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NON-INTEGRABILITY OF SOME HAMILTONIANS WITH RATIONAL POTENTIALS

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Abstract. In this paper we give a mechanism to compute the families of classical hamiltonians of two degrees of freedom with an invariant plane and normal variational equations of Hill-Schrödinger type selected in a suitable way. In particular we deeply study the case of these equations with polynomial or trigonometrical potentials, analyzing their integrability in the Picard-Vessiot sense using Kovacic's algorithm and introducing an algebraic method (algebrization) that transforms equations with transcendental coefficients in equations with rational coefficients without changing the Galoisian structure of the equation. We compute all Galois groups of Hill-Schrödinger type equations with polynomial and trigonometric (Mathieu equation) potentials, obtaining Galoisian obstructions to integrability of hamiltonian systems by means of meromorphic or rational first integrals via Morales-Ramis theory.

1. **Introduction.** In a joint work of Simó with J. Morales-Ruiz the integrability of families of two degrees of freedom potentials with an invariant plane $\Gamma = \{x_2 = y_2 = 0\}$ and normal variational equations of Lamé type along generic curves in Γ was studied (see [13]). In such computations they used systematically differential equations that satisfied the coefficients of the Lamé equation. We present here a generalization of the Morales-Simó method to list the families of hamiltonians with invariant plane Γ and the normal variational equations (the NVEs) selected in a suitable way. As motivation of this general method, we give some particular cases such as the Mathieu equations, the Hill-Schrödinger equations with polynomial potential of odd degree, the quantum harmonic oscillator, and other examples. A similar approach to this inverse problem has been studied by Baider, Churchill and Rod [3].

The use of techniques of Differential Galois theory, such as Kovacic's algorithm (see Appendix A), to determine the non-integrability of hamiltonian systems, appeared independently for first time in [10, 12] and [5], followed by [2], [6] and [13]. A common limitation presented in these works is that they only analyzed cases of fuchsian monodromy groups, avoiding cases of irregular singularities of linear

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differential equations. The case of the NVEs with irregular singularities can be approached from the Morales-Ramis [14, 15] framework.

Along this paper we will consider the NVEs with irregular singularities. To study those families of linear differential equations we use Kovacic's algorithm for equations with rational coefficients. In particular we give a complete description of Galois groups of Hill-Schrödinger type equations with polynomial potential (Theorem 2.5).

We develop a new method to transform a linear differential equations with transcendental coefficients in its algebraic form (differential equation with rational coefficient). This method is called *algebrization* and is based in the concept of *hamiltonian change of variables* (See Section 2.1). This change of variables comes from a solution of a classical hamiltonian of one degree of freedom. We characterize equations that can be algebrized in such way. We prove the following result.

Algebrization algorithm. *The differential equation $\ddot{y} = r(t)y$ is algebrizable through a hamiltonian change of variable $x = x(t)$ if and only if there exists f, α such that $\frac{\alpha'}{\alpha}$, $\frac{f}{\alpha} \in \mathbb{C}(x)$, where $f(x(t)) = r(t)$, $\alpha(x) = 2(H - V(x)) = \dot{x}^2$. Furthermore, the algebraic form of the equation $\ddot{y} = r(t)y$ is*

$$y'' + \frac{1}{2}\frac{\alpha'}{\alpha}y' - \frac{f}{\alpha}y = 0.$$

Once we get the complete study of linearized equations, we apply one of the Morales-Ramis theorems and we obtain the following results on the non-integrability of those hamiltonians for generic values of the parameters.

Non-integrability results. *Let H be a hamiltonian system of two degrees of freedom given by $H = T + V$. If the potential V is written such as follows:*

1. $V = \frac{\lambda_4}{(\lambda_2 + 2\lambda_3 x_1)^2} + \lambda_0 - \lambda_1 x_2^2 - \lambda_2 x_1 x_2^2 - \lambda_3 x_1^2 x_2^2 + \beta(x_1, x_2)x_2^3$, $\lambda_3 \neq 0$,
2. $V = \lambda_0 + Q(x_1)x_2^2 + \beta(x_1, x_2)x_2^3$, being $Q(x_1)$ a non-constant polynomial,
3. $V = \mu_0 + \mu_1 x_1 + \frac{\omega^2 x_1^2}{2} - \lambda_0 x_2^2 - \lambda_1 x_1 x_2^2 + \beta(x_1, x_2)x_2^3$, $\omega \neq 0$,
4. $V = \mu_0 + \frac{\mu_1}{(\lambda_1 + 2\lambda_2 x_1)^2} + \frac{\lambda_1 \omega^2 x_1}{8\lambda_2} + \frac{\omega^2 x_1^2}{8} - \lambda_0 x_2^2 - \lambda_1 x_1 x_2^2 - \lambda_2 x_1^2 x_2^2 + \beta(x_1, x_2)x_2^3$, $\omega \cdot \lambda_2 \neq 0$,

being ω, λ_i, μ_i complex numbers and $\beta(x_1, x_2)$ an analytic function defined in some neighborhood of $\{x_2 = 0\}$, then the hamiltonian system X_H does not admit an additional rational first integral.

Corollary. *Every integrable (by rational functions) hamiltonian system with polynomial potential, constant on the invariant plane $\Gamma = \{x_2 = y_2 = 0\}$, can be written in the following form*

$$V = Q_1(x_1, x_2)x_2^3 + \lambda_1 x_2^2 + \lambda_0, \quad \lambda_0, \lambda_1 \in \mathbb{C}.$$

We note that we can fall in the case of homogeneous polynomial potentials when $\beta(x_1, x_2)$ is a polynomial and some values of the parameters are satisfied. Such cases had been deeply studied in [9, 17].

1.1. **Picard-Vessiot theory.** The Picard-Vessiot theory is the Galois theory of linear differential equations. In the classical Galois theory, the main object is a group of permutations of the roots, while in the Picard-Vessiot theory it is a linear algebraic group. For polynomial equations we want a solution in terms of radicals.

From classical Galois theory it is well known that this is possible if and only if the Galois group is a solvable group.

An analogous situation holds for linear homogeneous differential equations (see [19]). The following definition is true in general dimension, but for simplicity we are restricting ourselves to matrices 2×2.

Definition 1.1. An algebraic group of matrices 2×2 is a subgroup $G \subset GL(2, \mathbb{C})$, defined by algebraic equations in its matrix elements. That is, there exists a set of polynomials

$$\{P_i(x_{11}, x_{12}, x_{21}, x_{22})\}_{i \in I},$$

such that

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \in G \quad \Leftrightarrow \quad \forall i \in I, P_i(x_{11}, x_{12}, x_{21}, x_{22}) = 0.$$

In this case we say that G is an algebraic manifold endowed with a group structure. From now on we will only consider linear differential equations of second order, that is,

$$y'' + ay' + by = 0, \quad a, b \in \mathbb{C}(x).$$

Suppose that $\{y_1, y_2\}$ is a fundamental system of solutions of the differential equation. This means that y_1 and y_2 are linearly independent over $\mathbb{C}$ and every solution is a linear combination of y_1 and y_2. Let $L = \mathbb{C}(x)\langle y_1, y_2\rangle = \mathbb{C}(x)(y_1, y_2, y_1', y_2')$ (the smallest differential field containing to $\mathbb{C}(x)$ and $\{y_1, y_2\}$).

Definition 1.2 (Differential Galois Group)**.** The group of all differential automorphisms of L over $\mathbb{C}(x)$ is called the *Galois group* of L over $\mathbb{C}(x)$ and is denoted by $Gal(L/\mathbb{C}(x))$ or also by $Gal^L_{\mathbb{C}(x)}$. This means that for $\sigma\colon L \to L$, $\sigma(a') = \sigma'(a)$ and $\forall a \in \mathbb{C}(x)$, $\sigma(a) = a$.

If $\sigma \in Gal(L/\mathbb{C}(x))$ then $\{\sigma y_1, \sigma y_2\}$ is another fundamental system of solutions of the linear differential equation. Hence there exists a matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}),$$

such that

$$\sigma \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sigma y_1 \\ \sigma y_2 \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

This defines a faithful representation $Gal(L/\mathbb{C}(x)) \to GL(2, \mathbb{C})$ and it is possible to consider $Gal(L/\mathbb{C}(x))$ as a subgroup of $GL(2, \mathbb{C})$. It depends on the choice of the fundamental system $\{y_1, y_2\}$, but only up to conjugacy.

One of the fundamental results of the Picard-Vessiot theory is the following theorem.

Theorem 1.3. *The Galois group $G = Gal(L/\mathbb{C}(x))$ is an algebraic subgroup of $GL(2, \mathbb{C})$.*

Now we are interested in the reduced linear differential equation (the RLDE)

$$\xi'' = r\xi, \quad r \in \mathbb{C}(x). \tag{1}$$

We recall that equation (1) can be obtained from the general second order linear differential equation

$$y'' + ay' + by = 0, \quad a, b \in \mathbb{C}(x),$$

through the change of variable

$$y = e^{-\frac{1}{2}\int a}\xi, \quad r = \frac{a^2}{4} + \frac{a'}{2} - b.$$

On the other hand, through the change of variable $v = \xi'/\xi$ we get the associated Riccati equation to equation (1)

$$v' = r - v^2, \quad v = \frac{\xi'}{\xi}. \tag{2}$$

For the differential equation (1), $G = Gal(G/\mathbb{C}(x))$ is an algebraic subgroup of $SL(2,\mathbb{C})$.

Recall that an algebraic group G has a unique connected normal algebraic subgroup G^0 of finite index. This means that the identity component G^0 is the largest connected algebraic subgroup of G containing the identity.

Definition 1.4. Let F be a differential extension of $\mathbb{C}(x)$, and let be η solution of the differential equation

$$y'' + ay' + by = 0, \quad a, b \in F$$

1. η is *algebraic* over F if η satisfies a polynomial equation with coefficients in F, that is, η is an algebraic function of one variable.
2. η is *primitive* over F if $\eta' \in F$, that is, $\eta = \int f$ for some $f \in F$.
3. η is *exponential* over F if $\eta'/\eta \in F$, that is, $\eta = e^{\int f}$ for some $f \in F$.

Definition 1.5. A solution η of the previous differential equation is said to be *Liouvillian* over F if there is a tower of differential fields

$$F = F_0 \subset F_1 \subset ... \subset F_m = L,$$

with $\eta \in L$ and for each $i = 1, ..., m$, $F_i = F_{i-1}(\eta_i)$ with η_i either algebraic, primitive, or exponential over F_{i-1}. In this case we say that the differential equation is integrable.

Thus, a Liouvillian solution is built up using algebraic functions, integrals and exponentials. In the case $F = \mathbb{C}(x)$ we get, for instance logarithmic and trigonometric functions, but not special functions such as Airy functions.

We recall that a group G is called solvable if and only if there exists a chain of normal subgroups

$$e = G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n = G$$

such that the quotient G_i/G_j is abelian for all $n \geq i \geq j \geq 0$.

Theorem 1.6. *The equation (1) is integrable (has Liouvillian solutions) if and only if for $G = Gal(L/\mathbb{C}(x))$, the identity component G^0 is solvable.*

Using Theorem 1.6, Kovacic in 1986 introduced an algorithm to solve the differential equation (1) and showed that (1) is integrable if and only if one solution of the equation (2) is one of the following types of functions: rational function (case 1), non-rational root of some polynomial of degree two (case 2) root of some polynomial (irreducible over $\mathbb{C}(x)$) of degree 4, 6, or 12 (case 3) (see Appendix A). Based in Kovacic's algorithm we have the following *key* result.

The Galois group of the RLDE (1) with $r = Q_k(x)$ (a polynomial of degree $k > 0$) is a non-abelian connected group.

For the full, details and proof see Section 2.2.

1.2. **Morales-Ramis theory.** Morales-Ramis theory relates the integrability of hamiltonian systems with the integrability of linear differential equations (see [14, 15] and see also [11]). In such approach the linearization (variational equations) of hamiltonian systems along some known particular solution is studied. If the hamiltonian system is integrable, then we expect that the linearized equation has good properties in the sense of Picard-Vessiot theory. To be more precise, for integrable hamiltonian systems, the Galois group of the linearized equation must be virtually abelian. This gives us the best non-integrability criterion known so far for hamiltonian systems. This approach has been extended to higher order variational equations in [16].

1.2.1. *Integrability of hamiltonian systems.* A symplectic manifold (real or complex), M_{2n} is a $2n$-dimensional manifold, provided with a non-degenerate closed 2-form ω_2. This closed 2-form gives us a natural isomorphism between vector bundles, $\flat\colon TM \to T^*M$. Given a function H on M, there is an unique vector field X_H such that,

$$\flat(X_H) = dH$$

this is *the hamiltonian vector field of* H. Furthermore, it has a structure of *Poisson algebra* over the ring of differentiable functions of M_{2n} by defining:

$$\{H, F\} := X_H F.$$

We say that H and F are *in involution* if and only if $\{H, F\} = 0$. From our definition, it is obvious that F is a *first integral* of X_H if and only if H and F are in involution. In particular H is always a first integral of X_H. Moreover, if H and F are in involution, then their flows commute.

The equations of the flow of X_H, in a system of canonical coordinates, $p_1, \dots, p_n$, $q_1, \dots, q_n$ (that is, such that $\omega_2 = \sum_{i=1}^n p_i \wedge q_i$), can be written in the form

$$\dot{q} = \frac{\partial H}{\partial p} \left(= \{H, q\}\right), \quad \dot{p} = -\frac{\partial H}{\partial q} \left(= \{H, p\}\right),$$

and they are known as *Hamilton equations*.

Theorem 1.7 (Liouville-Arnold)**.** *Let X_H be a hamiltonian defined on a real symplectic manifold M_{2n}. Assume that there are n functionally independent first integrals $F_1, \dots, F_n$ in involution. Let M_a be a non-singular (that is, $dF_1, \dots, dF_n$ are independent over each point of M_a) level manifold,*

$$M_a = \{p\colon F_1(p) = a_1, \dots, F_n(p) = a_n\}.$$

1. *If M_a is compact and connected, then it is a torus $M_a \simeq \mathbb{R}^n/\mathbb{Z}^n$.*
2. *In a neighborhood of the torus M_a there are functions $I_1, \dots I_n, \phi_1, \dots, \phi_n$ such that*

$$\omega_2 = \sum_{i=1}^{n} dI_i \wedge d\phi_i,$$

   and $\{H, I_j\} = 0$ for $j = 1, \dots, n$.

From now on, we will consider $\mathbb{C}^{2n}$ as a complex symplectic manifold. Liouville-Arnold theorem gives us a notion of integrability for hamiltonian systems. A hamiltonian H in $\mathbb{C}^{2n}$ is called *integrable in the Liouville's sense* if and only if there exists n independent first integrals of X_H in involution. We will say that H is integrable *by rational functions* if and only we can find a complete set of first integrals within the family of rational functions.

1.2.2. *Variational equations.* We want to relate integrability of hamiltonian systems with Picard-Vessiot theory. We deal with non-linear hamiltonian systems. But, given a hamiltonian H in $\mathbb{C}^{2n}$ and Γ an integral curve of X_H, we can consider the *first variational equation* (VE), such as

$$\mathcal{L}_{X_H}\xi = 0,$$

in which the linear equation is induced over the tangent bundle (ξ represents a vector field supported on Γ).

Let Γ be parameterized by $\gamma\colon t \mapsto (x(t), y(t))$ in such way that

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \quad \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}.$$

Then the VE along Γ is the linear system,

$$\begin{pmatrix} \dot\xi_i \\ \dot\eta_i, \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 H}{\partial y_i \partial x_j}(\gamma(t)) & \frac{\partial^2 H}{\partial y_i \partial y_j}(\gamma(t)) \\ -\frac{\partial^2 H}{\partial x_i \partial x_j}(\gamma(t)) & -\frac{\partial^2 H}{\partial x_i \partial y_j}(\gamma(t)) \end{pmatrix} \begin{pmatrix} \xi_i \\ \eta_i \end{pmatrix}.$$

From the definition of Lie derivative, it follows that

$$\xi_i(t) = \frac{\partial H}{\partial y_i}(\gamma(t)), \quad \eta_i(t) = -\frac{\partial H}{\partial x_i}(\gamma(t)),$$

is a solution of the VE. We can use a generalization of D'Alambert's method to reduce our VE (see [14, 15] and see also [11]), obtaining the so-called *normal variational equation* (the NVE). We can see that the NVE is a linear system of rank $2(n-1)$. In the case of hamiltonian systems of 2-degrees of freedom, their NVE can be seen as second order linear homogeneous differential equation.

1.2.3. *Non-integrability tools.* Morales-Ramis theory is conformed by several results relating the existence of first integrals of H with the Galois group of the variational equations (see for example [14], [15] and see also [11]).

Most applications of Picard-Vessiot theory to integrability analysis are studied considering meromorphic functions, but for every equation considered throughout this paper, the point at infinity, of our particular solution, plays a transcendental role: the Galois group is mainly generated by Stokes phenomenon in the irregular singularity at infinity and by the exponential torus. So that we will only work with particular solutions in the context of meromorphic functions with certain properties of regularity near to the infinity point, that is, rational functions of the *positions and momenta*. Along this paper we will use the following result:

Theorem 1.8 ([14])**.** *Let H be a hamiltonian in $\mathbb{C}^{2n}$, and γ a particular solution such that the* NVE *has irregular singularities at points of γ at infinity. Then, if H is completely integrable by rational functions, then the identity component of Galois Group of the* NVE *is abelian.*

Remark 1. Here, the field of coefficients of the NVE is the field of meromorphic functions on γ.

2. **Some results on linear differential equations.**

In this section we present two new results related with second order linear differential equations.

2.1. **Algebrization of linear differential equations.** For some differential equations it is useful, if it is possible, to replace the original differential equation over a compact Riemann surface by a new differential equation over the Riemann sphere $\mathbb{P}^1$ (that is, with rational coefficients). To do this, we use a change of the independent variable. The equation over $\mathbb{P}^1$ is called the *algebraic form* or *algebrization* of the original equation.

This algebraic form dates back to the 19th century (Liouville, Darboux), but the problem of obtaining the algebraic form (if it exists) of a given differential equation is in general not an easy task. Here we develop a new algorithm using the concept of *hamiltonian change of variables*. This change of variables allow us to compute the algebraic form of a large number of differential equations of different types. We can see that Kovacic's algorithm can be applied over the algebraic form to solve the original equation.

The geometric mechanism behind the algebrization is a ramified covering of compact Riemann surfaces. We will use the following theorem of [14].

Theorem 2.1 (Morales-Ramis [14], see also [11])**.** *Let X be a Riemann surface, denote $\mathcal{M}(X)$ its field of meromorphic functions, and consider a linear differential equation*

$$\frac{d}{dx}\xi = A(x)\xi, \quad A \in Mat(m, \mathbf{C}(x)),$$

and a finite ramified covering of the projective line $x\colon X \to \mathbb{P}_1$ (t is a local parameter of X). Let

$$\frac{d}{dt}\xi = x^*(A)(t)\xi, \quad x^*(A) \in Mat(m, \mathcal{M}(X))$$

be the pullback of the equation by x (that is, the equation obtained by the change of variables $x = x(t)$). Then the identity components of the Galois group of both equations are the same.

Proposition 1 (Change of the independent variable)**.** *Let us consider the following equation, with coefficients in $\mathbb{C}(x)$:*

$$y'' + a(x)y' + b(x)y = 0, \quad y' = \frac{dy}{dx} \tag{3}$$

and $\mathbb{C}(x) \hookrightarrow L$ the corresponding Picard-Vessiot extension. Let (K, δ) be a differential field with $\mathbb{C}$ as field of constants. Let $\theta \in K$ be a non-constant element. Then, by the change of variable $x = \theta$, the equation (3) is transformed in

$$\ddot{y} + \left(a(\theta)\dot{\theta} - \frac{\ddot{\theta}}{\dot{\theta}}\right)\dot{y} + b(\theta)(\dot{\theta})^2 y = 0, \quad \dot{z} = \delta z. \tag{4}$$

Let $K_0 \subset K$ be the smallest differential field containing ξ and $\mathbb{C}$. Then the equation (4) is a differential equation with coefficients in K_0. Let $K_0 \hookrightarrow L_0$ be the corresponding Picard-Vessiot extension. Assume that

$$\mathbb{C}(x) \to K_0, \quad x \mapsto \theta$$

is an algebraic extension, then

$$Gal(L_0/K_0)^0 = Gal(L/\mathbb{C}(x))^0.$$

Proof. By the chain rule we have

$$\frac{d}{dx} = \frac{1}{\dot\theta}\delta,$$

and

$$\frac{d^2}{dx^2} = \frac{1}{(\dot\theta)^2}\delta^2 - \frac{\ddot\theta}{(\dot\theta)^3}\delta,$$

now, changing y', y'' in (3) and making monic this equation we have (4)

$$\ddot y + \left(a(\theta)\dot\theta - \frac{\ddot\theta}{\dot\theta}\right)\dot y + b(\theta)(\dot\theta)^2 y = 0.$$

In the same way we can obtain (3) through (4).

By assumption, K_0 is an algebraic extension of $\mathbb{C}(x)$. Therefore we can identify K_0 with the ring of meromorphic functions over a compact Riemann surface X. Furthermore, we can see that θ is a finite ramified covering of the Riemann sphere,

$$X \xrightarrow{\theta} \mathbb{P}^1,$$

then by Theorem 2.1, we conclude the proof. □

Recently Manuel Bronstein and Anne Fredet in [7] have implemented an algorithm to solve differential equation over $\mathbb{C}(t, e^{\int f(t)})$ without algebrizing the equation. As an immediate consequence of Proposition 1 we have the following corollary.

Corollary 1 (Linear differential equation over $\mathbb{C}(t, e^{\int f})$)**.** *Let $f \in \mathbb{C}(t)$ be a rational function. Then, the differential equation*

$$\ddot y - \left(f + \frac{\dot f}{f} - f e^{\int f} a\left(e^{\int f}\right)\right)\dot y + \left(f\left(e^{\int f}\right)\right)^2 b\left(e^{2\int f}\right) y = 0, \tag{5}$$

is algebrizable by the change $x = e^{\int f}$ and its algebraic form is given by

$$y'' + a(x)y' + b(x)y = 0.$$

Remark 2. In this corollary, we have the following cases[1].

1. $f = n\frac{h'}{h}$, for a rational function h, $n \in \mathbb{Z}_+$, we have the trivial case, both equations are over the Riemann sphere and they have the same differential field, so that does not need to be algebrized.
2. $f = \frac{1}{n}\frac{h'}{h}$, for a rational function h, $n \in \mathbb{Z}^+$, (5) is defined over an algebraic extension of $\mathbb{C}(t)$ and so that this equation is not necessarily over the Riemann sphere.
3. $f \neq q\frac{h'}{h}$, for any rational function h, $q \in \mathbb{Q}$, (5) is defined over a transcendental extension of $\mathbb{C}(t)$ and so that this equation is not over the Riemann sphere.

In the first and the second case, we can apply Proposition 1, taking $K_0 = \mathbb{C}(t, e^{\int f})$, so that the identity component of the algebrized equation is conserved. The preservation of the Galois group for the third case, requires further analysis, and will not be discussed here. We just remark that the Galois group corresponding to the original equation is a subgroup of the Galois group of the algebrized equation.

[1] Throughout this paper the reader should keep in mind the following notations:

$$\mathbb{Z}^+ = \{n \in \mathbb{Z} : \quad n \geq 1\}, \qquad \mathbb{Z}_+ = \{n \in \mathbb{Z} : \quad n \geq 0\}.$$

Definition 2.2 (Regular and irregular singularity)**.** The point $x = x_0$ is called regular singular point, or *regular singularity*, of the equation (3) if and only if $x = x_0$ is not an ordinary point and

$$(x - x_0)a(x), \quad (x - x_0)^2 b(x),$$

are both analytic in $x = x_0$. If $x = x_0$ is not a regular singularity, then it is an *irregular singularity.*

Remark 3 (Change to infinity)**.** To study asymptotic behaviors (that is, the point $x = \infty$) in (3) we can take $x(t) = \frac{1}{t}$ and analyze the behavior in $t = 0$ of (4). That is, to study the behavior in (3) in $x = \infty$ we should study the behavior in $t = 0$ of the equation

$$\ddot{y} + \left(\frac{2}{t} - \left(\frac{1}{t^2}\right) a \left(\frac{1}{t}\right)\right) \dot{y} + \frac{1}{t^4} b \left(\frac{1}{t}\right) y = 0. \tag{6}$$

In this way, by Definition 2.2, we say that $x = \infty$ is a regular singularity of the equation (3) if and only if $t = 0$ is a regular singularity of the equation (6).

To algebrize second order linear differential equations it is easier when the term in $\dot{y}$ is absent and the change of variable is *hamiltonian*, that is, the RLDE $\ddot{y} = r(t)y$.

Definition 2.3 (hamiltonian change of variable)**.** A change of variable $x = x(t)$ is called hamiltonian if and only if $(x(t), \dot{x}(t))$ is a solution curve of the autonomous hamiltonian system

$$H = H(x, y) = \frac{y^2}{2} + V(x),$$

for some $V \in \mathbb{C}(x)$.

Assume that we algebrize equation (4) through a hamiltonian change of variables, $x = \xi(t)$. Then, $K_0 = \mathbb{C}(\xi, \dot{\xi}, \ldots)$, but, we have the algebraic relation,

$$(\dot{\xi})^2 = 2h - 2V(\xi), \quad h = H(\xi, \dot{\xi}) \in \mathbb{C},$$

so that $K_0 = \mathbb{C}(\xi, \dot{\xi})$ is an algebraic extension of $\mathbb{C}(x)$. We can apply Proposition 1, and then the identity component of the Galois group is conserved.

Proposition 2 (Algebrization algorithm)**.** *The differential equation*

$$\ddot{y} = r(t)y$$

is algebrizable through a hamiltonian change of variable $x = x(t)$ if and only if there exist f, α such that

$$\frac{\alpha'}{\alpha}, \quad \frac{f}{\alpha} \in \mathbb{C}(x), \ \textit{where}\ f(x(t)) = r(t), \quad \alpha(x) = 2(H - V(x)) = \dot{x}^2.$$

Furthermore, the algebraic form of the equation $\ddot{y} = r(t)y$ is

$$y'' + \frac{1}{2}\frac{\alpha'}{\alpha} y' - \frac{f}{\alpha} y = 0. \tag{7}$$

Proof. Since $x = x(t)$ is a hamiltonian change of variable for the differential equation $\ddot{y} = r(t)y$ so that $\dot{x} = y$, $\dot{y} = \ddot{x} = -V'(x)$ and there exists f, α such that $\ddot{y} = f(x(t))y$ and $\dot{x}^2 = 2(H - V(x)) = \alpha(x)$. By Proposition 1 we have $-f(x) = b(x)\dot{x}^2$ and $a(x)\dot{x} - \ddot{x}/\dot{x} = 0$, therefore $a(x) = \ddot{x}/\dot{x}^2$ and $b(x) = -f(x)/\alpha(x)$. In this way $\ddot{y} = r(t)y$ is algebrizable if and only if $a(x), b(x) \in \mathbb{C}(x)$. As $2(H - V(x)) = \alpha(x)$, we have $\alpha'(x) = -2V'(x) = 2\ddot{x}$, and therefore $a(x) = \frac{1}{2}\frac{\alpha'(x)}{\alpha(x)}$. In this way, we obtain the equation (7). □

As a consequence of Proposition 2 we have the following result.

Corollary 2. *Let us consider* $r(t) = g(x_1, \cdots, x_n)$*, where* $x_i = e^{\lambda_i t}$*,* $\lambda_i \in \mathbb{C}$*. The differential equation* $\ddot{y} = r(t)y$ *is algebrizable if and only if*

$$\frac{\lambda_i}{\lambda_j} \in \mathbb{Q}, \quad 1 \le i \ne j \le n, \quad g \in \mathbb{C}(x).$$

Furthermore, we have $\lambda_i = c_i\lambda$*, where* $\lambda \in \mathbb{C}$ *and* $c_i \in \mathbb{Q}$ *and one change of variable is*

$$x = e^{\frac{\lambda t}{q}}, \text{ where } c_i = \frac{p_i}{q_i}, \ \gcd(p_i, q_i) = 1 \text{ and } q = mcm(q_1, \cdots, q_n).$$

Remark 4 (Using the algebrization algorithm)**.** To algebrize the RLDE $\ddot{y} = r(t)y$ we should keep in mind the following steps.

Step 1: Find a *hamiltonian change of variable* $x = x(t)$.
Step 2: Find f and α such that $r(t) = f(x(t))$ and $(\dot{x}(t))^2 = \alpha(x(t))$.
Step 3: Write $f(x)$ and $\alpha(x)$.
Step 4: Verify whether or not $f(x)/\alpha(x) \in \mathbb{C}(x)$ and $\alpha'(x)/\alpha(x) \in \mathbb{C}(x)$.
Step 5: If the answer of Step 4 is "yes" (that is, RLDE is algebrizable), write the algebraic form of the original equation as follows

$$y'' + \frac{1}{2}\frac{\alpha'}{\alpha}y' - \frac{f}{\alpha}y = 0.$$

When we have algebrized the RLDE, we study its integrability and its Galois group.

Remark 5 (Monic Polynomials)**.** Let us consider the RLDE

$$\ddot{y} = \left(\sum_{k=0}^{n} c_k t^k\right) y, \quad c_k \in \mathbb{C}, \quad k = 0, \cdots, n.$$

By the algebrization algorithm we can take $x = \mu t$, $\mu \in \mathbb{C}$, so that

$$\dot{x} = \mu, \quad \alpha(x) = \mu^2, \quad \alpha'(x) = 0 \text{ and } f(x) = \sum_{k=0}^{n} c_k \left(\frac{x}{\mu}\right)^k, \quad c_k, \mu \in \mathbb{C}.$$

Now, by (7) the new differential equation is

$$y'' = \left(\sum_{k=0}^{n} \left(\frac{c_k}{\mu^{k+2}}\right) x^k\right) y, \quad c_k, \mu \in \mathbb{C}.$$

In general, for $\mu = \sqrt[n+2]{c_n}$ we can obtain the equation

$$y'' = \left(x^n + \sum_{k=0}^{n-1} d_k x^k\right) y, \quad d_k = \left(\frac{c_k}{\mu^{k+2}}\right), \quad k = 0, \cdots, n-1.$$

Furthermore, we can observe, by Definition 2.2 and (6), that the point at ∞ is an irregular singularity for the differential equation with non-constant polynomial coefficients because using (6) we can see that zero is not an ordinary point and also it is not a regular singularity for the differential equation.

Remark 6 (Extended Mathieu)**.** The Mathieu differential equation, with phase dependance on parameters (see 3.2.3), is

$$\ddot{y} = (a + b\sin t + c\cos t)y. \tag{8}$$

Applying Corollary 2 and the steps of the algorithm we have $x = e^{it}$, $\dot{x} = ix$, therefore

$$f(x) = \frac{(b+c)x^2 + 2ax + c - b}{2x}, \quad \alpha(x) = -x^2, \quad \alpha'(x) = -2x,$$

so that the algebraic form of (8) is

$$y'' + \frac{1}{x}y' + \frac{(b+c)x^2 + 2ax + c - b}{2x^3}y = 0. \tag{9}$$

Making the change $x = 1/z$ in (9) we obtain

$$\ddot{\zeta} + \left(\frac{1}{z}\right)\dot{\zeta} + \left(\frac{(c-b)z^2 + 2az + (b+c)}{2z^3}\right)\zeta = 0. \tag{10}$$

We can observe, by Definition 2.2 and (6), that $z = 0$ is an irregular singularity for (10) and therefore $x = \infty$ is an irregular singularity for (9) and (8).

Now, we compute the Galois group and the integrability in (9). So that, the RLDE is given by

$$\xi'' = -\left(\frac{(b+c)x^2 + (2a+1)x + c - b}{2x^3}\right)\xi. \tag{11}$$

Applying Kovacic's algorithm, see appendix A, we can see that for $b \neq -c$ this equation falls in Case 2: (c_3, ∞_3), $E_0 = \{3\}$, $E_\infty = \{1\}$ and therefore $D = \emptyset$ because $m = -1 \notin \mathbb{Z}_+$. In this way we have that (11) is not integrable, the Galois Group is the connected group $SL(2,\mathbb{C})$, and finally, by Theorem 2.1 the identity component of the Galois group for (8) is exactly $SL(2,\mathbb{C})$, which is a non-abelian group. In the same way, for $b = -c$ we have the equations

$$\ddot{y} = (a - be^{-it})y, \quad y'' + \frac{1}{x}y' + \frac{2ax - 2b}{2x^3}y = 0,$$

in which ∞ continues being an irregular singularity. Now, the RLDE is given by

$$\xi'' = -\left(\frac{(2a+1)x - 2b}{2x^3}\right)\xi,$$

and applying Kovacic's algorithm we can see that this equation falls in Case 2: (c_3, ∞_2), $E_0 = \{3\}$, $E_\infty = \{0, 2, 4\}$, for instance $D = \emptyset$ because $1/2(e_\infty - e_0) \notin \mathbb{Z}_+$. This means that the Galois group continues being $SL(2,\mathbb{C})$. Using this result, taking ϵ instead of i, we can say that in the case of *harmonic oscillator with exponential waste*

$$\ddot{y} = (a + be^{-\epsilon t})y, \quad \epsilon > 0,$$

the point at ∞ is an irregular singularity and the identity component of the Galois group is $SL(2,\mathbb{C})$.

2.2. Galois groups of Hill-Schrödinger equations with polynomial potential.

Here and in the rest of the paper, we consider polynomials in $\mathbb{C}[x]$. Kovacic in [8] remarked that the Galois group of the RLDE with polynomial coefficient of odd degree is exactly $SL(2,\mathbb{C})$. For instance, we present here the complete result for the RLDE with non-constant polynomial coefficient (Theorem 2.5).

Lemma 2.4 (Completing Squares)**.** *Every monic polynomial of even degree can be written in one only way completing squares, that is,*

$$Q_{2n}(x) = x^{2n} + \sum_{k=0}^{2n-1} q_k x^k = \left(x^n + \sum_{k=0}^{n-1} a_k x^k\right)^2 + \sum_{k=0}^{n-1} b_k x^k. \tag{12}$$

Proof. Firstly, we can see that

$$\left(x^n+\sum_{k=0}^{n-1}a_kx^k\right)^2=x^{2n}+2\sum_{k=0}^{n-1}a_kx^{n+k}+\sum_{k=0}^{n-1}\sum_{j=0}^{n-1}a_ka_jx^{k+j},$$

so that by indeterminate coefficients we have

$$a_{n-1}=\frac{q_{2n-1}}{2},\quad a_{n-2}=\frac{q_{2n-2}-a_{n-1}^2}{2},\quad a_{n-3}=\frac{q_{2n-3}-2a_{n-1}a_{n-2}}{2},\cdots,$$

$$a_0=\frac{q_n-2a_1a_{n-1}-2a_2a_{n-2}-\cdots}{2},\quad b_0=q_0-a_0^2,\quad b_1=q_1-2a_0a_1,\quad\cdots,$$

$$b_{n-1}=q_{n-1}-2a_0a_{n-1}-2a_1a_{n-2}-\cdots.$$

In this way, we conclude the proof. □

Theorem 2.5 (Galois groups in polynomial case)**.** *Let us consider the equation,*

$$\ddot{\xi}=Q(x)\xi,$$

with $Q(x)\in\mathbb{C}[x]$ a polynomial of degree $k>0$. Then, its Galois group G falls in one of the following cases:

1. *$G=SL(2,\mathbb{C})$ (non-abelian, non-solvable, connected group).*
2. *$G=\mathbb{C}^*\ltimes\mathbb{C}$ (non-abelian, solvable, connected group).*

Furthermore, the second case is given if and only if the following conditions hold:

1. *$Q(x)$ is a polynomial of degree $k=2n$.*
2. *$\pm b_{n-1}-n$ is a positive even number $2m$, $m\in\mathbb{Z}_+$.*
3. *There exist a monic polynomial P_m of degree m, satisfying*

$$P_m''+2\left(x^n+\sum_{k=0}^{n-1}a_kx^k\right)P_m'+\left(nx^{n-1}+\sum_{k=0}^{n-2}(k+1)a_{k+1}x^k-\sum_{k=0}^{n-1}b_kx^k\right)P_m=0,\ \text{or}$$

$$P_m''-2\left(x^n+\sum_{k=0}^{n-1}a_kx^k\right)P_m'-\left(nx^{n-1}+\sum_{k=0}^{n-2}(k+1)a_{k+1}x^k+\sum_{k=0}^{n-1}b_kx^k\right)P_m=0.$$

In such cases, Liouvillian solutions are given by

$$\xi_1=P_me^{\frac{x^{n+1}}{n+1}+\sum_{k=0}^{n-1}\frac{a_kx^{k+1}}{k+1}},\quad \xi_2=\xi_1\int\frac{e^{-2\left(\frac{x^{n+1}}{n+1}+\sum_{k=0}^{n-1}\frac{a_kx^{k+1}}{k+1}\right)}}{P_m^2}dx,\ \text{or,}$$

$$\xi_1=P_me^{-\frac{x^{n+1}}{n+1}-\sum_{k=0}^{n-1}\frac{a_kx^{k+1}}{k+1}},\qquad \xi_2=\xi_1\int\frac{e^{2\left(\frac{x^{n+1}}{n+1}+\sum_{k=0}^{n-1}\frac{a_kx^{k+1}}{k+1}\right)}}{P_m^2}dx.$$

Proof. Let us consider the RLDE

$$\ddot{\xi}=Q(x)\xi,\quad Q(x)=\sum_{i=0}^{k}c_ix^i,\quad c_i\in\mathbb{C},\quad c_k\neq 0,\quad k>0.$$

This equation without poles only can fall in Case 1, Condition (c_0) of Kovacic's algorithm (see appendix A). Now if $k=2n+1$, then $Q(x)$ does not satisfies Step 1, and this imply that the RLDE has not Liouvillian solutions and its Galois Group is $G=SL(2,\mathbb{C})$ (non-abelian, non-solvable, connected group). On the other hand, if we consider $k=2n$, then the RLDE falls in Case 1, specifically in (c_0) (because it has not poles) and (∞_3) (because $\circ r_\infty=-2n$), that is, Conditions $\{c_0,\infty_3\}$ of

Kovacic's algorithm (see appendix A). By Remark 5, through the change of variable $x \mapsto \sqrt[n+2]{c_k}x$, the RLDE with polynomial coefficients is transformed in

$$\xi'' = Q_{2n}(x)\xi, \quad Q_{2n}(x) = x^{2n} + \sum_{k=0}^{2n-1} q_k x^k.$$

By Lemma 2.4 we have that

$$Q_{2n}(x) = \left(x^n + \sum_{k=0}^{n-1} a_k x^k\right)^2 + \sum_{k=0}^{n-1} b_k x^k.$$

Setting $Q_{2n}(x) = r$, by Step 1 we have $[\sqrt{r}]_c = 0$, $\alpha_c^{\pm} = 0$, $\circ(r_\infty) = -2n$,

$$[\sqrt{r}]_\infty = x^n + \sum_{k=0}^{n-1} a_k x^k \quad \text{and} \quad \alpha_\infty^{\pm} = \frac{1}{2}(\pm b_{n-1} - n).$$

By Step 2 of Kovacic's algorithm (see appendix A), $D = \{m \in \mathbb{Z}_+ : \quad m = \alpha_\infty^{\pm}\}$, therefore $D = \emptyset$ which means that the Galois group of the RLDE is $SL(2,\mathbb{C})$ (the RLDE has not Liouvillian solutions) or $\#D = 1$ because $2m = b_{n-1} - n > 0$ or $2m = b_{n-1} + n < 0$, in this way for $m \in D$ we have ω_+ or $\omega_- \in \mathbb{C}(x)$ given by

$$\omega = \omega_\pm = \pm [\sqrt{r}]_\infty = \pm \left(x^n + \sum_{k=0}^{n-1} a_k x^k\right).$$

Now, by Step 3, we search a monic polynomial P_m of degree m satisfying

$$P_m'' + 2\left(x^n + \sum_{k=0}^{n-1} a_k x^k\right) P_m' + \left(nx^{n-1} + \sum_{k=0}^{n-2}(k+1)a_{k+1}x^k - \sum_{k=0}^{n-1} b_k x^k\right) P_m = 0, \ or,$$

$$P_m'' - 2\left(x^n + \sum_{k=0}^{n-1} a_k x^k\right) P_m' - \left(nx^{n-1} + \sum_{k=0}^{n-2}(k+1)a_{k+1}x^k + \sum_{k=0}^{n-1} b_k x^k\right) P_m = 0.$$

If P_m does not exist, then the Galois group of the RLDE is $SL(2,\mathbb{C})$ (the RLDE has not Liouvillian solutions). On the other hand, if there exists such P_m, then Kovacic's algorithm can provide us only one solution. This solution is given by

$$\xi_1 = P_m e^{\frac{x^{n+1}}{n+1} + \sum_{k=0}^{n-1} \frac{a_k x^{k+1}}{k+1}}, \quad or, \quad \xi_1 = P_m e^{-\frac{x^{n+1}}{n+1} - \sum_{k=0}^{n-1} \frac{a_k x^{k+1}}{k+1}}$$

and we can see that ξ_1 is not an algebraic function. This means, by Remark 11, case [I5] that the Galois group of the RLDE is the non-abelian, solvable and connected group $G = \mathbb{C}^* \ltimes \mathbb{C}$. The second solution is obtained using D'Alambert reduction and is given by

$$\xi_2 = P_m e^{\frac{x^{n+1}}{n+1} + \sum_{k=0}^{n-1} \frac{a_k x^{k+1}}{k+1}} \int \frac{e^{-2\left(\frac{x^{n+1}}{n+1} + \sum_{k=0}^{n-1} \frac{a_k x^{k+1}}{k+1}\right)}}{P_m^2} dx, \ \textit{or respectively},$$

$$\xi_2 = P_m e^{-\frac{x^{n+1}}{n+1} - \sum_{k=0}^{n-1} \frac{a_k x^{k+1}}{k+1}} \int \frac{e^{2\left(\frac{x^{n+1}}{n+1} + \sum_{k=0}^{n-1} \frac{a_k x^{k+1}}{k+1}\right)}}{P_m^2} dx.$$

in this way we have proven the theorem. □

Remark 7 (Quadratic case)**.** Let consider the case where r is a polynomial of degree two:

$$\ddot{y} = \left(At^2 + Bt + C\right) y.$$

There are no poles and the order at ∞, $\circ r_\infty$, is -2, so we need to follow Case 1 in $\{c_0, \infty_3\}$ of Kovacic's algorithm. Now, by Remark 5 and by Lemma 2.4 we have

$$y'' = \left((x+a)^2 + b\right) y.$$

We find that

$$[\sqrt{r}]_\infty = x + a,$$
$$\alpha_\infty^\pm = \frac{1}{2}\left(\pm b - 1\right),$$
$$m = \alpha_\infty^+ \quad \text{or} \quad \alpha_\infty^-.$$

If b is not an odd integer, then m is not an integer. Therefore Case 1 does not hold, which means that the RLDE has no Liouvillian solutions. If b is an odd integer, then we can complete Steps 2 and 3 and actually (only) we can find a solution given by

$$y = P_m e^{\frac{x^2}{2} + ax}, \ \textit{or,} \ y = P_m e^{-\frac{x^2}{2} - ax}.$$

In particular, the quantum harmonic oscillator

$$y'' = (x^2 - \lambda)y$$

is integrable when λ is an odd integer and the only one solution obtained by means of Kovacic's algorithm is given by

$$y = H_m e^{\frac{x^2}{2}}, \ \textit{or,} \ y = \widehat{H}_m e^{-\frac{x^2}{2}}$$

where H_m and $\widehat{H}_m$ denotes the classical Hermite's polynomials.

For another approach to this problem see Vidunas [20] and Zoladek in [21].

3. **Determining families of hamiltonians with specific NVE.** Let us consider a two degrees of freedom classical hamiltonian,

$$H = \frac{y_1^2 + y_2^2}{2} + V(x_1, x_2).$$

V is the *potential function*, and it is assumed to be analytical in some open subset of $\mathbb{C}^2$. The evolution of the system is determined by Hamilton equations:

$$\dot{x}_1 = y_1, \quad \dot{x}_2 = y_2, \quad \dot{y}_1 = -\frac{\partial V}{\partial x_1}, \quad \dot{y}_2 = -\frac{\partial V}{\partial x_2}.$$

Let us assume that the plane $\Gamma = \{x_2 = 0, y_2 = 0\}$ is an invariant manifold of the hamiltonian. We keep in mind that the family of integral curves lying on Γ is parameterized by the energy $h = H|_\Gamma$, but we do not need to use it explicitly. We are interested in studying the linear approximation of the system near Γ. Since Γ is an invariant manifold, we have

$$\left.\frac{\partial V}{\partial x_2}\right|_\Gamma = 0,$$

so that the NVE for a particular solution

$$t \mapsto \gamma(t) = (x_1(t), y_1 = \dot{x}_1(t), x_2 = 0, y_2 = 0),$$

is written,

$$\dot{\xi} = \eta, \quad \dot{\eta} = -\left[\frac{\partial^2 V}{\partial x_2^2}(x_1(t), 0)\right]\xi.$$

Let us define,

$$\phi(x_1) = V(x_1, 0), \quad \alpha(x_1) = -\frac{\partial^2 V}{\partial x_2^2}(x_1, 0),$$

and then we write the second order Taylor series in x_2 for V, obtaining the following expression for H

$$H = \frac{y_1^2 + y_2^2}{2} + \phi(x_1) - \alpha(x_1)\frac{x_2^2}{2} + \beta(x_1, x_2)x_2^3, \tag{13}$$

which is the *general form of a classical analytic hamiltonian, with invariant plane* Γ, providing that a Taylor expansion of the potential around $\{x_2 = 0\}$ exist. The NVE associated to any integral curve lying on Γ is,

$$\ddot{\xi} = \alpha(x_1(t))\xi. \tag{14}$$

3.1. **General method.** We are interested in computing hamiltonians of the family (13), such that its NVE (14) belongs to a specific family of Linear Differential Equations. Then we can apply our results about the integrability of this LDE, and Morales-Ramis theorem to obtain information about the non-integrability of such hamiltonians.

From now on, we will write $a(t) = \alpha(x_1(t))$, for a generic curve γ lying on Γ, parameterized by t. Then, the NVE is written

$$\ddot{\xi} = a(t)\xi. \tag{15}$$

Problem. *Consider a differential polynomial* $Q(\eta, \dot{\eta}, \ddot{\eta}, \ldots) \in \mathbb{C}[\eta, \dot{\eta}, \ddot{\eta}, \ldots]$*, being* η *a differential indeterminate (*Q *is polynomial in* η *and a finite number of the succesive derivatives of* η*). Compute all hamiltonians in the family* (13) *verifying: for all any particular solution in* Γ*, the coefficient* $a(t)$ *of the corresponding NVE is a differential zero of* Q*, in the sense that* $Q(a, \dot{a}, \ddot{a}, \ldots) = 0$.

In this section we give a method to compute the above family of hamiltonians by solving certain differential equations. This method was based in the computations done by J. Morales and C. Simó in [13].

We should notice that, for a generic integral curve $\gamma(t) = (x_1(t), y_1 = \dot{x}_1(t))$ lying on Γ, (15) depends only of the values of functions α, and ϕ. It depends on $\alpha(x_1)$, since $a(t) = \alpha(x_1(t))$. We observe that the curve $\gamma(t)$ is a solution of the restricted hamiltonian,

$$h = \frac{y_1^2}{2} + \phi(x_1) \tag{16}$$

whose associated hamiltonian vector field is,

$$X_h = y_1\frac{\partial}{\partial x_1} - \frac{d\phi}{dx_1}\frac{\partial}{\partial y_1}, \tag{17}$$

thus $x_1(t)$ is a solution of the differential equation, $\ddot{x}_1 = -\frac{d\phi}{dx_1}$, and then, the relation of $x_1(t)$ is given by ϕ.

Since $\gamma(t)$ is an integral curve of X_h, for any function $f(x_1, y_1)$ defined in Γ we have

$$\frac{d}{dt}\gamma^*(f) = \gamma^*(X_h f),$$

where γ^* denote the usual pullback of functions. Then, using $a(t) = \gamma^*(\alpha)$, we have for each $k \geq 0$,

$$\frac{d^k a}{dt^k} = \gamma^*(X_h^k \alpha), \tag{18}$$

so that,

$$Q(a, \dot{a}, \ddot{a}, \ldots) = Q(\gamma^*(\alpha), \gamma^*(X_h \alpha), \gamma^*(X_h^2 \alpha), \ldots).$$

There is an integral curve of the hamiltonian passing through each point of Γ, so that we have proven the following.

Proposition 3. *Let H be a hamiltonian of the family (13), and $Q(a, \dot{a}, \ddot{a}, \ldots)$ a differential polynomial with constants coefficients. Then, for each integral curve lying on Γ, the coefficient $a(t)$ of the NVE (15) verifies $Q(a, \dot{a}, \ddot{a}, \ldots,) = 0$, if and only if the function*

$$\hat{Q}(x_1, y_1) = Q(\alpha, X_h\alpha, X_h^2\alpha, \ldots),$$

vanishes on Γ.

Remark 8. In fact, the NVE of a integral curve depends on the parameterization. Our criterion does not depend on any choice of parameterization of the integral curves. This is simple, the NVE corresponding to different parameterizations of the same integral curve are related by a translation of time t. We just observe that a polynomial $Q(a, \dot{a}, \ddot{a}, \ldots)$ with constant coefficients is invariant of the group by translations of time. So that, if the coefficient $a(t)$ of the NVE (15) for a certain parameterization of an integral curve $\gamma(t)$ satisfied $\{Q = 0\}$, then it is also satisfied for any other right parameterization of the curve.

Next, we will see that $\hat{Q}(x_1, y_1)$ is a polynomial in y_1 and its coefficients are differential polynomials in α, ϕ. If we write down the expressions for successive Lie derivatives of α, we obtain

$$X_h\alpha = y_1 \frac{d\alpha}{dx_1}, \tag{19}$$

$$X_h^2\alpha = y_1^2 \frac{d^2\alpha}{dx_1^2} - \frac{d\phi}{dx_1}\frac{d\alpha}{dx_i} \tag{20}$$

$$X_h^3\alpha = y_1^3 \frac{d^3\alpha}{dx_1^3} - y_1\left(\frac{d}{dx_1}\left(\frac{d\phi}{dx_1}\frac{d\alpha}{dx_1}\right) + 2\frac{d\phi}{dx_1}\frac{d^2\alpha}{dx_1^2}\right) \tag{21}$$

$$\begin{aligned} X_h^4\alpha =& y_1^4\frac{d^4\alpha}{dx_1^4} - y_1^2\left(\frac{d}{dx_1}\left(\frac{d}{dx_1}\left(\frac{d\phi}{dx_1}\frac{d\alpha}{dx_1}\right) + 2\frac{d\phi}{dx_1}\frac{d^2\alpha}{dx_1^2}\right) + 3\frac{d^3\alpha}{dx_1^3}\frac{d\phi}{dx_1}\right) \\ &+ \left(\frac{d}{dx_1}\left(\frac{d\phi}{dx_1}\frac{d\alpha}{dx_1}\right) + 2\frac{d\phi}{dx_1}\frac{d^2\alpha}{dx_1^2}\right)\frac{d\phi}{dx_1}. \end{aligned} \tag{22}$$

In general form we have,

$$X_h^{n+1}\alpha = y_1 \frac{\partial X_h^n \alpha}{dx_1} - \frac{d\phi}{dx_1}\frac{\partial X_h^n}{\partial y_1}, \tag{23}$$

it inductively follows that they all are polynomials in y_1, in which their coefficients are differential polynomials in α and ϕ. If we write it down explicitly,

$$X_h^n\alpha = \sum_{n\geq k\geq 0} E_{n,k}(\alpha, \phi) y_1^k \tag{24}$$

we can see that the coefficients $E_{n,k}(\alpha,\phi) \in \mathbb{C}\left[\alpha, \phi, \frac{d^r\alpha}{dx_1^r}, \frac{d^s\phi}{dx_1^s}\right]$, satisfy the following recurrence law,

$$E_{n+1,k}(\alpha,\phi) = \frac{d}{dx_1}E_{n,k-1}(\alpha,\phi) - (k+1)E_{n,k+1}(\alpha,\phi)\frac{d\phi}{dx_1} \tag{25}$$

with initial conditions,

$$E_{1,1}(\alpha,\phi) = \frac{d\alpha}{dx_1}, \quad E_{1,k}(\alpha,\phi) = 0 \ \forall k \neq 1. \tag{26}$$

Remark 9. The recurrence law (25) and the initial conditions (26), determine the coefficients $E_{n,k}(\alpha,\phi)$. We can compute the value of some of them easily:

- $E_{n,n}(\alpha,\phi) = \frac{d^n\alpha}{dx_1^n}$ for all $n \geq 1$.
- $E_{n,k}(\alpha,\phi) = 0$ if $n-k$ is odd, or $k<0$, or $k>n$.

3.2. **Some examples.** Here we compute families of hamiltonians (13) containing specific NVEs. Although, in order to do these computations, we need to solve polynomial differential equations, we will see that we can deal with this in a branch of cases. Particularly, when Q is a linear differential operator, we will obtain equations that involve products of few linear differential operators.

Example 1. The *harmonic oscillator* equation is given by

$$\ddot{\xi} = c_0\xi, \tag{27}$$

with c_0 constant. So that, a hamiltonian of type (13) give us such NVE whether $\dot{a} = 0$. Looking at (19), it follows that $\frac{d\alpha}{dx_1} = 0$, for instance α is a constant. We conclude that the general form of a hamiltonian (13) with NVEs of type (27) is

$$H = \frac{y_1^2 + y_2^2}{2} + \phi(x_1) + \lambda_0 x_2^2 + \beta(x_1,x_2)x_2^3,$$

being λ_0 a constant, and ϕ, β arbitrary analytical functions.

Example 2. In [1], M. Audin notice that the hamiltonian,

$$\frac{y_1^2+y_2^2}{2} + x_1x_2^2$$

is an example of a simple non-integrable classical hamiltonian, since its NVE along any integral curve in Γ is an *Airy equation*. Here we compute the family of classical hamiltonians containing NVEs of type Airy for integral curves lying on Γ. The general form of the Airy equation is

$$\ddot{\xi} = (c_0 + c_1 t)\xi \tag{28}$$

with $c_0, c_1 \neq 0$ two constants. It follows that a hamiltonian contains NVE of this type whether $\ddot{a} = 0$ and $\dot{a} \neq 0$. By Proposition 3 and (20), the equation $\ddot{a} = 0$ give us the following system:

$$\frac{d^2\alpha}{dx_1^2} = 0, \quad \frac{d\phi}{dx_1}\frac{d\alpha}{dx_1} = 0. \tag{29}$$

It splits in two independent systems,

$$\frac{d\alpha}{dx_1} = 0, \quad \begin{cases} \frac{d^2\alpha}{dx_1^2} = 0 \\ \frac{d\phi}{dx_1} = 0 \end{cases} \tag{30}$$

Solutions of the first one fall into the previous case of *harmonic oscillator*. Then, taking the general solution of the second system, we conclude that the general form of a classical hamiltonian of type (13) with Airy NVE is:

$$H = \frac{y_1^2 + y_2^2}{2} + \lambda_0 + \lambda_1 x_2^2 + \lambda_2 x_1 x_2^2 + \beta(x_1, x_2) x_2^3, \tag{31}$$

with $\lambda_2 \neq 0$.

3.2.1. *The NVE of type quantum harmonic oscillator.* Let us consider now equations with $\frac{d^3a}{dt^3} = 0$, and $\frac{d^2a}{dt^2} \neq 0$, the NVE is given by

$$\ddot{\xi} = (c_0 + c_1 t + c_2 t^2)\xi \tag{32}$$

with $c_2 \neq 0$. Those equation can be reduced to a *quantum harmonic oscillator equation* by an affine change of t, and its integrability has been studied using Kovacic's algorithm. Using Proposition 3 and (21), we obtain the following system of differential equations for α and ϕ:

$$\frac{d^3\alpha}{dx_1^3} = 0, \quad \frac{d^\alpha}{dx_1}\frac{d^2\phi}{dx_1^2} + 3\frac{d^2\alpha}{dx_1^2}\frac{d\phi}{dx_1} = 0.$$

The general solution of the first equation is

$$\alpha = \frac{\lambda_1}{2} + \frac{\lambda_2}{2}x_1 + \frac{\lambda_3}{2}x_1^2,$$

and substituting it into the second equation we obtain a linear differential equation for ϕ,

$$\frac{d^2\phi}{dx_1^2} + 3\frac{2\lambda_3}{\lambda_2 + 2\lambda_3 x_1}\frac{d\phi}{dx_1} = 0,$$

this equation is integrated by two quadratures, and its general solution is

$$\phi = \frac{\lambda_4}{(\lambda_2 + 2\lambda_3 x_1)^2} + \lambda_0.$$

We conclude that the general formula for hamiltonians of type (13) with NVE (32) for any integral curve lying on Γ is

$$H = \frac{y_1^2 + y_2^2}{2} +$$

$$\frac{\lambda_4}{(\lambda_2 + 2\lambda_3 x_1)^2} + \lambda_0 - \lambda_1 x_2^2 - \lambda_2 x_1 x_2^2 - \lambda_3 x_1^2 x_2^2 + \beta(x_1, x_2) x_2^3, \tag{33}$$

with $\lambda_3 \neq 0$.

Remark 10. The first example in this paper in which we find non-linear dynamics over the invariant plane Γ is given by (33). Notice that this dynamic is continuously deformed to linear dynamics when λ_4 tends to zero. In general case, for a fixed energy h, we have the general integral of the equation:

$$8\lambda_3^2 h^2 (t - t_0)^2 = h(\lambda_2 + 2\lambda_3 x_1)^2 - \lambda_4.$$

3.2.2. *The NVE with polynomial coefficient.* Let us consider for $n > 0$ the following differential polynomial,

$$Q_m(a, \dot{a}, \ldots) = \frac{d^m a}{dt^m}.$$

It is obvious that $a(t)$ is a polynomial of degree n if and only if $Q_n(a, \dot{a}, \ldots) \neq 0$ and $Q_{n+1}(a, \dot{a}, \ldots) = 0$.

Looking at Proposition 3, we see that a hamiltonian (13) has NVE along a generic integral curve lying on Γ,

$$\ddot{\xi} = P_n(t)\xi, \tag{34}$$

with $P_n(t)$ polynomial of degree n, if and only if $X_h^n \alpha \neq 0$ and $X_h^{n+1}\alpha$ vanishes on Γ. Let us remind expression (24), $X_h^{n+1}\alpha$ vanishes on Γ if and only if (α, ϕ) is a solution of the differential system,

$$R_{n+1} = \{E_{n+1,0}(\alpha, \phi) = 0, \ldots, E_{n+1,n+1}(\alpha, \phi) = 0\}.$$

A particular solution of R_{n+1} not verifying R_n, is given by $\phi = \lambda_0$, $\alpha(x_1) = Q_n(x_1)$, a polynomial of degree n. Therefore, the following hamiltonians,

$$H = \frac{y_1^2 + y_2^2}{2} + \lambda_0 + Q_n(x_1)x_2^2 + \beta(x_1, x_2)x_2^3, \tag{35}$$

have NVE, along a generic integral curve lying on Γ, of the form (34).

If n is an even number, there are more solutions of the differential system R_{n+1} not verifying R_n, being a particular case the potentials with generic quantum harmonic oscillators, computed above. We will prove, using the recurrence law (25), that for odd n, the above family is the only solution of R_{n+1} not verifying R_n.

Lemma 3.1. *Let (α, ϕ) be a solution of R_{2m}. Then, if $\frac{d\phi}{dx_1} \neq 0$, then (α, ϕ) is a solution of R_{2m-1}.*

Proof. By Remark 9 we have that $E_{2m-1,2k}(\alpha, \phi) = 0$ for all $m - 1 \geq k \geq 0$. Now, let us prove that $E_{2m-1,2k+1}(\alpha, \phi)$ for all $m - 2 \geq k \geq 0$.

In the first step of the recurrence law defining R_{2m},

$$0 = E_{2m,0}(\alpha, \phi) = \frac{dE_{2m-1,1}}{dx_1}(\alpha, \phi) - \frac{d\phi}{dx_1} E_{2m-1,1}(\alpha, \phi).$$

We use $\frac{d\phi}{dx_1} \neq 0$, and Remark 9, $E_{2m-1,-1}(\alpha, \phi) = 0$ to obtain,

$$E_{2m-1,1}(\alpha, \phi) = 0.$$

If we assume $E_{2m-1,2k+1}(\phi, \alpha) = 0$, substituting it in the recurrence law

$$E_{2m,2k+1}(\alpha, \phi) = \frac{dE_{2m-1,2k}}{dx_1}(\alpha, \phi) - 2(k+1)\frac{d\phi}{dx_1} E_{2m-1,2(k+1)}(\alpha, \phi),$$

we obtain that

$$E_{2m-1,2(k+1)}(\alpha, \phi) = 0,$$

and we conclude by finite induction. □

Corollary 3. *Let H be a classical hamiltonian of type (13), then the following statements are equivalent,*

1. *The NVE for a generic integral curve given by (15) lying on Γ, has a polynomial coefficient $a(t)$ of degree $2m - 1$.*

2. The hamiltonian H can be written as
$$H = \frac{y_1^2 + y_2^2}{2} + \lambda_0 - P_{2m-1}(x_1)x_2^2 + \beta(x_1, x_2)x_2^3, \tag{36}$$
for λ_0 constant, and $P_{2m-1}(x_1)$ a polynomial of degree $2m-1$.

Proof. It is clear that condition *1.* is satisfied if and only if (α, ϕ) is a solution of R_{2m} and it is not a solution of R_{2m-1}. By the previous lemma, it implies $\frac{d\phi}{dx_1} = 0$, and then the system R_{2m} is reduced to $\frac{d^{2m}\alpha}{dx_1^{2m}}$ and then, ϕ is a constant and α is a polynomial of degree at most $2m-1$. □

3.2.3. *The NVE of type Mathieu extended.* This is the standard Mathieu equation,
$$\ddot{\xi} = (c_0 + c_1 \cos(\omega t))\xi, \quad \omega \neq 0. \tag{37}$$
We can not apply our method to compute the family of hamiltonians corresponding to this equation, because $\{c_0 + c_1\cos(\omega t)\}$ is not the general solution of any differential polynomial with constant coefficients. Now, considering
$$Q(a) = \frac{d^3 a}{dt^3} + \omega^2 \frac{da}{dt}, \tag{38}$$
we can see that the general solution of $\{Q(a) = 0\}$ is
$$a(t) = c_0 + c_1\cos(\omega t) + c_2 \sin(\omega t).$$
Just notice that,
$$c_1\cos(\omega t) + c_2\sin(\omega t) = \sqrt{c_1^2 + c_2^2}\cos\left(\omega t + \arctan\frac{c_2}{c_1}\right),$$
thus the NVE (15), when a is a solution of (38), is reducible to Mathieu equation (37) by a translation of time.

Using Proposition 3, we find the system of differential equations that determine the family of hamiltonians,
$$\frac{d^3\alpha}{dx_1^3} = 0, \quad \frac{d\alpha}{dx_1}\frac{d^2\phi}{dx_1^2} + 3\frac{d^2\alpha}{dx_1}\frac{d\phi}{dx_1} - \omega^2\frac{d\alpha}{dx_1} = 0.$$
The general solution of the first equation is
$$\alpha = \lambda_0 + \lambda_1 x_1 + \lambda_2 x_1^2.$$
substituting it in the second equation, and writing $y = \frac{d\phi}{dx_1}$, we obtain a non-homogeneous linear differential equation for y,
$$\frac{dy}{dx_1} + \frac{6\lambda_2 y}{\lambda_1 + 2\lambda_2 x_1} = \omega^2. \tag{39}$$
We must distinguish two cases depending on the parameter. If $\lambda_2 = 0$, then we just integrate the equation by trivial quadratures, obtaining
$$\phi = \mu_0 + \mu_1 x_1 + \frac{\omega^2 x_1^2}{2}$$
and then,
$$H = \frac{y_1^2 + y_2^2}{2} + \mu_0 + \mu_1 x_1 + \frac{\omega^2 x_1^2}{2} - \lambda_0 x_2^2 - \lambda_1 x_1 x_2^2 + \beta(x_1, x_2)x_2^3, \tag{40}$$
If $\lambda_2 \neq 0$, then we can reduce the equation to a separable equation using
$$u = \frac{6\lambda_2 y}{\lambda_1 + 2\lambda_2 x_1},$$

obtaining

$$\frac{3du}{3\omega^2 - 4u} = \frac{6\lambda_2 dx}{\lambda_1 + 2\lambda_2 x_1}, \quad u = \frac{3\omega^2}{4} + \frac{3\mu_1}{4(\lambda_1 + 2\lambda_2 x_1)^4},$$

and then

$$y = \frac{1}{8\lambda_2}\left(\omega^2\lambda_1 + 2\omega^2\lambda_2 x_1 + \frac{\mu_1}{(\lambda_1 + 2\lambda_2 x_1)^3}\right),$$

and finally we integrate it to obtain ϕ,

$$\phi = \int y dx_1 = \mu_0 - \frac{\mu_1}{32\lambda_2^2}\frac{1}{(\lambda_1 + 2\lambda_2 x_1)^2} + \frac{\omega^2\lambda_1 x_1}{8\lambda_2} + \frac{\omega^2 x_1^2}{8}.$$

Scaling the parameters adequately we write down the general formula for the hamiltonian,

$$\begin{aligned} H = &\frac{y_1^2 + y_2^2}{2} + \mu_0 + \frac{\mu_1}{(\lambda_1 + 2\lambda_2 x_1)^2} + \frac{\lambda_1\omega^2 x_1}{8\lambda_2} + \frac{\omega^2 x_1^2}{8} \\ &- \lambda_0 x_2^2 - \lambda_1 x_1 x_2^2 - \lambda_2 x_1^2 x_2^2 + \beta(x_1, x_2) x_2^3. \end{aligned} \tag{41}$$

3.3. **Application of non-integrability criteria.**

3.3.1. *The NVE with $a(t)$ polynomial.* According to Theorem 2.5, the Galois group corresponding to equations,

$$\ddot{\xi} = P(t)\xi$$

where $P(t)$ is a non-constant polynomial, is a connected non-abelian group. So that we can apply Theorem 1.8 to the NVE of a generic integral curve in Γ of hamiltonians in the family (35). We get the following result:

Proposition 4. *hamiltonians*

$$H = \frac{y_1^2 + y_2^2}{2} + x_2^2 Q(x_1) + \beta(x_1, x_2) x_2^3$$

where $Q(x_1)$ is a non-constant polynomial, and $\beta(x_1, x_2)$ analytic function around Γ, do no admit any additional rational first integral.

Corollary 4. *Every integrable (by rational functions) polynomial potential constant on the invariant plane $\Gamma = \{x_2 = y_2 = 0\}$ is written in the following form,*

$$V = Q(x_1, x_2)x_2^3 + \lambda_1 x_2^2 + \lambda_0, \quad \lambda_0, \lambda_1 \in \mathbb{C},$$

with $Q(x_1, x_2)$ polynomial.

Proof. As we have seen, a polynomial potential V with invariant plane Γ is written

$$V = P_0(x_1) + P_2(x_1)x_2^2 + Q(x_1, x_2)x_2^3,$$

if V is constant in Γ, then $P_0(x_1) = \lambda_0 \in \mathbb{C}$ and then V falls in the above family, of non-integrable potentials, unless $P_2(x_1) = \lambda_1 \in \mathbb{C}$. ∎

3.3.2. *The NVE reducible to quantum harmonic oscillator.* We have seen that hamiltonians (33), have generic NVE along curves in Γ of type (32). Once again, we apply Theorem 2.5.

Proposition 5. *hamiltonians of the family* (33) *do not admit any additional rational first integral.*

We can also discuss the Picard-Vessiot integrability of (32). First, we shall notice that by just a scaling of t,

$$t = \frac{\tau}{\sqrt[4]{c_2}} - \frac{c_1}{2c_2}$$

we reduce it to a *quantum harmonic oscillator equation*,

$$\frac{d^2\xi}{d\tau^2} = (\tau^2 - E)\xi, \qquad E = \frac{c_1^2 - 4c_0c_1}{4\sqrt{c_2^3}}. \tag{42}$$

In Remark 7 we analyzed this equation. It is *Picard-Vessiot integrable* if and only if E is an odd positive number. Then, let us compute the parameter E associated to the NVE of integral curves of hamiltonians (33).

Let us keep in mind that the family of those curves is parameterized by

$$h = \frac{y_1^2}{2} + \frac{\lambda_4}{(\lambda_2 + 2\lambda_3 x_1)^2}.$$

In order to fix the parameterization of those curves, let us assume that time $t = 0$ corresponds to $x_1 = 0$. The NVE corresponding to a curve, depending on energy h, is written:

$$\ddot{\xi} = (c_0(h) + c_1(h)t + c_2(h)t^2)\xi.$$

We compute these coefficients $c_i(h)$ using,

$$y_1 = \frac{\sqrt{2h(\lambda_2 + 2\lambda_3 x_1)^2 - 2\lambda_4}}{\lambda_2 + 2\lambda_3 x_1} \xrightarrow{t\to 0} \frac{\sqrt{2h\lambda_2^2 - 2\lambda_4}}{\lambda_2},$$

and then, by applying the hamiltonian field,

$$c_0(h) = \frac{\lambda_1}{2}, \quad c_1(h) = \sqrt{\frac{h\lambda_2^2 - \lambda_4}{2}}, \quad c_2(h) = \lambda_3 h,$$

and then, it is reducible to (42) with parameter,

$$E = \frac{1}{8\sqrt{\lambda_3^3}}\left(\frac{\lambda_2^2 - 4\lambda_1\lambda_3}{\sqrt{h}} - \frac{\lambda_4}{\sqrt{h^3}}\right).$$

If $\lambda_2 = 4\lambda_1\lambda_3$ and $\lambda_4 = 0$, then the parameter E vanishes for every integral curve in Γ. For any other case, E is a non-constant analytical function of h.

We have proven that those NVE are generically not Picard-Vessiot integrable for any hamiltonian of the (33) *family.*

3.3.3. *The NVE of type Mathieu.* In order to apply Theorem 1.8, we just need to make some remarks on the field of coefficients. Let γ be a generic integral curve of (40), or (41). Those curves are, in general, Riemann spheres. The field of coefficients $\mathcal{M}_\gamma$, is generated by x_1, y_1, so that it is $\mathbb{C}(x_1, \dot{x}_1)$. We also have,

$$a = \lambda_0 + \lambda_1 x_1 + \lambda_2 x_1^2, \quad \dot{a} = \dot{x}_1(\lambda_1 + 2\lambda_2 x_1),$$

so that, for $\lambda_2 = 0$, we have

$$\mathcal{M}_\gamma = \mathbb{C}(\alpha, \dot{\alpha}) = \mathbb{C}(\sin t, \cos t) = \mathbb{C}(e^{it}).$$

and, for $\lambda_2 \neq 0$,

$$\mathbb{C}(e^{it}) \hookrightarrow \mathcal{M}_\gamma$$

is an algebraic extension.

So that in our algebrization algorithm we take $\mathbb{C}(e^{it})$ as the field of coefficients of Mathieu equation. For $\lambda_2 = 0$, we can apply directly Theorem 1.8. For $\lambda_2 \neq 0$, we can apply Theorem 2.1 and Theorem 1.8.

Non-trivial equations of type Mathieu, with field of coefficients $\mathbb{C}(e^{it})$, analyzed in Remark 6, have Galois group $SL(2, \mathbb{C})$. Thus for computed families of hamiltonians with NVE of type Mathieu, we get:

Proposition 6. *hamiltonians of the families* (40) *and* (41) *if* $\lambda_1 \neq 0$ *and* $(\lambda_1, \lambda_2) \neq (0, 0)$ *respectively, do not admit any additional rational first integral.*

Final comments and open questions. The algebrization method of linear differential equations presented here can be seen as an easy algorithm. Is possible to give a classification of algebrizable higher order linear differential equations generalizing our algorithm? We do not know yet.

The problem of determining families of classical hamiltonians with an invariant plane and NVE of Hill-Schrödinger type with polynomial coefficient of even degree greater than two is still open.

The problem of analyzing the monodromy of the NVE of integral curves of a two degrees of freedom hamiltonian (both, classical and general) has been studied by Baider, Churchill and Rod at the beginning of the 90's (see [3]). Their method is quite different, they have imposed the monodromy group to verify some special properties that were translated as algebraic conditions in the hamiltonian functions. Their theory were restricted to the case of fuchsian groups, which in terms of Galois theory means regular singularities, while we work in the general case. The comparison of both methods should be done.

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Appendix.

Appendix A. **Kovacic's Algorithm.** This algorithm is devoted to solve the RLDE $\xi'' = r\xi$ and is based on the algebraic subgroups of $SL(2, \mathbb{C})$. For more details see [8]. Improvements for this algorithm are given in [18], where it is not necessary to reduce the equation. Here, we follow the original version given by Kovacic in [8] .

Theorem A.1. *Let G be an algebraic subgroup of $SL(2, \mathbb{C})$. Then one of the following four cases can occur.*

1. *G is triangularizable.*
2. *G is conjugate to a subgroup of infinite dihedral group (also called meta-abelian group) and case 1 does not hold.*

3. *Up to conjugation G is one of the following finite groups: Tetrahedral group, Octahedral group or Icosahedral group, and cases 1 and 2 do not hold.*
4. $G = SL(2,\mathbb{C})$.

Each case in Kovacic's algorithm is related with each one of the algebraic subgroups of $SL(2,\mathbb{C})$ and the associated Riccatti equation

$$\theta' = r - \theta^2 = \left(\sqrt{r} - \theta\right)\left(\sqrt{r} + \theta\right), \quad \theta = \frac{\xi'}{\xi}.$$

According to Theorem A.1, there are four cases in Kovacic's algorithm. Only for cases 1, 2 and 3 we can solve the differential equation the RLDE, but for the case 4 we have not Liouvillian solutions for the RLDE. It is possible that Kovacic's algorithm can provide us only one solution (ξ_1), so that we can obtain the second solution (ξ_2) through

$$\xi_2 = \xi_1 \int \frac{dx}{\xi_1^2}. \tag{43}$$

Notations. For the RLDE given by

$$\frac{d^2\xi}{dx^2} = r\xi, \qquad r = \frac{s}{t}, \quad s, t \in \mathbb{C}[x],$$

we use the following notations.

1. Denote by Γ' be the set of (finite) poles of r, $\Gamma' = \{c \in \mathbb{C} : t(c) = 0\}$.
2. Denote by $\Gamma = \Gamma' \cup \{\infty\}$.
3. By the order of r at $c \in \Gamma'$, $\circ(r_c)$, we mean the multiplicity of c as a pole of r.
4. By the order of r at ∞, $\circ(r_\infty)$, we mean the order of ∞ as a zero of r. That is $\circ(r_\infty) = deg(t) - deg(s)$.

A.1. **The four cases. Case 1.** In this case $[\sqrt{r}]_c$ and $[\sqrt{r}]_\infty$ means the Laurent series of $\sqrt{r}$ at c and the Laurent series of $\sqrt{r}$ at ∞ respectively. Furthermore, we define $\varepsilon(p)$ as follows: if $p \in \Gamma$, then $\varepsilon(p) \in \{+,-\}$. Finally, the complex numbers $\alpha_c^+, \alpha_c^-, \alpha_\infty^+, \alpha_\infty^-$ will be defined in the first step. If the differential equation has not poles it only can fall in this case.

Step 1. Search for each $c \in \Gamma'$ and for ∞ the corresponding situation as follows:

(c_0): If $\circ(r_c) = 0$, then

$$\left[\sqrt{r}\right]_c = 0, \quad \alpha_c^\pm = 0.$$

(c_1): If $\circ(r_c) = 1$, then

$$\left[\sqrt{r}\right]_c = 0, \quad \alpha_c^\pm = 1.$$

(c_2): If $\circ(r_c) = 2$, and

$$r = \cdots + b(x-c)^{-2} + \cdots, \quad \text{then}$$

$$\left[\sqrt{r}\right]_c = 0, \quad \alpha_c^\pm = \frac{1 \pm \sqrt{1+4b}}{2}.$$

(c_3): If $\circ(r_c) = 2v \geq 4$, and

$$r = (a\,(x-c)^{-v} + \ldots + d\,(x-c)^{-2})^2 + b(x-c)^{-(v+1)} + \cdots, \quad \text{then}$$

$$\left[\sqrt{r}\right]_c = a\,(x-c)^{-v} + \ldots + d\,(x-c)^{-2}, \quad \alpha_c^\pm = \frac{1}{2}\left(\pm\frac{b}{a} + v\right).$$

(∞_1)**:** If $\circ(r_\infty) > 2$, then

$$\left[\sqrt{r}\right]_\infty = 0, \quad \alpha_\infty^+ = 0, \quad \alpha_\infty^- = 1.$$

(∞_2)**:** If $\circ(r_\infty) = 2$, and $r = \cdots + bx^2 + \cdots$, then

$$\left[\sqrt{r}\right]_\infty = 0, \quad \alpha_\infty^\pm = \frac{1 \pm \sqrt{1+4b}}{2}.$$

(∞_3)**:** If $\circ(r_\infty) = -2v \leq 0$, and

$$r = (ax^v + ... + d)^2 + bx^{v-1} + \cdots, \quad \text{then}$$

$$\left[\sqrt{r}\right]_\infty = ax^v + ... + d, \quad \text{and} \quad \alpha_\infty^\pm = \frac{1}{2}\left(\pm\frac{b}{a} - v\right).$$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ m \in \mathbb{Z}_+ : m = \alpha_\infty^{\varepsilon(\infty)} - \sum_{c \in \Gamma'} \alpha_c^{\varepsilon(c)}, \forall \left(\varepsilon\left(p\right)\right)_{p \in \Gamma} \right\}.$$

If $D = \emptyset$, then we should start with the case 2. Now, if $\#D > 0$, then for each $m \in D$ we search $\omega \in \mathbb{C}(x)$ such that

$$\omega = \varepsilon\left(\infty\right)\left[\sqrt{r}\right]_\infty + \sum_{c \in \Gamma'} \left(\varepsilon\left(c\right)\left[\sqrt{r}\right]_c + \alpha_c^{\varepsilon(c)}(x-c)^{-1} \right).$$

Step 3. For each $m \in D$, search for a monic polynomial P_m of degree m with

$$P_m'' + 2\omega P_m' + (\omega' + \omega^2 - r)P_m = 0.$$

If success is achieved then $\xi_1 = P_m e^{\int \omega}$ is a solution of the differential equation the RLDE. Else, Case 1 cannot hold.

Case 2. Search for each $c \in \Gamma'$ and for ∞ the corresponding situation as follows:

Step 1. Search for each $c \in \Gamma'$ and ∞ the sets $E_c \neq \emptyset$ and $E_\infty \neq \emptyset$. For each $c \in \Gamma'$ and for ∞ we define $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows:

(c_1)**:** If $\circ(r_c) = 1$, then $E_c = \{4\}$
(c_2)**:** If $\circ(r_c) = 2$, and $r = \cdots + b(x-c)^{-2} + \cdots$, then

$$E_c = \left\{ 2 + k\sqrt{1+4b} : k = 0, \pm 2 \right\}.$$

(c_3)**:** If $\circ(r_c) = v > 2$, then $E_c = \{v\}$
(∞_1)**:** If $\circ(r_\infty) > 2$, then $E_\infty = \{0, 2, 4\}$
(∞_2)**:** If $\circ(r_\infty) = 2$, and $r = \cdots + bx^2 + \cdots$, then

$$E_\infty = \left\{ 2 + k\sqrt{1+4b} : k = 0, \pm 2 \right\}.$$

(∞_3)**:** If $\circ(r_\infty) = v < 2$, then $E_\infty = \{v\}$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ m \in \mathbb{Z}_+ : \quad m = \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma'} e_c \right), \forall e_p \in E_p,\ p \in \Gamma \right\}.$$

If $D = \emptyset$, then we should start the case 3. Now, if $\#D > 0$, then for each $m \in D$ we search a rational function θ defined by

$$\theta = \frac{1}{2}\sum_{c\in\Gamma'} \frac{e_c}{x-c}.$$

Step 3. For each $m \in D$, search a monic polynomial P_m of degree m, such that

$$P_m''' + 3\theta P_m'' + (3\theta' + 3\theta^2 - 4r)P_m' + \left(\theta'' + 3\theta\theta' + \theta^3 - 4r\theta - 2r'\right) P_m = 0.$$

If P_m does not exist, then Case 2 cannot hold. If such a polynomial is found, set $\phi = \theta + P'/P$ and let ω be a solution of

$$\omega^2 + \phi\omega + \frac{1}{2}\left(\phi' + \phi^2 - 2r\right) = 0.$$

Then $\xi_1 = e^{\int \omega}$ is a solution of the differential equation the RLDE.

Case 3. Search for each $c \in \Gamma'$ and for ∞ the corresponding situation as follows:

Step 1. Search for each $c \in \Gamma'$ and ∞ the sets $E_c \neq \emptyset$ and $E_\infty \neq \emptyset$. For each $c \in \Gamma'$ and for ∞ we define $E_c \subset \mathbb{Z}$ and $E_\infty \subset \mathbb{Z}$ as follows:

(c_1)**:** If $\circ(r_c) = 1$, then $E_c = \{12\}$
(c_2)**:** If $\circ(r_c) = 2$, and $r = \cdots + b(x-c)^{-2} + \cdots$, then

$$E_c = \left\{ 6 + k\sqrt{1+4b} : \quad k = 0, \pm1, \pm2, \pm3, \pm4, \pm5, \pm6 \right\}.$$

(∞)**:** If $\circ(r_\infty) = v \geq 2$, and $r = \cdots + bx^2 + \cdots$, then

$$E_\infty = \left\{ 6 + \frac{12k}{n}\sqrt{1+4b} : \quad k = 0, \pm1, \pm2, \pm3, \pm4, \pm5, \pm6 \right\}, \quad n \in \{4, 6, 12\}.$$

Step 2. Find $D \neq \emptyset$ defined by

$$D = \left\{ m \in \mathbb{Z}_+ : \quad m = \frac{n}{12}\left(e_\infty - \sum_{c\in\Gamma'} e_c \right), \forall e_p \in E_p,\ p \in \Gamma \right\}.$$

In this case we start with $n = 4$ to obtain the solution, afterwards $n = 6$ and finally $n = 12$. If $D = \emptyset$, then the differential equation has not Liouvillian solution because it falls in the case 4. Now, if $\#D > 0$, then for each $m \in D$ with its respective n, search a rational function

$$\theta = \frac{n}{12}\sum_{c\in\Gamma'} \frac{e_c}{x-c}$$

and a polynomial S defined as

$$S = \prod_{c\in\Gamma'} (x - c).$$

Step 3. Search for each $m \in D$, with its respective n, a monic polynomial $P_m = P$ of degree m, such that its coefficients can be determined recursively by

$$P_{-1} = 0, \quad P_n = -P,$$

$$P_{i-1} = -SP_i' - ((n-i)\,S' - S\theta)\,P_i - (n-i)\,(i+1)\,S^2 r P_{i+1},$$

where $i \in \{0, 1 \ldots, n-1, n\}$. If P does not exist, then the differential equation has not Liouvillian solution because it falls in Case 4. Now, if P exists search ω such that

$$\sum_{i=0}^{n} \frac{S^i P}{(n-i)!}\omega^i = 0,$$

then a solution of the differential equation the RLDE is given by

$$\xi = e^{\int \omega},$$

where ω is solution of the previous polynomial of degree n.

A.2. **Some remarks on Kovacic's algorithm.** Along this section we assume that the RLDE falls only in one of the four cases.

Remark 11 (Case 1)**.** If the RLDE falls in case 1, then its Galois group is given by one of the following groups:

I1: e when the algorithm provides two rational solutions or only one rational solution and the second solution obtained by (43) has not logarithmic term.

$$e = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

this group is connected and abelian.

I2: $\mathbb{G}_k$ when the algorithm provides only one algebraic solution ξ such that $\xi^k \in \mathbb{C}(x)$ and $\xi^{k-1} \notin \mathbb{C}(x)$.

$$\mathbb{G}_k = \left\{ \begin{pmatrix} \lambda & d \\ 0 & \lambda^{-1} \end{pmatrix} : \quad \lambda \text{ is a } k\text{-root of the unity, } d \in \mathbb{C} \right\},$$

this group is disconnected and its identity component is abelian.

I3: $\mathbb{C}^*$ when the algorithm provides two non-algebraic solutions.

$$\mathbb{C}^* = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} : c \in \mathbb{C}^* \right\},$$

this group is connected and abelian.

I4: $\mathbb{C}^+$ when the algorithm provides one rational solution and the second solution is not algebraic.

$$\mathbb{C}^+ = \left\{ \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} : d \in \mathbb{C} \right\}, \quad \xi \in \mathbb{C}(x),$$

this group is connected and abelian.

I5: $\mathbb{C}^* \ltimes \mathbb{C}^+$ when the algorithm only provides one solution ξ such that ξ and its square are not rational functions.

$$\mathbb{C}^* \ltimes \mathbb{C}^+ = \left\{ \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} : c \in \mathbb{C}^*, d \in \mathbb{C} \right\}, \quad \xi \notin \mathbb{C}(x), \quad \xi^2 \notin \mathbb{C}(x).$$

This group is connected and non-abelian.

I6: $SL(2,\mathbb{C})$ if the algorithm does not provide any solution. This group is connected and non-abelian.

Remark 12 (Case 2)**.** If the RLDE falls in case 2, then Kovacic's Algorithm can provide us one or two solutions. This depends on r as follows:

II1: if r is given by
$$r=\frac{2\phi'+2\phi-\phi^2}{4},$$
then there exist only one solution,

II2: if r is given by
$$r\neq\frac{2\phi'+2\phi-\phi^2}{4},$$
then there exists two solutions.

II3: The identity component of the Galois group for this case is abelian.

Remark 13 (Case 3)**.** If the RLDE falls in case 3, then its Galois group is given by one of the following groups:

III1: Tetrahedral group when ω is obtained with $n=4$. This group of order 24 is generated by
$$\begin{pmatrix} e^{\frac{k\pi i}{3}} & 0\\ 0 & e^{-\frac{k\pi i}{3}}\end{pmatrix},\quad \frac{1}{3}\left(2e^{\frac{k\pi i}{3}}-1\right)\begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix},\quad k\in\mathbb{Z}.$$

III2: Octahedral group when ω is obtained with $n=6$. This group of order 48 is generated by
$$\begin{pmatrix} e^{\frac{k\pi i}{4}} & 0\\ 0 & e^{-\frac{k\pi i}{4}}\end{pmatrix},\quad \frac{1}{2}e^{\frac{k\pi i}{4}}\left(e^{\frac{k\pi i}{2}}+1\right)\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix},\quad k\in\mathbb{Z}.$$

III3: Icosahedral group when ω is obtained with $n=12$. This group of order 120 is generated by
$$\begin{pmatrix} e^{\frac{k\pi i}{5}} & 0\\ 0 & e^{-\frac{k\pi i}{5}}\end{pmatrix},\quad \begin{pmatrix}\phi & \psi\\ \psi & -\phi\end{pmatrix},\quad k\in\mathbb{Z},$$
being ϕ and ψ defined as
$$\phi=\frac{1}{5}\left(e^{\frac{3k\pi i}{5}}-e^{\frac{2k\pi i}{5}}+4e^{\frac{k\pi i}{5}}-2\right),\quad \psi=\frac{1}{5}\left(e^{\frac{3k\pi i}{5}}+3e^{\frac{2k\pi i}{5}}-2e^{\frac{k\pi i}{5}}+1\right)$$

III4: The identity component of the Galois group for this case is abelian.

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