Corrigendum to ‘Algebraic characterizations of regularity properties in bipartite graphs’ [European Journal of Combinatorics 34 (2013) 1223–1231] *

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The authors regret a flaw in the proof of a result in our paper [1] cited in the title. They would like to apologize for any inconvenience caused. This corrigendum provides corrected statements and proofs of such a result. (For theoretical background on the subject, see, for instance, Cvetković, Doob, and Sachs [5], Biggs [2], Brouwer, Cohen, and Neumaier [3], Brouwer and Haemers [4], Van Dam, Koolen, and Tanaka [6], Fiol [7], and Godsil and Royle [8].)

The result we are referring to is Theorem 4.2(a), where it was stated that a regular bipartite graph Γ with adjacency matrix $\boldsymbol{A}$, $d+1$ different eigenvalues, diameter $D=d$, and predistance polynomial p_{d-2} is distance-regular if and only if

$$\boldsymbol{A}_{d-2} = p_{d-2}(\boldsymbol{A}). \tag{1}$$

Then a part of the proof goes as follows: “Under the assumption that $\mathrm{dist}(u,v)=i$ and d have the same parity, and if $i<d-2$, it follows that $(\boldsymbol{A}p_{d-1}(\boldsymbol{A}))_{uv} = \sum_{w\in V} a_{uw}(p_{d-1}(\boldsymbol{A}))_{wv}$ $= \sum_{w\in\Gamma(u)}(p_{d-1}(\boldsymbol{A}))_{wv} = 0$, since $\mathrm{dist}(v,w)$ and i have distinct parity.” But, if fact, to reach such a conclusion, we need that $\mathrm{dist}(v,w)$ and $d-1$ have distinct parity, which is not the case. Anyway, to our knowledge it seems unclear whether the result is really false. So far we have no counterexample, because it seems that non-distance-regular graphs are hard to satisfy the condition (1). Thus, we could leave it as an open problem.

What we managed to prove is that the result holds if the former condition (a) in Theorem 4.2 becomes either ($a1$) or ($a2$) (see below). Then, the theorem is now stated and proved as follows.

*DOI of original article: 10.1016/j.ejc.2013.05.006
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Theorem 4.2 *A regular bipartite graph Γ with diameter $D = d$, idempotent $\boldsymbol{E}_1$, and predistance polynomials $p_0, \dots, p_d$ is distance-regular if and only if any of the following conditions holds:*

$(a1)$ $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ for $i = d-3, d-2$ $\quad (d \geq 3)$.

$(a2)$ $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ for $i = d-4, d-2$ $\quad (d \geq 4)$.

(b) $\boldsymbol{E}_1 \in \mathcal{D}$.

(c) $a_{uv}^{(\ell)} = a_i^{(\ell)}$ for $\ell = i \leq D-2$.

Proof. We only prove sufficiency because necessity is straightforward.

$(a1)$ From Theorem 4.1(a), it suffices to prove that $p_{d-1}(\boldsymbol{A}) = \boldsymbol{A}_{d-1}$. Let u, v be two vertices at distance $\operatorname{dist}(u,v) = i$. First notice that, if i and j have distinct parity, then $(p_j(\boldsymbol{A}))_{uv} = 0$. In particular, this is the case when $i = d-1$ and $j = d$, so that $(p_{d-1}(\boldsymbol{A}))_{uv} = (H(\boldsymbol{A}))_{uv} = 1$. Moreover, if $i = d-3$ we have $(p_{d-3}(\boldsymbol{A}))_{uv} + (p_{d-1}(\boldsymbol{A}))_{uv} = 1$, where, from the hypothesis, $(p_{d-3}(\boldsymbol{A}))_{uv} = 1$. Hence, $(p_{d-1}(\boldsymbol{A}))_{uv} = 0$. Thus, the only case left is when $i \leq d-5$ and i has the same parity as $d-1$. In this case the two-term recurrence $xp_j = \beta_{j-1}p_{j-1} + \gamma_{j+1}p_{j+1}$, for $j = 0, \dots, d$, satisfied by the predistance polynomials (of a bipartite graph) yields for $j = d-2$

$$(\boldsymbol{A}p_{d-2}(\boldsymbol{A}))_{uv} = \beta_{d-3}(\boldsymbol{A}_{d-3})_{uv} + \gamma_{d-1}(p_{d-1}(\boldsymbol{A}))_{uv} = \gamma_{d-1}(p_{d-1}(\boldsymbol{A}))_{uv},$$

with first term, using again the hypothesis,

$$(\boldsymbol{A}p_{d-2}(\boldsymbol{A}))_{uv} = \sum_{w \in V} a_{uw}(\boldsymbol{A}_{d-2})_{wv} = \sum_{w \in \Gamma(u)} (\boldsymbol{A}_{d-2})_{wv} = 0,$$

since $\operatorname{dist}(v,w) \leq d-4$. Consequently, as $\gamma_{d-1} \neq 0$, $(p_{d-1}(\boldsymbol{A}))_{uv} = 0$ and $p_{d-1}(\boldsymbol{A}) = \boldsymbol{A}_{d-1}$, as claimed.

$(a2)$ Now, from Theorem 4.1(a), it suffices to prove that $p_d(\boldsymbol{A}) = \boldsymbol{A}_d$. Then the proof is similar to the previous one. Indeed, with the same notation as before, if $i = d$, $(p_d(\boldsymbol{A}))_{uv} = (H(\boldsymbol{A}))_{uv} = 1$, and if i and d have distinct parity, $(p_d(\boldsymbol{A}))_{uv} = 0$. The cases $i = d-2, d-4$ are proved again by using that $p_0 + \cdots + p_d = H$. Then, the only case left is when $i \leq d-6$ and i and d have the same parity. Now, by applying two times the above two-term recurrence of the predistance polynomials, we have

$$x^2 p_{d-2} = \beta_{d-3}\beta_{d-4}p_{d-4} + (\beta_{d-3}\gamma_{d-2} + \gamma_{d-1}\beta_{d-2})p_{d-2} + \gamma_{d-1}\gamma_d p_d.$$

Hence, from the hypothesis,

$$\begin{aligned}(\boldsymbol{A}^2\boldsymbol{A}_{d-2})_{uv} &= \beta_{d-3}\beta_{d-4}(\boldsymbol{A}_{d-4})_{uv} + (\beta_{d-3}\gamma_{d-2} + \gamma_{d-1}\beta_{d-2})((\boldsymbol{A})_{d-2})_{uv} + \gamma_{d-1}\gamma_d(p_d(\boldsymbol{A}))_{uv} \\ &= \gamma_{d-1}\gamma_d(p_d(\boldsymbol{A}))_{uv},\end{aligned}$$

where
$$(\boldsymbol{A}^2\boldsymbol{A}_{d-2})_{uv} = \sum_{w\in V}(\boldsymbol{A}^2)_{uw}(\boldsymbol{A}_{d-2})_{wv} = \sum_{\mathrm{dist}(w,u)\le 2}(\boldsymbol{A}_{d-2})_{wv} = 0,$$
since $\mathrm{dist}(w,v) \le d-4$. Then, as $\gamma_{d-1}\gamma_d \neq 0$, $(p_d(\boldsymbol{A}))_{uv} = 0$ and $p_d(\boldsymbol{A}) = \boldsymbol{A}_d$, as required.

For the proofs of (b) and (c) see the paper cited. □

Under the new hypotheses $(a1)$ and $(a2)$, Corollary 4.3 still holds. However, the hypotheses for Theorem 4.5 to hold are now

$$\overline{\delta}_i = \frac{p_i(\lambda_0)}{\omega_i^2[\overline{a}_i^{(i)}]^2},$$
for either $i = d-3, d-2$ or $i = d-4, d-2$.

Acknowledgments. We would like to thank Guang-Siang Lee for bringing the flaw in Theorem 4.2 of the former paper to our attention. We also thank the anonymous referee of this note for helpful comments.
Research supported by the Ministerio de Educación y Ciencia (Spain) and the European Regional Development Fund under project MTM2011-28800-C02-01, and by the Catalan Research Council under project 2009SGR1387.

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