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Symmetries of the Free Schrödinger Equation in the Non-Commutative Plane*

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Abstract. We study all the symmetries of the free Schrödinger equation in the non-commutative plane. These symmetry transformations form an infinite-dimensional Weyl algebra that appears naturally from a two-dimensional Heisenberg algebra generated by Galilean boosts and momenta. These infinite high symmetries could be useful for constructing non-relativistic interacting higher spin theories. A finite-dimensional subalgebra is given by the Schrödinger algebra which, besides the Galilei generators, contains also the dilatation and the expansion. We consider the quantization of the symmetry generators in both the reduced and extended phase spaces, and discuss the relation between both approaches.

Key words: non-commutative plane; Schrödinger equation; Schrödinger symmetries; higher spin symmetries

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1 Introduction and results

The symmetries of a free massive non-relativistic particle and the associated Schrödinger equation have been investigated. The projective symmetries of the Schrödinger equation induced by the transformation on the coordinates $(t, \vec{x})$ are well known. They form the Schrödinger group [12, 19, 20, 23] that, apart from the Galilei symmetries, contains the dilatation and the expansion. Recently Valenzuela [24] (see also [4]) discussed higher-order symmetries of the free Schrödinger equation. These symmetry transformations form an infinite-dimensional Weyl algebra constructed from the generators of space-translation and the ordinary commuting Galilean boost. The extra symmetries that do not belong to the Schrödinger group correspond to higher spin symmetries. These transformations are not induced by the transformations on the coordinates but they map solutions into solutions of the Schrödinger equation.

In the case of 2+1 dimensions, the Galilei group admits two central extensions [2, 5, 14, 15, 21], one associated to the exotic non-commuting boost and other appearing in the commutator of the ordinary boost and spatial translations. The non-relativistic particle in the non-commutative plane was introduced in [22] by considering a higher order Galilean invariant Lagrangian for the coordinates $(t, \vec{x})$ of the particle. A first order Lagrangian depending on the coordinates $(t, \vec{x})$

*This paper is a contribution to the Special Issue on Deformations of Space-Time and its Symmetries. The full collection is available at http://www.emis.de/journals/SIGMA/space-time.html and extra coordinates $\vec{v}$ was introduced in [9]. For these Lagrangians there are two possible realizations, one with non-commuting (exotic) boosts, and the other with ordinary commuting boosts [5, 16] (see [15] for a review).

In this paper we study all the infinitesimal Noether symmetries of a massive free particle in the $(2+1)$-dimensional non-commutative plane. The Noether symmetries are constructed from the Heisenberg algebra of commuting boosts X_i and the generators of translations P_i, $\{X_i, P_j\} = \delta_{ij}$, $i, j = 1, 2$, all of which are constants of motion and are written explicitly in terms of the initial conditions. The algebra of these symmetries is the infinite-dimensional Weyl algebra associated with the Heisenberg algebra. A general element of the Weyl algebra is given by $\mathfrak{G}(X_i, P_j)$. The generators given by higher degree polynomials do not form a closed algebra for any finite degree. These infinite symmetries are the non-relativistic counterpart of all the symmetries of the free massless Klein–Gordon equation [10]. There is no known realization of this Weyl algebra for an Schrödinger equation with interaction. These symmetries could be useful to construct a non-relativistic analogue of Vasiliev's higher spin theories [25].

The subset of generators constructed up to quadratic polynomials of (X_i, P_j) form a finite-dimensional sub-algebra, which in turn contains the 9-dimensional Schrödinger algebra. We study the realization of this algebra in the classical unreduced phase-space, as well as in the reduced one, the later appearing due to the presence of second class constraints. We also study all the symmetries of the free Schrödinger equation in the non-commutative plane. The symmetries are in one to one correspondence with the Noether symmetries of the free particle in the non-commutative plane. This analysis is done in the quantum reduced phase space, as well as in the extended one. In the extended space we impose non-hermitian combinations of the second class constraints. In this case we consider two representations for the physical states, namely a Fock representation [16] and a coordinate representation. We study the Schrödinger subalgebra in detail, and we show the equivalence between the reduced and extended space formulations. We show that, in general, the quadratic (and higher) generators in the extended space contain second order derivatives and hence do not generate point transformations for the coordinates.

The organization of the paper is as follows. In Section 2 we construct all Noether symmetries of the massive particle in the non-commutative plane. Section 3 is devoted to the study of the quantum symmetries of the Schrödinger equation.

2 Classical symmetries of the non-relativistic particle Lagrangian in the non-commutative plane

In this section we introduce a first order Lagrangian describing a particle in the non-commutative plane [9], and present the corresponding Hamiltonian formalism. The main result of the section is the construction of all the Noether symmetries of the non-relativistic particle in the non-commutative plane (equations (2.11)–(2.14) and the ensuing discussion). For the sake of completeness, we review the construction of the standard and exotic Galilei algebras and of the Schödinger generators [3, 5, 15, 16, 17]. We also perform the reduction of the second class constraints of the system for later use in the quantization in the reduced phase space.

The first order Lagrangian of a non-relativistic particle in the non-commutative plane, see for example [9], is given by

$$\mathcal{L}_{\rm nc} = m\left(v_i\dot{x}_i - \frac{v_i^2}{2}\right) + \frac{\kappa}{2}\epsilon_{ij}v_i\dot{v}_j, \qquad i, j = 1, 2, \tag{2.1}$$

where the overdot means derivative with respect to "time" t. This Lagrangian can be obtained using the NLR method [7, 6] applied to the exotic Galilei group in $2+1$ dimensions[1]; see [1]

[1]Note that this Lagrangian is not dynamically equivalent to the higher order Lagrangian for a non-relativistic

for the case of exotic Newton–Hooke whose flat limit gives (2.1). The coordinates x_i's are the Goldstone bosons of the transverse translations and v_i's are the Goldstone bosons of the broken boost. The v_i's and κ are dimensionless.

The Lagrangian (2.1) gives two primary second class constraints

$$\Pi_i = \pi_i + \frac{\kappa}{2}\epsilon_{ij}v_j \approx 0, \qquad V_i = p_i - mv_i \approx 0, \tag{2.2}$$

where p_i and π_i are the momenta canonically conjugate to x_i and v_i. The constraints (2.2) satisfy relations

$$\{\Pi_i, \Pi_j\} = \kappa\epsilon_{ij}, \qquad \{V_i, V_j\} = 0, \qquad \{\Pi_i, V_j\} = m\delta_{ij},$$

and the Dirac Hamiltonian is

$$H = \frac{p_i^2}{2m}, \tag{2.3}$$

up to quadratic terms in the constraints.

From the canonical pairs (x, v, p, π) we can get a new set of canonical pairs $(\tilde{x}, \tilde{v}, \tilde{p}, \tilde{\pi})$ given by

$$\begin{pmatrix} \tilde{x} \\ \tilde{p} \\ \tilde{v} \\ \tilde{\pi} \end{pmatrix} = \begin{pmatrix} 1 & -\frac{\kappa}{2m^2}\epsilon & \frac{\kappa}{2m}\epsilon & -\frac{1}{m} \\ & 1 & & \\ & -\frac{1}{m} & 1 & \\ & \frac{\kappa}{2m}\epsilon & & 1 \end{pmatrix} \begin{pmatrix} x \\ p \\ v \\ \pi \end{pmatrix}. \tag{2.4}$$

In terms of the new variables the constraints (2.2) become a canonical pair,

$$\tilde{v}_i = -\frac{1}{m}V_i \approx 0, \qquad \tilde{\pi}_i = \Pi_i + \frac{\kappa}{2m}\epsilon_{ij}V_j \approx 0. \tag{2.5}$$

The position and momentum of the particle are expressed as

$$x_i = \tilde{x}_i - \frac{\kappa}{2m^2}\epsilon_{ij}\tilde{p}_j - \frac{\kappa}{2m}\epsilon_{ij}\tilde{v}_j + \frac{1}{m}\tilde{\pi}_i, \qquad p_i = \tilde{p}_i, \tag{2.6}$$

and the Dirac Hamiltonian (2.3) is written as

$$H = \frac{1}{2m}\tilde{p}_i^2. \tag{2.7}$$

The phase space is a direct product of two spaces. One is spanned by $(\tilde{v}, \tilde{\pi})$ with the constraints (2.5)

$$\tilde{v}_i \approx 0, \qquad \tilde{\pi}_i \approx 0 \tag{2.8}$$

and thus classically trivial. The other one is spanned by $(\tilde{x}, \tilde{p})$ with the Hamiltonian (2.7). It is a system of a free non-relativistic particle in $2+1$ dimensions but with the coordinates $\tilde{x}_i$. In the classical reduced phase space defined by the second class constraints (2.8) the coordinates x_i become non-commutative (see also Subsection 2.1),

$$\{x_i, x_j\}^* = \left\{\tilde{x}_i - \frac{\kappa}{2m^2}\epsilon_{ik}\tilde{p}_k, \tilde{x}_j - \frac{\kappa}{2m^2}\epsilon_{j\ell}\tilde{p}_\ell\right\} = \frac{\kappa}{m^2}\epsilon_{ij}. \tag{2.9}$$

particle proposed in [22]. It can be obtained from (2.1) using the inverse Higgs mechanism [18].

If we consider a point transformation $(x, v) \to (y, u)$

$$y_i = x_i + \frac{\kappa}{2m}\epsilon_{ij}v_j, \qquad u_i = v_i, \tag{2.10}$$

in the Lagrangian (2.1) it becomes

$$\mathcal{L} = m\left(u_i\dot{y}_i - \frac{u_i^2}{2}\right),$$

which is the Lagrangian of a free non-relativistic particle with the commutative coordinates y_i. Although it has a form of free particle we keep x_i as the "position coordinates" of this system. Local interactions would be introduced at the position x_i rather than y_i. The coordinates y_i in (2.10) are identified with the commuting coordinates $\tilde{x}_i$ in (2.6), while x_i are non-commutative as in (2.9).

All the Noether symmetries are generated by constants of motion which are arbitrary functions $\mathfrak{G}(X_i, P_j)$ of

$$X_i = \tilde{x}_i(0) = \tilde{x}_i(t) - \frac{t}{m}\tilde{p}_i(t) \qquad \text{and} \qquad P_i = \tilde{p}_i(0) = \tilde{p}_i(t), \tag{2.11}$$

verifying

$$\{P_i, P_j\} = 0, \qquad \{X_i, P_j\} = \delta_{ij}, \qquad \{X_i, X_j\} = 0.$$

The Lagrangian (2.1) is quasi-invariant under the transformation generated by $\mathfrak{G}(X_i, P_j)$. The canonical variations of (x, v) are

$$\delta x_i = \frac{\partial \mathfrak{G}}{\partial p_i} = \frac{\partial \mathfrak{G}}{\partial P_i} - \frac{t}{m}\frac{\partial \mathfrak{G}}{\partial X_i} + \frac{\kappa}{2m^2}\epsilon_{ij}\frac{\partial \mathfrak{G}}{\partial X_j}, \qquad \delta v_i = \frac{\partial \mathfrak{G}}{\partial \pi_i} = -\frac{1}{m}\frac{\partial \mathfrak{G}}{\partial X_i}. \tag{2.12}$$

When computing the variation of the Lagrangian (2.1) under (2.12), the (p_i, π_i) are replaced, using the definition of momenta (2.2), by

$$p_i \to mv_i, \qquad \pi_i \to -\frac{\kappa}{2}\epsilon_{ij}v_j, \qquad X_i \to x_i - tv_i + \frac{\kappa}{2m}\epsilon_{ij}v_j. \tag{2.13}$$

It follows that the variation of the Lagrangian becomes a total derivative,

$$\begin{aligned} \delta\mathcal{L}_{\rm nc} &= \frac{\mathrm{d}}{\mathrm{d}\tau}\mathfrak{F}(x, v, t), \\ \mathfrak{F}(x, v, t) &= \left[p_i\delta x_i + \pi_i\delta v_i - \mathfrak{G}\right]_{p_i = mv_i,\, \pi_i = -\frac{\kappa}{2}\epsilon_{ij}v_j} \\ &= \left[mv_i\left(\frac{\partial \mathfrak{G}}{\partial P_i} - \frac{t}{m}\frac{\partial \mathfrak{G}}{\partial X_i}\right) - \mathfrak{G}\right]_{p_i = mv_i,\, \pi_i = -\frac{\kappa}{2}\epsilon_{ij}v_j}. \end{aligned} \tag{2.14}$$

All these Noether symmetries generate an infinite-dimensional Weyl algebra. The Weyl algebra, denoted by $[\mathfrak{h}_2^*]$, can be defined [24] as the one generated by (the Weyl ordered) polynomials of the Heisenberg algebra generators, (X_i, P_i), that we indicate by $\mathfrak{G}(X_i, P_j)$. $[\mathfrak{h}_2^*]$ is the infinite-dimensional algebra of a particle in the non-commutative plane. These infinite symmetries are the non-relativistic counterpart of the complete set of symmetries of the free massless Klein–Gordon equation [10]. The existence of a realization of this Weyl algebra for an interacting Schrödinger equation is an interesting open question.

There are finite-dimensional subalgebras of the higher spin algebra whose generators are constructed from the product of generators X_i, P_j up to second order:

$$\mathfrak{h}_2 \subset \text{Galilei} \subset \text{Sch}(2) \subset \mathfrak{h}_2 \oplus \mathfrak{sp}(4) \subset [\mathfrak{h}_2^*].$$

Sch(2) is the Schrödinger algebra[2] in 2D, whose generators are those of the Galilean algebra X_i, P_i, H, J, together with the dilatation, D, and the expansion, C.

[2]A field theory realization of this algebra was given in [14].

Let us restrict now to Galilean and Schrödinger symmetries. We start by considering the Galilean symmetries of (2.1). The action is invariant under translations,

$$x'_i = x_i + \alpha_i, \qquad v'_i = v_i,$$

boosts,

$$x'_i = x_i - \beta_i t, \qquad v'_i = v_i - \beta_i,$$

rotations,

$$x'_i = x_i \cos\varphi + \epsilon_{ij} x_j \sin\varphi, \qquad v'_i = v_i \cos\varphi + \epsilon_{ij} v_j \sin\varphi,$$

and time translations

$$t' = t - \gamma.$$

The corresponding Noether charges of translations and boosts are given by

$$P_i = p_i, \qquad K_i = m x_i - p_i t - \pi_i + \frac{\kappa}{2}\epsilon_{ij} v_j = m X_i + \frac{\kappa}{2m}\epsilon_{ij} P_j,$$

while the angular momentum is

$$J = \epsilon_{ij}(x_i p_j + v_i \pi_j) = \epsilon_{ij}(X_i P_j + \tilde{v}_i \tilde{\pi}_j). \tag{2.15}$$

Together with the total Hamiltonian (2.3), they generate the exotic Galilei algebra [2, 5, 14, 15, 21]

$$\begin{aligned} &\{H, J\} = 0, \qquad \{H, K_i\} = -P_i, \qquad \{H, P_i\} = 0, \qquad \{J, P_i\} = \epsilon_{ij} P_j, \\ &\{J, K_i\} = \epsilon_{ij} K_j, \qquad \{K_i, P_j\} = m\delta_{ij}, \qquad \{K_i, K_j\} = -\kappa\epsilon_{ij}, \qquad \{P_i, P_j\} = 0. \end{aligned}$$

From this, it may seem that the Lagrangian (2.1) gives a phase space realization of the $(2+1)$-dimensional Galilei group with two central charges m, κ. However, one of the central charges is trivial since, if we modify the generator of the boost as in [5, 13],

$$\tilde{K}_i = K_i - \frac{\kappa}{2m}\epsilon_{ij} P_j = m X_i = m x_i - \pi_i + \frac{1}{2}\kappa\epsilon_{ij} v_j - \frac{\kappa}{2m}\epsilon_{ij} p_j - p_i t,$$

one gets that $(H, P, \tilde{K}, J)$ verifies the standard Galilean algebra without κ.[3] Physically, the result of changing the boost generators is a shift in the parameter of the translations

$$\alpha_i \to \alpha_i + \frac{\kappa}{2m}\epsilon_{ij}\beta_j.$$

Note that the modified boost generators $\tilde{K}_i$ are proportional to the coordinates at $t = 0$, $X_i = \tilde{x}^i(0)$, that verify $\{X_i, X_j\} = 0$, and we have a realization with only one non-trivial central charge associated to the mass of the particle[4].

The Schrödinger generators are those of the Galilean algebra X_i, P_i, H, J, and the dilatation, D, and the expansion, C, given by

$$D = X_i P_i = x_i p_i - \frac{t}{m} p_i^2 - \frac{1}{m}\pi_i p_i + \frac{\kappa}{2m}\epsilon_{ij} p_i v_j,$$

[3] If we introduce an interaction with a background field this statement is no longer true, since it depends on which coordinates (commutative or non-commutative) are used to define the interaction; see [8, 9, 15, 17]. Notice however that the background field will break, in general, part of the symmetries of the Galilei group.

[4] Note however that $\delta_{K_i} L = \delta_{\tilde{K}_i} L = \frac{\mathrm{d}}{\mathrm{d}t}(-m x_i - \frac{\kappa}{2}\epsilon_{ij} v_j)\beta_i$, where β_i is boost parameter.

$$C = mX_iX_i = mx_i^2 + \frac{1}{m}t^2p_i^2 + \frac{1}{m}\pi_i^2 + \frac{\kappa^2}{4m}v_i^2 + \frac{\kappa^2}{4m^3}p_i^2 - 2tx_ip_i - 2x_i\pi_i + \kappa\epsilon_{ij}x_iv_j$$
$$- \frac{\kappa}{m}\epsilon_{ij}x_ip_j + \frac{2}{m}tp_i\pi_i - \frac{\kappa}{m}t\epsilon_{ij}p_iv_j - \frac{\kappa}{m}\epsilon_{ij}\pi_iv_j + \frac{\kappa}{m^2}\epsilon_{ij}\pi_ip_j - \frac{\kappa^2}{2m^2}v_ip_i.$$

In the same spirit, we also redefine the generator of rotations as

$$J = \epsilon_{ij}X_iP_j = \epsilon_{ij}x_ip_j - \frac{\kappa}{2m^2}p_i^2 + \frac{\kappa}{2m}v_ip_i + \frac{1}{m}\epsilon_{ij}p_i\pi_j,$$

which, up to square of constraints, coincides with (2.15).

The new, non-zero Poisson brackets are

$$\{D,C\} = -2C, \qquad \{D,H\} = 2H, \qquad \{H,C\} = -2D,$$
$$\{D,P_i\} = P_i, \qquad \{D,X_i\} = -X_i, \qquad \{C,P_i\} = 2mX_i.$$

The transformations of the coordinates x_i, v_i under dilatation and expansion are obtained from (2.12) as

$$\delta_D x_i = \frac{\alpha}{m}(mx_i - 2mtv_i + \kappa\epsilon_{ij}v_j), \qquad \delta_D v_i = -\alpha v_i,$$
$$\delta_C x_i = \frac{\lambda}{m}\left(2mt^2v_i - 2mtx_i + \kappa\epsilon_{ij}x_j - 2\kappa t\epsilon_{ij}v_j - \frac{\kappa^2}{2m}v_i\right),$$
$$\delta_C v_i = \frac{\lambda}{m}(-2mx_i + 2mtv_i - \kappa\epsilon_{ij}v_j),$$

where α and λ are the corresponding infinitesimal parameters.

2.1 Reduction of second class constraints

The classical symmetry algebra is also realized in the reduced phase space defined by the second class constraints $\Pi_i = V_i = 0$. The Dirac bracket is

$$\{A,B\}^* = \{A,B\} + \{A,\Pi_i\}\frac{1}{m}\{V_i,B\} - \{A,V_i\}\frac{1}{m}\{\Pi_i,B\} - \{A,V_i\}\frac{\kappa\epsilon_{ij}}{m^2}\{V_j,B\}$$

and yields

$$\{x_i,x_j\}^* = \frac{\kappa}{m^2}\epsilon_{ij}, \qquad \{x_i,p_j\}^* = \delta_{ij}, \qquad \{p_i,p_j\}^* = 0. \tag{2.16}$$

In this space, the symmetry transformations are generated using the Dirac bracket and the reduced generators, which can be obtained by substituting $v_i = p_i/m$, $\pi_i = -\kappa/(2m)\epsilon_{ij}p_j$ into the standard ones.

The infinite Weyl symmetries are generated by

$$\mathfrak{G}^{(R)}(x_i,p_j) = \mathfrak{G}(X_i,P_j)|_{v_i=p_i/m,\,\pi_i=-\kappa/(2m)\epsilon_{ij}p_j}.$$

In particular the Schrödinger generators are given by [3]

$$P_i^{(R)} = p_i, \tag{2.17}$$
$$K_i^{(R)} = mx_i - tp_i + \frac{\kappa}{m}\epsilon_{ij}p_j \qquad \text{(exotic Galilei)}, \tag{2.18}$$
$$\tilde{K}_i^{(R)} = K_i^{(R)} - \frac{\kappa}{2m}\epsilon_{ij}P_j^{(R)} = mx_i - tp_i + \frac{\kappa}{2m}\epsilon_{ij}p_j \qquad \text{(standard Galilei)}, \tag{2.19}$$
$$H^{(R)} = \frac{1}{2m}p_i^2, \tag{2.20}$$

$$J^{(R)} = \epsilon_{ij} x_i p_j + \frac{\kappa}{2m^2} p_i^2, \tag{2.21}$$

$$D^{(R)} = p_i x_i - \frac{1}{m} t p_i^2, \tag{2.22}$$

$$C^{(R)} = m x_i^2 + \frac{1}{m} t^2 p_i^2 + \frac{\kappa^2}{4m^3} p_i^2 - 2t x_i p_i + \frac{\kappa}{m} \epsilon_{ij} x_i p_j. \tag{2.23}$$

They generate the Schrödinger algebra with the Dirac bracket, since $\tilde{K}_i^{(R)}$, $P_i^{(R)}$ generate a Heisenberg algebra:

$$\left\{ \tilde{K}_i^{(R)}, P_j^{(R)} \right\}^* = m\delta_{ij}, \qquad \left\{ P_i^{(R)}, P_j^{(R)} \right\}^* = 0, \qquad \text{and} \qquad \left\{ \tilde{K}_i^{(R)}, \tilde{K}_j^{(R)} \right\}^* = 0.$$

Symmetry transformations are generated either using the Poisson brackets in the original phase space or using the Dirac brackets with the reduced generators, (2.17)–(2.23). For example the "exotic Galilei" generators K_i satisfy

$$\{K_i, K_j\} = \left\{ K_i^{(R)}, K_j^{(R)} \right\}^* = -\kappa \epsilon_{ij},$$

and generate "standard(covariant) Galilei" transformation of (x_i, p_i) as

$$\delta x_i = \{x_i, \beta \cdot K\} = \left\{x_i, \beta \cdot K^{(R)}\right\}^* = -t\beta_i,$$
$$\delta p_i = \{p_i, \beta \cdot K\} = \left\{p_i, \beta \cdot K^{(R)}\right\}^* = -m\beta_i.$$

The "standard Galilei" generators $\tilde{K}_i$ satisfy

$$\{\tilde{K}_i, \tilde{K}_j\} = \left\{ \tilde{K}_i^{(R)}, \tilde{K}_j^{(R)} \right\}^* = 0.$$

and generate "exotic Galilei" (non-covariant) transformations of x_i, p_i,

$$\delta x_i = \{x_i, \beta \cdot \tilde{K}\} = \left\{x_i, \beta \cdot \tilde{K}^{(R)}\right\}^* = -t\beta_i + \frac{\kappa}{2m} \epsilon_{ij} \beta_j,$$
$$\delta p_i = \{p_i, \beta \cdot \tilde{K}\} = \left\{p_i, \beta \cdot \tilde{K}^{(R)}\right\}^* = -m\beta_i.$$

3 Quantum symmetries of free Schrödinger equation in the non-commutative plane

In this section we will study the quantization of the model at the level of the Schrödinger equation and their symmetries. We will quantize it in two approaches, one in the reduced phase space and the other in the extended phase space.

3.1 Quantization in the reduced phase space

In the classical theory, x_i has a nonzero Dirac bracket $\{x_i, x_j\}^*$ as in (2.16) in the reduced phase space. Since Dirac brackets are replaced by commutators in the canonical quantization, one cannot have a x_i-coordinate representation of quantum states[5]. To discuss symmetries of Schrödinger equations we introduce new coordinates

$$y_i \equiv x_i + \frac{\kappa}{2m^2} \epsilon_{ij} p_j, \qquad q_i = p_i, \tag{3.1}$$

[5] Since p_i's are commuting the momentum representation is possible [9].

such that

$$\{y_i, y_j\}^* = 0, \qquad \{y_i, q_j\}^* = \delta_{ij}, \qquad \{q_i, q_j\}^* = 0.$$

The coordinate y_i is the one introduced in (2.10) and q_i is its conjugate. In these coordinates, the Schrödinger equation $(\mathrm{i}\partial_t - H)|\Psi(t)\rangle = 0$ takes the form corresponding to a free particle for the wave function

$$\Psi(y,t) = \langle y|\Psi(t)\rangle, \qquad \hat{y}_i|y\rangle = y_i|y\rangle, \qquad \langle y|y'\rangle = \delta^2(y-y'),$$

i.e.

$$\left(\mathrm{i}\partial_t - \frac{1}{2m}(-\mathrm{i}\partial_y)^2\right)\Psi\,(y,t) = 0,$$

and the inner product is

$$\langle\Psi|\Psi\rangle = \int \mathrm{d}y\,\overline{\Psi\,(y,t)}\Psi\,(y,t)\,.$$

Note that y_i are not covariant under exotic Galilei transformation generated by K_i

$$\delta y_i = \{y_i, \beta\cdot K\} = \left\{y_i, \beta\cdot K^{(R)}\right\}^* = -\beta_i t - \frac{\kappa}{2m}\epsilon_{ij}\beta_j,$$

but covariant under the Galilei transformation generated by $\tilde{K}_i$

$$\delta y_i = \{y_i, \beta\cdot \tilde{K}\} = \left\{y_i, \beta\cdot \tilde{K}^{(R)}\right\}^* = -\beta_i t.$$

The position operators, covariant under K_i, are

$$\hat{x}_i = y_i - \frac{\kappa}{2m^2}\epsilon_{ij}(-\mathrm{i}\partial_{y_j}).$$

They are hermitian since $\hat{y}_i = y_i$, $\hat{q}_i = -\mathrm{i}\partial_{y_i}$, with appropriate boundary conditions on $\Psi\,(y,t)$, are hermitian.

Although in the free theory we are able to work with both the commutative $\hat{y}_i = y_i$ and the non-commutative $\hat{x}_i = y_i - \frac{\kappa}{2m^2}\epsilon_{ij}(-\mathrm{i}\partial_{y_j})$ position operators, this may not be the case in an interacting theory. For example, if we consider an interaction with a background electromagnetic field, which introduces couplings with a source at position x_i, the non-commutative coordinates are naturally selected (see, for example, [8, 9, 15, 17]). If we denote generically by $\mathfrak{G}^{(R)}(t,x,p) = \mathfrak{G}(X,P)|_{\Pi=V=0}$ the generators of the Weyl algebra in the reduced classical space, the generators in this quantization are given by

$$\hat{\mathfrak{G}}_i^{(1)}(t,y,\hat{q}) = \mathfrak{G}_i^{(R)}\Big|_{x_j = y_j - \frac{\kappa}{2m^2}\epsilon_{jl}\hat{q}_l,\, p_j=\hat{q}_j} = \mathfrak{G}_i\left(y - \frac{t}{m}\hat{q}, \hat{q}\right), \tag{3.2}$$

with $\hat{q}_i = -\mathrm{i}\partial/\partial y_i$ and with the appropriate dealing of operator ordering.

The knowledge of all the symmetries of the Schrödinger equation in terms of the coordinates y^i, $\hat{y}_i$ is the non-commutative analog in $2+1$ dimensions of the high spin symmetries of the relativistic massless Klein Gordon equation [10]. The Vasiliev [25] non-linear theory has these high spin symmetries. In this sense these high spin-nonrelativistic symmetries could be useful in order to construct a non-relativistic Vasiliev theory [24].

We consider next in detail the Schrödinger generators, given by

$$\hat{P}_i^{(1)} = \hat{q}_i = -\mathrm{i}\frac{\partial}{\partial y_i}, \tag{3.3}$$

$$\hat{\hat{K}}_i^{(1)} = my_i - t\hat{q}_i = my_i + \mathrm{i}t\frac{\partial}{\partial y_i}, \tag{3.4}$$

$$\hat{H}^{(1)} = \frac{1}{2m}\hat{q}_i^2 = -\frac{1}{2m}\frac{\partial^2}{\partial {y_i}^2}, \tag{3.5}$$

$$\hat{J}^{(1)} = \epsilon_{ij}y_i\hat{q}_j = -\mathrm{i}\epsilon_{ij}y_i\frac{\partial}{\partial y_j}, \tag{3.6}$$

$$\hat{D}^{(1)} = y_i\hat{q}_i - \mathrm{i} - \frac{1}{m}t\hat{q}_i^2 = -\mathrm{i}y_i\frac{\partial}{\partial y_i} + \frac{1}{m}t\frac{\partial^2}{\partial {y_i}^2} - \mathrm{i}, \tag{3.7}$$

$$\hat{C}^{(1)} = my_i^2 - 2ty_i\hat{q}_i + 2\mathrm{i}t + \frac{1}{m}t^2\hat{q}_i^2 = my_i^2 + 2\mathrm{i}ty_i\frac{\partial}{\partial y_i} - \frac{1}{m}t^2\frac{\partial^2}{\partial {y_i}^2} + 2\mathrm{i}t, \tag{3.8}$$

where a Weyl ordering has been used for $\hat{D}^{(1)}$ and $\hat{C}^{(1)}$. These generators are hermitian operators when acting on the wave functions $\Psi(t, y)$. Furthermore, they obey the abstract quantum Schrödinger algebra *off shell*, with non-zero commutators given by

$$\begin{aligned}
&\big[\hat{\hat{K}}_i, \hat{P}_j\big] = \mathrm{i}m\delta_{ij}, \qquad \big[\hat{J}, \hat{P}_i\big] = \mathrm{i}\epsilon_{ij}\hat{P}_j, \qquad \big[\hat{J}, \hat{\hat{K}}_i\big] = \mathrm{i}\epsilon_{ij}\hat{\hat{K}}_j, \qquad \big[\hat{H}, \hat{\hat{K}}_i\big] = -\mathrm{i}\hat{P}_i,\\
&\big[\hat{D}, \hat{H}\big] = 2\mathrm{i}\hat{H}, \qquad \big[\hat{D}, \hat{P}_i\big] = \mathrm{i}\hat{P}_i, \qquad \big[\hat{D}, \hat{\hat{K}}_i\big] = -\mathrm{i}\hat{\hat{K}}_i,\\
&\big[\hat{D}, \hat{C}\big] = -2\mathrm{i}\hat{C}, \qquad \big[\hat{H}, \hat{C}\big] = -2\mathrm{i}\hat{D}, \qquad \big[\hat{C}, \hat{P}_i\big] = 2\mathrm{i}\hat{\hat{K}}_i.
\end{aligned} \tag{3.9}$$

Using these, together with

$$\big[\mathrm{i}\partial_t, \hat{\hat{K}}_i^{(1)}\big] = -\mathrm{i}\hat{P}_i^{(1)}, \qquad \big[\mathrm{i}\partial_t, \hat{D}^{(1)}\big] = -2\mathrm{i}\hat{H}^{(1)}, \qquad \big[\mathrm{i}\partial_t, \hat{C}^{(1)}\big] = -2\mathrm{i}\hat{D}^{(1)}, \tag{3.10}$$

one can show that

$$\left[\mathrm{i}\partial_t - \hat{H}^{(1)}, \hat{\mathfrak{G}}_i^{(1)}\right] = 0$$

for all the generators $\hat{\mathfrak{G}}_i^{(1)}$, which proves the invariance of the Schrödinger equation under the Schrödinger transformations in this reduced space quantization.

Under a general Weyl transformation, the wave functions transform as

$$\Psi'(y, t) = e^{\mathrm{i}\alpha_i\hat{\mathfrak{G}}_i^{(1)}(t, y, (-\mathrm{i}\partial_y))}\Psi(y, t),$$

where the α_i are the parameters of the transformations. In particular, for the *on-shell* Schrödinger transformations one has

$$\Psi'(y, t) = e^{A+iB}\Psi\left(y', t'\right),$$

where the coordinate transformations of (y, t) are those of the $N = 1$ conformal Galilean transformation, and the multiplicative factor is e^{A+iB}, with A and B real functions of the coordinates and of the parameters of the transformation given by (see, for instance, [11, 23])

1) H (time translation),

$$t' = t + a, \qquad y' = y, \qquad A = B = 0,$$

2) D (dilatation),

$$t' = e^{\lambda}t, \qquad y' = e^{\frac{\lambda}{2}}y, \qquad A = \frac{\lambda}{2}, \qquad B = 0,$$

3) C (expansion),

$$t' = \frac{t}{1-\kappa t}, \qquad y_i' = \frac{y_i}{1-\kappa t}, \qquad e^A = \frac{1}{(1-\kappa t)}, \qquad B = -\frac{\kappa m y^2}{2(1-\kappa t)},$$

4) (spatial translations and boost)

$$t' = t, \quad y_i' = y_i + \left(\beta^0 + t\frac{\beta^1}{m}\right)_i, \qquad A = 0, \quad B = -m\left(y_i + \frac{1}{2}\left(\beta_i^0 + t\frac{\beta_i^1}{m}\right)\right)\frac{\beta_i^1}{m},$$

with $[\beta_i^0] = L,\ [\beta_i^1] = L^{-1}$.

The difference with respect to the transformation of the ordinary Schrödinger equation is that in the non-commutative case the coordinates that are transformed by conformal Galilean transformations are the canonical ones y_i, and not the physical position of the particle, x_i.

The invariance of the solutions of the Schrödinger equation under a general element of the Weyl algebra can be proved using the invariance under the generators of the Heisenberg algebra and the commutators (3.10).

3.2 Quantization in the extended phase space

3.2.1 Fock representation

In order to quantize the model in the extended phase space the second class constraints (2.2) are imposed as physical state conditions by taking their non-hermitian combinations as in [1]. We first consider the canonical transformation (2.4) that separates the second class constraints as new coordinates. It is realized at quantum level as a unitary transformation

$$\tilde{q} = U^\dagger q U, \qquad U = e^{\frac{\mathrm{i}}{m}p_i(\pi_i - \frac{\kappa}{2}\epsilon_{ij}v_j)}. \tag{3.11}$$

For example,

$$\tilde{x}_i = U^\dagger x_i U = x_i - \frac{1}{m}\left(\pi_i - \frac{\kappa}{2}\epsilon_{ij}v_j\right) + \frac{1}{2}\frac{\kappa}{m}\epsilon_{ij}\left(-\frac{p_j}{m}\right).$$

It is useful to introduce the complex combinations of the phase space variables $\tilde{\pi}_\pm = \tilde{\pi}_1 \pm \mathrm{i}\tilde{\pi}_2$ and $\tilde{v}_\pm = \tilde{v}_1 \pm \mathrm{i}\tilde{v}_2$, which allow us to introduce two pairs of annihilation and creation operators

$$\tilde{a}_\pm = \frac{\mathrm{i}}{\sqrt{2\kappa}}\left(\tilde{\pi}_\pm - \mathrm{i}\frac{\kappa}{2}\tilde{v}_\pm\right), \qquad \tilde{a}_\pm^\dagger = \frac{-\mathrm{i}}{\sqrt{2\kappa}}\left(\tilde{\pi}_\mp + \mathrm{i}\frac{\kappa}{2}\tilde{v}_\mp\right),$$

with nonzero commutators $[\tilde{a}_\pm, \tilde{a}_\pm^\dagger] = 1$. Using the Fock representation for $(\tilde{v}, \tilde{\pi})$ and coordinate representation for $(\tilde{x}, \tilde{p})$, any state of this system is described by

$$|\Psi(t)\rangle = \sum_{n_+\geq 0, n_-\geq 0}\int \mathrm{d}y\, |n_+, n_-\rangle \otimes |y\rangle \Phi_{n_+n_-}(y,t),$$

where $|n_+, n_-\rangle$ is the eigenstate of $\tilde{N}_\pm = \tilde{a}_\pm^\dagger \tilde{a}_\pm$ with eigenvalues $n_\pm \in \mathbb{N}\cup\{0\}$ and $|y\rangle$ is the eigenstate of commuting operators $\tilde{x}_i$ with eigenvalue y_i. They are normalized as

$$\langle n_+, n_- | n_+', n_-'\rangle = \delta_{n_+n_+'}\delta_{n_-n_-'}, \qquad \langle y|y'\rangle = \delta^2(y-y').$$

The scalar product is given by

$$\langle \Psi|\Psi'\rangle = \sum_{n_\pm}\int \mathrm{d}y\, \overline{\Phi_{n_+n_-}(y,t)}\Phi'_{n_+n_-}(y,t).$$

In the quantization in the extended phase space the second class constraints (2.2) are imposed as physical state conditions by taking their non-hermitian combination,

$$\tilde{a}_\pm|\Psi_{\rm phys}(t)\rangle = 0. \tag{3.12}$$

This means that physical states are minimum uncertainty states in $(\tilde{v},\tilde{\pi})$. Condition (3.12) selects out only the $n_+ = n_- = 0$ state, so that $\Phi_{n_+n_-}(y,t) = 0$ except for $\Phi_{0,0}(y,t) \equiv \Phi_0(y,t)$,

$$|\Psi_{\rm phys}(t)\rangle = \int {\rm d}y\,|0,0\rangle \otimes |y\rangle\Phi_0(y,t).$$

The Schrödinger equation is

$$({\rm i}\partial_t - H)|\Psi_{\rm phys}(t)\rangle = 0, \qquad H = \frac{\hat{\tilde{p}}^2}{2m},$$

and thus

$$({\rm i}\partial_t - H)\Phi_0(y,t) = 0, \qquad H = \frac{1}{2m}(-{\rm i}\partial_{y_i})^2.$$

The generators of the Weyl algebra are given in the extended space as polynomials $\mathfrak{G}(X,P)$ of the operator equivalent of (2.11), and, since they commute with $\tilde{a}_\pm$ and $\tilde{a}^\dagger_\pm$, physical states remain physical[6]. They act on the physical states as

$$\Psi_{\rm phys}(t)\rangle \quad\to\quad |\Psi'_{\rm phys}(t)\rangle = e^{{\rm i}\mathfrak{G}(X,P)}|\Psi_{\rm phys}(t)\rangle$$

and it turns out that the transformation of the wave function $\Phi_0(y,t)$ is

$$\Phi'_0(y,t) = e^{{\rm i}\mathfrak{G}(X,P)}\Phi_0(y,t) = e^{{\rm i}\mathfrak{G}(y-t(-{\rm i}\partial_y),(-{\rm i}\partial_y))}\Phi_0(y,t).$$

This transformation has the same form as the one in the reduced phase space generated by (3.2)–(3.8). Then the wave function in the reduced space $\Psi(y,t) = \langle y|\Psi(t)\rangle$ and $\Phi_0(y,t) = \langle y|\otimes\langle 00|\Psi(t)\rangle$ that appear in the extended space quantization are identified. Note that in the former $\langle y|$ is eigenstate of $\hat{y}_i = x_i + \frac{\kappa}{2m^2}\epsilon_{ij}p_j$ in (3.1) but $\langle y|$ in the latter is eigenstate of $\hat{\tilde{x}}_i$ that are commuting in the extended space.

We can see now how the non-commutativity of the position operators appears. $\hat{x}_\pm = x_1 \pm {\rm i}x_2$ are commuting in the extended phase space. Using (2.4) we write

$$x_+ = \tilde{x}_+ + {\rm i}\frac{\kappa}{2m^2}\tilde{p}_+ + {\rm i}\sqrt{\frac{2\kappa}{m^2}}\tilde{a}^\dagger_-, \qquad x_- = \tilde{x}_- - {\rm i}\frac{\kappa}{2m^2}\tilde{p}_- - {\rm i}\sqrt{\frac{2\kappa}{m^2}}\tilde{a}_- = x^\dagger_+.$$

In the reduced space quantization procedure, the $\tilde{a}_\pm$ are effectively put to zero and $x_\pm$ becomes a non-commutative operator on $|\Psi(t)\rangle$. On the other hand in the quantization in the extended space, expectation values of the position operators between two physical states are given by

$$\begin{aligned}\langle\Psi|\hat{x}_\pm|\Psi'\rangle &= \int {\rm d}y{\rm d}y'\,\overline{\Phi_0(y,t)}\langle y|\langle 0|\left(\tilde{x}_\pm \pm {\rm i}\frac{\kappa}{2m^2}\tilde{p}_\pm \pm {\rm i}\sqrt{\frac{2\kappa}{m^2}}\begin{pmatrix}\tilde{a}^\dagger_-\\ \tilde{a}_-\end{pmatrix}\right)|0\rangle|y'\rangle\Phi'_0(y',t)\\ &= \int {\rm d}y\,\overline{\Phi_0(y,t)}\left(y_\pm \pm {\rm i}\frac{\kappa}{2m^2}(-2{\rm i}\partial_{y_\pm})\right)\Phi'_0(y,t).\end{aligned}$$

Commutative position operators $\hat{x}_\pm$ on states $|\Psi\rangle$ act as non-commutative operators $(y_\pm \pm {\rm i}\frac{\kappa}{2m^2}(-2{\rm i}\partial_{y_\pm}))$ on the wave function $\Phi_0(y,t)$.

[6] The angular momentum J in (2.15) contains a term depending on (v,π), but it commutes with $\tilde{a}_\pm$, $\tilde{a}^\dagger_\pm$.

It is useful to consider the unitary transformation U in (3.11) on the creation and annihilation operators $\tilde{a}_\pm$, $\tilde{a}^\dagger_\pm$,

$$\tilde{a}_+ = U^\dagger a_+ U = a_+, \qquad \tilde{a}_- = U^\dagger a_- U = a_- - \sqrt{\frac{\kappa}{2m^2}}p_-. \tag{3.13}$$

The quantization in the extended phase space can be also done by considering the constraint equations (3.12) in terms of the operators $a_\pm$, $a^\dagger_\pm$. The physical state conditions (3.12) are

$$a_+|\Psi_{\rm phys}(t)\rangle = 0, \qquad \left(p_- - \sqrt{\frac{2m^2}{\kappa}}a_-\right)|\Psi_{\rm phys}(t)\rangle = 0,$$

and $|\Psi_{\rm phys}\rangle$ is a coherent state of a_- with eigenvalue $\sqrt{\frac{\kappa}{2m^2}}p_-$ [16]. In this representation, the Schrödinger generators are

$$\begin{aligned}
X^{(2)}_\pm &= \left(x_\pm \mp {\rm i}\frac{\kappa}{2m^2}p_\pm\right) - \frac{t}{m}p_\pm \pm {\rm i}\frac{\kappa}{m^2}\left(p_\pm - \sqrt{\frac{2m^2}{\kappa}}\begin{pmatrix} a^\dagger_- \\ a_-\end{pmatrix}\right),\\
P^{(2)}_\pm &= p_\pm = -2{\rm i}\partial_{x_\mp}, \qquad [x_\pm, p_\mp] = 2{\rm i},\\
D^{(2)} &= \frac{1}{2}\Bigg(\left(x_+p_- + p_+x_- - \frac{2t}{m}p_+p_-\right) + {\rm i}\frac{\kappa}{m^2}\left(p_+ - \sqrt{\frac{2m^2}{\kappa}}a^\dagger_-\right)p_-\\
&\quad - {\rm i}\frac{\kappa}{m^2}p_+\left(p_- - \sqrt{\frac{2m^2}{\kappa}}a_-\right)\Bigg),\\
C^{(2)} &= \frac{1}{2}\Bigg(\left(x_+ - {\rm i}\frac{\kappa}{2m^2}p_+\right)\left(x_- + {\rm i}\frac{\kappa}{2m^2}p_-\right)\\
&\quad - \frac{t}{m}\left(\left(x_+ - {\rm i}\frac{\kappa}{2m^2}p_+\right)p_- + p_+\left(x_- + {\rm i}\frac{\kappa}{2m^2}p_-\right)\right)\\
&\quad + \frac{t^2}{2m^2}p_+p_- + \frac{1}{2}\left(\left(x_+ - {\rm i}\frac{\kappa}{2m^2}p_+\right) - \frac{t}{m}p_+\right)\left(-{\rm i}\frac{\kappa}{m^2}\right)\left(p_- - \sqrt{\frac{2m^2}{\kappa}}a_-\right)\\
&\quad + \frac{1}{2}{\rm i}\frac{\kappa}{m^2}\left(p_+ - \sqrt{\frac{2m^2}{\kappa}}a^\dagger_-\right)\left(\left(x_- + {\rm i}\frac{\kappa}{2m^2}p_-\right) - \frac{t}{m}p_-\right)\\
&\quad + \frac{1}{2}\left({\rm i}\frac{\kappa}{m^2}\left(p_+ - \sqrt{\frac{2m^2}{\kappa}}a^\dagger_-\right)\right)\left(-{\rm i}\frac{\kappa}{m^2}\left(p_- - \sqrt{\frac{2m^2}{\kappa}}a_-\right)\right)\Bigg).\\
J^{(2)} &= \frac{\rm i}{2}\Bigg(\left(x_+p_- - p_+x_- - {\rm i}\frac{\kappa}{m^2}p_+p_-\right) + {\rm i}\frac{\kappa}{m^2}\left(p_+ - \sqrt{\frac{2m^2}{\kappa}}a^\dagger_-\right)p_-\\
&\quad + {\rm i}\frac{\kappa}{m^2}p_+\left(p_- - \sqrt{\frac{2m^2}{\kappa}}a_-\right)\Bigg).
\end{aligned}$$

These generators commute with the constraint equations and with the Schrödinger operator ${\rm i}\partial_t - H$. Notice that the set of generators do not depend on a_+, $a^\dagger_+$, and therefore the transition to the Fock space used in [16] is recovered.

The Fock expression of a generic element of the Weyl algebra $\mathfrak{G}(X,P)$ can be obtained using the expression of the operators X and P given by (2.11).

3.2.2 Coordinate representation

In the representation of coordinates the time Schrödinger equation and the constraint equations (3.13) in the non-commutative plane becomes [1]

$$\hat{S}_1\Psi \equiv \left(\frac{\partial}{\partial v_-} + \frac{\kappa}{4}v_+\right)\Psi(x,v,t) = 0,$$
$$\hat{S}_2\Psi \equiv \left(\frac{\partial}{\partial x_+} - \mathrm{i}\frac{m}{4}v_- - \mathrm{i}\frac{m}{\kappa}\frac{\partial}{\partial v_+}\right)\Psi(x,v,t) = 0,$$
$$\hat{S}_3\Psi \equiv \left(\mathrm{i}\frac{\partial}{\partial t} + \frac{2}{m}\frac{\partial^2}{\partial x_+\partial x_-}\right)\Psi(x,v,t) = 0.$$

In this representation, the operators associated to the generators of the Heisenberg algebra are

$$\hat{P}_1 = -\mathrm{i}\frac{\partial}{\partial x_+} - \mathrm{i}\frac{\partial}{\partial x_-}, \qquad \hat{P}_2 = \frac{\partial}{\partial x_+} - \frac{\partial}{\partial x_-},$$
$$\hat{\hat{K}}_1 = \frac{m}{2}(x_+ + x_-) + \left(\mathrm{i}t - \frac{\kappa}{2m}\right)\frac{\partial}{\partial x_+} + \left(\mathrm{i}t + \frac{\kappa}{2m}\right)\frac{\partial}{\partial x_-} + \frac{\kappa}{4\mathrm{i}}(v_+ - v_-) + \mathrm{i}\frac{\partial}{\partial v_+} + \mathrm{i}\frac{\partial}{\partial v_-},$$
$$\hat{\hat{K}}_2 = \frac{m}{2\mathrm{i}}(x_+ - x_-) - \left(t + \mathrm{i}\frac{\kappa}{2m}\right)\frac{\partial}{\partial x_+} + \left(t - \mathrm{i}\frac{\kappa}{2m}\right)\frac{\partial}{\partial x_-} - \frac{\kappa}{4}(v_+ + v_-) - \frac{\partial}{\partial v_+} + \frac{\partial}{\partial v_-},$$

or, in covariant form,

$$\hat{P}_i = -\mathrm{i}\frac{\partial}{\partial x_i}, \qquad \hat{\hat{K}}_i = m x_i + \mathrm{i}t\frac{\partial}{\partial x_i} + \mathrm{i}\frac{\kappa}{2m}\epsilon_{ij}\frac{\partial}{\partial x_j} + \frac{\kappa}{2}\epsilon_{ij}v_j + \mathrm{i}\frac{\partial}{\partial v_i},$$

which, indeed, satisfy $[\hat{P}_i, \hat{\hat{K}}_j] = -\mathrm{i}m\delta_{ij}$, with all the other commutators equal to zero.

It is immediate to check that the operators $\hat{P}_i$, $\hat{\hat{K}}_i$ commute with all of $\hat{S}_1$, $\hat{S}_2$ and $\hat{S}_3$, and hence that they generate Schrödinger symmetries for the free particle in the non-commutative plane. The rest of generators of the Schrödinger algebra are given by

$$\hat{H} = -\frac{2}{m}\frac{\partial^2}{\partial x_+\,\partial x_-} = -\frac{1}{2m}\frac{\partial^2}{\partial {x_i}^2},$$
$$\hat{J} = -\mathrm{i}\epsilon_{ij}x_i\frac{\partial}{\partial x_j} + \frac{\kappa}{2m^2}\frac{\partial^2}{\partial {x_i}^2} - \mathrm{i}\frac{\kappa}{2m}v_i\frac{\partial}{\partial x_i} - \frac{1}{m}\epsilon_{ij}\frac{\partial^2}{\partial x_i\,\partial v_j},$$
$$\hat{D} = -\mathrm{i}x_i\frac{\partial}{\partial x_i} + \frac{1}{m}t\frac{\partial^2}{\partial {x_i}^2} + \frac{1}{m}\frac{\partial^2}{\partial x_i\,\partial v_i} + \mathrm{i}\frac{\kappa}{2m}\epsilon_{ij}v_i\frac{\partial}{\partial x_j} - \mathrm{i},$$
$$\begin{aligned}\hat{C} = {} & 2\mathrm{i}tx_i\frac{\partial}{\partial x_i} + \mathrm{i}\frac{\kappa^2}{2m^2}v_i\frac{\partial}{\partial x_i} + \mathrm{i}\frac{\kappa}{m}\epsilon_{ij}x_i\frac{\partial}{\partial x_j} - \mathrm{i}\frac{\kappa}{m}t\epsilon_{ij}v_i\frac{\partial}{\partial x_j} \\ & - \mathrm{i}\frac{\kappa}{m}\epsilon_{ij}v_i\frac{\partial}{\partial v_j} + 2\mathrm{i}x_i\frac{\partial}{\partial v_i} - \frac{1}{m}t^2\frac{\partial^2}{\partial {x_i}^2} - \frac{\kappa^2}{4m^3}\frac{\partial^2}{\partial {x_i}^2} - \frac{1}{m}\frac{\partial^2}{\partial {v_i}^2} \\ & - \frac{2}{m}t\frac{\partial^2}{\partial x_i\,\partial v_i} + \frac{\kappa}{m^2}\epsilon_{ij}\frac{\partial^2}{\partial x_i\,\partial v_j} + m x_i^2 + \kappa\epsilon_{ij}x_i v_j + \frac{\kappa^2}{4m}v_i^2 + 2\mathrm{i}t.\end{aligned}$$

Using these expressions, one can check explicitly the commutators (3.9), and also that these quadratic generators commute with $\hat{S}_1$, $\hat{S}_2$ and $\hat{S}_3$ (this also follows from the derivation properties of the commutators and the corresponding commutation of the linear generators $\hat{P}_i$, $\hat{\hat{K}}_i$, and this proves that the Schrödinger equation for the free particle in the noncommutative plane has the Schrödinger algebra as a symmetry. Notice, however, that in this coordinate representation of the non-reduced quantum space the quadratic operators contain second order derivatives, and hence do not generate point transformations for the coordinates x, v. This is in agreement with the results obtained in the reduced space quantization and the Fock space representation. In any case, the fact that the linear generators commute with $\hat{S}_1$, $\hat{S}_2$ and $\hat{S}_3$ allows to prove that the quadratic ones also commute, and thus generate symmetries of the Schrödinger equation of the free particle in the non-commutative plane.

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