

ON THE BACKWARD IN TIME PROBLEM FOR THE THERMOELASTICITY WITH TWO TEMPERATURES

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ABSTRACT. This paper is devoted to the study of the existence, uniqueness, continuous dependence and spatial behaviour of the solutions for the backward in time problem determined by the Type III with two temperatures thermoelastodynamic theory. We first show the existence, uniqueness and continuous dependence of the solutions. Instability of the solutions for the Type II with two temperatures theory is proved later. For the one-dimensional Type III with two temperatures theory, the exponential instability is also pointed-out. We also analyze the spatial behaviour of the solutions. By means of the exponentially weighted Poincaré inequality, we are able to obtain a function that defines a measure on the solutions and, therefore, we obtain the usual exponential type alternative for the solutions of the problem defined in a semi-infinite cylinder.

1. Introduction. The classical theory of heat conduction based on Fourier's law [29] predicts that perturbations occurring at any point spread instantly to any other point in the space. This fact is not compatible with the impossibility of overcoming the speed of light. Solving this “paradox” has been and still remains a question of intense study. Several thermoelastic theories mainly based on the heat conduction theory of Cattaneo and Maxwell [29] (see Hetnarski and Ignaczak [14, 15], the books by Ignaczak and Ostoj-Starzewski [21], Straughan [46] and the references cited therein) have been proposed since the decade of the 1960's. Recently, Green and Naghdi [11, 12, 13] postulated other theories that are currently the subject of deep research (see Iesan [18, 19]; Iesan and Quintanilla [20]; Lazzari and Nibbi [22]; Leseduarde *et al.* [23, 24]; Liu and Quintanilla [27]; Liu and Lin [26], Messaoudi and Soufyane [30]; Puri and Jordan [32]; Qin *et al.* [33]; Quintanilla [34, 35, 36, 38]; Quintanilla and Racke [41]; Quintanilla and Straughan [43, 44]; Yang and Wang [49], among others). These theories are based on well accepted thermomechanical axioms and therefore they are consistent from the mathematical standpoint.

Other theories based on intuition have been also considered. This is not a problem from the mathematical point of view. But a deep mathematical analysis is required to decide their adequacy to describe real phenomena. A good example of this procedure is the theory proposed by Tzou [47] which introduces a delay parameter in the classical version of Fourier's law. This theory has been generalized by Choudhuri [45] with some proposals that sought to recover the theories introduced by Green

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and Naghdi [11, 12, 13]. Unfortunately, they bring to ill-posed problems in the sense of Hadamard (see [6]) and therefore they are not good candidates to describe heat conduction. However, when we replace in these proposals the constitutive equations by their Taylor approximations of different orders, we can recover the Cattaneo-Maxwell theory and the Green-Naghdi theories. They also get other proposals that should be studied from the physical and mathematical standpoint to clarify its applicability.

In [39, 40], Quintanilla suggested an alternative way to obtain a well-posed problem (in the sense of Hadamard) when the constitutive equation contains a delay parameter. The basic idea is to consider the same approach of Tzou or Choudhuri, but in the context of the two temperatures theory proposed by Chen *et al.* [2, 3, 4, 48]. Alternative ideas are currently being studied, like the ones obtained replacing the exact constitutive equations by their Taylor approximations (see the references cited in [46]). In this paper we analyze the theory which is obtained as an approximation. It was suggested by Quintanilla in [40]. Other recent contributions concerning this theory are [7, 50, 31].

In the first part of the paper we study some basic results of the *anisotropic Type III with two temperatures thermoelastodynamic theory*. We consider the backward in time problem and we obtain several qualitative results on the existence, the uniqueness, the continuous dependence and the instability of the solutions.

Spatial behaviour of the solutions for partial differential equations is related with the Saint-Venant's principle. Both questions are under intensive investigation nowadays, from mathematics and also from thermomechanics. Spatial decay estimates for elliptic [8], parabolic [16, 17], hyperbolic [9] equations and/or combination of them [35] have been obtained. In particular, these studies describe how the perturbations on a part of the boundary are damped for the points located far away from this part of the boundary. It is usual to consider a semi-infinite cylinder whose finite end is perturbed and to see what happens when the spatial variable increases. We recall several spatial decay contributions for the theory of Chen *et al.* [1, 37], or more recently, for the Type II with two temperatures [42]. Our present contribution intends to overcome at least two mathematical difficulties. On the one side, we do not assume the positivity of the parameter k (see system ((1),(2)); on the other side, we consider the backward in time problem that has not been previously considered in the case of the Type III thermoelasticity with two temperatures.

The plan of this work is the following. Section 2 recalls the field equations and states the basic assumptions that will be used in the paper. Existence, uniqueness and continuous dependence of the solutions are proved in Section 3. Section 4 is devoted to the Type II with two temperatures theory and we show the instability of the solutions. For the one-dimensional Type III with two temperatures theory, the exponential instability is pointed-out in Section 5. With the help of the exponentially weighted Poincaré inequality, we show later that a certain function defines a measure for the solutions of our problem. In fact, this point is the most relevant contribution of Section 6. We use this function to obtain an exponential growth/decay alternative of the Phragmén-Lindelöf type for the solutions.

2. Preliminaries. In this section we set down the basic equations and the assumptions that we consider in this article. We restrict our attention to anisotropic but homogeneous materials. We will work in the anisotropic Type III with two temperatures thermoelastodynamic theory (T.III+2TT).

The general system of field equations is

$$\rho \ddot{u}_i = (C_{ijkl} u_{k,h} - a_{ij} \theta)_{,j} + \rho f_i, \tag{1}$$

$$c(\ddot{\alpha} - d_{ij} \ddot{\alpha}_{,ij}) = -a_{ij} \dot{u}_{i,j} + k d_{rj} \alpha_{,rj} + b d_{rj} T_{,rj} + h, \tag{2}$$

where the sub-indices can take values from one to three, α is the conductive thermal displacement which is defined by

$$\alpha(\mathbf{x}, t) = \int_0^t T(\mathbf{x}, s) ds + \alpha_0(\mathbf{x}), \tag{3}$$

T is the conductive temperature, $\theta = T - d_{ij} T_{,ij}$ is the thermodynamic temperature, $\Omega = \alpha - d_{ij} \alpha_{,ij}$ is the thermodynamic thermal displacement, u_i is the displacement vector, ρ is the mass density, c is the thermal capacity, (C_{ijkl}) is the elasticity tensor which is assumed to satisfy the major symmetry¹

$$C_{ijkl} = C_{klij},$$

d_{ij} is the typical tensor that related the two-temperatures in this theory, it is assumed to be symmetric ($d_{ij} = d_{ji}$), a_{ij} is the coupling tensor, k is a typical parameter of the theory which multiplies the thermal displacement and b is related with the thermal conductivity². The functions f_i and h are the supply terms.

When the parameter b vanishes, we obtain the *anisotropic Type II with two temperatures thermoelastodynamic theory (T.II+2TT)*. Otherwise, when the parameter k vanishes, we recover the theory of Chen *et al.* [2, 3, 4, 48]. Thus, the results we will prove have a direct application for the classical theory with two temperatures which was proposed from a “well accepted” axiomatic thermomechanical frame.

As we restrict our attention to homogeneous materials, the coefficients are independent of the spatial variable. It is worth noting that we do not have *a priori* any thermomechanical reason to assume that k is positive. In fact, for the Green and Naghdi theories the sign of the parameter k is not a consequence of the basic axioms. Therefore, the sign of the parameter k is not definite and, for the moment, we do not assume that k is positive³.

We will assume that:

(A1) The mass density ρ and the heat capacity c are positive numbers. That is,

$$\rho > 0, \quad c > 0.$$

(A2) The tensor (d_{ij}) is positive definite. That is, there exists a positive constant C such that

$$d_{ij} \xi_i \xi_j \geq C \xi_i \xi_i, \tag{4}$$

for every vector (ξ_i) .

(A3) The elasticity tensor is positive definite. That is, there exists a positive constant C^* such that

$$C_{ijkl} \zeta_{ij} \zeta_{kl} \geq C^* \zeta_{ij} \zeta_{ij}, \tag{5}$$

for every tensor (ζ_{ij}) .

¹We do not impose the other symmetries which are usual in the linear case. Thus our studies apply for linearized initially stressed solids.

²This expression is used by Chen *et al.* in the reference [4] p. 110.

³It is suitable to analyze the consequences of the positivity of k . However, in this paper we only assume that k is positive in Section 5.

It is natural to impose assumptions (A1)-(A2) and the physical meaning is clear. Assumption (A3) is related with the theory of elastic stability. When the T.II+2TT theory is considered, as in Section 4, the parameter b must be zero, but in the remaining sections of the paper we shall assume that the thermal conductivity b is strictly positive ($b > 0$). When the linear theory is considered, the elasticity tensor satisfies other symmetries $C_{ijkl} = C_{jikl} = C_{ijlk}$. In that case, we can restrict the assumption (A3) to symmetric tensors (ζ_{ij}) .

Our aim of study is the backward in time problem. Therefore, the system we want to analyze is

$$\rho \ddot{u}_i = (C_{ijkl} u_{k,h} + a_{ij} \theta)_{,j} + \rho f_i, \quad (6)$$

$$c(\ddot{\alpha} - d_{ij} \ddot{\alpha}_{,ij}) = a_{ij} \dot{u}_{i,j} + k d_{rj} \alpha_{,rj} - b d_{rj} T_{,rj} + h. \quad (7)$$

The backward in time problem is relevant from the mathematical and the thermomechanical points of view because it provides information about what happened in the past by means of the information we have at the present time. We here restrict our attention to a couple of problems. We first consider the case when the domain is bounded and we prove the existence, uniqueness, instability and continuous dependence of the solutions. We study later the spatial behavior of the solutions of the homogeneous version ($f_i = h = 0$) of the system (6), (7). In this second situation we will deal with an unbounded prismatic cylinder.

From Sections 3 to 5 we study the solutions of the system (6), (7) in $B \times I$, where I is a time interval and B is a bounded domain. Let Γ be the boundary of B . To set a well defined problem, we must adjoin boundary and initial conditions to the field equations. We consider the homogeneous Dirichlet boundary conditions

$$u_i(\mathbf{x}, t) = 0, \quad \alpha(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \Gamma, \quad t \in I, \quad (8)$$

together with the initial conditions

$$u_i(\mathbf{x}, 0) = u_i^0, \quad \dot{u}_i(\mathbf{x}, 0) = v_i^0, \quad \Omega(\mathbf{x}, 0) = \Omega^0, \quad \dot{\Omega}(\mathbf{x}, 0) = \theta^0, \quad \mathbf{x} \in B. \quad (9)$$

The operator $(I - d_{ij} \partial^2 / \partial x_i \partial x_j)$ acts on $W^{2,2} \cap W_0^{1,2}$ and takes values in L^2 (the usual Sobolev spaces). We denote by $\Phi(\Omega) = \alpha$ the inverse operator. In view of the boundary conditions, we have that

$$\|\Omega\|^2 = \|\alpha\|^2 + 2(d_{ij} \alpha_{,i} \alpha_{,j}) + \|d_{ij} \alpha_{,ij}\|^2. \quad (10)$$

In the last section of the paper we obtain a Phragmén-Lindelöf alternative for the solutions of the homogeneous version of our system (6), (7). Thus, we will consider the problem determined by this system in a semi-infinite cylinder $B = [0, \infty) \times D$, where D is a two dimensional bounded domain with sufficiently smooth boundary ∂D to apply the divergence theorem.

As before, we must also adjoin boundary and initial conditions to the field equations. We consider the following boundary conditions

$$u_i(\mathbf{x}, t) = 0, \quad \alpha(\mathbf{x}, t) = 0, \quad \mathbf{x} \in [0, \infty) \times \partial D, \quad t \in I, \quad (11)$$

$$u_i(\mathbf{x}, t) = u_i^*, \quad \alpha(\mathbf{x}, t) = \alpha^*, \quad \mathbf{x} \in \{0\} \times D, \quad t \in I. \quad (12)$$

We assume null initial conditions

$$u_i(\mathbf{x}, 0) = 0, \quad \dot{u}_i(\mathbf{x}, 0) = 0, \quad \Omega(\mathbf{x}, 0) = 0, \quad \theta(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in B. \quad (13)$$

The starting point for the analysis of the spatial behavior is the following equality⁴:

$$\begin{aligned} & ((C_{ijkh}u_{k,h} + a_{ij}\theta)\dot{u}_i + (kd_{rj}\alpha_{,r} - bd_{rj}T_{,r})T)_{,j} \\ &= \frac{1}{2} \frac{d}{dt} \phi - bd_{ij}T_{,i}T_{,j} - b(d_{ij}T_{,ij})^2, \end{aligned} \tag{14}$$

where

$$\phi = \rho \dot{u}_i \dot{u}_i + C_{ijkh}u_{i,j}u_{k,h} + c\theta^2 + kd_{ij}\alpha_{,i}\alpha_{,j} + k(d_{ij}\alpha_{,ij})^2 \tag{15}$$

and

$$(d_{ij}T_{,i}T)_{,j} = d_{ij}T_{,i}T_{,j} + d_{ij}T_{,ij}T. \tag{16}$$

They are satisfied for every solution for the backward in time system.

We will also use the following fact (see the Appendix of [25] for a proof):

Exponentially weighted Poincaré inequality. Assume that $f(s)$ is a differentiable function in $[0, t]$ such that $f(0) = 0$. Then the following inequality

$$\int_0^t \exp(-2\omega s) f^2(s) ds \leq \frac{4t^2}{\pi^2 + 4t^2\omega^2} \int_0^t \exp(-2\omega s) (\dot{f}(s))^2 ds \tag{17}$$

holds, for every $\omega > 0$. We note that $\varphi(t) = \frac{4t^2}{\pi^2 + 4t^2\omega^2}$ is a growing function and so

$$\begin{aligned} & \int_0^t \exp(-2\omega s) f^2(s) ds \\ & \leq \frac{4t_0^2}{\pi^2 + 4t_0^2\omega^2} \int_0^t \exp(-2\omega s) (\dot{f}(s))^2 ds, \text{ for } 0 < t \leq t_0. \end{aligned} \tag{18}$$

3. Existence, uniqueness and continuous dependence. In this section we obtain the existence, uniqueness and continuous dependence of the solutions for the proposed problem. We state a suitable frame where the problem is well posed.

We transform the boundary-initial-value problem into an abstract problem on a suitable Hilbert space. We denote by \mathcal{Z} the vectorial space

$$\{(\mathbf{u}, \mathbf{v}, \Omega, \theta), \mathbf{u} \in \mathbf{W}_0^{1,2}, \mathbf{v} \in \mathbf{L}^2, \Omega, \theta \in L^2\},$$

where $\mathbf{W}_0^{1,2} = [W_0^{1,2}]^3$ and $\mathbf{L}^2 = [L^2]^3$. We define

$$\begin{aligned} A_i(\mathbf{u}) &= \rho^{-1}(C_{ijkh}u_{k,h})_{,j}, \quad B_i(\theta) = \rho^{-1}(a_{ij}\theta)_{,j}, \quad \mathbf{A} = (A_i), \quad \mathbf{B} = (B_i), \\ C(\mathbf{v}) &= c^{-1}a_{ij}v_{i,j}, \quad D(\Omega) = c^{-1}kd_{rj}(\Phi(\Omega))_{,rj}, \quad E(\theta) = -c^{-1}kd_{rj}(\Phi(\theta))_{,rj} \end{aligned}$$

and

$$\mathcal{A} = \begin{pmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{A} & \mathbf{0} & \mathbf{0} & \mathbf{B} \\ 0 & 0 & 0 & I \\ 0 & C & D & E \end{pmatrix}, \tag{19}$$

where \mathbf{I} is the identity operator.

Our problem can be transformed into the following abstract Cauchy problem in the Hilbert space \mathcal{Z} :

$$\frac{d\omega}{dt} = \mathcal{A}\omega(t) + \mathcal{F}(t), \quad \omega(0) = \omega_0, \tag{20}$$

where

$$\omega_0 = (\mathbf{u}^0, \mathbf{v}^0, \Omega^0, \theta^0), \quad \mathcal{F} = (\mathbf{0}, \mathbf{f}, 0, c^{-1}h).$$

⁴See Appendix 2 for a proof.

We introduce in \mathcal{Z} the inner product

$$\begin{aligned} &< (\mathbf{u}, \mathbf{v}, \Omega, \theta), (\mathbf{u}^*, \mathbf{v}^*, \Omega^*, \theta^*) >_{\mathcal{Z}} \\ &= \frac{1}{2} \int_B (\rho v_i \bar{v}_i^* + C_{ijkl} u_{i,j} \bar{u}_{k,l}^* + c\theta \bar{\theta}^* + \epsilon \Omega \bar{\Omega}^*) dv, \end{aligned} \tag{21}$$

where ϵ is an arbitrary positive constant and the bar denotes the conjugate. In fact, the summand $\epsilon \Omega \bar{\Omega}^*$ in the inner product is the key point to overcome the difficulties. We note that the associated norm is

$$\|(\mathbf{u}, \mathbf{v}, \Omega, \theta)\|_{\mathcal{Z}}^2 = \frac{1}{2} \int_B (\rho v_i \bar{v}_i + C_{ijkl} u_{i,j} \bar{u}_{k,l} + c\theta \bar{\theta} + \epsilon \Omega \bar{\Omega}) dv. \tag{22}$$

In view of the assumptions on the constitutive coefficients and the Poincaré inequality, we have that this inner product is equivalent to the usual one in \mathcal{Z} . The domain \mathcal{D} of \mathcal{A} is the set of $\omega \in \mathcal{Z}$ such that $\mathcal{A}\omega \in \mathcal{Z}$. It is a dense subspace of \mathcal{Z} . The next two lemmas are useful in the proof of the existence of solutions.

Lemma 3.1. *There exists a positive constant C_1 such that the estimate*

$$\langle \mathcal{A}\omega, \omega \rangle \leq C_1 \|\omega\|_{\mathcal{Z}}^2 \tag{23}$$

is satisfied, for every $\omega \in \mathcal{D}$.

Proof. In view of the system of equations and the boundary conditions, we see that

$$\begin{aligned} \langle \mathcal{A}\omega, \omega \rangle &= \frac{\epsilon}{2} \int_B \bar{\Omega} \theta dv - \frac{k}{2} \int_B d_{rj} \alpha_{,r} \bar{T}_{,j} dv - \frac{k}{2} \int_B d_{rj} \alpha_{,rj} d_{il} \bar{T}_{,il} dv \\ &\quad + \frac{b}{2} \int_B d_{rj} T_{,r} \bar{T}_{,j} dv + \frac{b}{2} \int_B d_{rj} T_{,rj} d_{ik} \bar{T}_{,ik} dv. \end{aligned}$$

From the definition of the inner product and the equality (10), we see that the lemma is proved. □

Lemma 3.2. *There exists a positive constant λ_0 such that the condition*

$$\text{Range} (\lambda_0 \mathcal{I} - \mathcal{A}) = \mathcal{Z}. \tag{24}$$

is satisfied.

Proof. Let $\tilde{\omega} = (\tilde{\mathbf{u}}, \tilde{\mathbf{v}}, \tilde{\Omega}, \tilde{\theta}) \in \mathcal{Z}$. We must prove that the equation

$$(\lambda_0 - \mathcal{A})\omega = \tilde{\omega}$$

has a solution $\omega = (\mathbf{u}, \mathbf{v}, \Omega, \theta) \in \mathcal{D}$ when λ_0 is sufficiently large. From the definition of the operator \mathcal{A} , we obtain the system

$$\lambda_0 \mathbf{u} - \mathbf{v} = \tilde{\mathbf{u}}, \quad \lambda_0 \mathbf{v} - \mathbf{A}\mathbf{u} - \mathbf{B}\theta = \tilde{\mathbf{v}}, \quad \lambda_0 \Omega - \theta = \tilde{\Omega}, \quad \lambda_0 \theta - C\mathbf{v} - D\Omega - E\theta = \tilde{\theta}. \tag{25}$$

By substituting the first and the third equations into the others two we obtain

$$\begin{aligned} \lambda_0^2 \mathbf{u} - \mathbf{A}\mathbf{u} - \lambda_0 \mathbf{B}\Omega &= \lambda_0 \tilde{\mathbf{u}} + \tilde{\mathbf{v}} - \mathbf{B}\tilde{\Omega}, \\ (\lambda_0^2 - (D + \lambda_0 E))\Omega - \lambda_0 C\mathbf{u} &= \tilde{\theta} + (\lambda_0 - E)\tilde{\Omega} - C\tilde{\mathbf{v}}. \end{aligned} \tag{26}$$

To study this system we introduce the following bilinear form

$$\begin{aligned} &\mathcal{B}[(\mathbf{u}_1, \Omega_1), (\mathbf{u}_2, \Omega_2)] \\ &= \left\langle \begin{pmatrix} \lambda_0^2 \mathbf{I} - \mathbf{A} & -\lambda_0 \mathbf{B} \\ -\lambda_0 C & (\lambda_0^2 - (D + \lambda_0 E)) \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \Omega_1 \end{pmatrix}, \begin{pmatrix} \mathbf{u}_2 & \Omega_2 \end{pmatrix} \right\rangle_{\mathbf{L}^2 \times \mathbf{L}^2}. \end{aligned} \tag{27}$$

This bilinear form is defined on $\mathbf{W}_0^{1,2} \times L^2$. After the use of the divergence theorem and the equality (10), we can see that \mathcal{B} is bounded on each component. On the other side,

$$\begin{aligned} &\mathcal{B}[(\mathbf{u}, \Omega), (\mathbf{u}, \Omega)] \\ &= \int_B (\lambda_0^2 u_i \bar{u}_i + C_{ijkl} u_{i,j} \bar{u}_{k,l} + \lambda_0^2 \Omega \bar{\Omega} + k d_{rj} \alpha_{,rj} \bar{\Omega} - \lambda_0 b d_{rj} \alpha_{,rj} \bar{\Omega}) dv. \end{aligned} \tag{28}$$

Recalling again the equality (10), we see that, for λ_0 large enough, the estimate

$$\lambda_0 \int_B (\lambda_0 \Omega \bar{\Omega} + \frac{k}{\lambda_0} d_{rj} \alpha_{,rj} \bar{\Omega} - b d_{rj} \alpha_{,rj} \bar{\Omega}) dv \geq C_2 \int_B \Omega \bar{\Omega} dv, \quad (C_2 > 0)$$

is satisfied. Thus, for λ_0 sufficiently large, the bilinear form \mathcal{B} becomes coercive. As the right-hand side of the system (26) lies in $\mathbf{W}^{-1,2} \times L^2$, the Lax-Milgram theorem implies the existence of a solution of the system (26). Thus, the equation has a unique solution in \mathcal{D} . \square

We can state the main results of this section:

Theorem 3.3. *The operator \mathcal{A} generates a quasi-contractive semigroup.*

Proof. The proof is a direct consequence of Lemmas 3.1 and 3.2 and the Lumer-Phillips corollary to the Hille-Yosida theorem. \square

Therefore, we conclude the next theorem (see [10], p. 84):

Theorem 3.4. *Assume that $f_i, h \in C^1([0, \infty), L^2)$ and $\omega_0 \in \mathcal{D}$. Then, there exists a unique solution $\omega(t) \in C^1([0, \infty), \mathcal{Z})$ that takes values in the domain to the initial value problem.*

Remark 1. Since the solutions are defined by means of a quasi-contractive semigroup, they satisfy the estimate

$$\|\omega(t)\| \leq \left(\|\omega_0\| + K_2 \int_0^t g(s) ds \right) \exp(K_1 t),$$

where K_1, K_2 are two computable positive constants and

$$g(s) = \left(\int_B (f_i \bar{f}_i + c^{-2} h \bar{h}) dv \right)^{1/2}.$$

This is a continuous dependence result for the solutions upon the initial data and the supply terms. Thus, we have proved that the proposed problem is *well posed*.

Remark 2. Theorem 3.2 has been proposed for the T.III+2TT theory, but it is clear that the same argument also applies for the T.II+2TT theory.

Remark 3. Theorem 3.2 can be extended without difficulties to the case where B is a semi-infinite cylinder, whenever we assume that the solutions tend to zero when the spatial variable becomes unbounded.

4. Instability for T.II+2TT. In this section we give a result of instability for the solutions of the homogeneous version of the T.II+2TT theory⁵. That is, when

⁵In this section the inner product is defined on the real Hilbert space.

the supply terms vanish and the thermal conductivity parameter b vanishes. In this case we have that the following energy type equality

$$J(t) = \frac{1}{2} \left[(\rho \ddot{u}_k, \ddot{u}_k) + (C_{ijkl} \dot{u}_{i,j}, \dot{u}_{k,l}) + \|c^{1/2} \dot{\theta}\|^2 + k (d_{ij} T_{,i}, T_{,j}) + k \|d_{ij} T_{,ij}\|^2 \right] = J(0) \tag{29}$$

is satisfied for every solution of the problem.

Theorem 4.1. *For the homogeneous problem determined by the system (6)–(7) with $b = 0$, the boundary conditions (8) and the initial conditions (9), the solution becomes unbounded in an exponential way, whenever $J(0) < 0$.*

Proof. We define the function

$$G_{\omega,t_0}(t) = \|c^{1/2} \theta\|^2 + (C_{ijkl} u_{i,j}, u_{k,l}) + \omega (t + t_0)^2, \tag{30}$$

where ω and t_0 are two positive constants to be selected later. After direct differentiation we get

$$G'_{\omega,t_0}(t) = 2(c\theta, \dot{\theta}) + 2(C_{ijkl} \dot{u}_{i,j}, \dot{u}_{k,l}) + 2\omega (t + t_0) \tag{31}$$

and

$$G''_{\omega,t_0}(t) = 2\|c^{1/2} \dot{\theta}\|^2 + 2(C_{ijkl} \ddot{u}_{i,j}, \ddot{u}_{k,l}) + 2(c\theta, \ddot{\theta}) + 2(C_{ijkl} u_{i,j}, \ddot{u}_{k,l}) + 2\omega. \tag{32}$$

We have that

$$(\rho \ddot{u}_k, \ddot{u}_k) + (C_{ijkl} u_{i,j}, \ddot{u}_{k,l}) = (a_{ij} \theta_{,i}, \ddot{u}_j) \tag{33}$$

and

$$(c\theta, \ddot{\theta}) + k (d_{ij} T_{,i}, T_{,j}) + k \|d_{ij} T_{,ij}\|^2 = -(a_{ij} \theta_{,i}, \ddot{u}_j). \tag{34}$$

Upon addition, we see

$$(c\theta, \ddot{\theta}) + (C_{ijkl} u_{i,j}, \ddot{u}_{k,l}) = -(\rho \ddot{u}_k, \ddot{u}_k) - k (d_{ij} T_{,i}, T_{,j}) - k \|d_{ij} T_{,ij}\|^2. \tag{35}$$

After the use of the previous equalities, we obtain

$$\begin{aligned} & G_{\omega,t_0} G''_{\omega,t_0} - (G'_{\omega,t_0})^2 \\ &= 4 \left[\|c^{1/2} \theta\|^2 + (C_{ijkl} u_{i,j}, u_{k,l}) + \omega (t + t_0)^2 \right] \\ & \quad \times \left[\|c^{1/2} \dot{\theta}\|^2 + (C_{ijkl} \dot{u}_{i,j}, \dot{u}_{k,l}) + \omega \right] \\ & \quad - 4 \left[(c\theta, \dot{\theta}) + (C_{ijkl} u_{i,j}, \dot{u}_{k,l}) + \omega (t + t_0) \right]^2 - 2G_{\omega,t_0} (2J(0) + \omega). \end{aligned} \tag{36}$$

From the Cauchy-Schwarz inequality, we get

$$G_{\omega,t_0} G''_{\omega,t_0} - (G'_{\omega,t_0})^2 \geq -2G_{\omega,t_0} (2J(0) + \omega). \tag{37}$$

When the initial conditions satisfy that $J(0) < 0$, we can select $\omega = -2J(0)$. Thus, a quadrature implies that

$$G_{\omega,t_0}(t) \geq G_{\omega,t_0}(0) \exp\left(\frac{G'_{\omega,t_0}(0)t}{G_{\omega,t_0}(0)}\right), \quad t \geq 0. \tag{38}$$

It is worth noting that we can always select t_0 large enough to guarantee that $G'_{\omega,t_0}(0) > 0$. Then, the instability of the solutions follows. In fact, we have seen the exponential instability of the solutions. \square

5. Instability for the one-dimensional problem. In this section we show the exponential stability for the forward in time problem of the one-dimensional T.III+2TT theory, when the parameter k is positive and the coupling term does not vanish. This result suggests the instability of the solutions for the backward in time problem. The system of equations can be written as:

$$\rho \ddot{u} = C u_{xx} - a \theta_x, \tag{39}$$

$$c(\ddot{\alpha} - d \ddot{\alpha}_{,xx}) = -a \dot{u}_x + k d \alpha_{xx} + b d T_{xx}. \tag{40}$$

The solutions for the problem determined by this system with the one dimensional initial and boundary conditions define a semigroup of contractions. The generator of the semigroup is the matrix operator

$$\mathcal{A}_1 = \begin{pmatrix} 0 & I & 0 & 0 \\ A & 0 & 0 & B_1 \\ 0 & 0 & 0 & I \\ 0 & C_1 & D & E_1 \end{pmatrix}, \tag{41}$$

which is defined at the corresponding one-dimensional version of the Hilbert space \mathcal{Z} . It is worth noting that in this case $E_1 = -E$, $B_1 = -B$ and $C_1 = -C$ and the inner product is

$$\begin{aligned} &< (u, v, \Omega, \theta), (u^*, v^*, \Omega^*, \theta^*) >_{\mathcal{Z}} \\ &= \frac{1}{2} \int_B (\rho v \bar{v}^* + C u_x \bar{u}_x^* + c \theta \bar{\theta}^* + k d (\alpha_x \bar{\alpha}_x^* + d \alpha_{xx} \bar{\alpha}_{xx}^*)) dx, \end{aligned} \tag{42}$$

where the bar denotes the conjugate.

We need the following lemmas:

Lemma 5.1. *Under the above notations, $0 \in \rho(\mathcal{A}_1)$, where $\rho(\mathcal{A}_1)$ is the set resolvent of the operator \mathcal{A}_1 .*

Proof. The proof is very similar to that of Lemma 3.3. □

Lemma 5.2. *The operator \mathcal{A}_1 generates a semigroup of contractions.*

Proof. In this case

$$\text{Re} \langle \mathcal{A}_1 \omega, \omega \rangle = -\frac{1}{2} (bd \|T_x\|^2 + bd^2 \|T_{xx}\|^2).$$

In view of the previous lemma and the fact that the domain is dense, the lemma is proved. □

To prove the exponential stability of the solutions, it is worth recalling the following result, which can be found in the book by Liu and Zheng [28].

Theorem 5.3. *Let $S(t) = e^{At}$ be a C_0 -semigroup of contractions generated by the operator \mathcal{A} in the Hilbert space \mathcal{H} . If $i\mathbb{R} \subseteq \rho(\mathcal{A})$, then $S(t)$ is exponentially stable if and only if*

$$\overline{\lim}_{|\beta| \rightarrow \infty} \|(i\beta I - \mathcal{A})^{-1}\| < \infty, \quad \beta \in \mathbb{R}. \tag{43}$$

Theorem 5.4. *Under the above conditions, the operator \mathcal{A}_1 generates a semigroup which is exponentially stable.*

Proof. Following the arguments in [28], the proof consists of the following steps:

(i) Since 0 is in the resolvent of \mathcal{A}_1 , using the contraction mapping theorem, we have that for any real λ such that $|\lambda| < \|\mathcal{A}_1^{-1}\|^{-1}$, the operator $i\lambda \mathcal{I} - \mathcal{A}_1 =$

$\mathcal{A}_1(i\lambda\mathcal{A}_1^{-1} - \mathcal{I})$ is invertible. Moreover, $\|(i\lambda\mathcal{I} - \mathcal{A}_1)^{-1}\|$ is a continuous function of λ in the interval $(-\|\mathcal{A}_1^{-1}\|^{-1}, \|\mathcal{A}_1^{-1}\|^{-1})$.

(ii) If $\sup\{\|(i\lambda\mathcal{I} - \mathcal{A}_1)^{-1}\|, |\lambda| < \|\mathcal{A}_1^{-1}\|^{-1}\} = M < \infty$, then by the contraction theorem, the operator

$$i\lambda\mathcal{I} - \mathcal{A}_1 = (i\lambda_0\mathcal{I} - \mathcal{A}_1)\left(\mathcal{I} + i(\lambda - \lambda_0)(i\lambda_0\mathcal{I} - \mathcal{A}_1)^{-1}\right)$$

is invertible for $|\lambda - \lambda_0| < M^{-1}$. It turns out that, by choosing λ_0 as close to $\|\mathcal{A}_1^{-1}\|^{-1}$ as we can, the set $\{\lambda, |\lambda| < \|\mathcal{A}_1^{-1}\|^{-1} + M^{-1}\}$ is contained in the resolvent of \mathcal{A}_1 and $\|(i\lambda\mathcal{I} - \mathcal{A}_1)^{-1}\|$ is a continuous function of λ in the interval $(-\|\mathcal{A}_1^{-1}\|^{-1} - M^{-1}, \|\mathcal{A}_1^{-1}\|^{-1} + M^{-1})$.

(iii) If the imaginary axis is not contained in the resolvent, then there exists a real number ϖ with $\|\mathcal{A}_1^{-1}\|^{-1} \leq |\varpi| < \infty$ such that the set $\{i\lambda, |\lambda| < |\varpi|\}$ is in the resolvent of \mathcal{A}_1 and $\sup\{\|(i\lambda\mathcal{I} - \mathcal{A}_1)^{-1}\|, |\lambda| < |\varpi|\} = \infty$. Therefore, there exists a sequence of real numbers λ_n with $\lambda_n \rightarrow \varpi$, $|\lambda_n| < |\varpi|$ and a sequence of unit vectors $\omega_n = (u_n, v_n, \Omega_n, \theta_n)$ in the domain of the operator \mathcal{A}_1 such that

$$\|(i\lambda_n\mathcal{I} - \mathcal{A}_1)\omega_n\| \rightarrow 0. \tag{44}$$

This is

$$i\lambda_n u_n - v_n \rightarrow 0 \text{ in } H^1, \tag{45}$$

$$i\rho\lambda_n v_n - (Cu_{n,xx} - a\theta_{n,x}) \rightarrow 0 \text{ in } L^2, \tag{46}$$

$$i\lambda_n \Omega_n - \theta_n \rightarrow 0 \text{ in } L^2, \tag{47}$$

$$ic\lambda_n \theta_n - (-av_{n,x} + kd\alpha_{n,xx} + bdT_{n,xx}) \rightarrow 0 \text{ in } L^2. \tag{48}$$

Considering the inner product of $(i\lambda_n\mathcal{I} - \mathcal{A}_1)\omega_n$ times ω_n in \mathcal{Z} and then taking its real part yields $\|T_{n,x}\|, \|T_{n,xx}\| \rightarrow 0$. It then follows that $\|\theta_n\|, \|\Omega_n\| \rightarrow 0$. From the last condition we obtain that $v_{n,x} \rightarrow 0$ and then it also follows that $u_{n,x} \rightarrow 0$, which contradicts the condition that the sequence is of unit norm.

Now, we prove (43) by a contradiction argument. If this condition is not true, there exist a sequence λ_n such that the absolute value tends to infinite and a sequence of unit vectors in the domain of the operator \mathcal{A} such that (44) holds.

Following the arguments used previously when λ_n is bounded, we can obtain that θ_n and Ω_n tend to zero in L^2 . However, the remaining arguments need some more attention. We also see that $\lambda_n^{-1}v_{n,x} \rightarrow 0$. As $\lambda_n^{-1}v_{n,x}$ is equivalent to $iu_{n,x}$, we obtain that $u_{n,x}$ tends to zero. From the second condition, we see that

$$i\rho < v_n, v_n > + \lambda_n^{-1}C < u_{n,x}, v_{n,x} > - \lambda_n^{-1}a < \theta_n, v_{n,x} > \rightarrow 0.$$

Therefore, we get that $v_n \rightarrow 0$ in L^2 , which contradicts the condition that the sequence is of unit norm. □

6. A Phragmén-Lindelöf alternative. The aim of this section is to obtain a Phragmén-Lindelöf alternative for the solutions of the homogeneous version for the backward in time problem of the T.III+2TT thermoelastodynamic theory. The problem we study in this section is ill-posed. That is, we cannot expect for the uniqueness of the solutions, neither to find a frame where the solutions lie with continuous dependence with respect to the data of the problem. However, if we assume suitable asymptotic conditions and sufficiently regular boundary conditions, we can adapt the argument proposed by Díaz and Quintanilla (see [5], pp. 5,6) to our situation. So, we obtain a result on existence and uniqueness of solutions with the help of the last remark to Theorem 3.2 (see Appendix 1). But, in the general case we should assume the existence of solutions. This means that the solutions

satisfy the regularity conditions imposed by the different manipulations proposed in the analysis⁶. Of course, the classical solutions are contained in the class of solutions we study.

The main idea is to obtain the inequality (63). This is the challenge of the problem. The use of the exponentially weighted Poincaré inequality allows us to overcome it. We shall denote

$$\Lambda_\omega(t) = \exp(-2\omega t)\phi(t) + \int_0^t \exp(-2\omega s)\psi(s)ds, \tag{49}$$

where

$$\psi = 2\omega\phi + 4\omega c(d_{ij}TT_{,j})_{,i} - 2b(d_{ij}T_{,i}T_{,j} + (d_{ij}T_{,ij})^2) \tag{50}$$

and ϕ has been defined in Section 2.

Lemma 6.1. *Assume that $\alpha(0) = 0$. Then, there exists a positive constant \tilde{C} such that*

$$\begin{aligned} \Lambda_\omega(t) \geq & 2\omega \int_0^t \exp(-2\omega s) [\rho\dot{u}_i\dot{u}_i + C_{ijkh}u_{k,h}u_{i,j} + cT^2] ds \\ & + \tilde{C} \int_0^t \exp(-2\omega s) [b(d_{ij}\alpha_{,i}\alpha_{,j} + (d_{ij}\alpha_{,ij})^2) \\ & + c(d_{ij}T_{,i}T_{,j} + (d_{ij}T_{,ij})^2)] ds \\ & + \exp(-2\omega t)\tilde{C} [c\theta^2(t) + \rho\dot{u}_i(t)\dot{u}_i(t) + C_{ijkh}u_{k,h}(t)u_{i,j}(t) \\ & + b(d_{ij}\alpha_{,i}\alpha_{,j} + (d_{ij}\alpha_{,ij})^2)], \end{aligned} \tag{51}$$

when ω is sufficiently large and for every $t \leq t_0$.

Proof. We first note that

$$2\omega c\theta^2 + 4\omega c(d_{ij}TT_{,i})_{,j} = 2\omega c(T^2 + 2d_{ij}T_{,i}T_{,j} + (d_{ij}T_{,ij})^2). \tag{52}$$

We have that

$$\begin{aligned} & k \left(\exp(-2\omega t)d_{ij}\alpha_{,i}\alpha_{,j} + 2\omega \int_0^t \exp(-2\omega s)d_{ij}\alpha_{,i}\alpha_{,j}ds \right) \\ = & k \left(\int_0^t \frac{d}{ds}(\exp(-2\omega s)d_{ij}\alpha_{,i}\alpha_{,j})ds + 2\omega \int_0^t \exp(-2\omega s)d_{ij}\alpha_{,i}\alpha_{,j}ds \right) \\ = & 2k \int_0^t \exp(-2\omega s)d_{ij}\alpha_{,i}T_{,j}ds \\ \leq & 2|k| \left(\int_0^t \exp(-2\omega s)d_{ij}\alpha_{,i}\alpha_{,j}ds \right)^{1/2} \left(\int_0^t \exp(-2\omega s)d_{ij}T_{,i}T_{,j}ds \right)^{1/2} \\ \leq & \frac{4|k|C_3t_0}{\sqrt{\pi^2 + 4t_0^2\omega^2}} \int_0^t \exp(-2\omega s)d_{ij}T_{,i}T_{,j}ds, \end{aligned} \tag{53}$$

⁶This is the usual framework in the study of general Phragmén-Lindelöf alternatives (see [8, 25, 26]).

where C_3 is a calculable positive constant⁷. In a similar way, we can see that

$$\begin{aligned} b \left(\exp(-2\omega t) d_{ij} \alpha_{,i} \alpha_{,j} + 2\omega \int_0^t \exp(-2\omega s) d_{ij} \alpha_{,i} \alpha_{,j} ds \right) \\ \leq \frac{4bC_3 t_0}{\sqrt{\pi^2 + 4t_0^2 \omega^2}} \int_0^t \exp(-2\omega s) d_{ij} T_{,i} T_{,j} ds. \end{aligned} \quad (54)$$

The following estimates can be obtained by means of similar arguments:

$$\begin{aligned} k \left(\exp(-2\omega t) (d_{ij} \alpha_{,ij})^2 + 2\omega \int_0^t \exp(-2\omega s) (d_{ij} \alpha_{,ij})^2 ds \right) \\ \leq \frac{4|k|t_0}{\sqrt{\pi^2 + 4t_0^2 \omega^2}} \int_0^t \exp(-2\omega s) (d_{ij} T_{,ij})^2 ds \end{aligned} \quad (55)$$

and

$$\begin{aligned} b \left(\exp(-2\omega t) (d_{ij} \alpha_{,ij})^2 + 2\omega \int_0^t \exp(-2\omega s) (d_{ij} \alpha_{,ij})^2 ds \right) \\ \leq \frac{4bt_0}{\sqrt{\pi^2 + 4t_0^2 \omega^2}} \int_0^t \exp(-2\omega s) (d_{ij} T_{,ij})^2 ds. \end{aligned} \quad (56)$$

Then, we get

$$\begin{aligned} \Lambda_\omega(t) - \tilde{C}b \left[\exp(-2\omega t) (d_{ij} \alpha_{,i} \alpha_{,j} + (d_{ij} \alpha_{,ij})^2) \right. \\ \left. + 2\omega \int_0^t \exp(-2\omega s) (d_{ij} \alpha_{,i} \alpha_{,j} + (d_{ij} \alpha_{,ij})^2) ds \right] \\ \geq 2\omega \int_0^t \exp(-2\omega s) \left[\rho \dot{u}_i \dot{u}_i + C_{ijkh} u_{i,j} u_{k,h} + cT^2 + 2cd_{ij} T_{,i} T_{,j} \right. \\ \left. + c(d_{ij} T_{,ij})^2 \right] ds \\ - \left(2b + \frac{4(|k| + b\tilde{C})C_3 t_0}{\sqrt{\pi^2 + 4t_0^2 \omega^2}} \right) \int_0^t \exp(-2\omega s) (d_{ij} T_{,i} T_{,j} + (d_{ij} T_{,ij})^2) ds \\ + \exp(-2\omega t) (\rho \dot{u}_i \dot{u}_i + C_{ijkh} u_{i,j} u_{k,h} + c\theta^2). \end{aligned} \quad (57)$$

It is clear that we can select ω sufficiently large and then the lemma is proved. \square

Now, we obtain a Phragmén-Lindelöf alternative for the solutions of the problem determined by the system (6), (7) with the boundary and null initial conditions (11)–(13). Our analysis follows from the ideas proposed by Flavin *et al.* [8]. They estimate the absolute value of a suitable function by means of its spatial derivative. The function we consider is

$$\begin{aligned} H_\omega(z, t) = - \int_0^t \int_{D(z)} \exp(-2\omega s) (a_{i1kh} u_{k,h} + a_{i1}\theta) \dot{u}_i da ds \\ - \int_0^t \int_{D(z)} \exp(-2\omega s) (kd_{1j} \alpha_{,j} + bd_{1j} T_{,j} + 4\omega cd_{1j} T_{,j}) T da ds. \end{aligned} \quad (58)$$

⁷We note that $C_3 = d^*/d_*$, where d^* and d_* are the maximum and the minimum of the eigenvalues of the matrix d_{ij} , respectively.

Here, $D(z)$ denotes the cross section of the cylinder at a distance z from the base. That is, $D(z) = \{\mathbf{x} \in B, x_3 = z\}$ and ω is a positive constant to be selected. After the use of the field equations, the boundary and the initial conditions, we obtain

$$H_\omega(z+h, t) = H_\omega(z, t) - \frac{1}{2} \int_{B(z+h, z)} \Lambda_\omega(t) dv, \tag{59}$$

for every $h > 0$, where $B(z+h, z) = \{\mathbf{x} \in B, z < x_3 < z+h\}$. When $H_\omega(z, t)$ tends to zero as z becomes unbounded, we see that

$$H_\omega(z, t) = \frac{1}{2} \int_{B(\infty, z)} \Lambda_\omega(t) dv, \tag{60}$$

where $B(\infty, z) = \{\mathbf{x} \in B, z < x_3\}$. We also note that from (59) we have

$$\frac{\partial H_\omega}{\partial z} = -\frac{1}{2} \int_{D(z)} \Lambda_\omega(t) da. \tag{61}$$

It is worth noting that for ω sufficiently large, Lemma 5.1 implies that

$$\begin{aligned} \frac{\partial H_\omega}{\partial z} &\leq -\omega \int_0^t \int_{D(z)} \exp(-2\omega s) (\rho \dot{u}_i \dot{u}_i + C_{ijkh} u_{k,h} u_{i,j} + cT^2) ds da \\ &\quad - \frac{\tilde{C}}{2} \int_0^t \int_{D(z)} \exp(-2\omega s) \left[b (d_{ij} \alpha_{,i} \alpha_{,j} + (d_{ij} \alpha_{,ij})^2) \right. \\ &\quad \left. + c (d_{ij} T_{,i} T_{,j} + (d_{ij} T_{,ij})^2) \right] ds da \\ &\quad - \exp(-2\omega t) \frac{\tilde{C}}{2} \int_{D(z)} \left[c\theta^2(t) + \rho \dot{u}_i(t) \dot{u}_i(t) + C_{ijkh} u_{k,h}(t) u_{i,j}(t) \right. \\ &\quad \left. + b(d_{ij} \alpha_{,i} \alpha_{,j} + (d_{ij} \alpha_{,ij})^2) \right] da. \end{aligned} \tag{62}$$

In view of the definition of H_ω and the estimate (62), we can find a positive constant κ such that the inequality

$$|H_\omega(z, t)| \leq -\kappa \frac{\partial H_\omega}{\partial z} \tag{63}$$

holds, for every $z \geq 0$. This inequality is usual in the study of spatial estimates. We get two inequalities:

$$-\frac{\partial H_\omega(z, t)}{\partial z} \geq \kappa^{-1} (-H_\omega(z, t)), \tag{64}$$

$$-\frac{\partial H_\omega(z, t)}{\partial z} \geq \kappa^{-1} H_\omega(z, t). \tag{65}$$

If there is some $z_0 \geq 0$ such that $H_\omega(z_0, t) < 0$, then $H_\omega(z, t) < 0$ for $z \geq z_0$. From (64) we obtain

$$-H_\omega(z, t) \geq -H_\omega(z_0, t) \exp\left(\frac{z-z_0}{\kappa}\right), \quad z \geq z_0. \tag{66}$$

Thus, when $H_\omega(z_0, t) < 0$, we conclude that⁸

$$\lim_{z \rightarrow \infty} \exp\left(-\frac{z}{\kappa}\right) F(z, t) \geq Q > 0, \tag{67}$$

where

$$F(z, t) = \int_{B(z)} \Lambda_\omega(t) dv, \tag{68}$$

⁸It is possible that the limit becomes unbounded.

where $B(z) = \{\mathbf{x} \in B : 0 \leq x_1 \leq z\}$. Otherwise, $H_\omega(z, t) \geq 0$ for all $z \geq 0$. A quadrature on (65) yields

$$H_\omega(z, t) \leq H_\omega(0, t) \exp\left(-\frac{z}{\kappa}\right), \quad z \geq 0. \quad (69)$$

We can summarize the results in the following way.

Theorem 6.2. *Let (u_i, Ω) be a solution of the problem determined by the homogeneous version of the system (6)–(7), the boundary conditions (11), (12) and the null initial conditions (13). Then, either the solution satisfies the asymptotic condition (67) or it satisfies the decay estimate (69).*

If we define the functions

$$\begin{aligned} \Xi(t) = & 2\omega \int_0^t (\rho \dot{u}_i \dot{u}_i + C_{ijkl} u_{k,h} u_{i,j} + cT^2) ds \\ & + \tilde{C} \int_0^t (b(d_{ij} \alpha_{,i} \alpha_{,j} + (d_{ij} \alpha_{,ij})^2) + cb(d_{ij} T_{,i} T_{,j} + (d_{ij} T_{,ij})^2)) ds \\ & + \tilde{C} \left[c\theta^2(t) + \rho \dot{u}_i(t) \dot{u}_i(t) + C_{ijkl} u_{k,h}(t) u_{i,j}(t) \right. \\ & \left. + b(d_{ij} \alpha_{,i} \alpha_{,j} + (d_{ij} \alpha_{,ij})^2) \right] \end{aligned} \quad (70)$$

and

$$\Omega_1(z, t) = \int_{B(z)} \Xi(t) dv, \quad \Omega_2(z, t) = \int_{B(\infty, z)} \Xi(t) dv, \quad (71)$$

the instability condition implies that⁹

$$\lim_{z \rightarrow \infty} \exp\left(-\frac{z}{\kappa}\right) \Omega_1(z, t) \geq Q^* > 0, \quad (72)$$

meanwhile the decay estimate yields

$$\Omega_2(z, t) \leq 2 \exp(2\omega t) H_\omega(0, t) \exp\left(-\frac{z}{\kappa}\right), \quad z \geq 0. \quad (73)$$

Remark 4. We point out that it is also possible to obtain an upper bound for the amplitude term $H_\omega(0, t)$ in terms of the boundary conditions. We do not develop this task here, because we only want to emphasize the main points of the analysis and we do not want to enlarge the paper.

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REFERENCES

- [1] E. S. Awad, [A note on the spatial decay estimates in non-classical linear thermoelastic semi-cylindrical bounded domains](#), *J. Thermal Stresses*, **34** (2011), 147–160.
- [2] P. J. Chen and M. E. Gurtin, [On a theory of heat involving two temperatures](#), *Jour. Appl. Math. Phys. (ZAMP)*, **19** (1968), 614–627.
- [3] P. J. Chen, M. E. Gurtin and W. O. Williams, [A note on non-simple heat conduction](#), *Jour. Appl. Math. Phys. (ZAMP)*, **19** (1968), 969–970.
- [4] P. J. Chen, M. E. Gurtin and W. O. Williams, [On the thermodynamics of non-simple materials with two temperatures](#), *Jour. Appl. Math. Phys. (ZAMP)*, **20** (1969), 107–112.

⁹This limit can also be unbounded.

- [5] J. I. Díaz and R. Quintanilla, [Spatial and continuous dependence estimates in linear viscoelasticity](#), *J. Math. Anal. Appl.*, **273** (2002), 1–16.
- [6] M. Dreher, R. Quintanilla and R. Racke, [Ill-posed problems in thermomechanics](#), *Appl. Math. Lett.*, **22** (2009), 1374–1379.
- [7] A. S. El-Karamany and M. A. Ezzat, [On the two-temperature Green-Naghdi thermoelasticity theories](#), *J. Thermal Stresses*, **34** (2011), 1207–1226.
- [8] J. N. Flavin, R. J. Knops and L. E. Payne, [Decay estimates for the constrained elastic cylinder of variable cross-section](#), *Quart. Appl. Math.*, **47** (1989), 325–350.
- [9] J. N. Flavin, R. J. Knops and L. E. Payne, [Energy bounds in dynamical problems for a semi-infinite elastic beam](#), in *Elasticity: Mathematical Methods and Applications* (eds. G. Eason and R.W. Ogden), Chichester: Ellis Horwood, (1989) pp. 101–111.
- [10] J. A. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford: Oxford Mathematical Monographs, Oxford, 1985.
- [11] A. E. Green and P. M. Naghdi, [On undamped heat waves in an elastic solid](#), *J. Thermal Stresses*, **15** (1992), 253–264.
- [12] A. E. Green and P. M. Naghdi, [Thermoelasticity without energy dissipation](#), *J. Elasticity*, **31** (1993), 189–208.
- [13] A. E. Green and P. M. Naghdi, [A unified procedure for construction of theories of deformable media, I. Classical continuum physics, II. Generalized continua, III. Mixtures of interacting continua](#), *Proc. Roy. Soc. London A*, **448** (1995), 379–388.
- [14] R. B. Hetnarski and J. Ignaczak, [Generalized thermoelasticity](#), *J. Thermal Stresses*, **22** (1999), 451–476.
- [15] R. B. Hetnarski and J. Ignaczak, [Nonclassical dynamical thermoelasticity](#), *Int. J. Solids Struct.*, **37** (1999), 215–224.
- [16] C. O. Horgan, L. E. Payne and L. T. Wheeler, [Spatial decay estimates in transient heat conduction](#), *Quart. Appl. Math.*, **42** (1984), 119–127.
- [17] C. O. Horgan and R. Quintanilla, [Spatial decay of transient end effects in functionally graded heat conducting materials](#), *Quart. Appl. Math.*, **59** (2001), 529–542.
- [18] D. Iesan, [On the theory of thermoelasticity without energy dissipation](#), *J. Thermal Stresses*, **21** (1998), 295–307.
- [19] D. Iesan, [Thermopiezoelectricity without energy dissipation](#), *Proc. Roy. Soc. London A*, **464** (2008), 631–656.
- [20] D. Iesan and R. Quintanilla, [On the thermoelastic bodies with inner structure and microtemperatures](#), *J. Math. Anal. Appl.*, **354** (2009), 12–23.
- [21] J. Ignaczak and M. Ostoj-Starzewski, *Thermoelasticity with Finite Wave Speeds*, Oxford: Oxford Mathematical Monographs, Oxford, 2010.
- [22] B. Lazzari and R. Nibbi, [On the exponential decay in thermoelasticity without energy dissipation and of type III in presence of an absorbing boundary](#), *J. Math. Anal. Appl.*, **338** (2008), 317–329.
- [23] M. C. Leseduarte, A. Magaña and R. Quintanilla, [On the time decay of solutions in porous-thermo-elasticity of type II](#), *Discrete Contin. Dyn. Sys., Ser. B*, **13** (2010), 375–391.
- [24] M. C. Leseduarte and R. Quintanilla, [On uniqueness and continuous dependence in type III thermoelasticity](#), *J. Math. Anal. Appl.*, **395** (2012), 429–436.
- [25] M. C. Leseduarte and R. Quintanilla, [Phragmén-Lindelöf alternative for an exact heat conduction equation with delay](#), *Commun. Pure Appl. Anal.*, **12** (2013), 1221–1235.
- [26] Y. Liu and C. Lin, [Phragmén-Lindelöf alternative and continuous dependence-type results for the thermoelasticity of type III](#), *Appl. Anal.*, **87** (2008), 431–449.
- [27] Z. Liu and R. Quintanilla, [Energy decay rates of mixed type II and type III thermoelastic system](#), *Discrete Contin. Dyn. Syst., Ser. B*, **14** (2010), 1433–1444.
- [28] Z. Liu and S. Zheng, *Semigroups Associated to Dissipative Systems*, Chapman & Hall/CRC Boca Raton, FL. Research Notes in Mathematics, vol. 398, 1999.
- [29] J. C. Maxwell, *Theory of Heat*, Cambridge University Press, Dover, Mineola, New York, 2011.
- [30] S. A. Messaoudi and A. Soufyane, [Boundary stabilization of memory type in thermoelasticity of type III](#), *Appl. Anal.*, **87** (2008), 13–28.
- [31] S. Mukhopadhyay, R. Prasad and R. Kumar, [On the theory of two-temperature thermoelasticity with two phase-lags](#), *J. Thermal Stresses*, **34** (2011), 352–365.
- [32] P. Puri and P. M. Jordan, [On the propagation of plane waves in type-III thermoelastic media](#), *Proc. Roy. Soc. London A*, **460** (2004), 3203–3221.

- [33] Y. Qin, S. Deng, L. Huang, Z. Ma and X. Su, Global existence for the three-dimensional thermoelastic equations of Type II, *Quart. Appl. Math.*, **68** (2010), 333–348.
- [34] R. Quintanilla, On the spatial behaviour in thermoelasticity without energy dissipation, *J. Thermal Stresses*, **22** (1999), 213–224.
- [35] R. Quintanilla, Damping of end effects in a thermoelastic theory, *Appl. Math. Lett.*, **14** (2001), 137–141.
- [36] R. Quintanilla, Convergence and structural stability in thermoelasticity, *Appl. Math. Comput.*, **135** (2003), 287–300.
- [37] R. Quintanilla, Exponential stability and uniqueness in thermoelasticity with two temperatures, *Dyn. Conti. Discrete Impuls. Syst. Ser. A: Math. Anal.*, **11** (2004), 57–68.
- [38] R. Quintanilla, On the impossibility of localization in linear thermoelasticity, *Proc. Roy. Soc. London A*, **463** (2007), 3311–3322.
- [39] R. Quintanilla, A well-posed problem for the dual-phase-lag heat conduction, *J. Thermal Stresses*, **31** (2008), 260–269.
- [40] R. Quintanilla, A well-posed problem for the three-dual-phase-lag heat conduction, *J. Thermal Stresses*, **32** (2009), 1270–1278.
- [41] R. Quintanilla and R. Racke, Stability in thermoelasticity of type III, *Discrete Contin. Dyn. Syst., Ser. B*, **3** (2003), 383–400.
- [42] R. Quintanilla and G. Saccomandi, Phragmén-Lindelöf alternative of exponential type for the solutions of a fourth order dispersive equation, *Rend. Lincei Mat. Appl.*, **23** (2012), 105–113.
- [43] R. Quintanilla and B. Straughan, Growth and uniqueness in thermoelasticity, *Proc. Roy. Soc. London A*, **456** (2000), 1419–1429.
- [44] R. Quintanilla and B. Straughan, Energy bounds for some non-standard problems in thermoelasticity, *Proc. Roy. Soc. London A*, **461** (2005), 1147–1162.
- [45] S. K. Roy Choudhuri, On a thermoelastic three-phase-lag model, *J. Thermal Stresses*, **30** (2007), 231–238.
- [46] B. Straughan, *Heat Waves*, Applied Mathematical Sciences, 177. Springer, New York, 2011.
- [47] D. Y. Tzou, A unified approach for heat conduction from macro to micro-scales, *ASME J. Heat Transfer*, **117** (1995), pp. 8–16.
- [48] W. E. Warren and P. J. Chen, Wave propagation in two temperatures theory of thermoelasticity, *Acta Mechanica*, **16** (1973), 21–33.
- [49] L. Yang and Y. G. Wang, Well-posedness and decay estimates for Cauchy problems of linear thermoelastic systems of type III in 3-D, *Indiana Univ. Math. J.*, **55** (2006), 1333–1361.
- [50] H. M. Youssef, Theory of two-temperature thermoelasticity without energy dissipation, *J. Thermal Stresses*, **34** (2011), 138–146.

Appendix 1. In this appendix we sketch how to show an existence and uniqueness result for the solutions of the problem determined by the homogeneous version of the system (6) with the boundary and initial conditions (11)–(13) under the extra condition

$$u_i(\mathbf{x}, t), \alpha(\mathbf{x}, t) \rightarrow 0 \text{ as } x_1 \rightarrow \infty.$$

We assume that the boundary conditions satisfy the following requirement:

There exist $\mathbf{U}(\mathbf{x}, t) \in C^3([0, \infty), \mathbf{W}^{2,2})$, $\Lambda(\mathbf{x}, t) \in C^3([0, \infty), W^{3,2})$, such that

$$(\mathbf{U}(\mathbf{x}, 0), \dot{\mathbf{U}}(\mathbf{x}, 0), \Lambda(\mathbf{x}, 0) - d_{ij}\Lambda_{,ij}(\mathbf{x}, 0), \dot{\Lambda}(\mathbf{x}, 0) - d_{ij}\dot{\Lambda}_{,ij}(\mathbf{x}, 0)) \in \mathcal{D}$$

and

$$U_i(0, x_2, x_3, t) = u_i^*, \quad \Lambda(0, x_2, x_3, t) = \alpha^*.$$

Let $u_i = w_i + U_i$, $\alpha = \delta + \Lambda$. Then, the functions (w_i, ν) (where $\nu = \delta - d_{ij}\delta_{,ij}$) satisfy the system (6), where

$$f_i(\mathbf{x}, t) = \rho^{-1}(C_{ijkh}U_{k,h} + a_{ij}(\Lambda - \delta_{ij}\Lambda_{,ij})_{,j} - \ddot{U}_i$$

and

$$h(\mathbf{x}, t) = a_{ij}\dot{U}_{i,j} + kd_{rj}\Lambda_{,rj} - bkd_{rj}\dot{\Lambda}_{,rj} - c(\ddot{\Lambda} - d_{ij}\ddot{\Lambda}_{,ij}).$$

The functions (w_i, ν) must satisfy the problem with null boundary conditions with suitable initial conditions:

$$w_i(\mathbf{x}, 0) = -U_i(\mathbf{x}, 0), \dot{w}_i(\mathbf{x}, 0) = -\dot{U}_i(\mathbf{x}, 0),$$

$$\nu(\mathbf{x}, 0) = -(\Lambda(\mathbf{x}, 0) - d_{ij}\Lambda_{,ij}(\mathbf{x}, 0)), \dot{\nu}(\mathbf{x}, 0) = -(\dot{\Lambda}(\mathbf{x}, 0) - d_{ij}\dot{\Lambda}_{,ij}(\mathbf{x}, 0)).$$

The regularity conditions are determined by the fact that f_i, h and the initial conditions satisfy the assumptions proposed in Theorem 3.2.

Appendix 2. In this appendix we obtain equality (14). We have

$$\begin{aligned} & ((C_{ijkh}u_{k,h} + a_{ij}\theta)\dot{u}_i + (kd_{rj}\alpha_{,r} - bd_{rj}T_{,r})T)_{,j} \\ &= \rho\ddot{u}_i\dot{u}_i + C_{ijkh}u_{k,h}\dot{u}_{i,j} + a_{ij}\theta\dot{u}_{i,j} + c\left(\dot{T} - d_{ij}\dot{T}_{,ij}\right)T - a_{ij}\dot{u}_{i,j}T \\ & \quad + (kd_{rj}\alpha_{,r} - bd_{rj}T_{,r})T_{,j} \\ &= \frac{1}{2}\frac{d}{dt}[\rho\dot{u}_i\dot{u}_i + C_{ijkh}u_{k,h}u_{i,j}] + a_{ij}\theta\dot{u}_{i,j} + c\left(\dot{T} - d_{ij}\dot{T}_{,ij}\right)\left(T - d_{kr}T_{,kr}\right) \\ & \quad + c\left(\dot{T} - d_{ij}\dot{T}_{,ij}\right)d_{kr}T_{,kr} - a_{ij}\dot{u}_{i,j}T + (kd_{rj}\alpha_{,r} - bd_{rj}T_{,r})T_{,j} \\ &= \frac{1}{2}\frac{d}{dt}[\rho\dot{u}_i\dot{u}_i + C_{ijkh}u_{k,h}u_{i,j} + c\theta^2] + a_{ij}\theta\dot{u}_{i,j} + a_{ij}\dot{u}_{i,j}d_{kr}T_{,kr} \\ & \quad + kd_{rj}\alpha_{,rj}d_{mn}T_{,mn} - bd_{rj}T_{,rj}d_{mn}T_{,mn} - a_{ij}\dot{u}_{i,j}T \\ & \quad + (kd_{rj}\alpha_{,r} - bd_{rj}T_{,j})T_{,r} \\ &= \frac{1}{2}\frac{d}{dt}[\rho\dot{u}_i\dot{u}_i + C_{ijkh}u_{k,h}u_{i,j} + c\theta^2] - b(d_{rj}T_{,rj})^2 - bd_{rj}T_{,r}T_{,j} \\ & \quad + kd_{rj}\alpha_{,rj}d_{mn}T_{,mn} + kd_{rj}\alpha_{,r}T_{,j} \\ &= \frac{1}{2}\frac{d}{dt}\phi - bd_{ij}T_{,i}T_{,j} - b(d_{ij}T_{,ij})^2, \end{aligned} \tag{74}$$

where $\phi = \rho\dot{u}_i\dot{u}_i + C_{ijkh}u_{i,j}u_{k,h} + c\theta^2 + kd_{ij}\alpha_{,i}\alpha_{,j} + k(d_{ij}\alpha_{,ij})^2$.

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