Section 4.1
Connectivity: Properties and Structure

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INTRODUCTION

Connectivity is one of the central concepts of graph theory, from both a theoretical and a practical point of view. Its theoretical implications are mainly based on the existence of nice *max-min* characterization results, such as Menger's theorems. In these theorems, one condition which is clearly necessary also turns out to be sufficient. Moreover, these results are closely related to some other key theorems in graph theory: Ford and Fulkerson's theorem about flows and Hall's theorem on perfect matchings. With respect to the applications, the study of connectivity parameters of graphs and digraphs is of great interest in the design of reliable and fault-tolerant interconnection or communication networks.

Since graph connectivity has been so widely studied, we limit ourselves here to the presentation of some of the key results dealing with finite simple graphs and digraphs. For results about infinite graphs and connectivity algorithms the reader can consult, for instance, Aharoni and Diestel [AhDi94], Gibbons [Gi85], Halin [Ha00], Henzinger, Rao, and Gabow [HeRaGa00], Wigderson [Wi92]. For further details, we refer the reader to some of the good textbooks and surveys available on the subject: Berge [Be76], Bermond, Homobono, and Peyrat [BeHoPe89], Frank [Fr90, Fr94, Fr95], Gross and Yellen [GrYe06], Hellwig and Volkmann [HeVo08], Lovász [Lo93], Mader [Ma79], Oellermann [Oe96], Tutte [Tu66].

4.1.1 Connectivity Parameters

In this first subsection the basic notions of connectivity and edge-connectivity of simple graphs and digraphs are reviewed.

NOTATION: Given a graph or digraph G, the vertex-set and edge-set are denoted $V(G)$ and $E(G)$, respectively. Often, when there is no ambiguity, we omit the argument and refer to these sets as V and E.

Preliminaries

DEFINITIONS

D1: A graph is ***connected*** if there exists a walk between every pair of its vertices. A graph that is not connected is called ***disconnected***.

D2: The subgraphs of G which are maximal with respect to the property of being connected are called the ***components*** of G.

D3: Let $G = (V, E)$ be a graph and $U \subset V$. The ***vertex-deletion subgraph*** $G - U$ is the graph obtained from G by deleting from G the vertices in U. That is, $G - U$ is the subgraph induced on the vertex subset $V - U$. If $U = \{u\}$, we simply write $G - u$.

D4: Let $G = (V, E)$ be a graph and $F \subset E$. The ***edge-deletion subgraph*** $G - F$ is the subgraph obtained from G by deleting from G the edges in F. Thus, $G - F = (V, E - F)$. As in the case of vertex deletion, if $F = \{e\}$, it is customary to write $G - e$ rather than $G - \{e\}$.

D5: A ***disconnecting (vertex-)set*** (or ***vertex-cut***) of a connected graph G is a vertex subset U such that $G - U$ has at least two different components.

D6: A vertex v is a ***cut-vertex*** of a connected graph G if $\{v\}$ is a disconnecting set of G.

D7: A ***disconnecting edge-set*** (or ***edge-cut***) of a connected graph G is an edge subset F such that $G - F$ has at least two different components.

D8: An edge e is a ***bridge*** (or ***cut-edge***) of a connected graph G if $\{e\}$ is a disconnecting edge-set of G.

FACTS

F1: Every nontrivial connected graph contains at least two vertices that are not cut-vertices.

F2: An edge is a bridge if and only if it lies on no cycle.

Vertex- and Edge-Connectivity

The simplest way of quantifying connectedness of a graph is by means of its parameters *vertex-connectivity* and *edge-connectivity*.

DEFINITIONS

D9: The ***(vertex-)connectivity*** $\kappa(G)$ of a graph G is the minimum number of vertices whose removal from G leaves a disconnected or a trivial graph.

D10: The ***edge-connectivity*** $\lambda(G)$ of a nontrivial graph G is the minimum number of edges whose removal from G results in a disconnected graph.

NOTATION: When the context is clear, we suppress the dependence on G and simply use κ and λ.

NOTATION: In some other sections of the *Handbook*, $\kappa_v(G)$ and $\kappa_e(G)$ are used instead of $\kappa(G)$ and $\lambda(G)$.

EXAMPLE

E1: Figure 4.1.1 shows an example of a graph with $\kappa = 2$ and $\lambda = 3$.

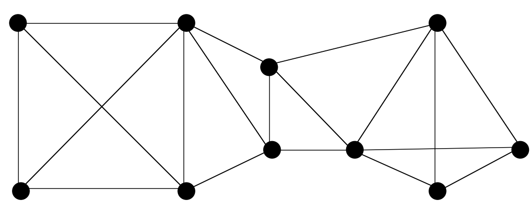

Figure 4.1.1: $\kappa = 2$ and $\lambda = 3$.

FACTS

F3: We have $\kappa = 0$ if and only if G is disconnected or $G = K_1$. If G has order n, then $\kappa = n - 1$ if and only if G is the complete graph K_n. In this case, the removal of $n - 1$ vertices results in the trivial graph K_1. Moreover, if $G \neq K_n$ is a connected graph, then $1 \leq \kappa \leq n - 2$ and there exists a disconnecting set U of κ vertices.

F4: If $G \neq K_1$ we have $\lambda = 0$ if G is disconnected. By convention, we set $\lambda(K_1) = 0$.

F5: If $G \neq K_1$ is connected, then the removal of λ edges results in a disconnected graph with precisely two components.

F6: The parameters κ and λ can be computed in polynomial time.

Relationships Among the Parameters

NOTATION: The ***minimum degree*** of a graph G is denoted $\delta(G)$. When the context is clear, we simply write δ. (In some other sections of the *Handbook*, the notation $\delta_{min}(G)$ is used.)

FACTS

F7: [Wh32] For any graph, $\kappa \leq \lambda \leq \delta$.

F8: [ChHa68] For all integers a, b, c such that $0 < a \leq b \leq c$, there exists a graph G with $\kappa = a$, $\lambda = b$, and $\delta = c$.

DEFINITIONS

D11: A graph G is ***maximally connected*** when $\kappa = \lambda = \delta$, and G is ***maximally edge-connected*** when $\lambda = \delta$.

D12: A graph G with connectivity $\kappa \geq k \geq 1$ is called ***k-connected***. Equivalently, G is k-connected if the removal of fewer than k vertices leaves neither a disconnected graph nor a trivial one. Analogously, if $\lambda \geq k \geq 1$, G is said to be ***k-edge-connected***.

D13: A connected graph G without cut-vertices ($\kappa > 1$ or $G = K_2$) is called a ***block***.

Some Simple Observations

The following facts are simply restatements of the definitions.

FACTS

F9: A nontrivial graph is 1-connected if and only if it is connected.

F10: If G is k-connected, either $G = K_{k+1}$ or it has at least $k+2$ vertices and $G - U$ is still connected for any $U \subset V$ with $|U| < k$.

F11: A graph G is k-edge-connected if the deletion of fewer than k edges does not disconnect it.

F12: Every block with at least three vertices is 2-connected.

Internally-Disjoint Paths and Whitney's Theorem

DEFINITIONS

D14: An ***internal vertex*** of a path is a vertex that is neither the initial nor the final vertex of that path.

D15: The paths $P_1, P_2, \ldots, P_k$ joining the vertices u and v are said to be ***internally-disjoint*** (or ***openly-disjoint***) u–v paths if no two paths in the collection have an internal vertex in common. Thus, $V(P_i) \cap V(P_j) = \{u, v\}$ for $i \neq j$.

FACTS

F13: [Wh32] A graph G with order $n \geq 3$ is 2-connected if and only if any two vertices of G are joined by at least two internally-disjoint paths.

F14: Fact F13 implies that every 2-connected graph is a block.

F15: A graph G with at least three vertices is a block if and only if every two vertices of G lie on a common cycle.

Strong Connectivity in Digraphs

For basic concepts on digraphs, see, for example, the textbooks of Bang-Jensen and Gutin [BaGu01], Chartrand, Lesniak, and Zhang [ChLeZh11], Harary, Norman, and Cartwright [HaNoCa68].

DEFINITIONS

D16: In a digraph G, vertices u and v are ***mutually reachable*** if G contains both a directed u–v walk and a directed v–u walk.

D17: A digraph G is said to be ***strongly connected*** if every two vertices u and v are mutually reachable.

D18: For a strongly connected digraph G, the ***(vertex) connectivity*** $\kappa = \kappa(G)$ is defined as the minimum number of vertices whose removal leaves a non-strongly connected or trivial digraph. Analogously, if G is not trivial, its ***edge-connectivity*** $\lambda = \lambda(G)$ is the minimum number of directed edges (or ***arcs***) whose removal results in a non-strongly connected digraph.

D19: Let G be an undirected graph. The ***associated symmetric digraph*** G^* is the digraph obtained from G by replacing each edge $uv \in E(G)$ by the two directed edges (u, v) and (v, u) forming a *digon*.

REMARKS

R1: In our context, the interest for studying digraphs is that we can deal with an undirected graph G by considering G^*. In particular, $\kappa(G^*) = \kappa(G)$, and, since a minimum edge-disconnecting set cannot contain digons, we also have $\lambda(G^*) = \lambda(G)$.

NOTATION: The symbols δ^+ and δ^- denote the minimum outdegree and indegree among the vertices of a digraph G. Then, the ***minimum degree*** of G is defined as $\delta = \min\{\delta^+, \delta^-\}$.

R2: Note that, if G is a strongly connected digraph, then $\delta \geq 1$. The following result, due to Geller and Harary, is the analogue of (and implies) Fact F7.

FACT

F16: [GeHa70] For any digraph, $\kappa \leq \lambda \leq \delta$.

TERMINOLOGY: A digraph G is said to be ***maximally connected*** when $\kappa = \lambda = \delta$, and G is ***maximally edge-connected*** when $\lambda = \delta$.

An Application to Interconnection Networks

The interconnection network of a communication or distributed computer system is usually modeled by a (directed) graph in which the vertices represent the switching elements or processors, and the communication links are represented by (directed) edges. *Fault-tolerance* is one of the main factors that have to be taken into account in the design of an interconnection network. See, for instance, the survey of Bermond, Homobono, and Peyrat [BeHoPe89] and the book by Xu [Xu01]. Indeed, it is generally expected that the system be able to work even if several of its elements fail. Thus, it is often required that the (di)graph associated with the interconnection network be sufficiently connected, and, in most cases, a good design requires that this (di)graph has maximum connectivity. Communication networks are discussed in §11.4 of the *Handbook*.

4.1.2 Characterizations

When a graph G is k-connected we need to delete at least k vertices to disconnect it. Clearly, if any pair u, v of vertices can be joined by k internally-disjoint $u{-}v$ paths, G is k-connected. In fact, it turns out that the converse statement is also true. That is, in a k-connected graph any two vertices can be joined by k internally-disjoint paths. We review in this subsection some key theorems of this type that characterize k-connectedness.

Menger's Theorems

DEFINITION

D20: Let u and v be two non-adjacent vertices of a connected graph $G \neq K_n$. A $(u|v)$***-disconnecting set*** X, or simply $(u|v)$***-set***, is a disconnecting set $X \subset V - \{u, v\}$ whose removal from G leaves u and v in different components.

NOTATION: For any pair of non-adjacent vertices u and v, $\kappa(u|v)$ denotes the minimum number of vertices in a $(u|v)$-set.

NOTATION: For any two vertices u and v, $\kappa(u{-}v)$ denotes the maximum number of internally-disjoint $u{-}v$ paths.

FACTS

F17: For any graph G, $\kappa(G) = \min\{\kappa(u|v) : u, v \in V, \text{ nonadjacent}\}$.

F18: (Menger's theorem) [Me27] For any pair of non-adjacent vertices u and v,

$$\kappa(u{-}v) = \kappa(u|v)$$

F19: Although $\kappa(u{-}v)$ can be arbitrarily smaller than the minimum of the degrees of u and v, Mader proved that every finite graph contains vertices for which equality holds:

F20: [Ma73] Every connected non-trivial graph contains adjacent vertices u and v for which $\kappa(u{-}v) = \min\{deg(u), deg(v)\}$.

NOTATION: For any pair of distinct vertices u and v, $\lambda(u|v)$ denotes the minimum number of edges whose removal from G (G non-trivial) leaves u and v in different components and $\lambda(u{-}v)$ denotes the maximum number of *edge-disjoint* $u{-}v$ paths.

F21: For any non-trivial graph G, $\lambda(G) = \min\{\lambda(u|v),\, u, v \in V\}$.

F22: (Edge-analogue of Menger's theorem) [ElFeSh56, FoFu56] For any pair of vertices u and v,

$$\lambda(u{-}v) = \lambda(u|v).$$

REMARKS

R3: Digraph versions of Menger's theorems are the same except that all paths are directed paths.

R4: The edge form and arc form of Menger's theorem were proved by Ford and Fulkerson [FoFu56] using *network-flow* methods. Network flow is discussed in Chapter 11 of this *Handbook*.

Other Versions and Generalizations of Menger's Theorem

In addition to the ones given below, there exist other versions and generalizations of Menger's theorem; see, for example, Diestel [Di00], Frank [Fr95], and McCuaig [McCu84]. A comprehensive survey about variations of Menger's theorem can be found in Oellermann [Oe12].

DEFINITIONS

D21: Given $A, B \subset V$, an $A{-}B$ ***path*** is a $u{-}v$ path P with $u \in A$, $v \in B$, $u \neq v$, and any other vertex of P is neither in A nor in B.

D22: A set $X \subset V$ ***separates*** A from B (or is $(A|B)$***-separating***) if every $A{-}B$ path in G contains a vertex of X.

D23: An A***-path*** is an $A{-}B$ path with $A = B$.

D24: A subset $X \subset V - A$ ***totally separates*** A if each component of $G - X$ contains at most one vertex of A (or, equivalently, every A-path between different vertices contains some vertex of X).

D25: A vertex subset is an ***independent set*** if no two of its vertices are adjacent.

NOTATION: The maximum number of (internally-)disjoint $A{-}B$ paths is denoted $\kappa(A{-}B)$, and the size of a minimum $(A|B)$-separating set is denoted $\kappa(A|B)$.

FACTS

F23: The minimum number of vertices separating A from B is equal to the maximum number of disjoint $A{-}B$ paths. That is,

$$\kappa(A{-}B) = \kappa(A|B).$$

F24: If A is an independent set, the maximum number of internally-disjoint A-paths is at most the minimum number of vertices in a totally A-separating set, that is,
$$\kappa(A{-}A) \leq \kappa(A|A).$$

F25: The corresponding Menger-type result does not hold and inequality can be strict. In fact, there exist examples for which $\kappa(A{-}A) = \kappa(A|A)/2$.

F26: Gallai [Ga61] conjectured that Fact F25 corresponds to the "extremal" situation and that always $\kappa(A{-}A) \geq \kappa(A|A)/2$, and Lovász [Lo76] conjectured that $\lambda(A{-}A) \geq \lambda(A|A)/2$. Both conjectures were proved by Mader.

F27: [Ma78b, Ma78c] $\kappa(A{-}A) \geq \kappa(A|A)/2$ and $\lambda(A{-}A) \geq \lambda(A|A)/2$.

REMARK

R5: The classical version of Menger's theorem (Fact F18) is easily derived from Fact F23 by taking A and B as the sets of vertices adjacent to u and v, respectively.

Another Menger-Type Theorem

NOTATION: For any pair of vertices u and v, $\kappa_n(u{-}v)$ denotes the maximum number of internally-disjoint $u{-}v$ paths of length less than or equal to n. For any pair of non-adjacent vertices u and v, $\kappa_n(u|v)$ denotes the minimum number of vertices of a set $X \subset V - \{u, v\}$ such that every $u{-}v$ path in $G - X$ has length greater than n.

FACTS

F28: There are examples for which we have the strict inequality $\kappa_n(u{-}v) < \kappa_n(u|v)$. However, for $n = d(u, v) \geq 2$ (i.e., for shortest $u{-}v$ paths), we have $\kappa_n(u{-}v) = \kappa_n(u|v)$. This Menger-type result is equivalently restated as Fact F29.

F29: [EnJaSl77, LoNePl78] The maximum number of internally-disjoint shortest $u{-}v$ paths is equal to the minimum number of vertices (different from u and v) necessary to destroy all shortest $u{-}v$ paths.

Whitney's Theorem

In a connected graph, there exists a path between any pair of its vertices, and if the graph is 2-connected, then there exist at least two internally-disjoint paths between two distinct vertices (Fact F13). As a corollary of Menger's theorem, we have the remarkable result that this property can be generalized to k-connected graphs, which was independently proved by Whitney. It provides a natural and intrinsic characterization of k-connected graphs.

FACTS

F30: (Whitney's theorem) [Wh32] A non-trivial graph G is k-connected if and only if for each pair u, v of distinct vertices there are at least k internally-disjoint $u{-}v$ paths (or, alternatively, if and only if every cut-set has at least k vertices).

F31: (Edge version of Whitney's theorem) A nontrivial graph G is k-edge-connected if and only if for each pair u, v of distinct vertices there exist at least k edge-disjoint $u{-}v$ paths.

F32: (The Fan Lemma) Let G be a k-connected graph ($k \geq 1$). Let $v \in V$ and let $B \subset V$, $|B| \geq k$, $v \notin B$. Then there exist distinct vertices $b_1, b_2, \ldots, b_k$ in B and a $v{-}b_i$ path P_i for each $i = 1, 2 \ldots, k$, such that the paths $P_1, P_2, \ldots, P_k$ are internally-disjoint (that is, with only vertex v in common) and $V(P_i) \cap B = \{b_i\}$ for $i = 1, 2, \ldots k$.

Other Characterizations

Another interesting characterization of k-connected graphs was independently conjectured by Frank and Maurer. The conjecture was proved by Lovász and by Györi (who worked independently), and it appears as Fact F33. Su proved a characterization of k-edge-connectivity for digraphs (Fact F34).

FACTS

F33: [Lo77, Gy78] A graph G with $n \geq k+1$ vertices is k-connected if and only if, for any distinct vertices $u_1, u_2, \ldots, u_k$ and any positive integers $n_1, n_2, \ldots, n_k$ such that $n_1 + n_2 + \cdots + n_k = n$, there is a partition $V_1, V_2, \ldots, V_k$ of $V(G)$ such that $u_i \in V_i$, $|V_i| = n_i$, and the induced subgraph $G(V_i)$ is connected, $1 \leq i \leq n$.

F34: [Su97] A digraph G with at least k edges is k-edge-connected if and only if, for any k distinct arcs $e_i = (u_i, v_i)$, $1 \leq i \leq k$, the digraph $G - \{e_1, e_2, \ldots, e_k\}$ contains k edge-disjoint spanning arborescences (rooted trees) $T_1, T_2, \ldots, T_k$ such that T_i is rooted at v_i, $1 \leq i \leq n$.

4.1.3 Structural Connectivity

Here our purpose is to give results about certain configurations that must be present in a k-connected or k-edge-connected graph.

Cycles Containing Prescribed Vertices

The first is a classical result by Dirac, which generalizes Fact F15.

FACTS

F35: [Di60] Let G be a k-connected graph, $k \geq 2$. Then G contains a cycle through any given k vertices.

F36: [WaMe67] Let G be a k-connected graph with $k \geq 3$. Then G has a cycle containing a given set H with $k+1$ vertices if and only if there is no set $T \subset V - H$ with $|T| = k$ vertices whose removal separates the vertices of H from each other.

The Lovász–Woodall Conjecture

Lovász [Lo74] and Woodall [Wo77] independently conjectured that every k-connected graph has a cycle containing a given set F of k *independent edges* (that is, no two edges have a vertex in common), if and only if F is not an edge-disconnecting set of odd cardinality. Partial results on this conjecture are given in Facts F37 $\rightarrow$ F39.

FACTS

F37: [Lo74, Lo77, ErGy85, Lo90, Sa96] The *Lovász–Woodall Conjecture* is true for $k = 3, 4, 5$.

F38: [HaTh82] The *Lovász–Woodall Conjecture* is true assuming that G is $(k+1)$-connected (without restriction on the edge set F).

F39: [Ka02] Under the same assumptions of the conjecture, F is either contained in a cycle or in two disjoint cycles.

TERMINOLOGY: A subset of independent edges is also called a ***matching***. Matchings are discussed in Section 11.3 of this *Handbook*.

Paths with Prescribed Initial and Final Vertices

Given any two subsets $A, B \subset V$ of k vertices of a k-connected graph, the existence of k disjoint paths P_i $(1 \leq i \leq k)$ connecting A and B is guaranteed by Menger's theorem. Menger's theorem does not, however, ensure that each of these paths can be so chosen to join a fixed u_i, v_i pair of vertices, $u_i \in A$, $v_i \in B$, $(1 \leq i \leq k)$. Now we consider the existence of paths with prescribed end-vertices.

DEFINITIONS

D26: A graph G is called ***k-linked*** if it has at least $2k$ vertices, and for every sequence $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$ of $2k$ different vertices, there exists a u_i–v_i path P_i, $i = 1, 2, \ldots, k$, such that the k paths are vertex-disjoint.

D27: A graph is ***weakly k-linked*** if it has at least $2k$ vertices, and for every k pairs of vertices (u_i, v_i), there exists a u_i–v_i path P_i, $1 \leq i \leq k$, such that the k paths are edge-disjoint.

D28: A graph is said to be ***k-parity-linked*** if one can find k disjoint paths with prescribed end-vertices and prescribed parities of the lengths.

D29: The ***bipartite index*** of a graph is the smallest number of vertices whose deletion creates a bipartite graph.

FACTS

F40: A k-linked graph is always $(2k-1)$-connected, but the converse is not true.

F41: [Ju70], [LaMa70] (independently) For each k, there exists an integer $f(k)$ such that if $\kappa \geq f(k)$ then G is k-linked.

F42: Thomassen [Th80a] and Seymour independently characterized the graphs that are not 2-linked. This is the first problem in the so-called ***k*-paths problem** that has been solved using the Robertson–Seymour theory [RoSe85].

NOTATION: For $k \geq 1$, $g(k)$ denotes the smallest integer such that every $g(k)$-edge-connected graph G is weakly k-linked.

CONJECTURE

[Th80a] For every integer $k \geq 1$, $g(2k+1) = g(2k) = 2k+1$.

FACTS

F43: [Ok84, Ok85, Ok87] If $k \geq 3$ is odd, $u_1, u_2, \ldots, u_k, v_1, v_2 \ldots, v_k$ are (not necessarily distinct) vertices from a set T with $|T| \leq 6$, and $\lambda(u_i, v_i) \geq k$ $(1 \leq i \leq k)$, then there exists a $u_i - v_i$ path for $1 \leq i \leq k$ such that the k paths are edge-disjoint.

F44: [Hu91] For every integer $k \geq 1$, $g(2k+1) \leq 2k+2$ and $g(2k) \leq 2k+2$.

F45: [Ok88, Ok90a] For every integer $k \geq 1$,

(a) $g(2k+1) \leq 3k$ and $g(2k+2) \leq 3k+2$,

(b) $g(3k) \leq 4k$ and $g(3k+2) \leq 4k+2$.

F46: [Th01] Every $f(k)$-connected graph (defined in Fact F41) with bipartite index at least $4k-3$ is k-parity-linked.

F47: [Su97] Let G be a k-edge-connected digraph, and let (u_1, f_1, v_1), (u_2, f_2, v_2), $\ldots$, (u_k, f_k, v_k) be any k triples, where $u_1, u_2 \ldots, u_k$, $v_1, v_2 \ldots, v_k$ are not necessarily distinct vertices, and $f_1, f_2, \ldots, f_k$ are k distinct arcs, either of the form $f_i = (u_i, t_i)$, $i = 1, \ldots, k$, or $f_i = (t_i, v_i)$, $i = 1, \ldots, k$. Then there exist k edge-disjoint u_i–v_i paths P_i in G such that $f_i \in E(P_i)$, $i = 1, \ldots, k$.

Subgraphs

High connectivity implies a large minimum degree (Fact F7). Conversely, a large minimum degree does not guarantee high connectivity (Fact F8). However, it does ensure the existence of a highly connected subgraph.

FACT

F48: [Ma72a] Every graph of minimum degree at least $4k$ contains a k-connected subgraph.

REMARK

R6: In fact, Mader [Ma72a] proved that if the average of the degrees of the vertices of G is at least $4k$, then G contains a k-connected subgraph. Concerning the proof of Fact F48, see also Thomassen [Th88].

4.1.4 Analysis and Synthesis

An interesting question in the study of graph connectivity is to describe how to obtain every k-(edge-)connected graph from a given "simple" one by a succession of elementary operations preserving k-connectedness. A classical result on this topic is Tutte's theorem, which states how to construct all 3-connected graphs, starting with a *wheel graph*. We also consider some relevant results dealing with deletion of edges or vertices. Finally, some facts concerning minimally and critically k-connected graphs, as well as a reference to connectivity augmentation problems, are considered.

Contractions and Splittings

DEFINITIONS

D30: The ***contraction*** of an edge uv consists of the identification of its endpoints u and v (keeping the old adjacencies but removing the self-loop from $u = v$ to itself). Let G be a k-connected graph. An edge of G is said to be ***k-contractible*** if its contraction results in a k-connected graph.

D31: The converse operation is called ***splitting***: A vertex w with degree δ is replaced by an edge uv in such a way that some of the vertices adjacent to w are now adjacent to u and the rest are adjacent to v. Moreover, if the new vertices u, v have degrees at least $k = \delta/2 + 1$ we speak about a ***k-vertex-splitting***.

D32: For any integer $n \geq 4$, the ***wheel graph*** W_n is the n-vertex graph obtained by joining a vertex to each of the $n - 1$ vertices of the cycle graph C_{n-1}.

FACTS

F49: If G is a k-connected graph, the operations of k-vertex splitting and edge addition always produce a graph that is also (at least) k-connected. In fact, as shown below, for $k = 3$ these operations suffice to derive all 3-connected graphs.

F50: [Th80b] Every 3-connected graph distinct from K_4 has a 3-contractible edge.

F51: [Th81] Every *triangle-free* (no 3-cycles) k-connected graph has a k-contractible edge.

F52: [Tu61] Every 3-connected graph can be obtained from a wheel by a finite sequence of 3-vertex-splittings and edge additions.

REMARK

R7: In general, k-connectedness does not ensure the existence of k-contractible edges.

EXAMPLE

E2: In Figure 4.1.2, the cube graph Q_3 is synthesized from the wheel graph W_5 in four steps. All but the second step are 3-vertex-splittings.

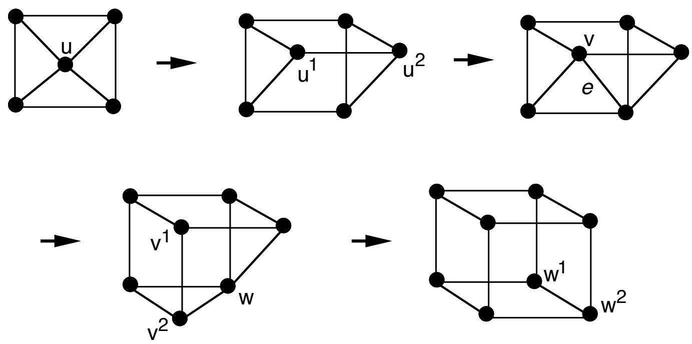

Figure 4.1.2: A 4-step Tutte synthesis of the cube graph Q_3.

REMARKS

R8: Thomassen used Fact F50 to give a short proof of Kuratowski's theorem on planarity. Fact F50 can also be derived from Tutte's theorem (Fact F52).

R9: Since Tutte's paper, the distribution of contractible edges in graphs of given connectivity has been extensively studied. For a comprehensive survey of this subject, we refer the reader to Kriesell [Kr02], where the author also considers subgraph contractions (see below).

R10: Fact F52 is a reformulation of the following proposition [Tu61]: a 3-connected graph is either a wheel, or it contains an edge whose removal leaves a 3-connected subgraph, or it contains a 3-contractible edge that is not in a cycle of length 3.

R11: Slater [Sl74] gave a similar result for constructing all 4-connected graphs starting from K_5, but in this case three more operations are required. For $k \geq 5$ the problem is still open. However, Lovász [Lo74] and Mader [Ma78a] managed to construct all k-edge-connected pseudographs (loops and multiple edges allowed) for every k even and odd, respectively.

Subgraph Contraction

The contraction of a subgraph is a natural generalization of edge contraction.

DEFINITION

D33: A connected subgraph H of a k-connected graph G is said to be k***-contractible*** if the contraction of H into a single vertex results in a k-connected graph.

FACTS

F53: [McOt94] Every 3-connected graph on $n \geq 9$ vertices has a 3-contractible path of length two.

F54: [ThTo81] Every 3-connected graph with minimum degree at least four contains a 3-contractible cycle.

F55: [Kr00] Every 3-connected graph of order at least eight has a 3-contractible subgraph of order four.

CONJECTURE

[McOt94] For every n, a 3-connected graph of sufficiently large order has a 3-contractible subgraph of order n.

Edge Deletion

DEFINITION

D34: A subgraph H of a k-edge-connected graph G is said to be ***ρ-reducible*** if the graph obtained from G by removing the edges of H is $(k-\rho)$-connected.

FACTS

F56: [Ma74] Every k-connected graph G with minimum degree at least $k+2$ contains a cycle C such that $G - E(C)$ is k-connected.

F57: [Ok88] Let G be a k-edge-connected graph with $k \geq 4$ even. Let $\{u, v\} \subset V$ and $\{e_1, e_2, f\} \subset E$, $e_i \neq f$ $(i = 1, 2)$. Then,

(a) There exists a 2-reducible cycle containing e_1 and e_2, but not f.

(b) There exists a 2-reducible $u{-}v$ path containing e_1, but not f.

F58: [Ok90b] Let G be a k-edge-connected graph with $k \geq 2$ even. If $\{u_1, v_1, u_2, v_2\}$ are distinct vertices, with edges $e_0 = v_1v_2$, $e_i = u_iv_i$ $(i = 1, 2)$, and there is no edge-cut with k or $k+1$ elements containing $\{e_0, e_1, e_2\}$, then there exists a 2-reducible cycle containing $\{e_0, e_1, e_2\}$.

F59: [HuOk92] For each odd $k \geq 3$, there exists a k-edge-connected graph containing two vertices u and v such that every cycle passing through u, v is ρ-reducible with $\rho \geq 3$.

REMARK

R12: For the case of three consecutive edges e_1, e_2, e_3 of a k-connected graph, Okamura [Ok95] also found a nontrivial equivalent reformulation of the condition that no cycle of G containing e_1, e_2, and e_3 is 2-reducible.

Vertex Deletion

FACTS

F60: [ChKaLi72] Every 3-connected graph of minimum degree at least 4 has a vertex v such that $G - v$ is 3-connected.

F61: [Th81] Every $(k + 3)$-connected graph has an induced (chordless) cycle whose deletion results in a k-connected graph.

F62: [Eg87] Every $(k + 2)$-connected triangle-free graph has an induced cycle whose deletion results in a k-connected graph.

REMARK

R13: Fact F61 was conjectured by Lovász, and Thomassen used Fact F51 to prove it.

Products of Graphs

DEFINITIONS

D35: Recall that the ***cartesian product*** of two graphs $G_i = (V_i, E_i)$, $i = 1, 2$, is the graph $G_1 \Box G_2$ with vertex set $V_1 \times V_2$, and for which vertices (x_1, x_2) and (y_1, y_2) are adjacent if $x_1 = y_1$ and $x_2y_2 \in E_2$, or $x_1y_1 \in E_1$ and $x_2 = y_2$.

D36: The ***Kronecker product*** of two graphs $G_i = (V_i, E_i)$, $i = 1, 2$, is the graph $G_1 \times G_2$ with vertex set $V_1 \times V_2$, and for which vertices (x_1, x_2) and (y_1, y_2) are adjacent if $x_1y_1 \in E_1$ and $x_2y_2 \in E_2$.

D37: [BeDeFa84] Let $G_i = (V_i, E_i)$, $i = 1, 2$, be two graphs with the edges of G_1 arbitrarily oriented, in such a way that an oriented edge from x_1 to y_1 is denoted by $e_{x_1y_1}$. For each arc $e_{x_1y_1}$, let $\pi_{e_{x_1y_1}}$ be a permutation of V_2. Then the ***twisted product*** $G_1 * G_2$ has $V_1 \times V_2$ as vertex set, with two vertices (x_1, x_2), (y_1, y_2) being adjacent if and only if either

$$x_1 = y_1 \quad \text{and} \quad x_2y_2 \in E_2$$

or

$$x_1y_1 \in E_1 \quad \text{and} \quad y_2 = \pi_{e_{x_1y_1}}(x_2).$$

D38: [BaDaFiMi09] Given two graphs $G_i = (V_i, E_i)$, $i = 1, 2$, and a non-empty vertex subset $U_1 \subset V_1$, the ***generalized hierarchical product*** $G_1(U_1) \sqcap G_2$ is the graph with vertex set $V_1 \times V_2$, and for which vertices (x_1, x_2) and (y_1, y_2) are adjacent if $x_1y_1 \in E_1$ and $x_2 = y_2$, or $x_1 = y_1 \in U_1$ and $x_2y_2 \in E_2$.

FACTS

F63: [XuYa06] For any nontrivial graphs G_1 and G_2,

$$\kappa(G_1 \Box G_2) \geq \min\{\kappa(G_1) + \delta(G_2), \kappa(G_2) + \delta(G_1)\}$$

and

$$\lambda(G_1 \Box G_2) \geq \min\{\lambda(G_1)|V_2|, \lambda(G_2)|V_1|, \delta(G_1) + \delta(G_2)\}.$$

F64: [Sp08] For any nontrivial graphs G_1 and G_2,

$$\kappa(G_1 \Box G_2) = \min\{\kappa(G_1)|V_2|, \kappa(G_2)|V_1|, \delta(G_1) + \delta(G_2)\}.$$

F65: [We62] If G_1 and G_2 are two connected graphs, then $G_1 \times G_2$ is connected if and only if G_1 and G_2 are not both bipartite graphs.

F66:

(a) [MaVu08] $\kappa(K_n \times K_m) = (n-1)(m-1)$ for any $n \geq m \geq 2$ and $n \geq 3$.

(b) [WaWu11] $\kappa(G \times K_n) = \min\{n\kappa(G), (n-1)\delta(G)\}$ for any nontrivial graph G and $n \geq 3$.

F67: [BaGVMa06, BaCeDiGVMa07]

(a) For any nontrivial graphs G_1 and G_2,

$$\min\{\kappa(G_1)|V_2|, (\delta_1 + 1)\kappa(G_2), \delta_1 + \delta_2)\} \leq \kappa(G_1 * G_2) \leq \delta_1 + \delta_2;$$

$$\min\{\lambda(G_1)|V_2|, (\delta_1 + 1)\lambda(G_2), \delta_1 + \delta_2)\} \leq \lambda(G_1 * G_2) \leq \delta_1 + \delta_2,$$

where $\delta_1 + \delta_2$ is the minimum degree of $G_1 * G_2$.

(b) If G_1 and G_2 are maximally connected, then $G_1 * G_2$ is also maximally connected.

(c) For every connected graph G, the graph $G * G$ is maximally connected.

F68: [BaDaFiMi09] The connectivity of the generalized hierarchical product satisfies

$$\kappa(G_1(U_1) \sqcap G_2) \leq \min\{\kappa(G_1)|V_2|, \kappa(U_1|U_1'), \delta(G_1(U_1) \sqcap G_2)\},$$

where $U_1' \subset V_1 - U_1$ and $\delta(G_1(U_1) \sqcap G_2) = \min\{\delta(G_1 - U_1), \delta(G_1(U_1)) + \delta_2\}$.

REMARKS

R14: The graph $G_1 * G_2$ can be viewed as formed by $|V_1|$ disjoint copies of G_2, each oriented edge x_1y_1 indicating that some perfect matching between the copies $G_1^{x_1}$, $G_1^{y_1}$ (respectively generated by the vertices x_1 and y_1 of G_1) is added. Moreover, $K_2 * G$ is a ***permutation graph*** [ChHa67].

R15: If in Definition D37, $\pi_{e_{x_1y_1}}$ is the identity permutation for any oriented edge $e_{x_1y_1}$, the twisted product $G_1 * G_2$ is the cartesian product $G_1 \Box G_2$.

R16: If U_1 is consits of only one vertex, then $G_1(U_1) \sqcap G_2$ is the standard hierarchical product [BaCoDaFi09], whereas if $U_1 = V_1$ we obtain the cartesian product $G_1 \Box G_2$.

R17: Fact F66(b) was previously proved for G bipartite in [GuVu09].

R18: Regarding Fact F66, the connectivity of Kronecker products by K_2 has been recently studied in [WaYa12].

Minimality and Criticality

A standard technique used to study a certain property $\mathcal{P}$ is to consider those graphs that are edge-minimal or vertex-minimal (critical) with respect to $\mathcal{P}$, in the sense that the removal of any vertex or edge produces a graph for which $\mathcal{P}$ does not hold.

DEFINITIONS

D39: A graph or digraph G is said to be ***minimally*** k***-connected*** if $\kappa(G) \geq k$ but, for each edge $e \in E$, $\kappa(G - e) < k$. Analogously, G is ***minimally*** k***-edge-connected*** if $\lambda(G) \geq k$, but for each $e \in E$, $\lambda(G - e) < k$.

D40: A vertex u of a digraph has ***half degree*** k if either $deg^+(u) = k$ or $deg^-(u) = k$.

FACTS

F69: [Ma71, Ma72b] Every minimally k-connected (or k-edge-connected) graph contains at least $k + 1$ vertices of degree k.

F70: [Ma72b] Every cycle of a minimally k-connected graph contains a vertex of degree k.

F71: Every cycle in a k-connected graph G contains either a vertex of degree k or an edge whose removal does not lower the connectivity of G.

F72: [Ha81] Every minimally k-connected digraph contains at least $k + 1$ vertices of half degree k.

F73: [Ma02] Every minimally k-connected digraph contains at least $k + 1$ vertices of outdegree k and at least $k + 1$ vertices of indegree k.

REMARKS

R19: Halin [Ha69, Ha00] proved the existence of a vertex of degree k in every minimally k-connected graph, and the corresponding theorem for minimally k-edge-connected graphs was proved by Lick [Li72]. Both results were then improved by Mader (Fact F69).

R20: Fact F72, a consequence of Mader's result Fact F73, is due to Hamidoune and is the digraph analogue of (and implies) Mader's theorem (Fact F69) about the existence of vertices of degree k. The existence of at least one vertex of half degree k had been previously asserted by Kameda [Ka74].

Vertex-Minimal Connectivity – Criticality

Maurer and Slater [MaSl77] introduced the general concept of *critically connected* and *critically edge-connected graphs*, graphs whose connectivity decreases when one or more vertices are removed.

DEFINITION

D41: A graph G is called ***k-critically n-connected***, or an ***(n, k)-graph***, if, for each vertex subset U with $|U| \leq k$, we have $\kappa(G - U) = n - |U|$. When $k = 1$, we simply refer to the graph as ***critically n-connected***.

FACTS

F74: [MaSl77] The only (n, n)-graph is the complete graph K_{n+1}.

F75: The "cocktail party graph" (obtained from K_{2n+2} by removing a 1-factor [perfect matching]) is a $(2n, n)$-graph but not a $(2n, n+1)$-graph.

F76: [Su88] The complete graph on $k+1$ vertices is the unique k-critically n-connected graph with $n < 2k$.

F77: [Ma77] If G is a $(n, 3)$-graph, then its order is at most $6n^2$. Thus, for each n, there are only finitely many of $(n, 3)$-critical graphs.

REMARKS

R21: An early survey about (n, k)-graphs can be found in [Ma84].

R22: Fact F75 led Slater to conjecture that, apart from K_{n+1}, there is no (n, k)-graph with $k > n/2$, which, after some partial results, was finally proved by Su (Fact F76).

R23: Fact F77 was generalized by Mader to the class of all finite n-connected graphs.

Connectivity Augmentation

We conclude the section by referring the reader to Frank [Fr94] for an in-depth discussion of *connectivity augmentation*. In the *edge-connectivity augmentation problem*, we are given a graph $G = (V, E)$ and a positive integer k, and the goal is to find the smallest set of edges F that we can add to G such that $G' = (V, E \cup F)$ is k-connected. Due to its applicability to the design of fault-tolerant networks, connectivity augmentation has also been widely investigated from an algorithmic point of view. Watanabe and Nakamura [WaNa87] gave the first polynomial-time algorithm solving the edge-connectivity augmentation problem. In the same paper, the authors formulated a necessary and sufficient condition to decide if a given graph G can be made k-connected by adding at most a certain number of edges. The same question for digraphs was solved in [Fr92].

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