The (Δ, D) and (Δ, N) Problems in Double-Step Digraphs with Unilateral Diameter *

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1 Preliminaries

We study the (Δ, D) and (Δ, N) problems for double-step digraphs considering the unilateral distance, which is the minimum between the distance in the digraph and the distance in its converse digraph, obtained by changing the directions of all the arcs.

The first problem consists of maximizing the number of vertices N of a digraph, given the maximum degree Δ and the unilateral diameter D^*, whereas the second one consists of minimizing the unilateral diameter given the maximum degree and the number of vertices. We solve the first problem for every value of the unilateral diameter and the second one for some infinitely many values of the number of vertices.

Miller and Sirán [4] wrote a comprehensive survey about (Δ, D) and (Δ, N) problems. In particular, for the double-step graphs considering the standard diameter, the first problem was solved by Fiol, Yebra, Alegre and Valero [3], whereas Bermond, Iliades and Peyrat [2], and also Beivide, Herrada, Balcázar and Arruabarrena [1] solved the (Δ, N) problem. In the case of the double-step digraphs, also with the standard diameter, Morillo, Fiol and Fàbrega [5] solved the (Δ, D) problem and provided some infinite families of digraphs which solve the (Δ, N) problem for their corresponding numbers of vertices.

1.1 Double-step digraphs

A *double-step digraph* $G(N; a, b)$ has set of vertices $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ and arcs from every vertex i to vertices $i + a \mod N$ and $i + b \mod N$, for $0 \le i \le N-1$, where

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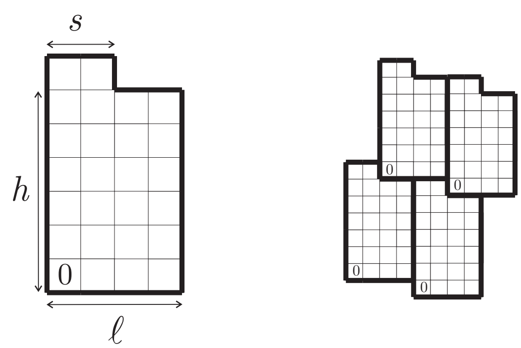

Figure 1: An L-shaped form and its tessellation.

a,b are some integers called *steps* such that $1 \leq a < b \leq N-1$. Because of the automorphisms $i \mapsto i+\alpha$ for $1 \leq \alpha \leq N-1$, the double-step digraphs are vertex-transitive. Moreover, they are strongly connected if and only if $\gcd(N,a,b)=1$. It is known that the maximum order N of a double-step digraph with diameter k is upper bounded by the Moore-like bound $N \leq M_{DSD}(2,k) = \binom{k+2}{2}$, where the equality would hold if all the numbers $ma+nb$ were different modulo N, with $m,n \geq 0$ and $m+n \leq k$. In fact, this bound cannot be attained for $k > 1$.

Every double-step digraph has an L-shaped form associated, which tessellates the plane. If one of the steps, say a, equals 1, we can choose an L-shaped tile with dimensions $\ell = b$, h being the quotient obtained dividing N by ℓ, $w = \ell - s$ with s being the remainder of such a division, and $y = 1$. Then, $N = \ell h + s$ with $0 \leq s < \ell$ (see Fig. 1).

1.2 Unilateral distance

Given a digraph $G = (V,A)$, the *unilateral distance* between two vertices $u,v \in V$ is defined as

$$\operatorname{dist}^*_G(u,v) = \min\{\operatorname{dist}_G(u,v), \operatorname{dist}_G(v,u)\} = \min\{\operatorname{dist}_G(u,v), \operatorname{dist}_{\overline{G}}(u,v)\},$$

where dist_G is the standard distance in digraph G and $\operatorname{dist}_{\overline{G}}$ is the distance in its *converse* digraph $\overline{G}$, that is, the digraph obtained by changing the directions of all the arcs of G. From this concept, we can define the *unilateral eccentricity* ecc^* from vertex u, the *unilateral radius* r^* of G, and the *unilateral diameter* D^* of G as follows:

$$\operatorname{ecc}^*(u) = \max_{v \in V}\{\operatorname{dist}^*_G(u,v)\}, \quad r^* = \min_{u \in V}\{\operatorname{ecc}^*(u)\}, \quad \text{and} \quad D^* = \max_{u \in V}\{\operatorname{ecc}^*(u)\}.$$

As an example, if we have $G = C_N$, the directed cycle on N vertices, then $D^* = \lfloor N/2 \rfloor$.

Note that, obviously, these parameters have as lower bounds the ones corresponding to the underlying graph, obtained from digraph G by changing the arcs for edges without direction.

2 The unilateral diameter of double-step digraphs with step $a = 1$

In this section we study the unilateral diameter of the double-step digraphs with $a = 1$ having 'small' b. Although we have not been able to prove that the optimal results can be obtained always by taking such values of the steps, computational experiments seem to support this claim. In fact, as we see in the next section, this approach allows us to solve the (Δ, D^*) problem for every value of D^*, and also to solve the $(\Delta, N)^*$ problem for infinitely many values of N.

As we have already seen, a double-step digraph $G(N; 1, b)$ with $N = \ell h + s$ and $0 \leq s < \ell$, can be described by an L-shaped form with dimensions $\ell = b$, $h = \lfloor N/\ell \rfloor$, $y = 1$, and $w = \ell - s$. See again Fig. 1 for both cases $s \neq 0$ and $s = 0$. In this context we have the following result for the unilateral diameter D^*.

Proposition 2.1. *For $N = \ell h + s$, where $1 < \ell \leq \lceil \sqrt{N} \rceil$ and $0 \leq s \leq \ell - 1$, a double-step digraph $G(N; a, b)$ with $a = 1$ and $b = \ell$ has unilateral diameter*

$$D^* = \begin{cases} \left\lfloor \dfrac{\ell + h + s - 1}{2} \right\rfloor & \text{if } 0 \leq s \leq \ell - 2, \\ \left\lfloor \dfrac{\ell + h - 1}{2} \right\rfloor & \text{if } s = \ell - 1. \end{cases} \tag{1}$$

If there is not restriction for the value of ℓ, then the values in Eq. (1) are an upper bound for the unilateral diameter. The first case is for $N = 430$, in which the unilateral diameter is 22 and the upper bound given by Eq. (1) is 23.

3 The (Δ, D^*) and $(\Delta, N)^*$ problems for double-step digraphs with unilateral diameter

3.1 The (Δ, D^*) problem

In our context, the (Δ, D^*) problem consists of finding the double-step digraph $G(N; a, b)$ with maximum number of vertices given a unilateral diameter D^* and the maximum degree $\Delta = 2$, that is, to find the steps that maximize the number of vertices for such a unilateral diameter. To get a Moore-like bound (see Miller and Sirán [4]), notice that at distance $k = 1, 2, \ldots, D^*$ from vertex 0 there are at most $2(k+1)$ vertices ($k+1$ of them going forward and the other $k+1$ going backwards). Then, this gives

$$N \leq M(2, D^*) = 2(1 + 2 + \cdots + D^* + 1) - 1 = (D^*)^2 + 3D^* + 1. \tag{2}$$

Moreover, if the maximum is attained, we get an 'optimal' X-shaped tile which tessellates the plane, and this allow us to solve the (Δ, D^*) problem, as shown in the following result.

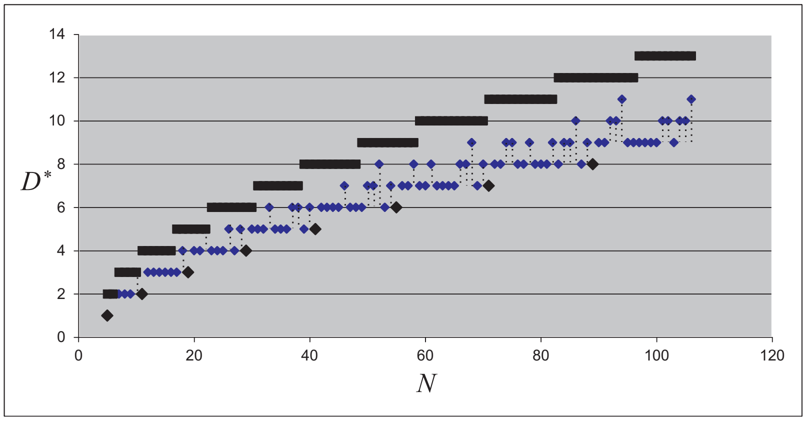

Figure 2: The minimum unilateral diameter D^* with respect to the number of vertices N, for $5 \leq N \leq 106$. (The largest points correspond to the (Δ, D^*) problem, and the thick lines to the upper bound given in Proposition 3.2.)

Proposition 3.1. *For each integer value $k \geq 0$, the double-step digraph $G(N;1,b)$, with $N = M(2,k) = k^2+3k+1$ and $b = k+1$ has unilateral diameter $D^* = k$.*

3.2 The $(\Delta, N)^*$ problem

In our context, the $(\Delta, N)^*$ problem consists of finding the minimum unilateral diameter D^* in double-step digraphs given a number of vertices N and their maximum degree $\Delta = 2$, that is, to find the steps that minimize the unilateral diameter for such a number of vertices. We begin with the following general upper bound for the unilateral diameter.

Proposition 3.2. *Given any number of vertices $N \geq 5$, there exists a double-step digraph with unilateral diameter D^* satisfying*

$$D^* \leq \left\lceil \sqrt{2(N+2)} \right\rceil - 2.$$

To solve the $(\Delta, N)^*$ problem for double-step digraphs with minimum unilateral diameter we consider the case $s = \ell - 1$ of Proposition 2.1. Moreover, to keep track of the excluded vertices from the maximum $M(2,k)$, we define r as the subindex of the triangular number $T_r = 1+2+\cdots+r = \binom{r+1}{2}$.

Proposition 3.3. (a) *If $0 \leq r < \frac{1}{2}(\sqrt{8k+9}-1)$, the double-step digraph $G(N;a,b)$, with number of vertices $N = k^2+3k+1-r(r+1)$ and steps $a = 1$ and $b = \ell = k-r+1$, has minimum unilateral diameter $D^* = k$.*
(b) *If $0 \leq r < \sqrt{k+1}$, the double-step digraph $G(N;a,b)$, with number of vertices $N = k^2+2k-r^2$ and steps $a = 1$ and $b = \ell = k-r+1$, has minimum unilateral diameter $D^* = k$.*

Table 1: Some results of the (Δ, D^*) and $(\Delta, N)^*$ problems solved with Proposition 3.3

Problem	$\ell+h$	r	ℓ	h	$N=\ell h+\ell-1$	D^*
(Δ, D^*)	even	0	$k+1$	$k+2$	k^2+3k+1	k
$(\Delta, N)^*$	even	1	k	$k+3$	k^2+3k-1	k
$(\Delta, N)^*$	even	2	$k-1$	$k+4$	k^2+3k-5	k
$(\Delta, N)^*$	even	3	$k-2$	$k+5$	$k^2+3k-11$	k
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$(\Delta, N)^*$	odd	0	$k+1$	$k+1$	k^2+2k	k
$(\Delta, N)^*$	odd	1	k	$k+2$	k^2+2k-1	k
$(\Delta, N)^*$	odd	2	$k-1$	$k+3$	k^2+2k-4	k
$(\Delta, N)^*$	odd	3	$k-2$	$k+4$	k^2+2k-9	k
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

As shown in Fig. 2, the unilateral diameter D^* does not increase monotonously with the number of vertices N.

Note that if we fix r for any k large enough, we get an infinite family of digraphs with minimum unilateral diameter for each number of vertices. See some examples of the cases of Proposition 3.3 in Table 1.

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