Aggregation Operators and the Lipschitzian Condition

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Abstract— **Lipschitzian and kernel aggregation operators with respect to the natural T-indistinguishability operator E_T and their powers are studied. A t-norm T is proved to be E_T-Lipschitzian, and is interpreted as a fuzzy point and a fuzzy map as well. Given an Archimedean t-norm T with additive generator t, the quasi-arithmetic mean generated by t is proved to be the most stable aggregation operator with respect to T.**

I. Introduction

Lipschitzian conditions are fulfilled by many maps and operators in fuzzy reasoning. They give stability to the system since the similarity or distance between the outputs is bounded by the corresponding one between the inputs. The Lipschitzian condition appears for example in the study of fuzzy maps [12], vague algebras [6], fuzzy modifiers and fuzzy logic in the narrow sense [16], fuzzy topology [8], extensionality [12] among others and therefore it deserves a deep study.

Lipschitzian aggregation operators have been studied in [4] [5] [13] [14] considering the usual metric on the unit interval. This paper studies the Lipschitzian condition of aggregation operators in a broader sense, i.e. with respect to the natural indistinguishability operator E_T and their powers E_T^p (see definitions below) so that an aggregation operator h is E_T^p-Lipschitzian when

$$T(E_T^p(x_1,y_1),...,E_T^p(x_n,y_n)) \leq$$

$$E_T(h(x_1,x_2,...,x_n),h(y_1,y_2,...,y_n))$$

for all $x_1,x_2,...,x_n,y_1,y_2,...,y_n \in [0,1]$. The meaning is that from similar values we obtain a similar aggregation. The use of E_T and E_T^p assumes the choice of a specific t-norm T and therefore the selection of a particular family of logics where the semantics of the conjunction and the biimplication are given by T and E_T.

As it will be seen in the paper, when T is the Lukasiewicz t-norm, the E_T-Lipschitzian condition coincides with the 1-Lipschitzian condition with the usual metric on [0,1]; not surprisingly, due to the relation between E_T and the usual metric on [0,1] in this case.

Easy to state, but interesting, the E_T^p-Lipschitzian condition is equivalent to the extensionality of the aggregation operator (Proposition 3.10).

Among other results, it will be proved that if T is a continuous Archimedean t-norm with additive generator t and h_t the quasi-arithmetic mean generated by t ($h_t(x_1,x_2,...,x_n) = t^{-1}\left(\frac{t(x_1)+t(x_2)+...,t(x_n)}{n}\right)$), then h_t is the most stable aggregation operator with respect to T (Proposition 3.22).

Also the t-norm T is not only Lipschitzian with respect to E_T, but it can be seen as a fuzzy point and a fuzzy map as well (Proposition 3.24, Proposition 3.26) and an aggregation operator h is greater or equal to T if and only if h is 1-E_T-Lipschitzian.

If in the definition of E_T-Lipschitzianity we replace the t-norm T by the minimum

$$(Min(E_T^p(x_1,y_1),...,E_T^p(x_n,y_n)) \leq$$

$$E_T(h(x_1,x_2,...,x_n),h(y_1,y_2,...,y_n)))$$

, we obtain a generalization of the kernel aggregation operators studied in [17] [13]. Again, if T is the Lukasiewicz t-norm this definition is equivalent to the one given in the above mentioned references.

II. Preliminaries

This section contains some results about t-norms and indistinguishability operators that will be needed later on in this paper. Besides well known definitions and theorems, the power T^n of a t-norm is generalized to irrational exponents in Definition 2.4. and it is given explicitly for continuous Archimedean t-norms in Proposition 2.5.

Though many results remain valid for arbitrary t-norms and especially for left continuous ones, for the sake of simplicity, we will assume continuity for the t-norms throughout the paper.

Definition 2.1: A continuous t-norm is a continuous map $T:[0,1]\times[0,1]\to[0,1]$ satisfying for all $x,y,z,x',y' \in [0,1]$

1) $T(x,T(y,z)) = T(T(x,y),z)$ (Associativity)
2) $T(x,y) = T(y,x)$ (Commutativity)
3) If $x \leq x'$ and $y \leq y'$, then $T(x,y) \leq T(x',y')$ (Monotonicity)
4) $T(1,x) = x$

Since a t-norm T is associative, we can extend it to an n-ary operation in the standard way:

$$T(x) = x$$
$$T(x_1,x_2,...x_n) = T(x_1,T(x_2,\dots,x_n)).$$

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In particular, $T(\overbrace{x,x,...x}^{n\ times})$ will be denoted by $x_T^{(n)}$.

If T is continuous, the n-th root $x_T^{(\frac{1}{n})}$ of x with respect to T is defined by

$$x_T^{(\frac{1}{n})} = sup\{z \in [0,1] \mid z_T^{(n)} \leq x\}$$

and for $m, n \in \mathbb{N}$, $x_T^{(\frac{m}{n})} = \left(x_T^{(\frac{1}{n})}\right)_T^{(m)}$.

Lemma 2.2: [18] If $k, m, n \in \mathbb{N},\ k, n \neq 0$ then $x_T^{(\frac{km}{kn})} = x_T^{(\frac{m}{n})}$.

Lemma 2.3: Let $x_1, \ldots, x_n \in (0,1]$ and $n \in \mathbb{N}$. $T(x_{1_T}^{(\frac{1}{n})}, \ldots, x_{n_T}^{(\frac{1}{n})}) \neq 0$.

Assuming continuity for the t-norm T, the powers $x_T^{(\frac{m}{n})}$ can be extended to irrational exponents in a straightforward way.

Definition 2.4: If $r \in \mathbb{R}^+$ is a positive real number, let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of rational numbers with $lim_{n\to\infty} a_n = r$. For any $x \in [0,1]$, the power $x_T^{(r)}$ is

$$x_T^{(r)} = lim_{n\to\infty} x_T^{(a_n)}.$$

Continuity assures the existence of the limit and its independence from the sequence $\{a_n\}_{n\in\mathbb{N}}$.

Proposition 2.5: Let T be an Archimedean t-norm with additive generator t, $x \in [0,1]$ and $r \in \mathbb{R}^+$. Then

$$x_T^{(r)} = t^{[-1]}(rt(x)).$$

Proof: Due to the continuity of t, we need to prove it only for rational values of r.

If r is a natural number m, then trivially $x_T^{(m)} = t^{[-1]}(mt(x))$.

If $r = \frac{1}{n}$ with $n \in \mathbb{N}$, then $x_T^{(\frac{1}{n})} = z$ with $z_T^{(n)} = x$ or $t^{[-1]}(nt(z)) = x$ and $x_T^{(\frac{1}{n})} = t^{[-1]}\left(\frac{t(x)}{n}\right)$.

For a rational number $\frac{m}{n}$,

$$x_T^{(\frac{m}{n})} = \left(x_T^{(\frac{1}{n})}\right)_T^{(m)} = t^{[-1]}\left(mt\left(x_T^{(\frac{1}{n})}\right)\right) =$$

$$t^{[-1]}\left(mt\left(t^{[-1]}\left(\frac{t(x)}{n}\right)\right)\right) = t^{[-1]}\left(\frac{m}{n}t(x)\right).$$

■

Let $E(T) = \{x \in [0,1] \mid x_T^{(2)} = x\}$ be the set of idempotent elements of T and $NIL(T) = \{x \in [0,1] \mid x_T^{(n)} = 0$ for some $n \in \mathbb{N}\}$ the set of nilpotent elements of T.

Definition 2.6: A continuous t-norm T is Archimedean if and only if $E(T) = \{0,1\}$. T is called non-strict or nilpotent when $NIL(T) = [0,1)$ Otherwise it is called strict and $NIL(T) = \{0\}$.

Theorem 2.7: Ling [15] A continuous t-norm T is Archimedean if and only if there exists a continuous decreasing map $t : [0,1] \to [0,\infty]$ with $t(1) = 0$ such that

$$T(x,y) = t^{[-1]}(t(x) + t(y))$$

where $t^{[-1]}$ stands for the pseudo-inverse of t defined by

$$t^{[-1]}(x) = \begin{cases} 1 & \text{if } x < 0 \\ t^{-1}(x) & \text{if } x \in [0, t(0)] \\ 0 & \text{otherwise.} \end{cases}$$

T is strict if $t(0) = \infty$ and non-strict otherwise.

t is called an additive generator of T and two additive generators of the same t-norm differ only by a multiplicative constant.

Definition 2.8: The residuation $\overrightarrow{T}$ of a t-norm T is defined by

$$\overrightarrow{T}(x|y) = sup\{\alpha \in [0,1] \mid T(x,\alpha) \leq y).$$

Definition 2.9: The natural T-indistinguishability E_T associated to a given t-norm T is the fuzzy relation on [0,1] defined by

$$E_T(x,y) = T(\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)) = Min(\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)).$$

Example 2.10:

1) If T is an Archimedean t-norm with additive generator t, then $E_T(x,y) = t^{-1}(|t(x) - t(y)|)$ for all $x, y \in [0,1]$.
2) If T is the Lukasiewicz t-norm, then $E_T(x,y) = 1 - |x-y|$ for all $x, y \in [0,1]$.
3) If T is the Product t-norm, then $E_T(x,y) = Min(\frac{x}{y}, \frac{y}{x})$ for all $x, y \in [0,1]$ where $\frac{z}{0} = 1$.
4) If T is the Minimum t-norm, then $E_T(x,y) = \begin{cases} Min(x,y) & \text{if } x \neq y \\ 1 & \text{otherwise.} \end{cases}$

E_T is indeed a special kind of (one-dimensional) T-indistinguishability operator (Definition 2.11) [3] and in a logical context where T plays the role of the conjunction, E_T is interpreted as the bi-implication associated to T [7].

The general definition of T-indistinguishability operator [22][21] is

Definition 2.11: Given a t-norm T, a T-indistinguishability operator E on a set X is a fuzzy relation $E : X \times X \to [0,1]$ satisfying for all $x, y, z \in X$

1) $E(x,x) = 1$ (Reflexivity)
2) $E(x,y) = E(y,x)$ (Symmetry)
3) $T(E(x,y), E(y,z)) \leq E(x,z)$ (T-transitivity).

Proposition 2.12: [21] Let μ be a fuzzy subset of X and T a continuous t-norm. The fuzzy relation E_μ on X defined for all $x, y \in X$ by

$$E_\mu(x,y) = E_T(\mu(x), \mu(y))$$

is a T-indistinguishability operator on X.

Definition 2.13: Let E be a T-indistinguishability operator on a set X. A fuzzy subset μ of X is extensional with respect to E if and only if for all $x, y \in X$

$$T(E(x,y), \mu(y)) \leq \mu(x).$$

Proposition 2.14: Let E be a T-indistinguishability operator on a set X. A fuzzy subset μ of X is extensional with respect to E if and only if for all $x, y \in X$

$$E(x,y) \leq E_T(\mu(x), \mu(y)).$$

Finally, let us recall in this preliminary section the definition of aggregation operator.

Definition 2.15: [4] An *aggregation operator* is a map $h : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ satisfying

1) $h(0,...,0) = 0$ and $h(1,...,1) = 1$
2) $h(x) = x \ \forall x \in [0,1]$
3) $h(x_1,...,x_n) \leq h(y_1,...,y_n)$
   if $x_1 \leq y_1, \ldots, x_n \leq y_n$ $(monotonicity)$.

The restriction of h to $[0,1]^n$ will be denoted by $h_{(n)}$ so that a global aggregation operator h can be split into the family of n-ary operators $(h_{(n)})_{n\in\mathbb{N}}$.

III. E_T-LIPSCHITZIAN AND E_T-KERNEL AGGREGATION OPERATORS

Lipschitzian and kernel aggregation operators with respect to the natural T-indistinguishability operator E_T and their powers are a special kind of aggregation operators that generalize the definitions of [13], [17]. Their interest lies in the fact that they are stable operators in the sense that the similarity between the aggregation of two n-tuples is bounded by the similarity between them.

It is interesting to point out that the Lipschitzian and kernel conditions are equivalent to extensionality (Proposition 3.10, Proposition 3.28).

Among other results, it will be proved that a t-norm T is E_T-Lipschitzian and moreover the maps $T_{(n)}$ can be interpreted as fuzzy points of $[0,1]^n$ and a fuzzy maps from $[0,1]^k$ to $[0,1]^{n-k}$.

Also quasi-arithmetic means are proved to be the more stable aggregation operators.

Proposition 3.1: Let E be a T indistinguishability operator on a set X. The fuzzy relation E^n defined by

$$E^n(x,y) = T(\overbrace{E(x,y), \ldots, E(x,y)}^{n \ times}) \ \forall x, y \in X$$

is a T-indistinguishability operator.

Corollary 3.2: [20] Let E_T be the natural T-indistinguishability operator on [0,1] associated to T. E_T^n is a T-indistinguishability operator.

The powers E_T^n of the natural T-indistinguishability operators have been studied in relation with antonymy and fuzzy partitions in [20].

Proposition 3.3: Let E be a T-indistinguishability operator on a set X. $E^{\frac{1}{n}}$ is a T-indistinguishability operator on X.

Proof: Reflexivity and symmetry are trivial.

Transitivity: If $E^{\frac{1}{n}} = F$, then $F^n = E$. Since E is a T-indistinguishability operator, $\forall x, y, z \in X$

$$F^n(x,z) \leq T(F^n(x,y), F^n(y,z)) = (T(F(x,y), F(y,z)))_T^{(n)}$$

$$(F^n(x,z))_T^{\frac{1}{n}} \leq \left((T(F(x,y), F(y,z)))_T^{(n)}\right)_T^{\frac{1}{n}}$$

and from Lemma 2.2

$$F(x,z) \leq T(F(x,y), F(y,z)).$$

■

Corollary 3.4: Let E_T be the natural T-indistinguishability operator on [0,1] associated to T. $E_T^{\frac{1}{n}}$ is a T-indistinguishability operator.

Corollary 3.5: Let E be a T-indistinguishability operator on a set X. $E^{\frac{m}{n}}$ is a T-indistinguishability operator on X.

Proof: Propositions 3.1. and 3.3. ■

Corollary 3.6: Let E_T be the natural T-indistinguishability operator on [0,1] associated to T. $E_T^{\frac{m}{n}}$ is a T-indistinguishability operator.

Continuity of the t-norm T allows us to extend the powers of a T-indistinguishability operator to positive irrational numbers in the same way as in Definition 2.4.

Example 3.7:

1) If T is continuous Archimedean with additive generator t, then $E_T^p(x,y) = t^{[-1]}(p|t(x) - t(y)|)$ for all $x, y \in [0,1]$.
2) If T is the Lukasiewicz t-norm, then $E_T^p(x,y) = Max(0, 1 - p|x - y|)$ for all $x, y \in [0,1]$.
3) If T is the Product t-norm, then $E_T^p(x,y) = \left(Min(\frac{x}{y}, \frac{y}{x})\right)^p$ for all $x, y \in [0,1]$ where $\frac{z}{0} = 1$.
4) If T is the Minimum t-norm, then $E_T^p(x,y) = E_T(x,y)$ for all $x, y \in [0,1]$.

With the previous results we can relax or strengthen the equivalence relations. Indeed, $E_T^p \leq E_T^q$ if and only if $p \geq q$.

Definition 3.8: Let E be a T-indistinguishability operator on $[0,1]$. An aggretation operator h is E-lipschitzian if and only if $\forall n \in \mathbb{N}$, $\forall x_1, ..., x_n, y_1, ..., y_n \in [0,1]$

$$T(E(x_1,y_1), \ldots, E(x_n,y_n)) \leq E_T(h(x_1,...,x_n), h(y_1,\ldots,y_n)).$$

Let us recall that if we have several T-indistinguishability operators $E_1, ..., E_n$ defined on different universes $X_1, ..., X_n$, there are several ways to define a T-indistinguishability operator on $X_1 \times \ldots \times X_n$.

Proposition 3.9: Let $E_1, ..., E_n$ be T-indistinguishability operators on $X_1, ..., X_n$ respectively. Then, the two fuzzy relations $T(E_1, ..., E_n)$ and $Min(E_1, \dots, E_n)$ on $X_1 \times \dots \times X_n$ defined for all $(x_1, \dots, x_n), (y_1, \dots, y_n) \in X_1 \times \dots \times X_n$ by

$$T(E_1, \dots, E_n)((x_1, \dots, x_n), (y_1, \dots, y_n)) = T(E_1(x_1, y_1), ..., E_n(x_n, y_n))$$

and

$$Min(E_1, \dots, E_n)((x_1, \dots, x_n), (y_1, \dots, y_n)) = Min(E_1(x_1, y_1), \dots, E_n(x_n, y_n))$$

are T-indistinguishability operators on $X_1 \times \dots \times X_n$.

Proposition 3.10: Let E be a T-indistinguishability on $[0,1]$ and h an aggregation operator. h is E-Lipschitzian if and only if $h_{(n)}$ (as a fuzzy subset of $[0,1]^n$) is extensional with respect to $T(\overbrace{E, \dots, E}^{n\ times})$ for all $n \in \mathbb{N}$.

Proof: Proposition 2.14 ∎

Lemma 3.11: [2] Let T be a continuous t-norm. The for all $x, y \in [0,1]$ $x \geq y$

$$T(x, \overrightarrow{T}(x|y)) = y$$

Next Proposition shows that a t-norm T is an E_T-Lipschitzian aggregation operator.

Proposition 3.12: Let T be a continuous t-norm. Then T is an E_T-Lipschitzian aggregation operator.

Note that if $x_i \leq y_i$ for all $i = 1, \dots n$, then $T(E_T(x_1, y_1), \dots, E_T(x_n, y_n)) = E_T(T(x_1, \dots, x_n), T(y_1, \dots, y_n))$. Since for every t-norm different from the Minimum $E_T^p < E_T^q$ if $p > q$, we have that $T \neq Min$ is not E_T^p-Lipschitzian for $p < 1$.

If T is a continuous Archimedean t-norm, the E_T^p-Lipschitzian property becomes a classical Lipschitzian condition.

Proposition 3.13: Let T be a continuous Archimedean t-norm with additive generator t, $p \in [0,1]$ and h an aggregation operator. h is E_T^p-Lipschitzian if and only if $\forall n \in \mathbb{N}$, $\forall x_1, \dots, x_n, y_1, \dots, y_n \in [0,1]$

$$p|t(x_1) - t(y_1)| + \dots + p|t(x_n) - t(y_n)| \geq |t\left(h(x_1, \dots, x_n)\right) - t\left(h(y_1, \dots, y_n)\right)| \quad (1).$$

Proof:

$$t^{[-1]}(t(t^{-1}(p|t(x_1) - t(y_1)|)) + \dots + t(t^{-1}(p|t(x_n) - t(y_n)|))) \leq t^{-1}(|t(h(x_1, \dots, x_n)) - t(h(x_1, \dots, x_n))|)$$

$$t^{[-1]}(p|t(x_1) - t(y_1)| + \dots + p|t(x_n) - t(y_n)|) \leq t^{-1}(|t(h(x_1, \dots, x_n)) - t(h(x_1, \dots, x_n))|)$$

$$p|t(x_1) - t(y_1)| + \dots + p|t(x_n) - t(y_n)| \geq |t(h(x_1, \dots, x_n)) - t(h(x_1, \dots, x_n))|.$$

∎

Last Proposition says that for all $n \in \mathbb{N}$, the map $H : [0, t(0)]^n \to [0, t(0)]$ defined by

$$H(x_1, \dots, x_n) = t(h(t^{-1}(x_1), \dots, t^{-1}(x_n)))$$

is a p-Lipschitzian map.

Also note that if T is the Lukasiewicz t-norm, then (1) is the definition of the Lipschitz property in [13], so that Definition 3.8 contains the one in [13] as a particular case.

If an aggregation operator h is E_T^p-Lipschitzian, it may happen that for different values of n the corresponding n-ary operators $h_{(n)}$ may satisfy the Lipschitzian conditions for different values of p ([4] p. 12).

Definition 3.14: An aggregation operator is sub-idempotent if and only if for all $x \in [0,1]$ and $n \in \mathbb{N}$, $h(\overbrace{x, \dots, x}^{n\ times}) \leq x$

Proposition 3.15: Let $T \neq Min$ be a t-norm, h a sub-idempotent aggregation operator and $n \in \mathbb{N}$. If $h_{(n)}$ is E_T^p-Lipschitzian, then $p \geq \frac{1}{n}$.

Proof: If $h_{(n)}$ is E_T^p-Lipschitzian, then in particular, for $x \in X$

$$T((\overbrace{E_T^p(1, x), \dots, E_T^p(1, x)}^{n\ times}) \leq E_T(h(\overbrace{1, \dots, 1}^{n\ times}), h(\overbrace{x, \dots, x}^{n\ times}))$$

and so

$$x_T^{(pn)} \leq h(\overbrace{x, \dots, x}^{n\ times}) \leq x$$

which holds if and only if $pn \geq 1$ or equivalently, if and only if $p \geq \frac{1}{n}$ ∎

For T is a strict continuous Archimedean t-norm the sub-idempotent property trivially holds.

Proposition 3.16: Let T be a strict continuous Archimedean t-norm with additive generator t, h an aggregation operator and $n \in \mathbb{N}$. If $h_{(n)}$ is E_T^p-Lipschitzian, then $p \geq \frac{1}{n}$.

Proof: Taking $x_i = 1$ and $y_i = 0$ for all $i = 1, \dots, n$ in Proposition 3.13, we get

$$p|t(1) - t(0)| + \dots + p|t(1) - t(0)| \geq |t(1) - t(0)|.$$

$$npt(0) \geq t(0)$$

or

$$p \geq \frac{1}{n}.$$

∎

In [4], it has been proved that the arithmetic mean is the only aggregation operator h whose n-ary maps $h_{(n)}$ are $\frac{1}{n}$-Lipschitzian. Proposition 3.22 generalizes this result to arbitrary quasi-arithmetic means.

Next Proposition is well known.

Proposition 3.17: [1], [18] m is a quasi-arithmetic mean in [0,1] if and only if there exists a continuous monotonic map $t : [0,1] \to [-\infty,\infty]$ such that for all $n \in \mathbb{N}$ and $x_1,\ldots,x_n \in [0,1]$

$$m(x_1,\ldots,x_n) = t^{-1}\left(\frac{t(x_1)+\ldots+t(x_n)}{n}\right).$$

m is continuous if and only if $Ran\ t \neq [-\infty,\infty]$.

Lemma 3.18: [11] Let $t,t' : [0,1] \to [-\infty,\infty]$ be two continuous strict monotonic maps with $Ran\ t, Ran\ t' \neq [-\infty,\infty]$ differing only by a non-zero multiplicative constant α ($t' = \alpha\ t$) and $m_t, m_{t'}$ the quasi-arithmetic means generated by them respectively. Then $m_t = m_{t'}$.

Lemma 3.19: [11] Let $t,t' : [0,1] \to [-\infty,\infty]$ be two continuous strict monotonic maps with $Ran\ t, Ran\ t' \neq [-\infty,\infty]$ differing only by an additive constant and $m_t, m_{t'}$ the quasi-arithmetic means generated by them respectively. Then $m_t = m_{t'}$.

Lemma 3.20: [11] Let $t : [0,1] \to [-\infty,\infty]$ be a continuous strict monotonic map. Then $m_t = m_{-t}$.

Proposition 3.21: [11] The map assigning to every continuous Archimedean t-norm T with generator t the mean m_t generated by t is a bijection between the set of continuous Archimedean t-norms and the set of continuous quasi-arithmetic means.

Proposition 3.22: Let T be a continuous Archimedean t-norm with additive generator t and m_t the quasi-arithmetic mean generated by t.

- (a) For every $n \in \mathbb{N}$ $m_{t(n)}$ is E_T^p-Lipschitzian if and only if $p \geq \frac{1}{n}$.
- (b) m_t is the only aggregation operator fulfilling (a)

In Proposition 3.12 we have proved that a t-norm T is E_T-Lipschitzian. In fact, $T_{(n)}$ can also be seen as a fuzzy point of $[0,1]^n$ and a fuzzy map from $[0,1]^{n-1}$ into $[0,1]$.

Definition 3.23: Let E be a T-indistinguishability operator on a set X, a fuzzy subset of X μ is a fuzzy point of X with respect to E if and only if for all $x,y \in X$

$$T(\mu(x),\mu(y)) \leq E(x,y).$$

Proposition 3.24: Let T be a continuous t-norm. $T_{(n)}$ is a fuzzy point of $[0,1]^n$ with respect to $T(\overbrace{E_T,\ldots,E_T}^{n\ times})$.

Proof: We have to prove that

$$T(T(x_1,\ldots,x_n),T(y_1,\ldots,y_n)) \leq$$

$$T(E_T(x_1,y_1),\ldots,E_T(x_n,y_n))$$

which is an immediate consequence of

$$T(x_i,y_i) \leq E_T(x_i,y_i) \text{ for all } i=1,\ldots,n.$$

■

Definition 3.25: Let E,F be two T-indistinguishability operators on X and Y respectively and R a fuzzy set of $X \times Y$ (i.e.: $R : X \times Y \to [0,1]$). R is a fuzzy map from X to Y if and only if for all $x,x' \in X$, $y,y' \in Y$

- (a) $T(E(x,x'),F(y,y'),R(x,y)) \leq R(x',y')$
- (b) $T(R(x,y),R(x,y')) \leq F(y,y')$.

Proposition 3.26: Let T be a continuous t-norm. $T_{(n)}$ is a fuzzy map from $[0,1]^{n-1}$ to $[0,1]$ endowed with the T indistinguishability operators $T(\overbrace{E_T,\ldots,E_T}^{n-1\ times})$ and E_T respectively.

In fact, it can be proved in the same way that $T_{(n)}$ is a fuzzy map from $[0,1]^k$ to $[0,1]^{n-k}$ $(2 \leq k \leq n-1)$ endowed with the T indistinguishability operators $T(\overbrace{E_T,\ldots,E_T}^{k\ times})$ and $T(\overbrace{E_T,\ldots,E_T}^{n-k\ times})$ respectively.

Kernel aggregation operators are a family of aggregation operators tightly related to Lipschitzian ones. They were introduced in [17] (see also [13], [4]). As the Lipschitzian condition, the condition for being a kernel operator was related to the usual metric on the unit interval. It can be extended using natural indistinguishability operators in the same way as it has been done in this paper with the Lipschitzian condition. Again, if the T norm is the Lukasiewicz one, the original definition of [17] is recovered.

Definition 3.27: Let E be a T-indistinguishability operator on [0,1] and h an aggregation operator. h is an E-kernel aggregation operator if and only if $\forall n \in \mathbb{N}$, $\forall x_1,\ldots,x_n,y_1,\ldots,y_n \in [0,1]$

$$Min(E(x_1,y_1),\ldots,E(x_n,y_n))$$
$$\leq E_T(h(x_1,\ldots,x_n),h(y_1,\ldots,y_n)).$$

Proposition 3.28: Let E be a T-indistinguishability operator on [0,1] and h an aggregation operator. h is an E-kernel aggregation operator if and only if $h_{(n)}$ (as a fuzzy subset of $[0,1]^n$) is extensional with respect to $Min(\overbrace{E,\ldots,E}^{n\ times})$ for all $n \in \mathbb{N}$.

Proof: Proposition 2.14 ■

For Archimedean t-norms, the kernel property can be written as follows.

Proposition 3.29: Let T be a continuous Archimedean t-norm with additive generator t, $p \in [0,1]$ and h an aggregation operator. h is E_T^p-kernel aggregation operator if and only if $\forall n \in \mathbb{N}$, $\forall x_1, \ldots, x_n, y_1, \ldots, y_n \in [0,1]$

$$Max(p|t(x_1) - t(y_1)|, \ldots, p|t(x_n) - t(y_n)|)$$
$$\geq |t(h(x_1, \ldots, x_n)) - t(h(y_1, \ldots, y_n))| \quad (4).$$

Proof:

$$Min(t^{-1}(p|t(x_1) - t(y_1)|), \ldots, t^{-1}(p|t(x_n) - t(y_n)|)) \leq$$
$$t^{-1}(|t(h(x_1, \ldots, x_n)) - t(h(x_1, \ldots, x_n))|)$$
$$t^{-1}(Max(p|t(x_1) - t(y_1)|, \ldots, p|t(x_n) - t(y_n)|)) \leq$$
$$t^{-1}(|t(h(x_1, \ldots, x_n)) - t(h(x_1, \ldots, x_n))|)$$
$$Max(p|t(x_1) - t(y_1)|, \ldots, p|t(x_n) - t(y_n)|) \geq$$
$$|t(h(x_1, \ldots, x_n)) - t(h(x_1, \ldots, x_n))|.$$

■

If T is the Lukasiewicz t-norm and $p = 1$, then (4) is the definition of the kernel aggregation operator introduced in [17].

IV. Conclusions

In this paper Lipschitzian and kernel aggregation operators with respect to the natural T-indistinguishability operator E_T and their powers have been studied.

It has been proved that a t-norm T is E_T-Lipschitzian, and it is also a fuzzy point and a fuzzy map as well.

Quasi-arithmetic means m_t play an important role since they are the more stable aggregation operator with respect to T, meaning that the corresponding n-ary operators $m_{t(n)}$ are $E_T^{\frac{1}{n}}$-Lipschitzian maps.

Lipschitzian and kernel properties are not only interesting for aggregation operators, but also in most of th areas where fuzzy reasoning is present. Therefore, they deserve a deeper study.

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