Finding close T-indistinguishability Operators to a given Proximity

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Abstract

Two ways to approximate a proximity relation R (i.e. a reflexive and symmetric fuzzy relation) by a T-transitive one where T is a continuous archimedean t-norm are given.

The first one aggregates the transitive closure $\overline{R}$ of R with a (maximal) T-transitive relation B contained in R.

The second one modifies the values of $\overline{R}$ or B to better fit them with the ones of R.

Keywords: Proximity, Transitive Closure, Opening, T-indistinguishability Operator, Aggregation Operator, Quasi Arithmetic Mean, Representation Theorem.

1 Introduction

A proximity matrix or relation on a finite universe X is a reflexive and symmetric fuzzy relation R on X. In many applications transitivity of R with respect to a t-norm T is required. In these cases, R must be replaced by a new relation E also satisfying transitivity, such relations called T-indistinguishability operators. Of course, it is desirable that E is as close as possible to R. This paper presents a couple of ways to find close transitive relations to R in a reasonable way - i.e.: easy and rapid to generate- when the t-norm is continuous archimedean.

There are of course several ways to calculate the closeness of two fuzzy relations, many of them related to some metric. In this paper we propose a way related to the natural indistinguishability operator E_T associated to T, so that the degree of closeness or similarity between two fuzzy relations R and S is calculated aggregating the similarity of their respective entries using the quasi-arithmetic mean generated by an additive generator of T.

Also the euclidean metric will be used as an alternative method to compare fuzzy relations.

Trying to find the closest E to R is very expensive. Indeed, if n is the cardinality of the universe X, the transitivity of T-indistinguishability operators can be modeled by $3\binom{n}{3}$ inequalities and they lay in the region of the $\binom{n}{2}$-dimensional space defined by them. The calculation of E becomes then a non-linear programming problem. Therefore, simpler methods to find a close E to R are desirable.

There are several algorithms to find the transitive closure $\overline{R}$ of a proximity relation R and it is well known that $\overline{R} \geq R$. There are also algorithms to find maximal T-indistinguishability operators B among the set of T-indistinguishability operators smaller or equal than R and also the Representation Theorem gives a T-indistinguishability operator $\underline{R}$ smaller or equal than R. It appears reasonable to aggregate $\overline{R}$ and B or $\underline{R}$ to obtain a new T-indistinguishability operator closer to R than $\overline{R}$, B or $\underline{R}$. This idea will be developed in Section 3.

If E is a T-indistinguishability operator, then the powers $E^{(p)}$ $p > 0$ are T-indistinguishability operators as well. This allows us to increase or decrease the values of E, since $E^{(p)} \leq E^{(q)}$ for $p \geq q$. So, we can decrease the values of the transitive closure or increase the ones of an operators smaller than R to find better approximations of it. Section 4 is devoted to this idea.

2 Preliminaries

This Section contains some results on t-norms and indistinguishability operators that will be needed later on in the paper. Besides well known definitions and theorems, the power T^n of a t-norm is generalized to irrational exponents in Definition 2.2. and given explicitly for continuous archimedean t-norms in Proposition 2.3.

Though many results remain valid for arbitrary t-norms and especially for left continuous ones, for the sake of simplicity we will assume continuity for the t-norms throughout the paper.

Since a t-norm T is associative, we can extend it to an n-ary operation in the standard way:

$$T(x) = x$$
$$T(x_1, x_2, ...x_n) = T(x_1, T(x_2, ..x_n)).$$

In particular, $T(\overbrace{x, x, ...x}^{n\ times})$ will be denoted by $x_T^{(n)}$ or simply by $x^{(n)}$ if the t-norm is clear.

If T is continuous, the n-th root $x_T^{(\frac{1}{n})}$ of x wrt T is defined by

$$x_T^{(\frac{1}{n})} = sup\{z \in [0,1] \mid z_T^{(n)} \leq x\}$$

and for $m, n \in N$, $x_T^{(\frac{m}{n})} = \left(x_T^{(\frac{1}{n})}\right)_T^{(m)}$.

Lemma 2.1. *[8] If $k, m, n \in N$, $k, n \neq 0$ then $x_T^{(\frac{km}{kn})} = x_T^{(\frac{m}{n})}$.*

Assuming continuity for the t-norm T, the powers $x_T^{(\frac{m}{n})}$ can be extended to irrational exponents in a straightforward way.

Definition 2.2. *If $r \in R^+$ is a positive real number, let $\{a_n\}_{n\in N}$ be a sequence of rational numbers with $lim_{n\to\infty} a_n = r$. For any $x \in [0,1]$, the power $x_T^{(r)}$ is*

$$x_T^{(r)} = lim_{n\to\infty} x_T^{(a_n)}.$$

Continuity assures the existence of last limit and independence of the sequence $\{a_n\}_{n\in N}$.

Proposition 2.3. *Let T be an archimedean t-norm with additive generator t, $x \in [0,1]$ and $r \in R^+$. Then*

$$x_T^{(r)} = t^{[-1]}(rt(x)).$$

Proof. Due to continuity of t we need to prove it only for rational r.

If r is a natural number m, then trivially $x_T^{(m)} = t^{[-1]}(mt(x))$.

If $r = \frac{1}{n}$ with $n \in N$, then $x_T^{(\frac{1}{n})} = z$ with $z_T^{(n)} = x$ or $t^{[-1]}(nt(z)) = x$ and $x_T^{(\frac{1}{n})} = t^{[-1]}\left(\frac{t(x)}{n}\right)$.

For a rational number $\frac{m}{n}$,

$$x_T^{(\frac{m}{n})} = \left(x_T^{(\frac{1}{n})}\right)_T^{(m)} = t^{[-1]}\left(mt\left(x_T^{(\frac{1}{n})}\right)\right) =$$

$$t^{[-1]}\left(mt\left(t^{[-1]}\left(\frac{t(x)}{n}\right)\right)\right) = t^{[-1]}\left(\frac{m}{n}t(x)\right).$$

□

Definition 2.4. *The residuation $\overrightarrow{T}$ of a t-norm T is defined by*

$$\overrightarrow{T}(x|y) = sup\{\alpha \in [0,1] \mid T(x, \alpha) \leq y).$$

Definition 2.5. *The natural T-indistinguishability E_T associated to a given t-norm T is the fuzzy relation on* [0,1] *defined by*

$$E_T(x,y) = T(\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)).$$

E_T is indeed a special kind of T-indistinguishability operator (Definition 2.6) [2] and in a logical context where T plays the role of the conjunction, E_T is interpreted as the bi-implication associated to T [5].

Definition 2.6. *Given a t-norm T, a T-indistinguishability operator E on a set X is a fuzzy relation $E : X \times X \to [0,1]$ satisfying for all $x, y, z \in X$*

1. *$E(x,x) = 1$ (Reflexivity)*
2. *$E(x,y) = E(y,x)$ (Symmetry)*
3. *$T(E(x,y), E(y,z)) \leq E(x,z)$ (T-transitivity).*

Example 2.7.

1. *If T is the Lukasiewicz t-norm, then $E_T(x,y) = 1 - |x-y|$ for all $x, y \in [0,1]$.*
2. *If T is the Product t-norm, then $E_T(x,y) = Min(\frac{x}{y}, \frac{y}{x})$ for all $x, y \in [0,1]$ where $\frac{z}{0} = 1$.*
3. *If T is the Minimum t-norm, then $E_T(x,y) = \begin{cases} Min(x,y) & \text{if } x \neq y \\ 1 & \text{otherwise.} \end{cases}$*

Theorem 2.8. *Representation Theorem [11]. Let R be a fuzzy relation on a set X and T a continuous t-norm. R is a T-indistinguishability operator if and only if there exists a family $(h_i)_{i\in I}$ of fuzzy subsets of X such that for all $x, y \in X$*

$$R(x,y) = inf_{i\in I} E_T(h_i(x), h_i(y)).$$

$(h_i)_{i\in I}$ is called a generating family of R.

In particular, given a proximity matrix or relation R on X (i.e. a reflexive and symmetric fuzzy relation), we can build the T-indistinguishability operator $\underline{R}$ generated by the set of the columns of R (i.e. the fuzzy subsets $R(x, \cdot)$, $x \in X$).

Proposition 2.9. $\underline{R} \leq R$.

Definition 2.10. *Let R be a proximity matrix or relation (i.e. a reflexive and symmetric fuzzy relation) on X and T a continuous t-norm. The T-transitive closure $\overline{R}$ of R is the smallest T-indistinguishability operator on X satisfying $R \leq \overline{R}$.*

Definition 2.11. *Let R and S be two fuzzy relations on X and T a continuous t-norm. The Sup-T product of R and S is the fuzzy relation $R \circ S$ on X defined for all $x, y \in X$ by*

$$(R \circ S)(x, y) = sup_{z \in X} T(R(x, z), S(z, y)).$$

Since the Sup-T product is associative or continuous t-norms, we can define for $n \in N$ the n^{th} power R_T^n of a fuzzy relation R:

$$R_T^n = \overbrace{R \circ ... \circ R}^{n\ times}.$$

Definition 2.12. *Let R be a fuzzy relation on a set X and T a continuous t-norm. The transitive closure of R with respect to T is the fuzzy relation*

$$R_T = sup_{n \in N} R_T^n.$$

Proposition 2.13. *Let R be a proximity relation on a finite set X of cardinality n. Then*

$$R^T = sup_{s \in \{1,...,n-1\}} R_T^s.$$

3 Aggregating the transitive closure and a T-indistinguishability smaller than R

Given a proximity relation R on X, it is necessary in many cases to replace it by a T-indistinguishability operator E, since T-transitivity is required. In these cases, we want to find E close to R, where the closeness or similarity between fuzzy relations can be defined in many different ways.

Let X be a finite set of cardinality n. Ordering its elements linearly, we can view the fuzzy subsets of X as vectors: $X = \{x_1, ..., x_n\}$ and a fuzzy set h is the vector $(h(x_1), ..., h(x_n))$. A proximity relation R on X can be represented by a matrix (also called R) determined by the $\binom{n}{2}$ entries r_{ij} $1 \leq i < j \leq n$ of R above the diagonal.

Proposition 3.1. *Let $E = (e_{ij})_{i,j=1,...,n}$ be a proximity matrix on a set X of cardinality n and T a continuous archimedean t-norm with additive generator t. E is a T-indistinguishability operator if and only if for all i, j, k $1 \leq i < j < k \leq n$*

$$t(e_{ij}) + t(e_{jk}) \geq t(e_{ik})$$
$$t(e_{ij}) + t(e_{ik}) \geq t(e_{jk})$$
$$t(e_{ik}) + t(e_{jk}) \geq t(e_{ij})$$

Example 3.2. *If T is the Lukasiewicz t-norm, then we can take $t(x) = 1 - x$ and last inequalities become*

$$e_{ij} + e_{jk} - e_{ik} \leq 1$$
$$e_{ij} + e_{ik} - e_{jk} \leq 1$$
$$e_{ik} + e_{jk} - e_{ij} \leq 1$$

Example 3.3. *If T is the Product t-norm, then we can take $t(x) = -log(x)$ and last inequalities become*

$$e_{ij} \cdot e_{jk} \leq e_{ik}$$
$$e_{ij} \cdot e_{ik} \leq e_{jk}$$
$$e_{ik} \cdot e_{jk} \leq e_{ij}$$

Given a proximity matrix R, we must then search for (one of) the closest matrices E satisfying the last $3\binom{n}{3}$ inequalities which is a non-linear programming problem.

Instead of this, we propose alternative methods to obtain not the best but reasonably good approximations of proximity relations by T-indistinguishability operators.

Definition 3.4. [1], [8] *Given a continuous monotonic map $t : [0, 1] \to [-\infty, \infty]$ and p, q positive integers with $p + q = 1$, the weighted quasi-arithmetic mean $m_t^{p,q}$ generated by t and weights p and q is defined for all $x, y \in [0, 1]$ by*

$$m_t^{p,q}(x, y) = t^{-1}(p \cdot t(x) + q \cdot t(y)).$$

m_t is continuous if and only if Ran $t \neq [-\infty, \infty]$.

Proposition 3.5. *Fixed the weights p and q, the map assigning to every continuous Archimedean t-norm T with generator t the weighted mean $m_t^{p,q}$ generated by t is a bijection between the set of continuous Archimedean t-norms and the set of continuous quasi-arithmetic means with these weights.*

Proposition 3.6. *Let T be a continuous archimedean t-norm with additive generator t, $p \in [0, 1]$ and E, F two T-indistinguishability operator on X. The weighted quasi-arithmetic mean $m_t^{p,1-p}$ with weights p and $1 - p$ of E and F is a T-indistinguishability operator.*

Thanks to this last proposition, given a proximity matrix R we can calculate its transitive closure $\overline{R}$ and a smaller T-indistinguishability operator than R, for example $\underline{R}$ and find the weights $p, 1 - p$ to obtain the closest average of $\overline{R}$ and $\underline{R}$ to R.

The similarity between two fuzzy relations on X will be calculated in the following way.

Definition 3.7. *Let T be a continuous archimedean t-norm with additive generator t and R, S two fuzzy* *relations on a finite set X of cardinality n. The degree $DS(R, S)$ of similarity or closeness between R and S is defined by*

$$DS(R,S) = t^{-1}\left(\frac{\sum_{1\le i,j\le n}|t(r_{ij})-t(s_{ij})|}{n}\right).$$

Proposition 3.8. *DS is a T-indistinguishability operator on the set of fuzzy relations on X.*

Corollary 3.9. *Let $R = (r_{ij})$ be a proximity matrix on a finite set X of cardinality n, T a continuous archimedean t-norm with additive generator t, $\overline{R} = (\overline{r}_{i,j})$ its transitive closure, $\underline{R} = (\underline{r}_{ij})$ the T-indistinguishability operator obtained from R with the Representation Theorem, $p \in [0,1]$ and $m_t^{p,1-p}(\overline{R}, \underline{R})$ the T-indistinguishability operator quasi-arithmetic mean of $\overline{R}$ and $\underline{R}$ with weights p and $1-p$. Then*

$$DS(R, m_t^{p,1-p}(E,F)) =$$

$$t^{-1}\left(\frac{\sum_{1\le i,j\le n}\left|p\cdot t(\overline{r}_{ij})+(1-p)\cdot t(\underline{r}_{ij})-t(r_{ij})\right|}{n}\right).$$

We are looking for the value (or values) of p that maximize the last equality. Since t^{-1} is a decreasing map, this is equivalent to minimize

$$\sum_{1\le i,j\le n}\left|p\cdot t(\overline{r}_{ij})+(1-p)\cdot t(\underline{r}_{ij})-t(r_{ij})\right|$$

and, since R is reflexive and symmetric, is equivalent to minimize

$$f(p) = \sum_{1\le i<j\le n}\left|p\cdot t(\overline{r}_{ij})+(1-p)\cdot t(\underline{r}_{ij})-t(r_{ij})\right|$$

Proposition 3.10. *Let $f_1, ..., f_n : [0,1] \to R$ be n concave functions. Then $\sum_1^n f_i$ is a concave function.*

Proof. By definition, given two points x_1, x_2 of $[0,1]$, the segments joining their images by f_i $i = 1, .., n$ are above f_i. $\sum_1^n f_i$ will then be bellow the sum of all the segments. □

Corollary 3.11. *$f(p)$ is a concave function.*

Proof. Each summand $\left|p\cdot t(\overline{r}_{ij})+(1-p)\cdot t(\underline{r}_{ij})-t(r_{ij})\right|$ of f is a concave function. □

Proposition 3.12. *The set of minima of $f(p)$ consists of a single point or of a closed interval.*

Proof. f is a concave function and its graphic is a polygonal line. □

Proposition 3.13. *The computation of $m_t(\overline{R}, \underline{R})$ with maximum $DS(R, m_t(\overline{R}, \underline{R}))$ can be done taking $O(n^3)$ time complexity.*

Proof:

The computation of $\overline{R}$ and $\underline{R}$ can be done in $O(n^3)$ complexity time [9].

The addition (aggregation of distances) takes $O(n^2)$ time complexity.

The minimization of f(p) takes at most $O(n^2)$ time complexity.

So the most complex part of this process is the computation of $\overline{R}$ and $\underline{R}$, which still takes $O(n^3)$ complexity time.

Example 3.14. *Let X be a set of cardinality 7 and R the proximity relation given by*

$$R = \begin{pmatrix} 1 & 1 & 0.3 & 0.3 & 0.1 & 0.3 & 0.4 \\ 1 & 1 & 0.6 & 0.4 & 0.5 & 0.4 & 0.2 \\ 0.3 & 0.6 & 1 & 0.1 & 0.3 & 0.2 & 0.5 \\ 0.3 & 0.4 & 0.1 & 1 & 1 & 1 & 1 \\ 0.1 & 0.5 & 0.3 & 1 & 1 & 1 & 1 \\ 0.3 & 0.4 & 0.2 & 1 & 1 & 1 & 1 \\ 0.4 & 0.2 & 0.5 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Then, for T the Lukasiewicz t-norm,

$$\overline{R} = \begin{pmatrix} 1 & 1 & 0.6 & 0.4 & 0.5 & 0.4 & 0.4 \\ 1 & 1 & 0.6 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.6 & 0.6 & 1 & 0.5 & 0.5 & 0.5 & 0.5 \\ 0.4 & 0.5 & 0.5 & 1 & 1 & 1 & 1 \\ 0.5 & 0.5 & 0.5 & 1 & 1 & 1 & 1 \\ 0.4 & 0.5 & 0.5 & 1 & 1 & 1 & 1 \\ 0.4 & 0.5 & 0.5 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and

$$\underline{R} = \begin{pmatrix} 1 & 0.6 & 0.3 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.6 & 1 & 0.3 & 0.2 & 0.1 & 0.2 & 0.2 \\ 0.3 & 0.3 & 1 & 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.2 & 0.1 & 1 & 0.8 & 0.9 & 0.6 \\ 0.1 & 0.1 & 0.1 & 0.8 & 1 & 0.8 & 0.7 \\ 0.1 & 0.2 & 0.1 & 0.9 & 0.8 & 1 & 0.7 \\ 0.1 & 0.2 & 0.1 & 0.6 & 0.7 & 0.7 & 1 \end{pmatrix}.$$

$$f(p) = |0.4p| + |0.3p-0.3| + |0.3p-0.1| + |0.4p-0.4| +$$
$$|0.3p-0.1| + |0.3p| + |0.3p| + |0.3p-0.1| + |0.4p| + |0.3p-0.1| +$$
$$|0.3p-0.3| + |0.4p-0.4| + |0.4p-0.2| + |0.4p-0.3| + |0.4p| +$$
$$|0.2p| + |0.1p| + |0.4p| + |0.2p| + |0.3p| + |0.3p|$$

which attains its minimum for $p = \frac{1}{3}$.

A good T-transitive approximation of R (for T the Lukasiewicz t-norm) is then

$$\begin{pmatrix} 1 & 0.733 & 0.4 & 0.2 & 0.233 & 0.2 & 0.2 \\ 0.733 & 1 & 0.4 & 0.3 & 0.233 & 0.3 & 0.3 \\ 0.4 & 0.4 & 1 & 0.233 & 0.233 & 0.233 & 0.233 \\ 0.2 & 0.3 & 0.233 & 1 & 0.867 & 0.933 & 0.733 \\ 0.233 & 0.233 & 0.233 & 0.867 & 1 & 0.867 & 0.8 \\ 0.2 & 0.3 & 0.233 & 0.933 & 0.867 & 1 & 0.8 \\ 0.2 & 0.3 & 0.233 & 0.733 & 0.8 & 0.8 & 1 \end{pmatrix}.$$

Example 3.15. *Let X be a set of cardinality 7 and R the proximity relation given by*

$$R = \begin{pmatrix} 1 & 0.5 & 0.7 & 0.7 & 0.5 & 0.7 & 0.8 \\ 0.5 & 1 & 1 & 0.8 & 0.9 & 0.8 & 0.6 \\ 0.7 & 1 & 1 & 0.5 & 0.7 & 0.6 & 0.9 \\ 0.7 & 0.8 & 0.5 & 1 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.9 & 0.7 & 0.5 & 1 & 0.5 & 0.5 \\ 0.7 & 0.8 & 0.6 & 0.5 & 0.5 & 1 & 0.5 \\ 0.8 & 0.6 & 0.9 & 0.5 & 0.5 & 0.5 & 1 \end{pmatrix}.$$

Then, for T the Product t-norm,

$$\overline{R} = \begin{pmatrix} 1 & 0.7 & 0.72 & 0.7 & 0.5 & 0.7 & 0.8 \\ 0.7 & 1 & 1 & 0.8 & 0.9 & 0.8 & 0.9 \\ 0.72 & 1 & 1 & 0.8 & 0.9 & 0.8 & 0.9 \\ 0.7 & 0.8 & 0.8 & 1 & 0.72 & 0.64 & 0.56 \\ 0.5 & 0.9 & 0.9 & 0.72 & 1 & 0.72 & 0.63 \\ 0.7 & 0.8 & 0.8 & 0.64 & 0.72 & 1 & 0.56 \\ 0.8 & 0.9 & 0.9 & 0.56 & 0.63 & 0.56 & 1 \end{pmatrix}$$

and

$$\underline{R} =$$

$$\begin{pmatrix} 1 & 0.5 & 0.5 & 0.625 & 0.5 & 0.625 & 0.714 \\ 0.5 & 1 & 0.625 & 0.5 & 0.625 & 0.555 & 0.555 \\ 0.5 & 0.625 & 1 & 0.5 & 0.555 & 0.555 & 0.6 \\ 0.625 & 0.5 & 0.5 & 1 & 0.5 & 0.5 & 0.5 \\ 0.5 & 0.625 & 0.555 & 0.5 & 1 & 0.5 & 0.5 \\ 0.625 & 0.555 & 0.555 & 0.5 & 0.5 & 1 & 0.5 \\ 0.714 & 0.555 & 0.6 & 0.5 & 0.5 & 0.5 & 1 \end{pmatrix}.$$

$f(p)$ attains its minimum for $p = 0.521$.

A good T-transitive approximation of R (for T the Product t-norm) is then

$$\begin{pmatrix} 1 & 0.587 & 0.595 & 0.660 & 0.5 & 0.660 & 0.754 \\ 0.587 & 1 & 0.783 & 0.626 & 0.744 & 0.662 & 0.700 \\ 0.595 & 0.783 & 1 & 0.626 & 0.700 & 0.662 & 0.729 \\ 0.660 & 0.626 & 0.626 & 1 & 0.595 & 0.563 & 0.528 \\ 0.5 & 0.744 & 0.700 & 0.595 & 1 & 0.595 & 0.559 \\ 0.660 & 0.662 & 0.662 & 0.563 & 0.595 & 1 & 0.528 \\ 0.754 & 0.700 & 0.729 & 0.528 & 0.559 & 0.528 & 1 \end{pmatrix}.$$

The degree of closeness between two fuzzy relations can also be calculated using the euclidean distance.

Definition 3.16. *Let $R = (r_{ij})$ and $S = (s_{ij})$ be two fuzzy relations on a finite set X of cardinality n. The euclidean distance D between R and S is*

$$D(R,S) = \left(\sum_{1 \le i,j \le n} (r_{ij} - s_{ij})^2 \right)^{\frac{1}{2}}$$

Corollary 3.17. *Let $R = (r_{ij})$ be a proximity matrix on a finite set X of cardinality n, T a continuous archimedean t-norm with additive generator t, $\overline{R} = (\overline{r}_{i,j})$ its transitive closure, $\underline{R} = (\underline{r}_{ij})$ the T-indistinguishability operator obtained from R with the Representation Theorem, $p \in [0,1]$ and $m_t(\overline{R}, \underline{R})$ the T-indistinguishability operator quasi-arithmetic mean of $\overline{R}$ and $\underline{R}$ with weights p and $1-p$. Then*

$$D(R, m_t(E,F)) =$$

$$\left(\sum_{1 \le i,j \le n} \left(t^{-1}\left(p \cdot t\left(\overline{r}_{ij}\right) + (1-p) \cdot t\left(\underline{r}_{ij}\right)\right) - t(r_{ij})\right)^2 \right)^{\frac{1}{2}}.$$

Proposition 3.18. *Let T be the Lukasiewicz t-norm and R a proximity on a set X of cardinality n. The closest $m_t(\overline{R}, \underline{R})$ to R is attained for*

$$p = \frac{\sum_{1 \le i < j \le n} \left(\overline{r}_{ij} - \underline{r}_{ij}\right)\left(r_{ij} - \underline{r}_{ij}\right)}{\sum_{1 \le i < j \le n} \left(\overline{r}_{ij} - \underline{r}_{ij}\right)^2}$$

Proof. Due to symmetry and reflexivity, it is enough to minimize

$$f(p) = \sum_{1 \le i < j \le n} \left(p\left(\overline{r}_{ij} - \underline{r}_{ij}\right) + \underline{r}_{ij} - r_{ij}\right)^2.$$

$$f'(p) = 2 \sum_{1 \le i < j \le n} \left(p\left(\overline{r}_{ij} - \underline{r}_{ij}\right) + \underline{r}_{ij} - r_{ij}\right)\left(\underline{r}_{ij} - r_{ij}\right) = 0$$

and

$$p = \frac{\sum_{1 \le i < j \le n} \left(\overline{r}_{ij} - \underline{r}_{ij}\right)\left(r_{ij} - \underline{r}_{ij}\right)}{\sum_{1 \le i < j \le n} \left(\overline{r}_{ij} - \underline{r}_{ij}\right)^2}.$$

□

Example 3.19. *Let X be a set of cardinality 4 and R the proximity relation on X given by*

$$R = \begin{pmatrix} 1 & 0.8 & 0.2 & 0.4 \\ 0.8 & 1 & 0.7 & 0.1 \\ 0.2 & 0.7 & 1 & 0.6 \\ 0.4 & 0.1 & 0.6 & 1 \end{pmatrix}.$$

If T is the Lukasiewicz t-norm, the closest T-indistinguishability operator of the type $m_t(\overline{R}, \underline{R})$ (with respect to the euclidean distance) is attained for $p = 0.6388889$.

A good T-approximation of R is then

$$\begin{pmatrix} 1 & 0.6917 & 0.3917 & 0.3639 \\ 0.6917 & 1 & 0.5917 & 0.2278 \\ 0.3917 & 0.5917 & 1 & 0.5278 \\ 0.3639 & 0.2278 & 0.5278 & 1 \end{pmatrix}$$

4 Applying a homotecy to a T-indistinguishability operator

In this Section, the fact that the power of a T-indistinguishability operator is again a T-indistinguishability operator will be exploited to modify the entries of $\overline{R}$ or $\underline{R}$ to find a better approximation of R.

Proposition 4.1. *Let T be a continuous t-norm, E a T-indistinguishability operator on X and $p > 0$. Then $E^{(p)}$ is a T-indistinguishability operator.*

Example 4.2.

- *If T is a continuous archimedean t-norm with additive generator t and E a T-indistinguishability operator, then $t^{[-1]}(p \cdot t(E))$ is a T-indistinguishability operator.*
- *If T is the Lukasiewicz t-norm and E a T-indistinguishability operator, then $Max(0, 1-p+p \cdot E)$ is a T-indistinguishability operator.*
- *If T is the Product t-norm and E a T-indistinguishability operator, then E^p is a T-indistinguishability operator.*

Let $R = (r_{ij})$ be a proximity matrix on a set X of cardinality X, $p > 0$ and $E = (e_{ij})$ a T-indistinguishability operator on X with T a continuous archimedean t-norm with additive generator t. Then

$$DS(R, E^{(p)}) = t^{-1}\left(\frac{\sum_{1\le i,j\le n} |t(r_{ij}) - p \cdot t(e_{ij}))|}{n}\right).$$

To maximize the previous expression is equivalent to minimize

$$\sum_{1\le i,j\le n} |t(r_{ij}) - p \cdot t(e_{ij}))|.$$

Since R is reflexive and symmetric, this is equivalent to minimize

$$g(p) = \sum_{1\le i<j\le n} |t(r_{ij}) - p \cdot t(e_{ij}))|.$$

Again g is a sum of concave functions in $[0,1]$ and therefore has a minimum or a close interval of minima.

Example 4.3. *Let us consider the same matrix*

$$R = \begin{pmatrix} 1 & 0.8 & 0.2 & 0.4 \\ 0.8 & 1 & 0.7 & 0.1 \\ 0.2 & 0.7 & 1 & 0.6 \\ 0.4 & 0.1 & 0.6 & 1 \end{pmatrix}.$$

Then, for T the Lukasiewicz t-norm,

$$\underline{R} = \begin{pmatrix} 1 & 0.5 & 0.2 & 0.3 \\ 0.5 & 1 & 0.4 & 0.1 \\ 0.2 & 0.4 & 1 & 0.4 \\ 0.3 & 0.1 & 0.4 & 1 \end{pmatrix}.$$

$$g(p) = |0.5 \cdot p - 0.2| + |0.8 \cdot p - 0.8| + |0.7 \cdot p - 0.6| + |0.6 \cdot p - 0.3| + |0.9 \cdot p - 0.9| + |0.6 \cdot p - 0.4|$$

which attains its minimum for $p = 0.857$.

A good approximation of R is then

$$\underline{R}^{(0.857)} = \begin{pmatrix} 1 & 0.5715 & 0.3144 & 0.4001 \\ 0.5715 & 1 & 0.4858 & 0.2287 \\ 0.3144 & 0.4858 & 1 & 0.4858 \\ 0.4001 & 0.2287 & 0.4858 & 1 \end{pmatrix}.$$

If we consider the euclidean distance between R and the power $E^{(p)}$ of a T-indistinguishability operator $E = (e_{ij})$, then

Proposition 4.4.

$$D(R, E^{(p)} = \left(\sum_{1\le i,j\le n} \left(t^{-1}\left(p \cdot t\left(e_{ij}\right)\right) - r_{ij}\right)^2\right)^{\frac{1}{2}}.$$

Example 4.5. *Continuing the last example, $D(R, \overline{R}^{(p)})$ is maximum for $p = 1.208633$ and $D(R, \underline{R}^{(p)})$ is maximum for $p = 0.821306$.*

Good approximations of R are therefore

$$\overline{R}^{(1.208633)} = \begin{pmatrix} 1 & 0.7583 & 0.3957 & 0.2748 \\ 0.7583 & 1 & 0.6374 & 0.1540 \\ 0.3957 & 0.6374 & 1 & 0.5165 \\ 0.2748 & 0.1540 & 0.5165 & 1 \end{pmatrix}.$$

and

$$\underline{R}^{(0.821306)} = \begin{pmatrix} 1 & 0.8357 & 0.5893 & 0.5072 \\ 0.8357 & 1 & 0.7536 & 0.4251 \\ 0.5893 & 0.7536 & 1 & 0.6715 \\ 0.5072 & 0.4251 & 0.6715 & 1 \end{pmatrix}.$$

5 Concluding Remarks

In this paper we have presented two ways to find good approximations of a proximity relation by T-transitive ones (T archimedean) in a reasonable computational way.

The obtained approximation R' is in general not comparable with R in the sense that neither $R' \geq R$ nor $R \geq R'$ must hold.

The simple examples show that in general these approximations are better than the transitive closure or openings of the proximity R.

The methods of the paper cannot be applied to the Minimum t-norm. Other ways to obtain similar results for this t-norm are therefore needed and the authors will work on it in forthcoming papers.

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