E_T-Lipschitzian Aggregation Operators

J. Jacas
Sec Matemàtiques i Informàtica
ETS Arquitecura de Barcelona
Universitat Politècnica de Catalunya
Diagonal 649
08028 Barcelona
Spain
joan.jacas@upc.edu

J. Recasens
Secció Matemàtiques i Informàtica
ETS Arquitectura del Vallès
Universitat Politècnica de Catalunya
Pere Serra 1-15
08190 Sant Cugat del Vallès
Spain
j.recasens@upc.edu

Abstract

Lipschitzian and kernel aggregation operators with respect to the natural T-indistinguishability operator E_T and their powers are studied. A t-norm T is proved to be E_T-lipschitzian, and is interpreted as a fuzzy point and a fuzzy map as well. Given an archimedean t-norm T with additive generator t, the quasi-arithmetic mean generated by t is proved to be the more stable aggregation operator with respect to T.

Keywords: Aggregation Opoerator, T-indistinguishability Operator, Lipschitzian, Kernel.

1 Introduction

Lipschitzian aggregation operators have been studied in [4] [5] [13] [14] by considering the usual metric on the unit interval. In this paper we study the lipschitzian condition of aggregation operators with respect to the natural indistinguishability operator E_T and their powers E_T^p (see definitions below) so that an aggregation operator h is E_T^p-lipschitzian when for all $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in [0,1]$ $T(E_T^p(x_1, y_1), ..., E_T^p(x_n, y_n)) \leq E_T(h(x_1, x_2, ..., x_n), h(y_1, y_2, ..., y_n))$. This means that from similar inputs we obtain similar aggregations. The use of E_T and E_T^p assumes the election of a specific t-norm T and therefore the selection of a particular family of logics where the semantics of the conjunction and the biimplication are given by T and E_T.

As it will be seen in this paper, when T is the Lukasiewicz t-norm, the E_T-lipschitzian condition coincides with the 1-lipschitzian condition with the usual metric on [0,1] and the definition of [13] is recovered. This is not a surprising result, due to the relation between E_T and the usual metric on [0,1] in this case.

It is worth noticing the relation between the lipschitzian condition of an aggregation operator E_T and its extensionality with respect to the integral powers $T(\overbrace{E_T, ..., E_T}^{n\ times})$ (Proposition 3.9).

Among other results, it will be proved in this paper that if T is a continuous archimedean t-norm with an additive generator t and m_t the quasi-arithmetic mean generated by t ($m_t(x_1, x_2, ..., x_n) = t^{-1}\left(\frac{t(x_1)+t(x_2)+...+t(x_n)}{n}\right)$), then m_t is the more stable aggregation operator with respect to T (Proposition 3.21).

Also the t-norm T is not only lipschitzian with respect to E_T, but it can be seen as a fuzzy point and a fuzzy map as well (Proposition 3.23, Proposition 3.25) and an aggregation operator h is greater than or equal to T if and only if h is 1-E_T-lipschitzian.

In the definition of E_T-lipschitzianity we replace the t-norm T by the minimum, i.e. $(Min(E_T^p(x_1, y_1), ..., E_T^p(x_n, y_n)) \leq E_T(h(x_1, x_2, ..., x_n), h(y_1, y_2, ..., y_n)))$, then we obtain a generalization of the kernel aggregation operators studied in [17] [13]. Again, if T is the Lukasiewicz t-norm this definition is equivalent to the one given in the above mentioned references.

2 Preliminaries

This section contains some results on t-norms and indistinguishability operators that will be needed later on in the paper. Besides well known definitions and theorems, the power T^n of a t-norm is generalized to irrational exponents in Definition 2.3 and given explicitly for continuous archimedean t-norms in Proposition 2.4.

For the sake of simplicity we will assume continuity for the t-norms throughout the paper.

Since a t-norm T is associative, we can extend it to an

n-ary operation in the standard way:

$$T(x) = x$$
$$T(x_1, x_2, ...x_n) = T(x_1, T(x_2, ..., x_n)).$$

In particular, following the notation in [18], $T(\overbrace{x, x, ..., x}^{n\ times})$ will be denoted by $x_T^{(n)}$.

The n-th root $x_T^{(\frac{1}{n})}$ of x with respect to T is defined by

$$x_T^{(\frac{1}{n})} = sup\{z \in [0,1] \mid z_T^{(n)} \leq x\}$$

and for $m, n \in N$, $x_T^{(\frac{m}{n})} = \left(x_T^{(\frac{1}{n})}\right)_T^{(m)}$.

Lemma 2.1. [18] *If* $k, m, n \in N$, $k, n \neq 0$ *then* $x_T^{(\frac{km}{kn})} = x_T^{(\frac{m}{n})}$.

Lemma 2.2. *Let* $x_1, ..., x_n \in (0,1]$ *and* $n \in N$. $T(x_{1T}^{(\frac{1}{n})}, ..., x_{nT}^{(\frac{1}{n})}) \neq 0$.

The powers $x_T^{(\frac{m}{n})}$ can be extended to irrational exponents in a straightforward way.

Definition 2.3. *If* $r \in R^+$ *is a positive real number, let* $\{a_n\}_{n \in N}$ *be a sequence of rational numbers with* $lim_{n\to\infty} a_n = r$. *For any* $x \in [0,1]$, *the power* $x_T^{(r)}$ *is*

$$x_T^{(r)} = lim_{n\to\infty} x_T^{(a_n)}.$$

Continuity assures the existence of limit and independence of limit from the selection of the sequence $\{a_n\}_{n \in N}$.

Proposition 2.4. *Let* T *be an archimedean t-norm with additive generator* t, $x \in [0,1]$ *and* $r \in R^+$. *Then*

$$x_T^{(r)} = t^{[-1]}(rt(x)).$$

Proof. Due to continuity of t we need to prove it only for rational r.

If r is a natural number m, then trivially $x_T^{(m)} = t^{[-1]}(mt(x))$.

If $r = \frac{1}{n}$ with $n \in N$, then $x_T^{(\frac{1}{n})} = z$ with $z_T^{(n)} = x$ or $t^{[-1]}(nt(z)) = x$ and $x_T^{(\frac{1}{n})} = t^{[-1]}\left(\frac{t(x)}{n}\right)$.

For a rational number $\frac{m}{n}$,

$$x_T^{(\frac{m}{n})} = \left(x_T^{(\frac{1}{n})}\right)_T^{(m)} = t^{[-1]}\left(mt\left(x_T^{(\frac{1}{n})}\right)\right) =$$

$$t^{[-1]}\left(mt\left(t^{[-1]}\left(\frac{t(x)}{n}\right)\right)\right) = t^{[-1]}\left(\frac{m}{n}t(x)\right).$$

□

Theorem 2.5. *Ling* [15] *A continuous t-norm* T *is archimedean if and only if there exists a continuous decreasing map* $t : [0,1] \to [0,\infty]$ *with* $t(1) = 0$ *such that*

$$T(x, y) = t^{[-1]}(t(x) + t(y))$$

where $t^{[-1]}$ *stands for the pseudo-inverse of* t *defined by*

$$t^{[-1]}(x) = \begin{cases} 1 & \text{if } x < 0 \\ t^{-1}(x) & \text{if } x \in [0, t(0)] \\ 0 & \text{otherwise.} \end{cases}$$

T *is strict if* $t(0) = \infty$ *and non-strict otherwise.*

t is called an additive generator of T and two additive generators of the same t-norm differ only by a multiplicative constant.

Definition 2.6. *The residuation* $\overrightarrow{T}$ *of a t-norm* T *is defined by*

$$\overrightarrow{T}(x|y) = sup\{\alpha \in [0,1] \mid T(x,\alpha) \leq y\}.$$

Definition 2.7. *The natural* T*-indistinguishability operator* E_T *associated to a given t-norm* T *is the fuzzy relation on* [0,1] *defined by*

$$E_T(x, y) = T(\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)) = Min(\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)).$$

Example 2.8.

1. *If* T *is an archimedean t-norm with additive generator* t, *then* $E_T(x, y) = t^{-1}(|t(x) - t(y)|)$ *for all* $x, y \in [0,1]$.
2. *If* T *is the Lukasiewicz t-norm, then* $E_T(x, y) = 1 - |x - y|$ *for all* $x, y \in [0,1]$.
3. *If* T *is the Product t-norm, then* $E_T(x, y) = \begin{cases} \frac{Min(x,y)}{Max(x,y)} & \text{if } x \neq y \\ 1 & \text{otherwise.} \end{cases}$
4. *If* T *is the Minimum t-norm, then* $E_T(x, y) = \begin{cases} Min(x, y) & \text{if } x \neq y \\ 1 & \text{otherwise.} \end{cases}$

E_T is indeed a special kind of (one-dimensional) T-indistinguishability operator (Definition 2.9) [3] and in a logical context where T plays the role of the conjunction, E_T is interpreted as the bi-implication associated to T [7].

The general definition of T-indistinguishability operator is

Definition 2.9. *Given a t-norm T, a T-indistinguishability operator E on a set X is a fuzzy relation $E : X \times X \rightarrow [0,1]$ satisfying for all $x, y, z \in X$*

1. $E(x,x) = 1$ *(Reflexivity)*
2. $E(x,y) = E(y,x)$ *(Symmetry)*
3. $T(E(x,y), E(y,z)) \leq E(x,z)$ *(T-transitivity).*

Proposition 2.10. [21] *Let μ be a fuzzy subset of X and T a continuous t-norm. The fuzzy relation E_μ on X defined for all $x, y \in X$ by*

$$E_\mu(x,y) = E_T(\mu(x), \mu(y))$$

is a T-indistinguishability operator on X.

Definition 2.11. *Let E be a T-indistinguishability operator on a set X. A fuzzy subset μ of X is extensional with respect to E if and only if for all $x, y \in X$*

$$T(E(x,y), \mu(y)) \leq \mu(x).$$

Proposition 2.12. *Let E be a T-indistinguishability operator on a set X. A fuzzy subset μ of X is extensional with respect to E if and only if for all $x, y \in X$*

$$E(x,y) \leq E_T(\mu(x), \mu(y)).$$

Finally, let us recall the definition of aggregation operator.

Definition 2.13. [4] *An aggregation operator is a map $h : \bigcup_{n \in N} [0,1]^n \rightarrow [0,1]$ satisfying*

1. $h(0,...,0) = 0$ and $h(1,...,1) = 1$
2. $h(x) = x \; \forall x \in [0,1]$
3. $h(x_1,...,x_n) \leq h(y_1,...,y_n)$ if $x_1 \leq y_1, ..., x_n \leq y_n$ *(monotonicity).*

The restriction of h to $[0,1]^n$ will be denoted by $h_{(n)}$ so that a global aggregation operator h can be split into the family of n-ary operators $(h_{(n)})_{n \in N}$.

3 E_T-Lipschitzian and E_T-kernel aggregation operators

Lipschitzian and kernel aggregation operators with respect to the natural T-indistinguishability operator E_T and their powers are a special kind of aggregation operators that generalize the definitions of [13], [17]. Their interest is in the fact that they are stable operators in the sense that the similarity between the aggregation of two n-tuples is bounded by the similarity between them.

It is interesting to point out that the lipschitzian and kernel conditions are equivalent to extensionality (Proposition 3.9, Proposition 3.27).

Among other results, it will be proved that a t-norm T is E_T-lipschitzian and moreover the maps $T_{(n)}$ can be interpreted as fuzzy points of $[0,1]^n$ and a fuzzy maps from $[0,1]^k$ to $[0,1]^{n-k}$.

Also quasi-arithmetic means are proved to be the more stable aggregation operators.

Proposition 3.1. *Let E be a T indistinguishability operator on a set X. The fuzzy relation E^n defined by*

$$E^n(x,y) = T(\overbrace{E(x,y), ...E(x,y)}^{n \ times}) \; \forall x, y \in X$$

is a T-indistinguishability operator.

The powers E_T^n of the natural T-indistinguishability operators have been studied in relation with antonymy and fuzzy partitions in [20].

Proposition 3.2. [11] *Let E be a T-indistinguishability operator on a set X. $E^{\frac{1}{n}}$ is a T-indistinguishability operator on X.*

Corollary 3.3. *Let E be a T-indistinguishability operator on a set X. $E^{\frac{m}{n}}$ is a T-indistinguishability operator on X.*

Proof. Propositions 3.1. and 3.2. □

Corollary 3.4. *Let E_T be the natural T-indistinguishability operator on [0,1] associated to T. $E_T^{\frac{m}{n}}$ is a T-indistinguishability operator.*

Continuity of the t-norm T allows us to extend the powers of a T-indistinguishability operator to positive irrational numbers in the same way as in Definition 2.3.

Example 3.5.

1. *If T is continuous archimedean with additive generator t, then $E_T^p(x,y) = t^{[-1]}(p|t(x) - t(y)|)$ for all $x, y \in [0,1]$.*
2. *If T is the Lukasiewicz t-norm, then $E_T^p(x,y) = Max(0, 1 - p|x-y|)$ for all $x, y \in [0,1]$.*
3. *If T is the Product t-norm, then* $E_T^p(x,y) = \begin{cases} \frac{Min(x^p,y^p)}{Max(x^p,y^p)} & \text{if } x \neq y \\ 1 & \text{otherwise.} \end{cases}$
4. *If T is the Minimum t-norm, then $E_T^p(x,y) = E_T(x,y)$ for all $x, y \in [0,1]$.*

Proposition 3.6. *Let T-be a t-norm and $p, q > 0$. $E_T^p \leq E_T^q$ if and only if $p \geq q$.*

Definition 3.7. *Let E be a T-indistinguishability operator on $[0,1]$. h is E-lipschitzian if and only if $\forall n \in N$, $\forall x_1, ..., x_n, y_1, ..., y_n \in [0,1]$*

$$T(E(x_1, y_1), ..., E(x_n, y_n)) \leq$$

$$E_T(h(x_1, ..., x_n), h(y_1, ..., y_n)).$$

If $E_1, ..., E_n$ are T-indistinguishability operators defined on the universes $X_1, ..., X_n$ respectively, there are at least two natural ways to define a T-indistinguishability operator on $X_1 \times ... \times X_n$.

Proposition 3.8. *Let $E_1, ..., E_n$ be T-indistinguishability operators on $X_1, ..., X_n$ respectively. Then the two fuzzy relations $T(E_1, ..., E_n)$ and $Min(E_1, ..., E_n)$ on $X_1 \times ... \times X_n$ defined for all $(x_1, ..., x_n), (y_1, , , , y_n) \in X_1 \times ... \times X_n$ by*

$$T(E_1, ..., E_n)((x_1, ..., x_n), (y_1, , , , y_n)) =$$

$$T(E_1(x_1, y_1), ..., E_n(x_n, y_n))$$

and

$$Min(E_1, ..., E_n)((x_1, ..., x_n), (y_1, , , , y_n)) =$$

$$Min(E_1(x_1, y_1), ..., E_n(x_n, y_n))$$

are T-indistinguishability operators on $X_1 \times ... \times X_n$.

Proposition 3.9. *Let E be a T-indistinguishability on $[0,1]$ and h an aggregation operator. h is E-lipschitzian if and only if $h_{(n)}$ (as a fuzzy subset of $[0,1]^n$) is extensional with respect to $T(\overbrace{E, ..., E}^{n\ times})$ for all $n \in N$.*

Proof. Proposition 2.12 □

Lemma 3.10. *Let T be a continuous t-norm. The for all $x, y \in [0,1]$ $x \geq y$*

$$T(x, \overrightarrow{T}(x|y) = y.$$

Next Proposition shows that a t-norm T is an E_T-lipschitzian aggregation operator.

Proposition 3.11. *Let T be a continuous t-norm. Then T is an E_T-lipschitzian aggregation operator.*

Note that if $x_i \leq y_i$ for all $i = 1, ...n$, then $T(E_T(x_1, y_1), ..., E_T(x_n, y_n)) = E_T(T(x_1, ..., x_n), T(y_1, ..., y_n))$. Since for every t-norm different from the Minimum $E_T^p < E_T^q$ if $p > q$, we have that $T \neq Min$ is not E_T^p-lipschitzian for $p < 1$.

If T is a continuous archimedean t-norm, the E_T^p-lipschitzian property translates to a classical lipschitzian condition.

Proposition 3.12. *Let T be a continuous archimedean t-norm with additive generator t, $p \in [0,1]$ and h an aggregation operator. h is E_T^p if and only if $\forall n \in N$, $\forall x_1, ..., x_n, y_1, ..., y_n \in [0,1]$*

$$p|t(x_1) - t(y_1)| + ... + p|t(x_n) - t(y_n)| \geq$$

$$|t\left(h(x_1, ..., x_n)\right) - t\left(h(y_1, ..., y_n)\right)| \quad (1).$$

Last Proposition says that for all $n \in N$ the map $H : [0, t(0)]^n \to [[0, t(0)]$ defined by

$$H(x_1, ..., x_n) = t(h(t^{-1}(x_1), ..., t^{-1}(x_n)))$$

is a p-lipschitzian map.

Also note that if T is the Lukasiewicz t-norm, then (1) is the definition of the Lipschitz property in [13], so that Definition 3.7 contains the one in [13] as a particular case.

If an aggregation operator h is E_T^p-lipschitzian, it may happen that for different values of n the corresponding n-ary operators $h_{(n)}$ may satisfy the lipschitzian conditions for different values of p ([4] p. 12).

Definition 3.13. *An aggregation operator is sub idempotent if and only if for all $x \in [0,1]$ and $n \in N$, $h(\overbrace{x, ..., x}^{n\ times}) \leq x$*

Proposition 3.14. *Let $T \neq Min$ be a t-norm, h a sub idempotent aggregation operator and $n \in N$. If $h_{(n)}$ is E_T^p-lipschitzian, then $p \geq \frac{1}{n}$.*

Proof. If $h_{(n)}$ is E_T^p-lipschitzian, then in particular, for $x \in X$

$$T((\overbrace{E_T^p(1,x), ..., E_T^p(1,x)}^{n\ times}) \leq E_T(h(\overbrace{1, ..., 1}^{n\ times}), h(\overbrace{x, ..., x}^{n\ times}))$$

and so

$$x_T^{(pn)} \leq h(\overbrace{x, ..., x}^{n\ times}) \leq x$$

which holds if and only if $pn \geq 1$ or equivalently, if and only of $p \geq \frac{1}{n}$ □

If T is a non-strict continuous archimedean t-norm the sub idempotent property can be dropped.

Proposition 3.15. *Let T be a non-strict continuous archimedean t-norm with additive generator t, h an aggregation operator and $n \in N$. If $h_{(n)}$ is E_T^p-lipschitzian, then $p \geq \frac{1}{n}$.*

Proof. Putting $x_i = 1$ and $y_i = 0$ for all $i = 1, ..., n$ in Proposition 3.12, we get

$$p|t(1) - t(0)| + ... + p|t(1) - t(0)| \geq |t(1) - t(0)|.$$

$$npt(0) \geq t(0)$$

or

$$p \geq \frac{1}{n}.$$

□

In [4] it has been proved that the arithmetic mean is the only aggregation operator h whose n-ary maps $h_{(n)}$ are $\frac{1}{n}$-lipschitzian. Proposition 3.21 generalizes this result to arbitrary quasi-arithmetic means.

Next Proposition is well known.

Proposition 3.16. [1], [18] *m is a quasi-arithmetic mean in [0,1] if and only if there exists a continuous monotonic map $t : [0,1] \to [-\infty, \infty]$ such that for all $n \in N$ and $x_1, ..., x_n \in [0,1]$*

$$m(x_1, ...x_n) = t^{-1}\left(\frac{t(x_1) + ... + t(x_n)}{n}\right).$$

m is continuous if and only if Ran $t \neq [-\infty, \infty]$.

t will be called a generator of m and if m is generated by t we will denote it by m_t.

Lemma 3.17. [11] *Let $t, t' : [0,1] \to [-\infty, \infty]$ be two continuous strict monotonic maps with Ran t, Ran $t' \neq [-\infty, \infty]$ differing only by a non-zero multiplicative constant α ($t' = \alpha\ t$) and $m_t, m_{t'}$ the quasi-arithmetic means generated by them respectively. Then $m_t = m_{t'}$.*

Lemma 3.18. [11] *Let $t, t' : [0,1] \to [-\infty, \infty]$ be two continuous strict monotonic maps with Ran t, Ran $t' \neq [-\infty, \infty]$ differing only by an additive constant and $m_t, m_{t'}$ the quasi-arithmetic means generated by them respectively. Then $m_t = m_{t'}$.*

Lemma 3.19. [11] *Let $t : [0,1] \to [-\infty, \infty]$ be a continuous strict monotonic map. Then $m_t = m_{-t}$.*

Proposition 3.20. [11] *The map assigning to every continuous Archimedean t-norm T with generator t the mean m_t generated by t is a bijection between the set of continuous Archimedean t-norms and the set of continuous quasi-arithmetic means.*

Proposition 3.21. *Let T be a continuous archimedean t-norm with additive generator t and m_t the quasi-arithmetic mean generated by t.*

- *(a) For every $n \in N$ $m_{t(n)}$ is E_T^p-lipschitzian if and only if $p \geq \frac{1}{n}$.*
- *(b) m_t is the only aggregation operator fulfilling (a)*

In Proposition 3.11 we have proved that a t-norm T is E_T-lipschitzian. In fact, $T_{(n)}$ can also be seen as a fuzzy point of $[0,1]^n$ and a fuzzy map from $[0,1]^{n-1}$ into $[0,1]$.

Definition 3.22. *Let E be a T-indistinguishability operator on a set X and μ a fuzzy subset of X. μ is a fuzzy point of X with respect to E if and only if for all $x, y \in X$*

$$T(\mu(x), \mu(y)) \leq E(x, y).$$

Proposition 3.23. *Let T be a continuous t-norm. $T_{(n)}$ is a fuzzy point of $[0,1]^n$ with respect to $T(\overbrace{E_T, ..., E_T}^{n\ times})$.*

Proof. We have to prove that

$$T(T(x_1, ..., x_n), T(y_1, ..., y_n))$$
$$\leq T(E_T(x_1, y_1), ..., E_T(x_n, y_n))$$

which is an immediate consequence of

$$T(x_i, y_i) \leq E_T(x_i, y_i) \text{ for all } i = 1, ..., n.$$

□

Definition 3.24. *Let E, F be two T-indistinguishability operators on X and Y respectively and R a fuzzy set of $X \times Y$ (i.e.: $R : X \times Y \to [0,1]$). R is a fuzzy map from X to Y if and only if for all $x, x' \in X$, $y, y' \in Y$*

- *(a) $T(E(x, x'), F(y, y'), R(x, y)) \leq R(x', y')$*
- *(b) $T(R(x, y), R(x, y')) \leq F(y, y')$.*

Proposition 3.25. *Let T be a continuous t-norm. $T_{(n)}$ is a fuzzy map from $[0,1]^{n-1}$ to $[0,1]$ endowed with the T indistinguishability operators $T(\overbrace{E_T, ..., E_T}^{n-1\ times})$ and E_T respectively.*

In fact, it can be proved in the same way that $T_{(n)}$ is a fuzzy map from $[0,1]^k$ to $[0,1]^{n-k}$ ($2 \leq k \leq n-1$) endowed with the T indistinguishability operators $T(\overbrace{E_T, ..., E_T}^{k\ times})$ and $T(\overbrace{E_T, ..., E_T}^{n-k\ times})$ respectively.

Kernel aggregation operators are a family of aggregation operators tightly related to lipschitzian ones. They were introduced in [17] (see also [13], [4]). As the lipschitzian condition, the condition for being a kernel operator was related to the usual metric on the unit interval. It can be extended using natural indistinguishability operators in the same way as it has been done in this paper with the lipschitzian condition. Again, if the T norm is the Lukasiewicz one, the original definition of [17] is recovered.

Definition 3.26. *Let E be a T-indistinguishability operator on [0,1] and h an aggregation operator. h is an E-kernel aggregation operator if and only if $\forall n \in N$, $\forall x_1, ..., x_n, y_1, ..., y_n \in [0,1]$*

$$Min(E(x_1, y_1), ..., E(x_n, y_n)) \leq$$

$$E_T(h(x_1, ..., x_n), h(y_1, ..., y_n)).$$

Proposition 3.27. *Let E be a T-indistinguishability operator on [0,1] and h an aggregation operator. h is an E-kernel aggregation operator if and only if $h_{(n)}$ (as a fuzzy subset of $[0,1]^n$) is extensional with respect to $Min(\overbrace{E, ..., E}^{n\ times})$ for all $n \in N$.*

Proof. Proposition 2.12 □

For archimedean t-norms, the kernel property can be written as in the follows.

Proposition 3.28. *Let T be a continuous archimedean t-norm with additive generator t, $p \in [0,1]$ and h an aggregation operator. h is E_T^p-kernel aggregation operator if and only if $\forall n \in N$, $\forall x_1, ..., x_n, y_1, ..., y_n \in [0,1]$*

$$Max(p|t(x_1) - t(y_1)|, ..., p|t(x_n) - t(y_n)|) \geq$$

$$|t\,(h(x_1, ..., x_n)) - t\,(h(y_1, ..., y_n))| \quad (2).$$

Proof.

$$Min(t^{-1}(p|t(x_1) - t(y_1)|), ..., t^{-1}(p|t(x_n) - t(y_n)|)) \leq$$

$$t^{-1}(|t(h(x_1, ..., x_n)) - t(h(x_1, ..., x_n))|)$$

$$t^{-1}\,(Max(p|t(x_1) - t(y_1)|, ..., p|t(x_n) - t(y_n)|)) \leq$$

$$t^{-1}(|t(h(x_1, ..., x_n)) - t(h(x_1, ..., x_n))|)$$

$$Max(p|t(x_1) - t(y_1)|, ..., p|t(x_n) - t(y_n)|) \geq$$

$$|t(h(x_1, ..., x_n)) - t(h(x_1, ..., x_n))|.$$

□

If T is the Lukasiewicz t-norm and $p = 1$, then (2) is the definition of the kernel aggregation operator in [17].

4 Concluding Remarks

In this paper Lipschitzian and kernel aggregation operators with respect to the natural T-indistinguishability operator E_T and their powers have been studied.

It has been proved that a t-norm T is E_T-lipschitzian, and a fuzzy point and a fuzzy map as well.

Quasi-arithmetic means m_t play an important role since they are the more stable aggregation operator with respect to T, meaning that the corresponding n-ary operators $m_{t(n)}$ are $E_T^{\frac{1}{n}}$-lipschitzian maps.

Lipschitzian and kernel properties are not only interesting for aggregation operators, but in almost any part of fuzzy reasoning and they deserve a deep study.

Acknowledgement

Research partially supported by DGICYT project number TIN2006-14311.

References

[1] J. Aczél, *Lectures on functional equations and their applications.* Academic Press, New York/London 1966.

[2] D. Boixader, Some Properties Concerning the Quasi-inverse of a t-norm, Mathware and Soft Computing 5 (1998) 5-12.

[3] D. Boixader, J. Jacas, J. Recasens, Fuzzy Equivalence Relations: Advanced Material. In Dubois, Prade Eds. *Fundamentals of Fuzzy Sets*, Kluwer, (2000), 261-290.

[4] T. Calvo, A. Kolesárova, M. Komorníková, R. Mesiar, Aggregation Operators: Properties, Classes and Construction Methods. In Mesiar, Calvo, Mayor Eds. *Aggregation Operators: New Trends and Applications.* Studies in Fuzziness and Soft Computing. Springer, (2002), 3-104.

[5] T. Calvo, R. Mesiar, Stability of aggregation operators. Proc EUSFLAT 2001, Leicester, (2001) 457-458.

[6] M. Demirci, Fundamentals of M-vague algebra and M-vague arithmetic operations, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 10 (2002) 25-75.

[7] P. Hájek, *Metamathematics of Fuzzy Logic.* Kluwer, Dordrecht, 1998.

[8] J. Jacas, Fuzzy topologies induced by S-metrics, The Journal of Fuzzy Mathematics 1 (1993) 173-191.

[9] J. Jacas, J. Recasens, One dimensional indistinguishability operators. Fuzzy Sets and Systems 109 (2000) 447-451.

[10] J. Jacas, J. Recasens, Aggregation of T-Transitive Relations, Int. J. of Intelligent Systems 18 (2003) 1193-1214.

[11] J. Jacas, J. Recasens, Aggregation Operators Based on Indistinguishability Operators, Int. J. of Intelligent Systems 21 (2006) 857-873.

[12] J. Jacas, J. Recasens, Linguistic Modifiers, Fuzzy Maps and the lipschitzian Condition. Proc IPMU 2006, Paris (2006).

[13] A. Kolesárová, J. Mordelová, 1-Lipschitz and kernel aggregation operators. Proc AGOP 2001, Oviedo (2001) 71-76.

[14] A. Kolesárová, J. Mordelová, E.Muel, Kernel aggregation. operators and their marginals, Fuzzy Sets and Systems (2004) 35-50.

[15] C. M. Ling, Representation of associative functions, Publ. Math. Debrecen 12 (1965) 189-212.

[16] V, Novák, R. Mesiar, Operations fitting triangular-norm-based biresiduation, Fuzzy Sets and Systems 104 (1999) 77-84.

[17] J. Mordelová, E. Muel, Kernel aggregation operators. Proc AGOP 2001, Oviedo (2001) 95-98.

[18] E. P. Klement, R. Mesiar, E. Pap, *Triangular norms.* Kluwer, Dordrecht, 2000.

[19] B. Schweizer, A. Sklar *Probabilistic Metric Spaces.* North-Holland, Amsterdam, 1983.

[20] A.R. De Soto, J. Recasens, Modeling a Linguistic Variable as a Hierarchical Family of Partitions induced by an Indistinguishability Operator, Fuzzy Sets and Systems 121 (2000) 427-437.

[21] L.Valverde, On the structure of F-indistinguishability operators, Fuzzy Sets and Systems 17 (1985) 313–328.

[22] L.A.Zadeh, Similarity relations and fuzzy orderings, Information Science 3 (1971) 177–200.