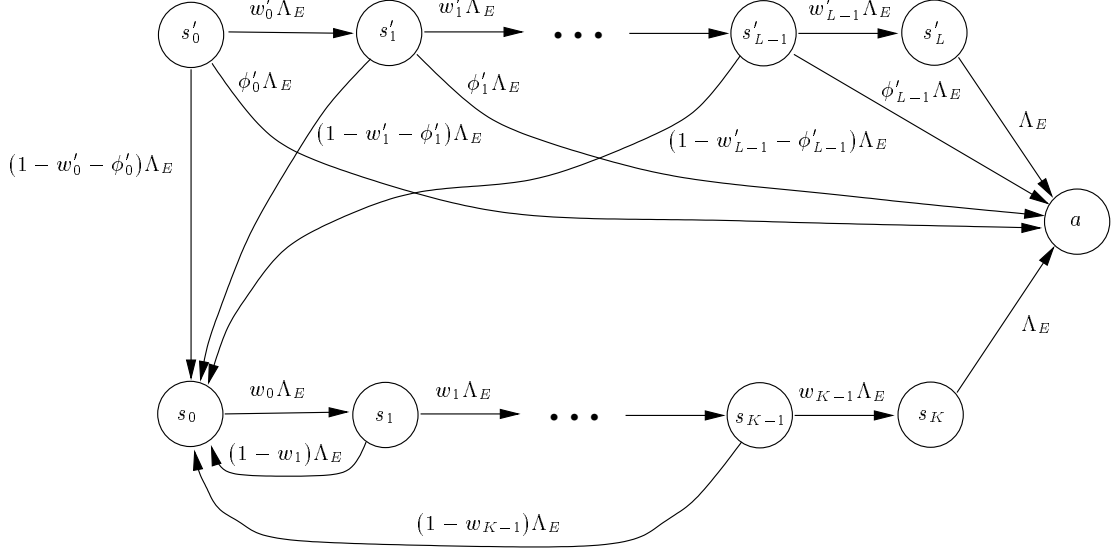


Figure 11: State transition diagram of the CTMC V_T'' for the case $\alpha_{E'} > 0$.

Let \widehat{V}_T'' be the DTMC obtained by randomizing V_T'' with rate Λ_E . The initial probability distribution of \widehat{V}_T'' is the same as the initial probability distribution of V_T'' . The state transition diagram of \widehat{V}_T'' for the case $\alpha_{E'} > 0$ is depicted in Figure 12. For the case $\alpha_{E'} = 0$, the states s'_k and the transitions from those states disappear.

Let $c(k) = P[\text{by step } k, \widehat{V}_T'' \text{ has entered } a \text{ through } s_K]$. Using the probabilistic identity of V_T'' and $\{(\widehat{V}_T'')_{Q(t)}; t \geq 0\}$, where Q is a Poisson process with arrival rate Λ_E independent of \widehat{V}_T'' ,

$$P[\text{by time } t, V_T'' \text{ has entered } a \text{ through } s_K] = \sum_{k=0}^{\infty} c(k) e^{-\Lambda_E t} \frac{(\Lambda_E t)^k}{k!}$$

and, therefore (49),

$$m_{K,L,M}^{e'''}(t) \leq r_{\max} \sum_{k=0}^{\infty} c(k) e^{-\Lambda_E t} \frac{(\Lambda_E t)^k}{k!}. \quad (50)$$

Consider now, for the case $\alpha_{E'} > 0$, the DTMC Y obtained from \widehat{V}_T'' by introducing an absorbing state f_1 with null initial probability and redirecting to f_1 the transitions from the states $s'_0, s'_1, \dots, s'_{L-1}$ to a . The DTMC Y is a particular instance of the DTMC $\widehat{V}_{K,L}$ considered in [9]. For the case $\alpha_{E'} = 0$, let Y be the DTMC obtained from \widehat{V}_T'' by introducing an absorbing state f_1 with null initial probability. In that case, Y is a particular instance of the DTMC \widehat{V}_K considered in [9]. Trivially, $c(k)$ is also the probability that, by step k , Y has entered a through s_K . Then, denoting by I_c the indicator function returning the value 1 if condition c is satisfied and the value 0 otherwise, applying Proposition 3 of [9] and using Proposition 2a, for the case $\alpha_{E'} > 0$,

$$\begin{aligned} c(k) &\leq I_{k>K} P[(\widehat{V}_T'')_0 \in \{s_0, s'_0\}] (k-K) \prod_{i=0}^{K-1} w_i \\ &= I_{k>K} (\alpha_r + \alpha_{\overline{E}} - a''(M) + \alpha_{E'}) (k-K) a(K) \\ &= I_{k>K} (\alpha_S - a''(M)) (k-K) a(K). \end{aligned}$$

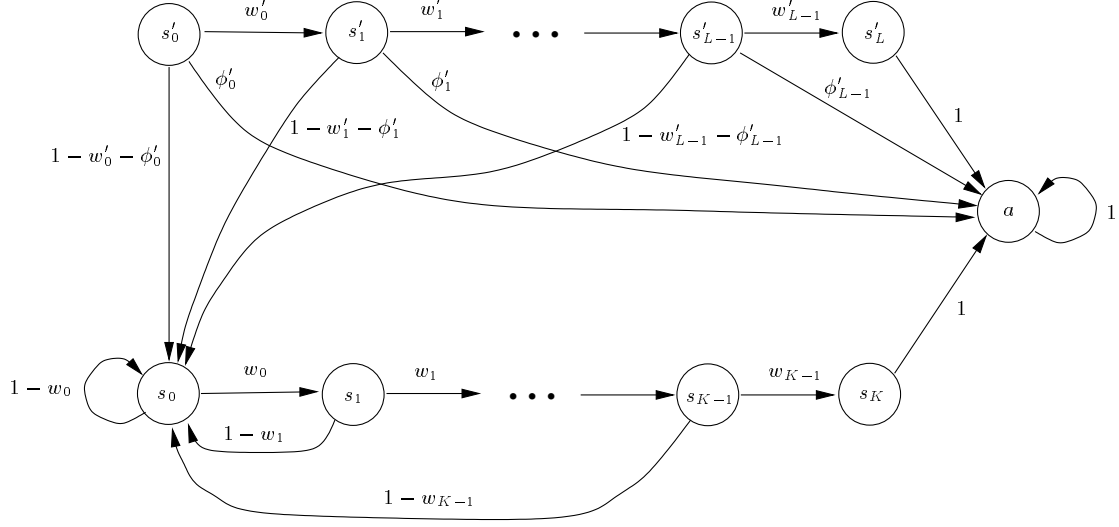


Figure 12: State transition diagram of the DTMC \hat{V}_T'' for the case $\alpha_{E'} > 0$.

Similarly, for the case $\alpha_{E'} = 0$,

$$\begin{aligned}
c(k) &\leq I_{k>K} P[(\hat{V}_T'')_0 = s_0] (k - K) \prod_{i=0}^{K-1} w_i \\
&= I_{k>K} (\alpha_r + \alpha_{\bar{E}} - a''(M)) (k - K) a(K) \\
&= I_{k>K} (\alpha_S - a''(M)) (k - K) a(K).
\end{aligned}$$

Combining $c(k) = I_{k>K} (\alpha_S - a''(M)) (k - K) a(K)$ with (50) gives the result asserted by the proposition. \square

Proof of Proposition 4. Consider, first, the CTMC $V_{T,k}'$ differing from V_T in that: (1) the teeth of the comb with the states s_k in the back hanging from states s_l , $k + 1 \leq l \leq K - 1$, have not been truncated (they are as in the CTMC V), (2) the transitions to the absorbing states f_i , $1 \leq i \leq A$, have been redirected to s_0 , and (3) the absorbing states f_i , $1 \leq i \leq A$, have been merged into state a . Consider, next, the CTMC $V_{T,k}''$ obtained from $V_{T,k}'$ by substituting the states with output rate $\Lambda_{\bar{E}}$ (states $s_{l,m}$, $l \in \gamma_K$, $0 \leq l \leq k$, $1 \leq m \leq K_l$; states $s_{l,m}$, $l \in \gamma_K$, $k + 1 \leq l \leq K - 1$, $m \geq 1$; if $\alpha_{E'} > 0$, states $s'_{l,m}$, $l \in \gamma'_L$, $1 \leq m \leq L_l$; and, if $\alpha_{\bar{E}} > 0$, states s'_l , $0 \leq l \leq M$) by instantaneous switches with jump probabilities to their successors equal to the jump probabilities of those states in $V_{T,k}'$. In a way similar as it was shown in the proof of Proposition 3 that $m_{K,L,M}^{e''}(t) \leq r_{\max} P$ [by time t , $V_{T,k}''$ has entered a through s_K], it can be shown that

$$m_{K,L,M,k}^{e''}(t) \leq r_{\max} P[\text{by time } t, V_{T,k}'' \text{ has entered } a \text{ through } s_k], \quad k \in \gamma_K. \quad (51)$$

We will derive next the initial probability distribution and the state transition diagram of $V_{T,k}''$. By construction, the initial probability distribution of $V_{T,k}'$ is:

$$\begin{aligned}
P[V_{T,k}'(0) = s_0] &= P[V_T(0) = s_0] = \alpha_r, \\
\text{for } \alpha_{E'} > 0, \quad P[V_{T,k}'(0) = s'_0] &= P[V_T(0) = s'_0] = \alpha_{E'},
\end{aligned}$$

$$\begin{aligned}
& \text{for } \alpha_{\overline{E}} > 0, \quad P[V'_{T,k}(0) = s''_0] = P[V_T(0) = s''_0] = \alpha_{\overline{E}}, \\
P[V'_{T,k}(0) = a] &= P[V_T(0) = a] + \sum_{i=1}^A P[V_T(0) = f_i] = \sum_{i=1}^A \alpha_{f_i}, \\
P[V'_{T,k}(0) = i] &= 0, \quad i \notin \{s_0, s'_0, s''_0, a\}.
\end{aligned}$$

As in the proof of Proposition 3, the initial probability distribution and state transition diagram of $V''_{T,k}$ can be obtained from the initial probability distribution and state transition diagram of $V'_{T,k}$ by looking at the states with output rate $\Lambda_{\overline{E}}$ as instantaneous switches and eliminating those switches in a way similar as vanishing markings are eliminated in GSPNs [6]. Figure 13 depicts the state transition diagram of $V'_{T,k}$ for the case $\alpha_{E'} > 0$, $\alpha_{\overline{E}} > 0$ and $A = 1$ with states with output rate $\Lambda_{\overline{E}}$ replaced by instantaneous switches (drawn as squares), indicating the switching probabilities. The initial probability distribution of $V''_{T,k}$ only differs from the initial probability distribution of $V'_{T,k}$ because of the elimination of the switches $s''_0, s''_1, \dots, s''_M$ and, similarly as in the proof of Proposition 3, can be shown to be

$$\begin{aligned}
P[V''_{T,k}(0) = s_0] &= P[V'_{T,k}(0) = s_0] + \alpha_{\overline{E}} - a''(M) = \alpha_r + \alpha_{\overline{E}} - a''(M), \\
& \text{for } \alpha_{E'} > 0, \quad P[V''_{T,k}(0) = s'_0] = P[V'_{T,k}(0) = s'_0] = \alpha_{E'}, \\
P[V''_{T,k}(0) = a] &= P[V'_{T,k}(0) = a] + a''(M) = \sum_{i=1}^A \alpha_{f_i} + a''(M), \\
P[V''_{T,k}(0) = i] &= 0, \quad i \notin \{s_0, s'_0, a\}.
\end{aligned}$$

As in the CTMC $V''_{T,k}$ considered in the proof of Proposition 3, if $\alpha_{E'} > 0$, elimination of the switches $s'_{l,m}$, $l \in \gamma'_L$, $1 \leq m \leq L_l$, will introduce transition rates from every state s'_l , $l \in \gamma'_L$, to a equal to $\phi'_l \Lambda_E$, with $\phi'_l = h'_l \prod_{i=1}^{L_l-1} w'_{l,i}$ and from every state s'_l , $l \in \gamma'_L$, to s_0 equal to $(1 - w'_l - \phi'_l) \Lambda_E$. Assuming conventionally that 0 by a non-defined quantity is 0, $V''_{T,k}$ will have those transition rates from every state s'_l , $0 \leq l \leq L - 1$. Elimination of the switches $s_{l,m}$, $l \in \gamma_K$, $0 \leq l \leq k$, will introduce a transition rate from every state s_l , $l \in \gamma_K$, $0 \leq l \leq k$, to state a of value $\phi_l \Lambda_E$, with

$$\phi_l = h_l \prod_{i=1}^{K_k-1} w_{l,i}. \tag{52}$$

Assuming conventionally that 0 by a non-defined quantity is 0, $V''_{T,k}$ will have a transition rate $\phi_l \Lambda_E$ from every state s_l , $0 \leq l \leq k$, to state a . Since the output rate of every state s_l , $0 < l \leq k$ is Λ_E , elimination of those switches will introduce a transition rate from every state s_l , $0 < l \leq k$ to state s_0 of value $\Lambda_E - w_l \Lambda_E - \phi_l \Lambda_E = (1 - w_l - \phi_l) \Lambda_E$. Elimination of the switches $s_{l,m}$, $l \in \gamma_K$, $k + 1 \leq l \leq K - 1$, $m \geq 1$, will introduce transition rates from s_l , $l \in \gamma_K$, $k + 1 \leq l \leq K - 1$, to state s_0 . The probability that, starting at a state $s_{l,1}$, $l \in \gamma_K$, $k + 1 \leq l \leq K - 1$, $V'_{T,k}$ will exit the subset of states $\{s_{l,1}, s_{l,2}, \dots\}$ through state s_0 is 1. This is a consequence of the fact that the limit for $m \rightarrow \infty$ of $\prod_{i=1}^m w_{l,i} = a(l, m + 1)/a(l, 1)$, by Proposition 2c, is 0 because, as shown in the proof of Theorem 4, $a(l, m)$ decreases geometrically fast with m . Then, since the output rate of the states s_l , $k + 1 \leq l \leq K - 1$, is Λ_E , elimination of the switches $s_{l,m}$, $l \in \gamma_K$, $k + 1 \leq l \leq K - 1$, $m \geq 1$, will result in a transition rate to state s_0 from every state s_l , $k + 1 \leq l \leq K - 1$, equal

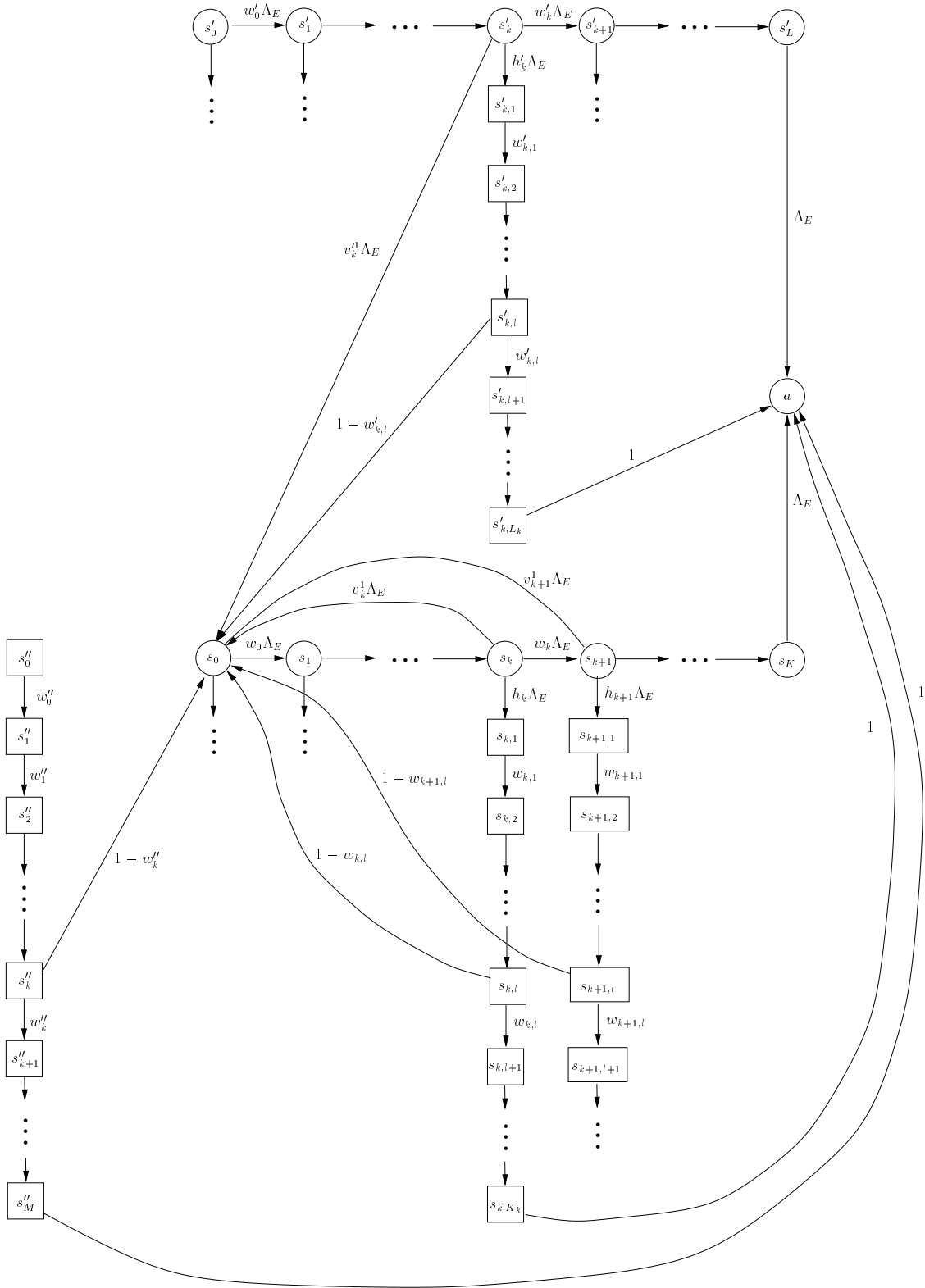


Figure 13: State transition diagram of $V'_{T,k}$ with states with outpurate $\Lambda_{\overline{E}}$ replaced by instantaneous switches for the case $\alpha_{E'} > 0$, $\alpha_{\overline{E}} > 0$, and $A = 1$ (there can exist transitions to s_0 from any state s_k , $0 < k \leq K - 1$, any state $s_{l,m}$, $0 \leq l \leq k$, $1 \leq m \leq K_k - 1$, any state $s_{l,m}$, $k + 1 \leq l \leq K - 1$, $m \geq 1$, any state s'_l , $0 \leq l \leq L - 1$, any state $s'_{l,m}$, $1 \leq m \leq L_l - 1$, and any state s''_l , $0 \leq l \leq M - 1$).

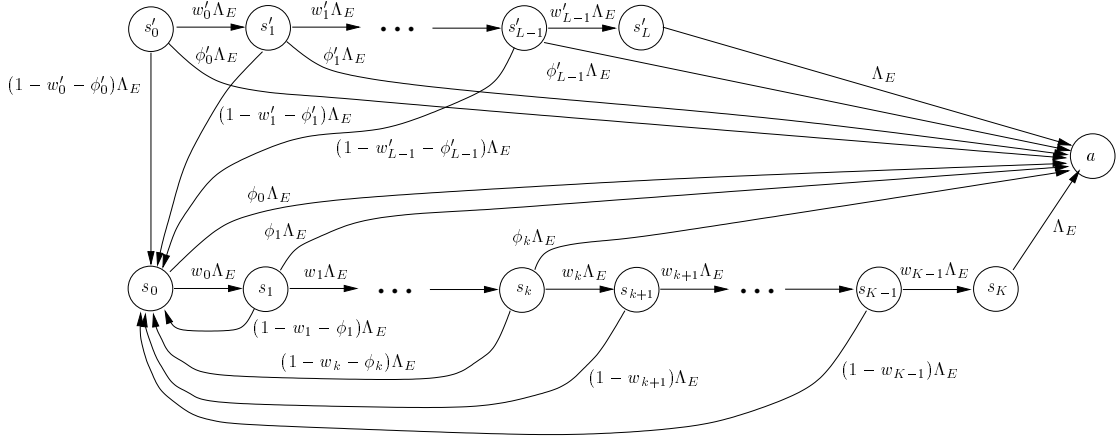


Figure 14: State transition diagram of the CTMC $V''_{T,k}$ for the case $\alpha_{E'} > 0$.

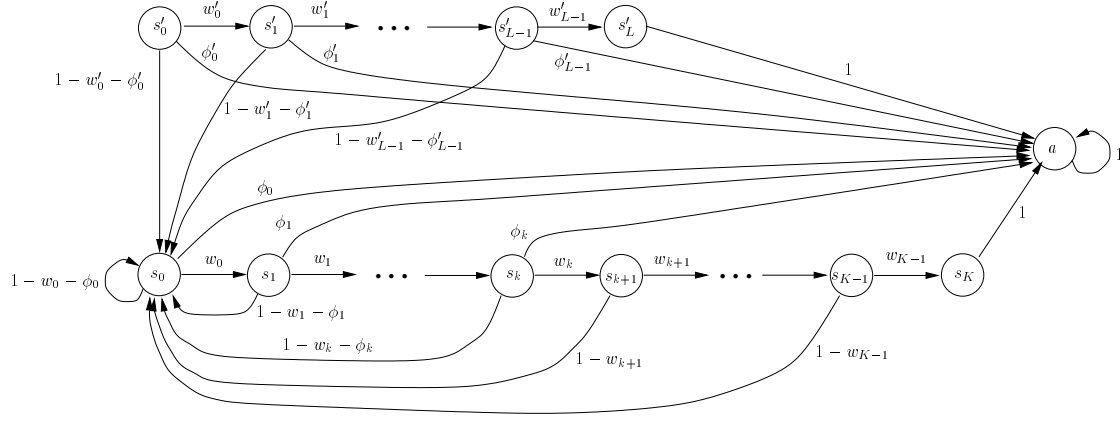


Figure 15: State transition diagram of the CTMC $\widehat{V}''_{T,k}$ for the case $\alpha_{E'} > 0$.

to $\Lambda_E - w_l \Lambda_E = (1 - w_l) \Lambda_E$. Figure 14 depicts the state transition diagram of $V''_{T,k}$ for the case $\alpha_{E'} > 0$. For the case $\alpha_{E'} = 0$, the states s'_l and the transitions from those states disappear.

Let $\widehat{V}''_{T,k}$, $k \in \gamma_K$, be the DTMC obtained by randomizing $V''_{T,k}$ with rate Λ_E . The initial probability distribution of $\widehat{V}''_{T,k}$ is the same as the initial probability distribution of $V''_{T,k}$. The state transition diagram of $\widehat{V}''_{T,k}$ for the case $\alpha_{E'} > 0$ is depicted in Figure 15. For the case $\alpha_{E'} = 0$, the states s'_l and the transitions from those states disappear.

Let $c_k(l) = P[\text{by step } l, \widehat{V}''_{T,k} \text{ has entered } a \text{ through } s_k]$, $k \in \gamma_K$. Using the probabilistic identity of $V''_{T,k}$ and $\{(\widehat{V}''_{T,k})_{Q(t)}; t \geq 0\}$, where Q is a Poisson process with arrival rate Λ_E independent of $\widehat{V}''_{T,k}$,

$$P[\text{by time } t, V''_{T,k} \text{ has entered } a \text{ through } s_k] = \sum_{l=0}^{\infty} c_k(l) e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!}, \quad k \in \gamma_K$$

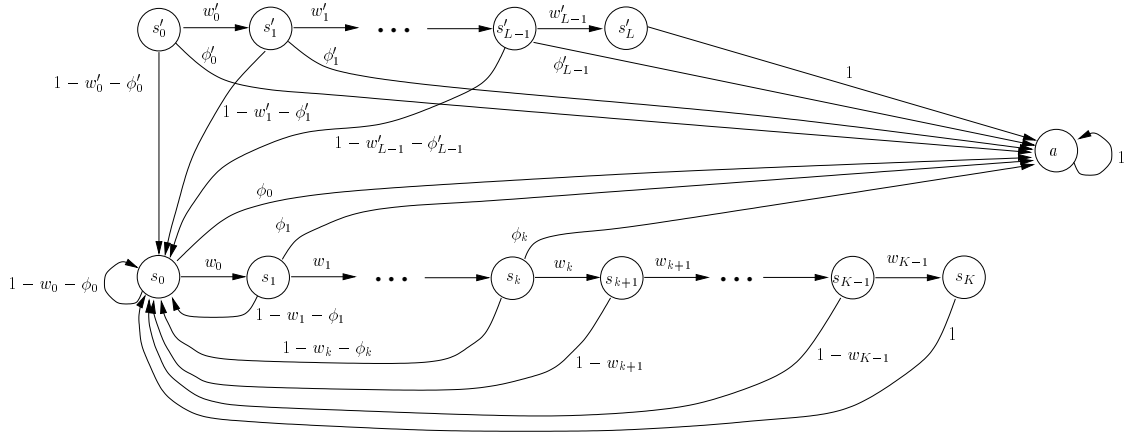


Figure 16: State transition diagram of the DTMC $\hat{V}_{T,k}^{III}$ for the case $\alpha_{E'} > 0$.

and, therefore (51),

$$m_{K,L,M,k}^{e'''}(t) \leq r_{\max} \sum_{l=0}^{\infty} c_k(l) e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!}, \quad k \in \gamma_K. \quad (53)$$

Let $\hat{V}_{T,k}^{III}$, $k \in \gamma_K$, be the DTMC obtained from $\hat{V}_{T,k}^{II}$ by directing the transition probability 1 from s_K to a to state s_0 . The state transition diagram of $\hat{V}_{T,k}^{III}$ for the case $\alpha_{E'} > 0$ is depicted in Figure 16. For the case $\alpha_{E'} = 0$, the states s'_l and the transitions from those states disappear. Let $c'_k(l) = P[\text{by step } l, \hat{V}_{T,k}^{III} \text{ has entered } a \text{ through } s_k]$, $k \in \gamma_K$. Let \mathcal{P}_k be the set of paths of $\hat{V}_{T,k}^{II}$ which enter state a through s_k . Every path p is a sequence $(x_0, x_1, \dots, x_{L(p)})$, where $L(p)$ is the length of the path, with $P[(\hat{V}_{T,k}^{II})_0 = x_0] > 0$, nonnull transition probability from every state x_l to x_{l+1} , $0 \leq l < L(p)$, $x_{L(p)-1} = s_k$ and $x_{L(p)} = a$. Let $P(p)$, $p \in \mathcal{P}_k$, be the probability that $\hat{V}_{T,k}^{II}$ will follow path p . We have

$$c_k(l) = \sum_{p \in \mathcal{P}_k, L(p) \leq l} P(p).$$

Let \mathcal{P}'_k be the set of paths of $\hat{V}_{T,k}^{III}$ which enter state a through s_k . Let $P'(p)$, $p \in \mathcal{P}'_k$ be the probability that $\hat{V}_{T,k}^{III}$ will follow path p . We have

$$c'_k(l) = \sum_{p \in \mathcal{P}'_k, L(p) \leq l} P'(p).$$

By construction, $\mathcal{P}_k \subset \mathcal{P}'_k$ and, for each $p \in \mathcal{P}_k$, $P'(p) = P(p)$. Then, $c_k(l) \leq c'_k(l)$.

Let $\hat{V}_{T,k}^{IV}$, $k \in \gamma_K$ be the DTMC obtained from $\hat{V}_{T,k}^{III}$ by replacing the states $s_{k+1}, s_{k+2}, \dots, s_K$ by instantaneous switches. The state transition diagram of $\hat{V}_{T,k}^{IV}$ for the case $\alpha_{E'} > 0$ is depicted in Figure 17. For the case $\alpha_{E'} = 0$, the states s'_l and the transitions from those states disappear. Let $c''_k(l) = P[\text{by step } l, \hat{V}_{T,k}^{IV} \text{ has entered } a \text{ through } s_k]$, $k \in \gamma_K$. For each $p \in \mathcal{P}'_k$, let $M(p)$ be the number of instances of states in the subset $\{s_{k+1}, \dots, s_K\}$ in path p . By construction,

$$c''_k(l) = \sum_{p \in \mathcal{P}'_k, L(p) - M(p) \leq l} P'(p)$$

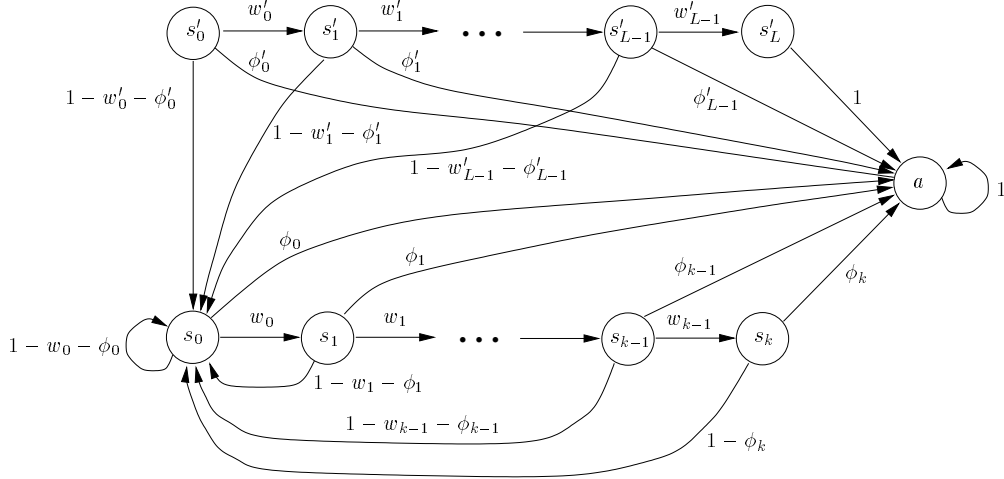


Figure 17: State transition diagram of the DTMC $\widehat{V}_{T,k}^{IV}$ for the case $\alpha_{E'} > 0$.

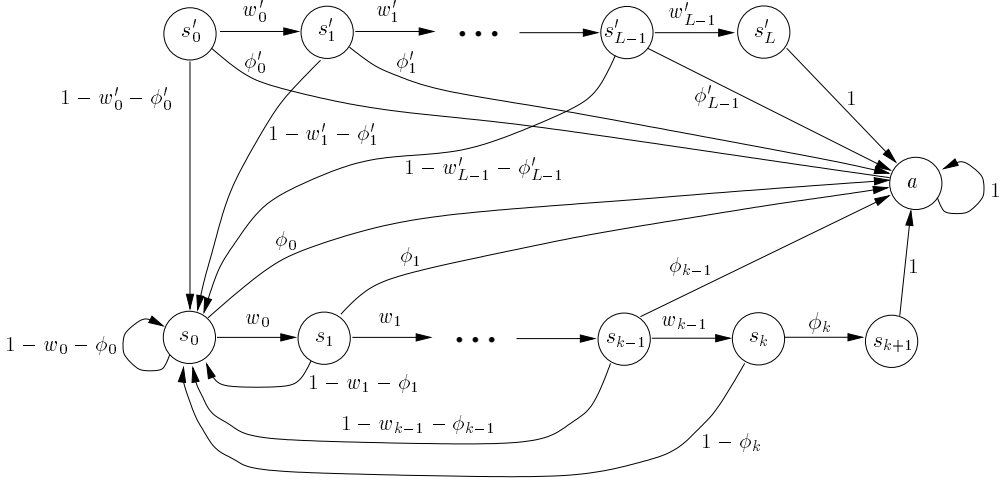


Figure 18: State transition diagram of the DTMC $\widehat{V}_{T,k}^V$ for the case $\alpha_{E'} > 0$.

and, then, $c'_k(l) \leq c''_k(l)$.

Finally, let $\widehat{V}_{T,k}^V$, $k \in \gamma_K$, be the DTMC obtained from $\widehat{V}_{T,k}^{IV}$ by delaying the entry from s_k to a by one step by adding a state s_{k+1} , redirecting to s_{k+1} the transition probability ϕ_k from s_k to a , and adding a transition probability 1 from s_{k+1} to a . The state transition diagram of $\widehat{V}_{T,k}^V$ for the case $\alpha_{E'} > 0$ is depicted in Figure 18. For the case $\alpha_{E'} = 0$, the states s'_l and the transitions from those states disappear. Let $c'''_k(l) = P[\text{by step } l, \widehat{V}_{T,k}^V \text{ has entered } a \text{ through } s_{k+1}]$, $k \in \gamma_K$. By construction of $\widehat{V}_{T,k}^V$, $c''_k(l) = c'''_k(l+1)$, $k \in \gamma_K$. Summarizing, we have $c_k(l) \leq c'''_k(l+1)$, $k \in \gamma_K$, which combined with (53) gives

$$m_{K,L,M,k}^{e'''}(t) \leq r_{\max} \sum_{l=0}^{\infty} c'''_k(l+1) e^{-\Lambda E t} \frac{(\Lambda E t)^l}{l!}, \quad k \in \gamma_K. \quad (54)$$

Consider now, for the case $\alpha_{E'} > 0$, the DTMC Y obtained from $\widehat{V}_{T,k}^V$ by introducing an absorbing state f_1 with null initial probability and redirecting to f_1 the transitions from the states $s'_0, s'_1, \dots, s'_{L-1}$ and the states s_0, s_1, \dots, s_{k-1} to a . The DTMC Y is a particular instance of the DTMC $\widehat{V}_{K,L}$ considered in [9]. For the case $\alpha_{E'} = 0$, let Y be the DTMC obtained from $\widehat{V}_{T,k}^V$ by introducing an absorbing state f_1 with null initial probability and directing to f_1 the transitions from the states s_0, s_1, \dots, s_{k-1} to a . In that case, Y is a particular instance of the DTMC \widehat{V}_K considered in [9]. Trivially, $c_k'''(l)$ is also the probability that, by step l , Y has entered a through s_{k+1} . Then, denoting by I_c the indicator function returning the value 1 if condition c is satisfied and the value 0 otherwise, applying Proposition 3 of [9], taking into account that the initial probability distribution of Y is the same as the initial probability distribution of $\widehat{V}_{T,k}''$, using (52), and using Proposition 2a, b and c, for the case $\alpha_{E'} > 0$,

$$\begin{aligned}
c_k'''(l) &\leq I_{l>k+1} P \left[(\widehat{V}_{T,k}'')_0 \in \{s_0, s'_0\} \right] (l-k-1) \left(\prod_{i=0}^{k-1} w_i \right) \phi_k \\
&= I_{l>k+1} (\alpha_r + \alpha_{\overline{E}} - a''(M) + \alpha_{E'}) (l-k-1) \left(\prod_{i=0}^{k-1} w_i \right) h_k \left(\prod_{i=1}^{K_k-1} w_{k,i} \right) \\
&= I_{l>k+1} (\alpha_S - a''(M)) (l-k-1) a(k) \frac{a(k,1)}{a(k)} \frac{a(k, K_k)}{a(k,1)} \\
&= I_{l>k+1} (\alpha_S - a''(M)) (l-k-1) a(k, K_k), \quad k \in \gamma_K.
\end{aligned}$$

Similarly, for the case $\alpha_{E'} = 0$,

$$\begin{aligned}
c_k'''(l) &\leq I_{l>k+1} P \left[(\widehat{V}_{T,k}'')_0 = s_0 \right] (l-k-1) \left(\prod_{i=0}^{k-1} w_i \right) \phi_k \\
&= I_{l>k+1} (\alpha_r + \alpha_{\overline{E}} - a''(M)) (l-k-1) \left(\prod_{i=0}^{k-1} w_i \right) h_k \left(\prod_{i=1}^{K_k-1} w_{k,i} \right) \\
&= I_{l>k+1} (\alpha_S - a''(M)) (l-k-1) a(k) \frac{a(k,1)}{a(k)} \frac{a(k, K_k)}{a(k,1)} \\
&= I_{l>k+1} (\alpha_S - a''(M)) (l-k-1) a(k, K_k), \quad k \in \gamma_K.
\end{aligned}$$

Combining $c_k'''(l) = I_{l>k+1} (\alpha_S - a''(M)) (l-k-1) a(k, K_k)$, $k \in \gamma_K$ with (54) gives the result asserted by the proposition. \square

Proof of Theorem 4. The states in \overline{E} of the DTMC Z'' are transient. Since the restriction of the transition probability matrix of Z'' to \overline{E} is (7) $\mathbf{P}_{\overline{E}, \overline{E}}^T$, this implies [37, Theorem 4.3] that the elements of $\mathbf{P}_{\overline{E}, \overline{E}}^l$ decrease geometrically fast with l and, therefore, that $\|(\mathbf{P}_{\overline{E}, \overline{E}}^T)^l\|_1 = \|(\mathbf{P}_{\overline{E}, \overline{E}}^l)^T\|_1$ decreases geometrically fast with l . We will start by proving $K_{\overline{E}} = \sum_{k \in \gamma_K} K_k = O((\log(\Lambda_E t / \varepsilon))^2)$. Using $\boldsymbol{\pi}(k, l)^T = (\mathbf{P}_{\overline{E}, \overline{E}}^T)^{l-1} \mathbf{P}_{\overline{E}, \overline{E}}^T \boldsymbol{\pi}(k)^T$, which follows from (10) and (11), $\|\boldsymbol{\pi}(k)^T\|_1 \leq 1$ and $\|\mathbf{P}_{\overline{E}, \overline{E}}^T\|_1 \leq 1$, we get (20)

$$a(k, l) = \|\boldsymbol{\pi}(k, l)^T\|_1 \leq \|(\mathbf{P}_{\overline{E}, \overline{E}}^T)^{l-1}\|_1 \|\mathbf{P}_{\overline{E}, \overline{E}}^T\|_1 \|\boldsymbol{\pi}(k)^T\|_1 \leq \|(\mathbf{P}_{\overline{E}, \overline{E}}^T)^{l-1}\|_1$$

and, then, $a(k, l)$ decreases geometrically fast with l , i.e. there exist $C > 0$, $0 < b < 1$, and $l_0 \geq 0$ such that

$$a(k, l) \leq C b^l, \quad l \geq l_0. \quad (55)$$

On the other hand, the upper bound for $m_{K,L,M,k}^{ell}$, $k \in \gamma_K$, given by Proposition 4 satisfies

$$\begin{aligned}
& r_{\max}(\alpha_S - a''(M))a(k, K_k) \sum_{l=k+1}^{\infty} (l-k)e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!} \\
&= r_{\max}(\alpha_S - a''(M))a(k, K_k) \sum_{l=1}^{\infty} l e^{-\Lambda_E t} \frac{(\Lambda_E t)^{k+l}}{(k+l)!} \\
&\leq r_{\max} \alpha_S a(k, K_k) \sum_{l=1}^{\infty} e^{-\Lambda_E t} \frac{(\Lambda_E t)^{k+l}}{(k+l-1)!} \\
&= r_{\max} \alpha_S a(k, K_k) \Lambda_E t \sum_{l=k}^{\infty} e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!} < r_{\max} \alpha_S a(k, K_k) \Lambda_E t. \tag{56}
\end{aligned}$$

Using (56), for each $k \in \gamma_k$, the required truncation parameter K_k is upper bounded by the minimum integer $l \geq 1$ satisfying $r_{\max} \alpha_S a(k, l) \Lambda_E t \leq \varepsilon_2 / |\gamma_K|$. Using (55), that integer is not greater than $K_{\text{ub}} = \max\{l_0, m\}$, where m is the minimum integer $l \geq 1$ satisfying $r_{\max} \alpha_S C b^l \Lambda_E t \leq \varepsilon_2 / |\gamma_K|$. But m is not greater than $\max\{1, \lfloor x \rfloor + 1\}$, where x satisfies

$$r_{\max} \alpha_S C b^x \Lambda_E t = \frac{\varepsilon_2}{|\gamma_K|}.$$

Taking logarithms

$$\begin{aligned}
\log r_{\max} + \log \alpha_S + \log C + x \log b + \log(\Lambda_E t) &= \log \left(\frac{\varepsilon_2}{|\gamma_K|} \right), \\
x &= \frac{\log(|\gamma_K| \Lambda_E t / \varepsilon_2) + \log r_{\max} + \log \alpha_S + \log C}{\log(1/b)}.
\end{aligned}$$

Being $0 < b < 1$, $\log(1/b) > 0$ and, then, the previous expression for x shows that x , m and K_{ub} are $O(\log(|\gamma_K| \Lambda_E t / \varepsilon_2)) = O(\log(K \Lambda_E t / \varepsilon)) = O(\log(\log(\Lambda_E t / \varepsilon) \Lambda_E t / \varepsilon))$, since $|\gamma_K| \leq K$, ε_2 is equal to ε except for a constant factor, and $K = O(\log(\Lambda_E t / \varepsilon))$. It is important to note that K_{ub} is independent of k and, therefore, we have $K_{\overline{E}} \leq K K_{\text{ub}}$. Then:

$$\begin{aligned}
K_{\overline{E}} &= O \left(\log \left(\frac{\Lambda_E t}{\varepsilon} \right) \log \left(\frac{\log \left(\frac{\Lambda_E t}{\varepsilon} \right) \Lambda_E t}{\varepsilon} \right) \right) \\
&= O \left(\log \left(\frac{\Lambda_E t}{\varepsilon} \right) \left(\log \left(\log \left(\frac{\Lambda_E t}{\varepsilon} \right) \right) + \log \left(\frac{\Lambda_E t}{\varepsilon} \right) \right) \right) \\
&= O \left(\left(\log \left(\frac{\Lambda_E t}{\varepsilon} \right) \right)^2 \right).
\end{aligned}$$

We will prove next $L_{\overline{E}} = \sum_{k \in \gamma_L} L_k = O((\log(1/\varepsilon))^2)$. Using $\boldsymbol{\pi}'(k, l)^T = (\mathbf{P}_{\overline{E}, \overline{E}}^T)^{l-1} \mathbf{P}_{E', \overline{E}}^T \boldsymbol{\pi}'(k)^T$, which follows from (14) and (15), $\|\boldsymbol{\pi}'(k)^T\|_1 \leq 1$ and $\|\mathbf{P}_{E', \overline{E}}^T\|_1 \leq 1$, we get (22)

$$a'(k, l) = \|\boldsymbol{\pi}'(k, l)^T\|_1 \leq \|(\mathbf{P}_{\overline{E}, \overline{E}}^T)^{l-1}\|_1 \|\mathbf{P}_{E', \overline{E}}^T\|_1 \|\boldsymbol{\pi}'(k)^T\|_1 \leq \|(\mathbf{P}_{\overline{E}, \overline{E}}^T)^{l-1}\|_1,$$

and, since, as we have proved, $\|(\mathbf{P}_{\overline{E}, \overline{E}}^T)^l\|_1$ decreases geometrically fast with l , so does $a'(k, l)$, and there exist $C > 0$, $0 < b < 1$, and $l_0 \geq 0$ such that

$$a'(k, l) \leq C b^l, \quad l \geq l_0. \tag{57}$$

On the other hand, the upper bound for $m_{L,k}^{e''}(t)$ given by (42) satisfies

$$r_{\max} a'(k, L_k) \sum_{l=k+1}^{\infty} e^{-\Lambda_E t} \frac{(\Lambda_E t)^l}{l!} < r_{\max} a'(k, L_k). \quad (58)$$

Using (58), for each $k \in \gamma'_L$, the required truncation parameter L_k is upper bounded by the minimum integer $l \geq 1$ satisfying $r_{\max} a'(k, l) \leq \varepsilon_3/|\gamma'_L|$. Using (57), that integer is not greater than $L_{\text{ub}} = \max\{l_0, m'\}$, where m' is the minimum integer $l \geq 1$ satisfying $r_{\max} C b^l \leq \varepsilon_3/|\gamma'_L|$. But m' is not greater than $\max\{1, \lfloor x' \rfloor + 1\}$, where x' satisfies

$$r_{\max} C b^{x'} = \frac{\varepsilon_3}{|\gamma'_L|}.$$

Taking logarithms

$$\begin{aligned} \log r_{\max} + \log C + x' \log b &= \log \left(\frac{\varepsilon_3}{|\gamma'_L|} \right), \\ x' &= \frac{\log(|\gamma'_L|/\varepsilon_3) + \log r_{\max} + \log C}{\log(1/b)}. \end{aligned}$$

Being $0 < b < 1$, $\log(1/b) > 0$ and, then, the previous expression for x' shows that x' , m' and L_{ub} are $O(\log(|\gamma'_L|/\varepsilon_3)) = O(\log(L/\varepsilon)) = O(\log(\log(1/\varepsilon))/\varepsilon)$, since $|\gamma'_L| \leq L$, ε_3 is equal to ε except for a constant factor, and $L = O(\log(1/\varepsilon))$. As before, it is important to note that L_{ub} is independent of k and, therefore, $L_{\overline{E}} \leq LL_{\text{ub}}$. Then

$$\begin{aligned} L_{\overline{E}} &= O \left(\log \left(\frac{1}{\varepsilon} \right) \log \left(\frac{\log(\frac{1}{\varepsilon})}{\varepsilon} \right) \right) \\ &= O \left(\log \left(\frac{1}{\varepsilon} \right) \left(\log \left(\log \left(\frac{1}{\varepsilon} \right) \right) + \log \left(\frac{1}{\varepsilon} \right) \right) \right) \\ &= O \left(\left(\log \left(\frac{1}{\varepsilon} \right) \right)^2 \right). \quad \square \end{aligned}$$

Proposition 5. *Conditions C1 and C3 are sufficient for all states in S of Z to be transient.*

Proof. States in S are transient in \widehat{X} if and only if they are transient in X . The recurrent classes of \widehat{X} in S are exactly the recurrent classes of X in S . Therefore, it is enough to prove the result with conditions C1 and C3 referred to \widehat{X} instead of to X .

If all states in S are transient in \widehat{X} , there will exist for each $x \in S$ a path in \widehat{X} from x to some absorbing state f_i . Besides x , the path may include r or not. Given the relationships between the state transition diagrams of \widehat{X} and Z , this implies the existence of a path in Z from x to either f_i (if, besides x , the path in \widehat{X} does not include r) or a (if, besides x , the path in \widehat{X} includes r), and that x is transient in Z . If \widehat{X} has a single recurrent class $C \subset S$ and $r \in C$, for each $x \in C$ there will exist a path in Z to a . For $x = r$, this is because $P_{r,r} > 0$. For $x \in C - \{r\}$, this is because, being C a recurrent class of \widehat{X} , there will exist a path in \widehat{X} from x to r . Consider now a state $x \in S - C$. Such state is transient in \widehat{X} and, therefore, either \widehat{X} has a path from x to C or \widehat{X} has a path from x to some state f_i not including states in C . In the first case, \widehat{X} has a path from x to r , which implies the existence in Z of a path from x to a . In the second case, Z has a path from x to the state f_i . In both cases $x \in S - C$ will be transient in Z . \square

Proposition 6. *Conditions C1 and C3 are sufficient for all states in S' of Z' to be transient.*

Proof. As in the proof of Proposition 5, it is enough to prove the sufficiency of conditions C1 and C3 referred to \widehat{X} . The result in the case in which all states in S are transient in \widehat{X} can be proved as it was proved Proposition 5 in the case in which all states in S are transient in \widehat{X} , except that only states $x \in S'$ have to be considered. Consider now the case in which \widehat{X} has a single recurrent class $C \subset S$ and $r \in C$. We have $S' = C - \{r\} \cup (S - C)$. Being C a recurrent class of \widehat{X} , there will exist a path in \widehat{X} from $x \in C - \{r\}$ to r , implying the existence of a path in Z' from $x \in C - \{r\}$ to a and that all states in $C - \{r\}$ will be transient in Z' . That the states $x \in S - C$ are transient in Z' can be proved as it was proved in the proof of Proposition 5 that the states $x \in S - C$ are transient in Z in the case in which \widehat{X} has a single recurrent class $C \subset S$ and $r \in C$. \square

Proposition 7. *Conditions C1, C3 and C7 are sufficient for all states in \overline{E} of Z'' to be transient.*

Proof. States in S are transient in \widehat{X} if and only if they are transient in X . The recurrent classes of \widehat{X} in S are exactly the recurrent classes of X in S . Also, for $i \in \overline{E}, j \in E'$, the transition probability of \widehat{X} $P_{i,j}$ is 0 if and only if the transition rate of X $\lambda_{i,j}$ is 0. Therefore, it is enough to prove the result with conditions C1 and C3 referred to \widehat{X} instead of to X and with condition C7 replaced by condition C7') $P_{i,j} = 0, i \in \overline{E}, j \in E'$.

Fulfillment of conditions C1 and C3 imply that either all states in S are transient in \widehat{X} or that \widehat{X} has a single recurrent class $C \subset S$ including state r . If all states in S are transient in \widehat{X} , for each $x \in \overline{E}$ there will exist a path in \widehat{X} from x to some absorbing state f_i . If, besides state f_i , that path only includes states in \overline{E} , there will exist a path in Z'' from x to f_i . If the path exits \overline{E} before entering f_i , by condition C7', the exit state will be state r and, then, there will exist a path in Z'' from x to a . In both cases, x will be transient in Z'' . It remains to deal with the case in which \widehat{X} has a single recurrent class $C \subset S$ including state r . Let $x \in \overline{E}$. If $x \notin C$, x will be transient in \widehat{X} and either there will exist a path in \widehat{X} from x to C or there will exist a path in \widehat{X} not including states in C from x to some absorbing state f_i . In the first case, \widehat{X} will have a path from x to r , implying, by condition C7', the existence of a path in Z'' from x to a . In the second case, Z'' will have a path from x to f_i , if the path in \widehat{X} does not exit \overline{E} , or, by condition C7', to a , if the path in \widehat{X} exits \overline{E} . If $x \in C$ ($x \neq r$, since $x \in \overline{E}$ and $r \in E$), being C a recurrent class of \widehat{X} , there will exist a path in \widehat{X} from x to r , implying, by condition C7', the existence of a path in Z'' from x to a . In summary, in the case in which \widehat{X} has a single recurrent class $C \subset S$ including state r , from every $x \in \overline{E}$ there will exist a path in Z'' to either state a or a state f_i and all states $x \in \overline{E}$ will be transient in Z'' . \square

Lemma 1. *Let $X = \{X(t); t \geq 0\}$ be a finite CTMC with state space Ω , transition rates $\lambda_{i,j}, i, j \in \Omega, i \neq j$ and output rates $\lambda_i = \sum_{j \in \Omega - \{i\}} \lambda_{i,j}, i \in \Omega$. Let \widehat{X} be the DTMC obtained by randomizing X with rate Λ_i in every state $i \in \Omega$. Let Y be the semi-Markov process obtained by composing the state visiting process \widehat{X} with independent visit durations with exponential distributions with parameters $\Lambda_s, s \in \Omega, s$ being the state visited by \widehat{X} . Then, for non-absorbing states $i \in \Omega$ in X , the kernel probabilities of Y are $q_{i,j}(t) = \lambda_{i,j}/\lambda_i(1 - e^{-\lambda_i t})$.*

Proof. Let $i \in \Omega$ be a non-absorbing state in X . The kernel probability $q_{i,j}(t)$, $j \in \Omega - \{i\}$ of Y is the probability that, after entering Y state i , Y will jump to state j in a time $\leq t$. The transition probabilities of \widehat{X} from state i have values $P_{i,j} = \lambda_{i,j}/\Lambda_i$, $j \neq i$, $P_{i,i} = 1 - \lambda_i/\Lambda_i$. Since Y is the result of composing the state visiting process \widehat{X} with independent visit durations with exponential distributions with parameters Λ_s , s being the state visited by \widehat{X} , letting W_k^i a random variable with Erlang distribution with k stages and parameter Λ_i , we have

$$\begin{aligned}
q_{i,j}(t) &= \sum_{k=0}^{\infty} P[\widehat{X}_{n+k+1} = j \wedge \widehat{X}_{n+1:n+k} = i \mid \widehat{X}_n = i] P[W_{k+1}^i \leq t] \\
&= \sum_{k=0}^{\infty} P_{i,i}^k P_{i,j} \sum_{l=k+1}^{\infty} e^{-\Lambda_i t} \frac{(\Lambda_i t)^l}{l!} = \sum_{l=1}^{\infty} \sum_{k=0}^{l-1} P_{i,i}^k P_{i,j} e^{-\Lambda_i t} \frac{(\Lambda_i t)^l}{l!} \\
&= \sum_{l=1}^{\infty} \frac{1 - P_{i,i}^l}{1 - P_{i,i}} P_{i,j} e^{-\Lambda_i t} \frac{(\Lambda_i t)^l}{l!} \\
&= \frac{P_{i,j}}{1 - P_{i,i}} \sum_{l=1}^{\infty} e^{-\Lambda_i t} \frac{(\Lambda_i t)^l}{l!} - \sum_{l=1}^{\infty} P_{i,i}^l \frac{P_{i,j}}{1 - P_{i,i}} e^{-\Lambda_i t} \frac{(\Lambda_i t)^l}{l!}.
\end{aligned}$$

Using, then, $P_{i,j}/(1 - P_{i,i}) = \lambda_{i,j}/\lambda_i$ and $\Lambda_i = \lambda_i + P_{i,i}\Lambda_i$,

$$\begin{aligned}
q_{i,j}(t) &= \frac{\lambda_{i,j}}{\lambda_i} (1 - e^{-\Lambda_i t}) - \frac{\lambda_{i,j}}{\lambda_i} \sum_{l=1}^{\infty} e^{-\lambda_i t} e^{-P_{i,i}\Lambda_i t} \frac{(P_{i,i}\Lambda_i t)^l}{l!} \\
&= \frac{\lambda_{i,j}}{\lambda_i} (1 - e^{-\Lambda_i t}) - \frac{\lambda_{i,j}}{\lambda_i} e^{-\lambda_i t} (1 - e^{-P_{i,i}\Lambda_i t}) \\
&= \frac{\lambda_{i,j}}{\lambda_i} - \frac{\lambda_{i,j}}{\lambda_i} e^{-\Lambda_i t} - \frac{\lambda_{i,j}}{\lambda_i} e^{-\lambda_i t} + \frac{\lambda_{i,j}}{\lambda_i} e^{-(\lambda_i + P_{i,i}\Lambda_i)t} \\
&= \frac{\lambda_{i,j}}{\lambda_i} - \frac{\lambda_{i,j}}{\lambda_i} e^{-\Lambda_i t} - \frac{\lambda_{i,j}}{\lambda_i} e^{-\lambda_i t} + \frac{\lambda_{i,j}}{\lambda_i} e^{-\Lambda_i t} \\
&= \frac{\lambda_{i,j}}{\lambda_i} (1 - e^{-\lambda_i t}). \quad \square
\end{aligned}$$

Lemma 2. *Let*

$$\mathbf{A} = \begin{pmatrix} 0 & \mathbf{0} \\ \boldsymbol{\beta} & \mathbf{B} \end{pmatrix},$$

where $\mathbf{0}$ is a row vector of appropriate dimension with all its elements equal to 0 and $\boldsymbol{\beta}$ is a column vector of appropriate dimension. Then $\rho(\mathbf{A}) = \rho(\mathbf{B})$.

Proof. The eigenvalues of \mathbf{A} are the roots of $\det(\lambda\mathbf{I} - \mathbf{A})$, where \mathbf{I} denotes an identity matrix of appropriate dimensions. But,

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \begin{vmatrix} \lambda & \mathbf{0} \\ -\boldsymbol{\beta} & \lambda\mathbf{I} - \mathbf{B} \end{vmatrix} = \lambda \det(\lambda\mathbf{I} - \mathbf{B}),$$

which shows that the set of eigenvalues of \mathbf{A} includes exactly 0 and the set of eigenvalues of \mathbf{B} , implying the result. \square

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