Technical Report

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Hybrid Linear Sewer Network Model
IRI Technical Report

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Abstract

This technical report presents a novel control-oriented hybrid linear sewer network model. It also provides the mathematical details of the *Mixed Logical Dynamic* (MLD) systems reformulation of the system equations to turn all the involved hybrid/logical statements into linear inequalities by means of the definition of binary variables. Using this reformulation a compact hybrid linear delayed expression is obtained to be used for simulation or optimal control purposes.

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1 Introduction

Physical models of sewer networks implemented in commercial simulators are based in the Saint-Venant equations. These equations are partial differential equations that precisely describe the flow and the water level at every point of the network. The presence of hydraulic structures such as weirs or overflow points is taken into account by means of boundary or interior conditions in the set of PDEs. Thus, the evaluation of such models becomes computationally very demanding and takes long times to be performed. In this context, control-oriented models are developed to come up with control solutions in real-time, which is a very desired feature in a problem that is strongly affected by a disturbance, the rain inflow, which is only predictable for short future horizons.

Early control models of sewer networks for optimal/predictive control date back to [7]. Following the linear approach of this work the *virtual reservoir* model was first presented in [1] and further developed into hybrid linear models in [4, 18, 16, 14]. The *virtual reservoir* is based on modeling entire areas of a city as linear reservoirs with an emptying parameter to be calibrated on-line and is therefore specially suitable for large networks. The improved hybrid linear version of the model was implemented in a software tool allowing for network design and closed-loop control simulation with MOUSE as reported in [6, 4].

Other works are based on nonlinear approximation of the model network elements [9, 10, 11, 19, 20, 5, 12, 13, 17, 21, 3]. These models, which are usually more precise, are more suitable for smaller networks due to the computational effort required for its evaluation.

The model developed in this work has its precedents in the *virtual reservoir* model but instead of simplifying city areas as reservoirs, each sewer between junctions is considered, which allows to explicitly take into account the two main properties of the water transport through sewers: the time delay and the wave attenuation. Other novelties in the model focus on the description of flow over weirs, overflows in sewer junctions and flood runoff in a hybrid linear framework that allows for a good trade-off between computational speed and precision. In a more structural sense, the model aims to be easily implemented and calibrated off-line and fastly modified for on-line calibration purposes, which will be developed in future research.

2 Model overview

The model describes only the flow values of the network, i.e., it does not take into account water levels. It has two main parts: the water transportation and mass balance equations and the weir, overflow and flood runoff equations. The water transportation and mass balance equations are linear equations that take into account delays and flow attenuation in the network sewers. On the other hand, the weir, overflow and flood runoff equations make use of logic/event-based decisions that allow the model to switch between different (linear) behaviors. This part of the model is therefore a hybrid linear model. The logic decisions of this part of the model are described by means of binary variables and linear inequalities following the *Mixed Logical Dynamic* systems approach described in [2].

All the variables and equations are considered in a discrete time setting where variable $t \in \mathbb{Z}$, is used as the discrete time variable with time step Δt. Thus, $x(t)$ means the value of x at time step t or, equivalently, $t\,\Delta t$ seconds after the computation start. To account for the delays in the sewers, Δt is expected to be of the order of 1 minute approximately, although the choice depends on the network properties.

To present the model equations the following notation will be used:

Description	Symbol	Units	Indexing
Flow upstream of each sewer	$q_i^{in}(t)$	m^3/s	$i = 1 \dots n_q$
Flow downstream of each sewer	$q_i^{out}(t)$	m^3/s	$i = 1 \dots n_q$
Volume stored in reservoirs	$v_i(t)$	m^3	$i = 1 \dots n_v$
Flow under gates	$g_i(t)$	m^3/s	$i = 1 \dots n_g$
Flow over weirs	$w_i(t)$	m^3/s	$i = 1 \dots n_w$
Overflows	$f_i(t)$	m^3/s	$i = 1 \dots n_f$
Flood runoff flow	$q_{t_i}(t)$	m^3/s	$i = 1 \dots n_f$
Overflow volume	$v_{t_i}(t)$	m^3/s	$i = 1 \dots n_f$
Rain inflow	$c_i(t)$	m^3/s	$i = 1 \dots n_c$

where n_* is the number of the corresponding elements in the network. The flood runoff flows and the overflows are defined associated with the overflow points, thus sharing the same number of elements n_f.

Overflow points, flood runoff flows and overflow volumes are defined in sewer junctions and therefore their number could be potentially the number of junctions. However, it can be observed from simulation or historical data that in most networks overflows occur only at some particular points, no matter how strong the rain events under consideration. Therefore, it is possible and worthwhile to define overflow variables only at those points to avoid the definition of a lot of useless variables.

Notation

For compact matrix notation, vectors of each kind of variables are defined with the same notation using capital letters, for example, $Q_{in}(t) = (q_1^{in}(t), \dots, q_{n_q}^{in}(t))^\top$.

In the description of each element the subindex will be dropped for clarity whenever possible.

2.1 Hybrid Linear Delayed Systems

Hybrid systems allow modeling systems involving both continuous and binary variables. These systems are defined not only by a set of dynamic equations, but also by a set of inequalities. If the system is properly defined, these inequalities define uniquely the values of the binary variables that are also involved in the dynamic equations [2].

In a delayed system setting both dynamic equations and inequalities involve the system variables at different time steps. In the discrete-time linear case the system reads

$$\begin{aligned} \sum_{i=0}^{T} M_i\, X(t-i) &= m(t), \\ \sum_{i=0}^{T} N_i\, X(t-i) &\leq n(t), \end{aligned} \tag{1}$$

where $t \in \mathbb{Z}$ is the discrete time variable and

$$X(t-i) = (x_1(t-i), \ldots, x_n(t-i))^\top,\ i = 0, \ldots, T,$$

with $x_j(t-i) \in \mathbb{R}$ for a subset of indices $j \in \mathcal{C} \subset \{1, \ldots, n\}$ and $x_j(t-i) \in \{0, 1\}$ for a subset of indices $j \in \mathcal{B} \subset \{1, \ldots, n\}$. Index sets $\mathcal{C}$ and $\mathcal{B}$ are such that $\mathcal{C} \cap \mathcal{B} = \varnothing$ and $\mathcal{C} \cup \mathcal{B} = \{1, \ldots, n\}$. M_i, $i = 0, \ldots, T$, are $n_{eq} \times n$ matrices, N_i, $i = 0, \ldots, T$, are $n_{ineq} \times n$ matrices, $m(t) \in \mathbb{R}^{n_{eq}}$ and $n(t) \in \mathbb{R}^{n_{ineq}}$.

Vectors $X(t-i),\ i = 0, \ldots, T$, include all system variables, making no distinction whether they are either state variables or controlled variables. The influence of any disturbance variable at any time step is included in vectors $m(t)$ and $n(t)$.

2.2 Mixed Logical Dynamical Systems Approach

The *Mixed Logical Dynamical* (MLD) systems is a framework for modeling and control of systems governed by linear dynamics together with switching behaviors arising from logical statements involving the variables of the system [2].

To this end, binary variables describing the truth value of the fulfillment of linear inequalities are defined. Using these variables, any statement constructed using the usual logical operators ('and', 'or', negation and implication) concerning the truth value of linear inequalities can be reformulated as further equalities and inequalities. Continuous variables resulting of the product of a binary variable and a continuous one can also be defined by means of linear inequalities and all the newly defined variables can be used in further linear equalities and inequalities.

Thus, the MLD framework allows to model any system behavior consisting in switching between different linear dynamics depending on the different combinations of the truth value of a set of linear inequalities. MLD systems have been shown to be equivalent to other system modeling formats including *linear complementarity systems*, *extended linear complementarity systems*, *piecewise affine systems*, and *max-min-plus-scaling systems* [8].

3 Sewer Network Model Equations and MLD Reformulation

3.1 Flow model

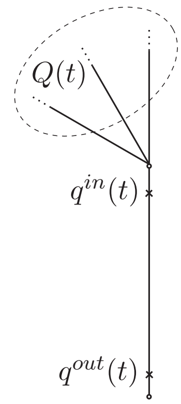

Figure 1: Pipe model diagram.

The basic flow model is based on a linear delayed expression. As mentioned before, two flows for each sewer are defined: $q_{in}(t)$ as the inflow and $q_{out}(t)$ as the outflow. The model is then based on the two following relations:

- Mass balance at each sewer junction:

$$\begin{aligned} q_i^{in}(t) = & \sum_{j=1}^{n_q} a_j^q q_j^{out}(t) + \sum_{j=1}^{n_w} a_j^w w_j(t) + \\ & \sum_{j=1}^{n_f} a_j^f f_j(t) + \sum_{j=1}^{n_c} a_j^{q_t} q_{t_j}(t) + \\ & \sum_{j=1}^{n_g} a_j^g g_j(t) + \sum_{j=1}^{n_c} a_j^c c_j(t), \end{aligned}$$

  where most of the a_j^* will be zeros since a junction does not usually involve more than three or four elements. This formulation will help representing the whole network equations in matrix form.

  If q_i^{in} is the only sewer leaving the junction, the nonzero a_j^* take the value 1 (-1 in the case of weirs or overflows leaving the junction). If there are several sewers leaving the junction they will take a value in $(0,1)$ depending on the proportion of flow that goes to each one. These values are obtained from simulation data.

- Flow routing through sewers accounting for delay and wave attenuation:

$$q_i^{out}(t) = a_i\, q_i^{in}(t - t_i) + (1 - a_i)\, q_i^{in}(t - t_i - 1),$$

  where $0 \le a_i \le 1$ are the attenuation coefficients, $t_i \in \{0, 1, 2, \ldots\}$ the sewer delays. These parameters are obtained from simulation data as the values a_i and t_i that minimize the difference between the outflow computed with the expression and the simulation outflow values.

  Notice that the use of a weighted sum of two inflows at consecutive time steps allows for a good approximation of sewers having a delay which is not a multiple of the simulation time step, which is mostly the case. Also, notice that, eventually, t_i can take the value 0 and then $q_i^{out}(t)$ depends on $q_i^{in}(t)$, that is, at the same time step. This is because for the shorter sewers the transport time can be smaller than the simulation time step and thus the transport in those sewers is computed using the weighted sum of inflows at the current and previous time steps.

The two previous expressions written in matrix form become:

$$\begin{aligned} Q_{in}(t) = A_Q Q_{out}(t) + A_W W(t) + A_F F(t) + \\ A_T Q_T(t) + A_G G(t) + A_C C(t), \end{aligned} \tag{2}$$

$$Q_{out}(t) = A_0 Q_{in}(t) + \sum_{i=1}^{T} A_i Q_{in}(t-i), \tag{3}$$

where matrices A_Q, A_W, A_G, A_C, A_F, A_T are dictated by the the network topology and A_i, $i = 1 \dots T$, are diagonal matrices containing the a_i or $1 - a_i$ coefficients for each sewer, where $T = \max_i\{t_i\}$.

In order to reduce the number of variables in the model, equations (2) and (3) can be combined. By substituting Q_{out} in (2) for its expression in (3) and solving for Q_{in}, the following expression is obtained

$$\boxed{\begin{aligned} Q_{in}(t) = \sum_{i=1}^{T} \tilde{A}_i Q_{in}(t-i) + \tilde{A}_W W(t) + \tilde{A}_G G(t) + \\ \tilde{A}_C C(t) + \tilde{A}_F F(t) + \tilde{A}_T Q_T(t), \end{aligned}} \tag{4}$$

with

$$\begin{aligned} \tilde{A}_i &= (I - A_Q A_0)^{-1} A_Q A_i, \; i = 1 \dots T, \\ \tilde{A}_W &= (I - A_Q A_0)^{-1} A_W, \\ \tilde{A}_G &= (I - A_Q A_0)^{-1} A_G, \\ \tilde{A}_C &= (I - A_Q A_0)^{-1} A_C, \\ \tilde{A}_F &= (I - A_Q A_0)^{-1} A_F, \\ \tilde{A}_T &= (I - A_Q A_0)^{-1} A_T. \end{aligned}$$

3.2 Reservoir model

The reservoir model follows the discrete-time volume equation

$$v(t) = v(t-1) + \Delta t\big(q_{in}(t-1) - q_{out}(t-1)\big),$$

where q_{in} and q_{out} are the total net inflow and outflow to the tank. These flows are outflows from sewers q_i^{out} or are controlled flows under gates g_i. Hence, the matrix expression for this part of the model is

$$\boxed{\begin{aligned} V(t) &= V(t-1) + \Delta t B_Q Q_{out}(t-1) + \Delta t B_G G(t-1) \\ &= V(t-1) + \Delta t B_Q \sum_{i=1}^{T+1} A_{i-1} Q_{in}(t-i) + \Delta t B_G G(t-1), \end{aligned}} \tag{5}$$

where B_Q and B_G are 0-1 matrices selecting the suitable variables and where (3) has been used to obtain the second expression.

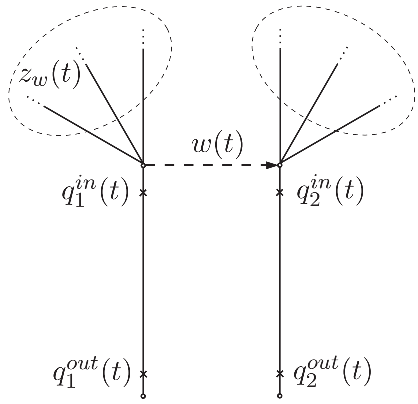

Figure 2: Weir model diagram.

3.3 Weir model

To keep the model structure, the flow over a weir is modeled as a function of the inflow to the junction where the weir is connected. In physical models, the flow over a weir is, in fact, a function of the square root of the difference between the water level and the weir crest level. Using this model would break the desired linear structure we want to use. However the hybrid structure can still be used to come up with a suitable approximation.

From simulation data, a flow value q_w^{max} is determined as the maximum inflow to the junction that does not produce any flow over the weir. Thus, while the inflow is below this threshold, the flow over the weir is zero. For inflow values higher than q_w^{max} we define the flow over the weir as a fraction $0 < a_w < 1$ of the difference between the inflow and q_w^{max}. Mathematically:

$$w(t) = \begin{cases} a_w\,(z_w(t) - q_w^{max}) & ,\text{ if } z_w(t) \geq q_w^{max} \\ 0 & ,\text{ otherwise,} \end{cases}$$

where $z_w(t)$ is the inflow to the junction where weir is connected. The previous expression is equivalent to

$$w(t) = \max\{0, a_w\,(z_w(t) - q_w^{max})\},$$

or

$$w(t) = a_w\,\delta_w(t)\,(z_w(t) - q_w^{max}), \tag{6}$$

where

$$\delta_w(t) = \begin{cases} 1 & ,\text{ if } z_w(t) \geq q_w^{max} \\ 0 & ,\text{ otherwise.} \end{cases} \tag{7}$$

The value of parameter a_w is also determined from simulation data as a the one that minimizes the error between prediction and data. The introduction of this parameter is new feature with respect of previous similar works like [15, 16, 14] and has proven to improve the predictions considerably.

3.3.1 Weir MLD formulation

According to the Mixed Linear Dynamic (MLD) systems approach developed in [2], condition

$$\delta_w(t) = \begin{cases} 1 & ,\text{ if } z_w(t) \geq q_w^{max} \\ 0 & ,\text{ otherwise} \end{cases},$$

and expression

$$w(t) = a_w\,\delta_w(t)\,(z_w(t) - q_w^{max}),$$

are equivalent to the following set of inequalities

$$
\begin{aligned}
z_w(t) - q_w^{max} &\geq m_w(1 - \delta_w(t)),\\
z_w(t) - q_w^{max} &\leq M_w\delta_w(t) + \varepsilon(\delta_w(t) - 1),\\
w(t) &\leq M_w\delta_w(t),\\
w(t) &\geq m_w\delta_w(t),\\
w(t) &\leq a_w\,(z_w(t) - q_w^{max}) - m_w(1 - \delta_w(t)),\\
w(t) &\geq a_w\,(z_w(t) - q_w^{max}) - M_w(1 - \delta_w(t)),
\end{aligned}
$$

where

$$
\begin{aligned}
m_w &= \min a_w\,(z_w(t) - q_w^{max}),\\
M_w &= \max a_w\,(z_w(t) - q_w^{max}).
\end{aligned}
$$

In matrix form

$$
\begin{pmatrix} -1\\ 1\\ 0\\ 0\\ -a_w\\ a_w \end{pmatrix} z_w(t) + \begin{pmatrix} -m_w\\ -M_w - \varepsilon\\ -M_w\\ m_w\\ -m_w\\ M_w \end{pmatrix} \delta_w(t) + \begin{pmatrix} 0\\ 0\\ 1\\ -1\\ 1\\ -1 \end{pmatrix} w(t) + \begin{pmatrix} m_w + q_w^{max}\\ \varepsilon - q_w^{max}\\ 0\\ 0\\ m_w + a_w\,q_w^{max}\\ -M_w - a_w\,q_w^{max} \end{pmatrix} \leq 0,
$$

Defining vectors

$$
p_j^w = \begin{pmatrix} 1\\ 1\\ 0\\ 0\\ -a_{w_j}\\ a_{w_j} \end{pmatrix}, \quad q_j^w = \begin{pmatrix} -m_{w_j}\\ -M_{w_j} - \varepsilon\\ -M_{w_j}\\ m_{w_j}\\ -m_{w_j}\\ M_{w_j} \end{pmatrix}, \quad r^w = \begin{pmatrix} 0\\ 0\\ 1\\ -1\\ 1\\ -1 \end{pmatrix}, \quad s_j^w = \begin{pmatrix} m_{w_j} - q_{w_j}^{max}\\ \varepsilon - q_{w_j}^{max}\\ 0\\ 0\\ m_{w_j} + a_{w_j}\,q_{w_j}^{max}\\ -M_{w_j} - a_{w_j}\,q_{w_j}^{max} \end{pmatrix},
$$

and matrices

$$
\underbrace{\begin{pmatrix} p_1^w & & & \\ & p_2^w & & \\ & & \ddots & \\ & & & p_{nw}^w \end{pmatrix}}_{E_{Z_W}} Z_W(t) + \underbrace{\begin{pmatrix} q_1^w & & & \\ & q_2^w & & \\ & & \ddots & \\ & & & q_{nw}^w \end{pmatrix}}_{E_{\Delta_W}} \Delta_W(t) + \underbrace{\begin{pmatrix} r^w & & & \\ & r^w & & \\ & & \ddots & \\ & & & r^w \end{pmatrix}}_{E_W} W(t) + \underbrace{\begin{pmatrix} s_1^w\\ s_2^w\\ \vdots\\ s_{n_w}^w \end{pmatrix}}_{E_{C_W}} \leq 0, \tag{8}
$$

where

$$
\begin{aligned}
W(t) &= \Big(w_1(t), \ldots, w_{n_w}(t)\Big)^\top,\\
Z_W(t) &= \Big(z_{w_1}(t), \ldots, z_{w_{n_w}}(t)\Big)^\top,\\
\Delta_W(t) &= \Big(\delta_{w_1}(t), \ldots, \delta_{w_{n_w}}(t)\Big)^\top,
\end{aligned}
$$

and n_w the number of weirs in the network, the set of inequalities defining the the weir flow variables W and Δ_W has the form:

$$
\boxed{E_{Z_W} Z_W(t) + E_{\Delta_W}\Delta_W(t) + E_W W(t) + E_{C_W} \leq 0} \tag{9}
$$

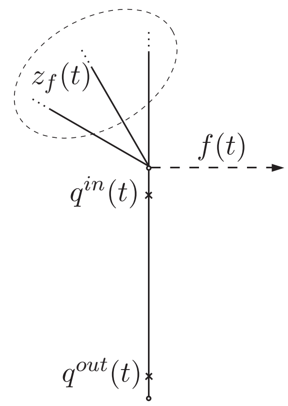

Figure 3: Overflow model diagram.

3.4 Overflow model

The overflow flow is defined in the same way as the flow over a weir:

$$f(t) = \max\{0, a_f(z_f(t) - q_f^{max})\} = a_f\, \delta_f(t)\, (z_f(t) - q_f^{max}), \tag{10}$$

with

$$\delta_f(t) = \begin{cases} 1 & , \text{ if } z_f(t) \geq q_f^{max} \\ 0 & , \text{ otherwise,} \end{cases} \tag{11}$$

with $z_f(t)$ the flow entering the junction where the overflow is considered to potentially occur.

3.5 Overflow MLD formulation

Since the overflow model is analogous to the weir one, only a few notation details are given. Starting from the definition of the overflow variable $f(t)$ and auxiliary boolean variable $\delta_f(t)$

$$f(t) = \max\{0, a_f(z_f(t) - q_f^{max})\} = a_f\, \delta_f(t)\, (z_f(t) - q_f^{max}),$$

with

$$\delta_f(t) = \begin{cases} 1 & , \text{ if } z_f(t) \geq q_f^{max} \\ 0 & , \text{ otherwise} \end{cases} ,$$

matrices E_{Z_F}, E_{Δ_F}, E_F and E_{C_F} are defined in the same way as E_{Z_W}, E_{Δ_W}, E_W and E_{C_W}, giving

$$\boxed{E_{Z_F}\, Z_F(t) + E_{\Delta_F}\, \Delta_F(t) + E_F F(t) + E_{C_F} \leq 0} \tag{12}$$

3.6 Flood runoff model

The flood runoff model is also a a novel contribution of the proposed modeling approach. In previous works, the overflow volume was considered to be leaving the network and not re-entering it anymore. In this new approach, the overflow volume is kept in a fictional reservoir which returns it to the network as soon as the overflow at the current junction finishes.

Using the previously defined overflow variable $f(t)$ a simple volume model is then used to keep track of the volume:

$$v_t(t) = v_t(t-1) + \Delta t\big(f(t-1) - q_t(t-1)\big),$$

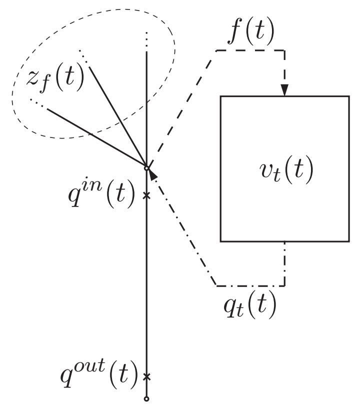

Figure 4: The new overflow model keeps track of the volume released to the enviroment and lets it re-enter the network when possible.

where $q_t(t)$ is the flow leaving the fictional reservoir and defined as follows:

$$q_t(t) = \min\left\{\max\left\{0, b_f\left(q_f^{max} - z_f(t)\right)\right\}, \frac{v_t(t)}{\Delta t}\right\}.$$

The reservoir can never provide more flow than that which would empty it in a single time step, thus the minimum is applied. If there is enough volume available and setting $b_f = 1$ the reservoir would provide the flow that together with the one entering the junction would make the outgoing pipe run full (zero in case it already is). The simulator works in a similar way but taking into account the shape of the fictional reservoir making its emptying rates vary depending on this shape. Parameter $b_f \in (0, 1]$ was introduced later to better approximate this phenomenon.

The fictional tank equations are expressed in matrix form as

$$\boxed{V_T(t) = V_T(t-1) + \Delta t\big(F(t-1) - Q_T(t-1)\big)} \tag{13}$$

3.6.1 Flood runoff MLD formulation

The flood runoff definition equation involving a maximum function within a minimum one may suggest that several boolean variables will be needed to reformulate it.

$$q_t(t) = \min\left\{\max\left\{0, b_f\left(q_f^{max} - z_f(t)\right)\right\}, \frac{v_t(t)}{\Delta t}\right\}, \tag{14}$$

However, making use of the already defined variables $f(t)$, only one such variable will be needed for each flood runoff one. First, recall the definition of $f(t)$

$$f(t) = \max\{0, a_f(z_f(t) - q_f^{max})\} = a_f\ \max\{0, z_f(t) - q_f^{max}\}.$$

Making use of the identity $\max\{0, a-b\} - \max\{0, b-a\} = a - b$

$$\max\{0, q_f^{max} - z_f(t)\} - \max\left\{0, z_f(t) - q_f^{max}\right\} = q_f^{max} - z_f(t),$$

the maximum function in (14) can be replaced by

$$\max\{0, q_f^{max} - z_f(t)\} = q_f^{max} - z_f(t) + \frac{f(t)}{a_f}.$$

In a last step, another boolean variable can be avoided by forcing one of the two arguments of minimum function to be zero:

$$
\begin{aligned}
q_t(t) &= \min\left\{ b_f \left(q_f^{max} - z_f(t) + \frac{f(t)}{a_f} \right), \frac{v_t(t)}{\Delta t} \right\} \\
&= \min\left\{ b_f \left(q_f^{max} - z_f(t) + \frac{f(t)}{a_f} \right) - \frac{v_t(t)}{\Delta t}, 0 \right\} + \frac{v_t(t)}{\Delta t}.
\end{aligned}
$$

Finally,

$$
q_t(t) = q_{aux}(t) + \frac{v_t(t)}{\Delta t},
$$

with

$$
q_{aux}(t) = \delta_t(t)\, f_{aux}(t),
$$

and

$$
f_{aux}(t) = b_f \left(q_f^{max} - z_f(t) + \frac{f(t)}{a_f} \right) - \frac{v_t(t)}{\Delta t},
$$

$$
\delta_t(t) = \begin{cases} 1 & , \text{ if } f_{aux}(t) \leq 0 \\ 0 & , \text{ otherwise} \end{cases} .
$$

Now, the MLD formulation of the previous expressions becomes

$$
\begin{aligned}
& f_{aux}(t) \leq M_t(1 - \delta_t(t)), \\
& f_{aux}(t) \geq m_t \delta_t(t) + \varepsilon\,(1 - \delta_t(t)), \\
& q_{aux}(t) \leq M_t \delta_t(t), \\
& q_{aux}(t) \geq m_t \delta_t(t), \\
& q_{aux}(t) \leq f_{aux(t)} - m_t(1 - \delta_t(t)), \\
& q_{aux}(t) \geq f_{aux(t)} - M_t(1 - \delta_t(t)).
\end{aligned}
$$

Expanding the terms in f_{aux} and regrouping

$$
\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} q_{aux}(t) + \begin{pmatrix} -b_f \\ b_f \\ 0 \\ 0 \\ b_f \\ -b_f \end{pmatrix} z_f(t) + \begin{pmatrix} b_f/a_f \\ -b_f/a_f \\ 0 \\ 0 \\ -b_f/a_f \\ b_f/a_f \end{pmatrix} f(t) +
$$

$$
\frac{1}{\Delta t} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} v_t(t) + \begin{pmatrix} M_t \\ m_t - \varepsilon \\ -M_t \\ m_t \\ -m_t \\ M_t \end{pmatrix} \delta_t(t) + \begin{pmatrix} b_f q_f^{max} - M_t \\ -b_f q_f^{max} + \varepsilon \\ 0 \\ 0 \\ -b_f q_f^{max} + m_t \\ b_f q_f^{max} - M_t \end{pmatrix} \leq 0.
$$

Defining vectors

$$
\begin{aligned}
V_T(t) &= \left(v_{t_1}(t), \ldots, v_{t_{n_f}}(t) \right)^\top, \\
Q_T(t) &= \left(q_{t_1}(t), \ldots, q_{t_{n_f}}(t) \right)^\top, \\
\Delta_T(t) &= \left(\delta_{t_1}(t), \ldots, \delta_{t_{n_f}}(t) \right)^\top, \\
Q_{aux}(t) &= \left(q_{aux_1}(t), \ldots, q_{aux_{n_f}}(t) \right)^\top,
\end{aligned}
$$

and repeating the set of inequalities in a block-diagonal way analogous to (8) with the corresponding parameters a_{f_j}, b_{f_j}, $q_{f_j}^{max}$, m_{t_j} and M_{t_j}, the matrix expression for the set of inequalities defining the flood runoff variables $Q_t(t)$ for the entire network is obtained

$$E_{Q_T} Q_{aux}(t) + E_{Z_T} Z_F(t) + E_{F_T} F(t) + E_{V_T} V_T(t) + E_{\Delta_T} \Delta_T + E_{C_T} \leq 0.$$

Finally, in order to reduce the number of variables to be used in the problem, variables $Q_{aux}(t)$ are substituted for their expression in terms of $Q_T(t)$ and $V_T(t)$

$$Q_{aux}(t) = Q_T(t) - \frac{1}{\Delta t} V_T(t),$$

to get

$$\boxed{E_{Q_T} Q_T(t) + E_{Z_T} Z_F(t) + E_{F_T} F(t) + \left(E_{V_T} - \frac{1}{\Delta t} E_{Q_T} \right) V_T(t) + E_{\Delta_T} \Delta_T + E_{C_T} \leq 0} \tag{15}$$

3.7 Inflow to Nodes

In (9), (12) and (15) the inflows to each weir $Z_W(t)$ and overflow junction $Z_F(t)$ have not been defined in terms of other elements flows in order to keep the notation clear and compact. However, it will be necessary to obtain the hybrid linear expression (4) for the system in terms of a reduced set of variables.

To this end, let variable $Z(t)$ be defined as the inflow to each sewer

$$\begin{aligned} Z(t) &= A_Q Q_{out}(t) + A_W^+ W(t) + A_G^+ G(t) + A_C C(t) \\ &= \sum_{i=0}^{T} A_Q A_i Q_{in}(t-i) + A_W^+ W(t) + A_G^+ G(t) + A_C C(t), \end{aligned} \tag{16}$$

where A_W^+ and A_G^+ collect only the positive terms in A_W and A_G respectively.

The components of $Z(t)$ corresponding to inflows to nodes connected to a weir can be selected using a matrix S_W defined as follows:

$$(S_W)_{ij} = \begin{cases} 1, & \text{if } w_i \text{ is connected upstream to the same junction as } q_j \\ 0, & \text{otherwise.} \end{cases}$$

A matrix S_F is defined in the same way, to select the components of $Z(t)$ corresponding to links connected upstream to a junction where overflow is considered to be possible.

Now, using S_W and S_F inflows $Z_W(t)$ and $Z_F(t)$ can be defined

$$\begin{aligned} Z_W(t) &= S_W Z(t), \\ Z_F(t) &= S_F Z(t). \end{aligned}$$

3.8 Hybrid Linear Delayed System Formulation

To obtain the Hybrid Linear Delayed System expression (1)

$$\begin{aligned} \sum_{i=0}^{T} M_i \, X(t-i) &= m(t), \\ \sum_{i=0}^{T} N_i \, X(t-i) &\leq n(t), \end{aligned}$$

for the sewer network model the first step is to define the vector of unknown variables. In this case, this vector includes all the systems variables at the current time step except the disturbance (rain inflow) ones:

$$X(t) = \left(V(t)^\top, Q_{in}(t)^\top, W(t)^\top, \Delta_W(t)^\top, F(t)^\top, \Delta_F(t)^\top, V_T(t)^\top, Q_T(t)^\top, \Delta_T(t)^\top, G(t)^\top\right)^\top,$$

Now equations (4), (5) and (13) and inequalities (9), (12) and (15) can be expressed in compact matrix form in terms of $X(t)$ as

$$\begin{aligned} M_0\, X(t) &= b_{eq}(t) \\ N_0\, X(t) &\leq b_{ineq}(t) \end{aligned} \tag{17}$$

with

$$M_0 = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & -\tilde{A}_W & 0 & -\tilde{A}_F & 0 & 0 & -\tilde{A}_T & 0 & -\tilde{A}_G \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & 0 \end{pmatrix},$$

$$b_{eq}(t) = \begin{pmatrix} V(t-1)+\sum_{i=1}^{T+1} \Delta t B_Q A_{i-1} Q_{in}(t-i)+\Delta t B_G G(t-1) \\ \sum_{i=1}^{T} \tilde{A}_i\, Q_{in}(t-i)+\tilde{A}_C\, C(t) \\ V_T(t-1+\Delta t F(t-1)-\Delta t Q_T(t-1), \end{pmatrix}$$

and

$$N_0 = \begin{pmatrix} 0 & E_{Z_W} S_W A_Q A_0 & E_W+E_{Z_W} S_W A_W^+ & E_{\Delta_W} & 0 & 0 & 0 & 0 & 0 & E_{Z_W} S_W A_G^+ \\ 0 & E_{Z_F} S_F A_Q A_0 & E_{Z_F} S_F A_W^+ & 0 & E_F & E_{\Delta_F} & 0 & 0 & 0 & E_{Z_F} S_F A_G^+ \\ 0 & E_{Z_T} S_F A_Q A_0 & E_{Z_T} S_F A_W^+ & 0 & E_{F_T} & 0 & E_{V_T} - \frac{1}{\Delta t} E_{Q_T} & E_{Q_T} & E_{\Delta_T} & E_{Z_T} S_F A_G^+ \end{pmatrix},$$

$$b_{ineq}(t) = -\begin{pmatrix} E_{Z_W} S_W \left(\sum_{i=1}^{T} A_Q A_i\, Q_{in}(t-i)+A_C\, C(t)\right)+E_{C_W} \\ E_{Z_F} S_F \left(\sum_{i=1}^{T} A_Q A_i\, Q_{in}(t-i)+A_C\, C(t)\right)+E_{C_F} \\ E_{Z_T} S_F \left(\sum_{i=1}^{T} A_Q A_i\, Q_{in}(t-i)+A_C\, C(t)\right)+E_{C_T} \end{pmatrix}.$$

Finally, to come up with an expression like (4) the left hand-sides of (17) are to be expressed in terms of the system variables at previous time steps $X(t-1), \ldots, X(t-T)$:

$$b_{eq}(t) = -\sum_{i=1}^{T+1} M_i X(t-i) + m(t),$$

$$b_{ineq}(t) = -\sum_{i=1}^{T} N_i X(t-i) + n(t),$$

with

$$M_1 = -\begin{pmatrix} I & \Delta t B_Q A_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta t B_G \\ 0 & \tilde{A}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Delta t I & 0 & I & -\Delta t I & 0 & 0 \end{pmatrix},$$

$$M_i = -\begin{pmatrix} 0 & \Delta t B_Q A_{i-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{A}_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad i = 2 \ldots T,$$

$$M_{T+1} = -\begin{pmatrix} 0 & \Delta t B_Q A_T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$m(t) = \begin{pmatrix} 0 \\ \tilde{A}_C C(t) \\ 0 \end{pmatrix}.$$

and

$$N_i = \begin{pmatrix} 0 & E_{Z_W} S_W A_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E_{Z_F} S_F A_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & E_{Z_T} S_F A_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad i = 1 \ldots T,$$

$$n(t) = -\begin{pmatrix} E_{Z_W} S_W A_C C(t)+E_{C_W} \\ E_{Z_F} S_F A_C C(t)+E_{C_F} \\ E_{Z_T} S_F A_C C(t)+E_{C_T} \end{pmatrix}.$$

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