Section 4.1
Connectivity: Properties and Structure

Camino Balbuena, Universitat Politècnica de Catalunya, Spain
Josep Fàbrega, Universitat Politècnica de Catalunya, Spain
Miquel Àngel Fiol, Universitat Politècnica de Catalunya, Spain

4.1.1 Connectivity Parameters ........................................ 235
4.1.2 Characterizations ................................................ 239
4.1.3 Structural Connectivity ........................................ 242
4.1.4 Analysis and Synthesis ........................................... 245
References .............................................................. 251

INTRODUCTION

Connectivity is one of the central concepts of graph theory, from both a theoretical and a practical point of view. Its theoretical implications are mainly based on the existence of nice max-min characterization results, such as Menger’s theorems. In these theorems, one condition which is clearly necessary also turns out to be sufficient. Moreover, these results are closely related to some other key theorems in graph theory: Ford and Fulkerson’s theorem about flows and Hall’s theorem on perfect matchings. With respect to the applications, the study of connectivity parameters of graphs and digraphs is of great interest in the design of reliable and fault-tolerant interconnection or communication networks.

Since graph connectivity has been so widely studied, we limit ourselves here to the presentation of some of the key results dealing with finite simple graphs and digraphs. For results about infinite graphs and connectivity algorithms the reader can consult, for instance, Aharoni and Diestel [AhDi94], Gibbons [Gi85], Halin [Ha00], Henzinger, Rao, and Gabow [HeRaGa00], Wigderson [Wi92]. For further details, we refer the reader to some of the good textbooks and surveys available on the subject: Berge [Be76], Bermond, Homobono, and Peyrat [BeHoPe89], Frank [Fr90, Fr94, Fr95], Gross and Yellen [GrYe06], Hellwig and Volkmann [HeVo08], Lovász [Lo93], Mader [Ma79], Oellermann [Oe96], Tutte [Tu66].
Section 4.1. Connectivity: Properties and Structure

4.1.1 Connectivity Parameters

In this first subsection the basic notions of connectivity and edge-connectivity of simple graphs and digraphs are reviewed.

NOTATION: Given a graph or digraph $G$, the vertex-set and edge-set are denoted $V(G)$ and $E(G)$, respectively. Often, when there is no ambiguity, we omit the argument and refer to these sets as $V$ and $E$.

Preliminaries

DEFINITIONS

D1: A graph is connected if there exists a walk between every pair of its vertices. A graph that is not connected is called disconnected.

D2: The subgraphs of $G$ which are maximal with respect to the property of being connected are called the components of $G$.

D3: Let $G = (V, E)$ be a graph and $U \subset V$. The vertex-deletion subgraph $G - U$ is the graph obtained from $G$ by deleting from $G$ the vertices in $U$. That is, $G - U$ is the subgraph induced on the vertex subset $V - U$. If $U = \{u\}$, we simply write $G - u$.

D4: Let $G = (V, E)$ be a graph and $F \subset E$. The edge-deletion subgraph $G - F$ is the subgraph obtained from $G$ by deleting from $G$ the edges in $F$. Thus, $G - F = (V, E - F)$. As in the case of vertex deletion, if $F = \{e\}$, it is customary to write $G - e$ rather than $G - \{e\}$.

D5: A disconnecting (vertex-)set (or vertex-cut) of a connected graph $G$ is a vertex subset $U$ such that $G - U$ has at least two different components.

D6: A vertex $v$ is a cut-vertex of a connected graph $G$ if $\{v\}$ is a disconnecting set of $G$.

D7: A disconnecting edge-set (or edge-cut) of a connected graph $G$ is an edge subset $F$ such that $G - F$ has at least two different components.

D8: An edge $e$ is a bridge (or cut-edge) of a connected graph $G$ if $\{e\}$ is a disconnecting edge-set of $G$.

FACTS

F1: Every nontrivial connected graph contains at least two vertices that are not cut-vertices.

F2: An edge is a bridge if and only if it lies on no cycle.
Vertex- and Edge-Connectivity

The simplest way of quantifying connectedness of a graph is by means of its parameters vertex-connectivity and edge-connectivity.

DEFINITIONS

D9: The (vertex-)connectivity \( \kappa(G) \) of a graph \( G \) is the minimum number of vertices whose removal from \( G \) leaves a disconnected or a trivial graph.

D10: The edge-connectivity \( \lambda(G) \) of a nontrivial graph \( G \) is the minimum number of edges whose removal from \( G \) results in a disconnected graph.

NOTATION: When the context is clear, we suppress the dependence on \( G \) and simply use \( \kappa \) and \( \lambda \).

NOTATION: In some other sections of the Handbook, \( \kappa_v(G) \) and \( \kappa_e(G) \) are used instead of \( \kappa(G) \) and \( \lambda(G) \).

EXAMPLE

E1: Figure 4.1.1 shows an example of a graph with \( \kappa = 2 \) and \( \lambda = 3 \).

![Figure 4.1.1: \( \kappa = 2 \) and \( \lambda = 3 \).](image)

FACTS

F3: We have \( \kappa = 0 \) if and only if \( G \) is disconnected or \( G = K_1 \). If \( G \) has order \( n \), then \( \kappa = n - 1 \) if and only if \( G \) is the complete graph \( K_n \). In this case, the removal of \( n - 1 \) vertices results in the trivial graph \( K_1 \). Moreover, if \( G \neq K_n \) is a connected graph, then \( 1 \leq \kappa \leq n - 2 \) and there exists a disconnecting set \( U \) of \( \kappa \) vertices.

F4: If \( G \neq K_1 \) we have \( \lambda = 0 \) if \( G \) is disconnected. By convention, we set \( \lambda(K_1) = 0 \).

F5: If \( G \neq K_1 \) is connected, then the removal of \( \lambda \) edges results in a disconnected graph with precisely two components.

F6: The parameters \( \kappa \) and \( \lambda \) can be computed in polynomial time.
Section 4.1. Connectivity: Properties and Structure

Relationships Among the Parameters

NOTATION: The minimum degree of a graph $G$ is denoted $\delta(G)$. When the context is clear, we simply write $\delta$. (In some other sections of the Handbook, the notation $\delta_{\text{min}}(G)$ is used.)

FACTS

F7: [Wh32] For any graph, $\kappa \leq \lambda \leq \delta$.

F8: [ChHa68] For all integers $a$, $b$, $c$ such that $0 < a \leq b \leq c$, there exists a graph $G$ with $\kappa = a$, $\lambda = b$, and $\delta = c$.

DEFINITIONS

D11: A graph $G$ is maximally connected when $\kappa = \lambda = \delta$, and $G$ is maximally edge-connected when $\lambda = \delta$.

D12: A graph $G$ with connectivity $\kappa \geq k \geq 1$ is called $k$-connected. Equivalently, $G$ is $k$-connected if the removal of fewer than $k$ vertices leaves neither a disconnected graph nor a trivial one. Analogously, if $\lambda \geq k \geq 1$, $G$ is said to be $k$-edge-connected.

D13: A connected graph $G$ without cut-vertices ($\kappa > 1$ or $G = K_2$) is called a block.

Some Simple Observations

The following facts are simply restatements of the definitions.

FACTS

F9: A nontrivial graph is 1-connected if and only if it is connected.

F10: If $G$ is $k$-connected, either $G = K_{k+1}$ or it has at least $k + 2$ vertices and $G - U$ is still connected for any $U \subset V$ with $|U| < k$.

F11: A graph $G$ is $k$-edge-connected if the deletion of fewer than $k$ edges does not disconnect it.

F12: Every block with at least three vertices is 2-connected.

Internally-Disjoint Paths and Whitney’s Theorem

DEFINITIONS

D14: An internal vertex of a path is a vertex that is neither the initial nor the final vertex of that path.

D15: The paths $P_1, P_2, \ldots, P_k$ joining the vertices $u$ and $v$ are said to be internally-disjoint (or openly-disjoint) $u-v$ paths if no two paths in the collection have an internal vertex in common. Thus, $V(P_i) \cap V(P_j) = \{u, v\}$ for $i \neq j$. 
Chapter 4. Connectivity and Traversability

FACTS

F13: [Wh32] A graph $G$ with order $n \geq 3$ is 2-connected if and only if any two vertices of $G$ are joined by at least two internally-disjoint paths.

F14: Fact F13 implies that every 2-connected graph is a block.

F15: A graph $G$ with at least three vertices is a block if and only if every two vertices of $G$ lie on a common cycle.

Strong Connectivity in Digraphs

For basic concepts on digraphs, see, for example, the textbooks of Bang-Jensen and Gutin [BaGu01], Chartrand, Lesniak, and Zhang [ChLeZh11], Harary, Norman, and Cartwright [HaNoCa68].

DEFINITIONS

D16: In a digraph $G$, vertices $u$ and $v$ are mutually reachable if $G$ contains both a directed $u \rightarrow v$ walk and a directed $v \rightarrow u$ walk.

D17: A digraph $G$ is said to be strongly connected if every two vertices $u$ and $v$ are mutually reachable.

D18: For a strongly connected digraph $G$, the (vertex) connectivity $\kappa = \kappa(G)$ is defined as the minimum number of vertices whose removal leaves a non-strongly connected or trivial digraph. Analogously, if $G$ is not trivial, its edge-connectivity $\lambda = \lambda(G)$ is the minimum number of directed edges (or arcs) whose removal results in a non-strongly connected digraph.

D19: Let $G$ be an undirected graph. The associated symmetric digraph $G^*$ is the digraph obtained from $G$ by replacing each edge $uv \in E(G)$ by the two directed edges $(u,v)$ and $(v,u)$ forming a digon.

REMARKS

R1: In our context, the interest for studying digraphs is that we can deal with an undirected graph $G$ by considering $G^*$. In particular, $\kappa(G^*) = \kappa(G)$, and, since a minimum edge-disconnecting set cannot contain digons, we also have $\lambda(G^*) = \lambda(G)$.

NOTATION: The symbols $\delta^+$ and $\delta^-$ denote the minimum outdegree and indegree among the vertices of a digraph $G$. Then, the minimum degree of $G$ is defined as $\delta = \min\{\delta^+, \delta^-\}$.

R2: Note that, if $G$ is a strongly connected digraph, then $\delta \geq 1$. The following result, due to Geller and Harary, is the analogue of (and implies) Fact F7.

FACT

F16: [GeHa70] For any digraph, $\kappa \leq \lambda \leq \delta$.

TERMINOLOGY: A digraph $G$ is said to be maximally connected when $\kappa = \lambda = \delta$, and $G$ is maximally edge-connected when $\lambda = \delta$. 
Section 4.1. Connectivity: Properties and Structure

An Application to Interconnection Networks

The interconnection network of a communication or distributed computer system is usually modeled by a (directed) graph in which the vertices represent the switching elements or processors, and the communication links are represented by (directed) edges. Fault-tolerance is one of the main factors that have to be taken into account in the design of an interconnection network. See, for instance, the survey of Bermond, Homobono, and Peyrat [BeHoPe89] and the book by Xu [Xu01]. Indeed, it is generally expected that the system be able to work even if several of its elements fail. Thus, it is often required that the (di)graph associated with the interconnection network be sufficiently connected, and, in most cases, a good design requires that this (di)graph has maximum connectivity. Communication networks are discussed in §11.4 of the Handbook.

4.1.2 Characterizations

When a graph \( G \) is \( k \)-connected we need to delete at least \( k \) vertices to disconnect it. Clearly, if any pair \( u, v \) of vertices can be joined by \( k \) internally-disjoint \( u-v \) paths, \( G \) is \( k \)-connected. In fact, it turns out that the converse statement is also true. That is, in a \( k \)-connected graph any two vertices can be joined by \( k \) internally-disjoint paths. We review in this subsection some key theorems of this type that characterize \( k \)-connectedness.

Menger’s Theorems

**Definition**

\[ \text{D20:} \] Let \( u \) and \( v \) be two non-adjacent vertices of a connected graph \( G \neq K_n \). A \((u|v)\)-disconnecting set \( X \), or simply \((u|v)\)-set, is a disconnecting set \( X \subset V - \{u,v\} \) whose removal from \( G \) leaves \( u \) and \( v \) in different components.

**Notation:** For any pair of non-adjacent vertices \( u \) and \( v \), \( \kappa(u|v) \) denotes the minimum number of vertices in a \((u|v)\)-set.

**Notation:** For any two vertices \( u \) and \( v \), \( \kappa(u-v) \) denotes the maximum number of internally-disjoint \( u-v \) paths.

**Facts**

\[ \text{F17:} \] For any graph \( G \), \( \kappa(G) = \min \{\kappa(u|v) : u, v \in V, \text{nonadjacent}\} \).

\[ \text{F18:} \] (Menger’s theorem) [Me27] For any pair of non-adjacent vertices \( u \) and \( v \),

\[ \kappa(u-v) = \kappa(u|v) \]

\[ \text{F19:} \] Although \( \kappa(u-v) \) can be arbitrarily smaller than the minimum of the degrees of \( u \) and \( v \), Mader proved that every finite graph contains vertices for which equality holds:

\[ \text{F20:} \] [Ma73] Every connected non-trivial graph contains adjacent vertices \( u \) and \( v \) for which \( \kappa(u-v) = \min \{\deg(u), \deg(v)\} \).
Chapter 4. Connectivity and Traversability

NOTATION: For any pair of distinct vertices \( u \) and \( v \), \( \lambda(u|v) \) denotes the minimum number of edges whose removal from \( G \) (\( G \) non-trivial) leaves \( u \) and \( v \) in different components and \( \lambda(u-v) \) denotes the maximum number of edge-disjoint \( u-v \) paths.

\[ F21: \text{For any non-trivial graph } G, \lambda(G) = \min\{\lambda(u|v), u, v \in V\}. \]

\[ F22: \text{(Edge-analogue of Menger’s theorem) [ElFeSh56, FoFu56] For any pair of vertices } u \text{ and } v, \lambda(u-v) = \lambda(u|v). \]

REMARKS

\[ R3: \text{Digraph versions of Menger’s theorems are the same except that all paths are directed paths.} \]

\[ R4: \text{The edge form and arc form of Menger’s theorem were proved by Ford and Fulkerson [FoFu56] using network-flow methods. Network flow is discussed in Chapter 11 of this Handbook.} \]

Other Versions and Generalizations of Menger’s Theorem

In addition to the ones given below, there exist other versions and generalizations of Menger’s theorem; see, for example, Diestel [Di00], Frank [Fr95], and McCuaig [McCu84]. A comprehensive survey about variations of Menger’s theorem can be found in Oellermann [Oe12].

DEFINITIONS

\[ D21: \text{Given } A, B \subseteq V, \text{ an } A-B \text{ path is a } u-v \text{ path } P \text{ with } u \in A, v \in B, u \neq v, \text{ and any other vertex of } P \text{ is neither in } A \text{ nor in } B. \]

\[ D22: \text{A set } X \subseteq V \text{ separates } A \text{ from } B \text{ (or is } (A|B)-\text{separating}) \text{ if every } A-B \text{ path in } G \text{ contains a vertex of } X. \]

\[ D23: \text{An } A-\text{path is an } A-B \text{ path with } A = B. \]

\[ D24: \text{A subset } X \subseteq V-A \text{ totally separates } A \text{ if each component of } G-X \text{ contains at most one vertex of } A \text{ (or, equivalently, every } A-\text{path between different vertices contains some vertex of } X). \]

\[ D25: \text{A vertex subset is an independent set if no two of its vertices are adjacent.} \]

NOTATION: The maximum number of (internally-)disjoint \( A-B \) paths is denoted \( \kappa(A-B) \), and the size of a minimum \( (A|B) \)-separating set is denoted \( \kappa(A|B) \).

FACTS

\[ F23: \text{The minimum number of vertices separating } A \text{ from } B \text{ is equal to the maximum number of disjoint } A-B \text{ paths. That is,} \]

\[ \kappa(A-B) = \kappa(A|B). \]
Section 4.1. Connectivity: Properties and Structure

F24: If $A$ is an independent set, the maximum number of internally-disjoint $A$-paths is at most the minimum number of vertices in a totally $A$-separating set, that is, $\kappa(A - A) \leq \kappa(A|A)$.

F25: The corresponding Menger-type result does not hold and inequality can be strict. In fact, there exist examples for which $\kappa(A - A) = \lambda(A|A)/2$.

F26: Gallai [Ga61] conjectured that Fact F25 corresponds to the “extremal” situation and that always $\kappa(A - A) \geq \kappa(A|A)/2$, and Lovász [Lo76] conjectured that $\lambda(A - A) \geq \lambda(A|A)/2$. Both conjectures were proved by Mader.

F27: [Ma78b, Ma78c] $\kappa(A - A) \geq \kappa(A|A)/2$ and $\lambda(A - A) \geq \lambda(A|A)/2$.

REMARK R5: The classical version of Menger’s theorem (Fact F18) is easily derived from Fact F23 by taking $A$ and $B$ as the sets of vertices adjacent to $u$ and $v$, respectively.

Another Menger-Type Theorem

NOTATION: For any pair of vertices $u$ and $v$, $\kappa_n(u-v)$ denotes the maximum number of internally-disjoint $u-v$ paths of length less than or equal to $n$. For any pair of non-adjacent vertices $u$ and $v$, $\kappa_n(u|v)$ denotes the minimum number of vertices of a set $X \subset V - \{u, v\}$ such that every $u-v$ path in $G - X$ has length greater than $n$.

FACTS

F28: There are examples for which we have the strict inequality $\kappa_n(u-v) < \kappa_n(u|v)$. However, for $n = d(u, v) \geq 2$ (i.e., for shortest $u-v$ paths), we have $\kappa_n(u-v) = \kappa_n(u|v)$. This Menger-type result is equivalently restated as Fact F29.

F29: [EnJaSl77, LoNePl78] The maximum number of internally-disjoint shortest $u-v$ paths is equal to the minimum number of vertices (different from $u$ and $v$) necessary to destroy all shortest $u-v$ paths.

Whitney’s Theorem

In a connected graph, there exists a path between any pair of its vertices, and if the graph is 2-connected, then there exist at least two internally-disjoint paths between two distinct vertices (Fact F13). As a corollary of Menger’s theorem, we have the remarkable result that this property can be generalized to $k$-connected graphs, which was independently proved by Whitney. It provides a natural and intrinsic characterization of $k$-connected graphs.

FACTS

F30: (Whitney’s theorem) [Wh32] A non-trivial graph $G$ is $k$-connected if and only if for each pair $u, v$ of distinct vertices there are at least $k$ internally-disjoint $u-v$ paths (or, alternatively, if and only if every cut-set has at least $k$ vertices).
Chapter 4. Connectivity and Traversability

F31: (Edge version of Whitney’s theorem) A nontrivial graph $G$ is $k$-edge-connected if and only if for each pair $u, v$ of distinct vertices there exist at least $k$ edge-disjoint $u-v$ paths.

F32: (The Fan Lemma) Let $G$ be a $k$-connected graph ($k \geq 1$). Let $v \in V$ and let $B \subseteq V, |B| \geq k, v \notin B$. Then there exist distinct vertices $b_1, b_2, \ldots, b_k$ in $B$ and a $v-b_i$ path $P_i$ for each $i = 1, 2, \ldots, k$, such that the paths $P_1, P_2, \ldots, P_k$ are internally-disjoint (that is, with only vertex $v$ in common) and $V(P_i) \cap B = \{b_i\}$ for $i = 1, 2, \ldots, k$.

Other Characterizations

Another interesting characterization of $k$-connected graphs was independently conjectured by Frank and Maurer. The conjecture was proved by Lovász and by Győri (who worked independently), and it appears as Fact F33. Su proved a characterization of $k$-edge-connectivity for digraphs (Fact F34).

FACTS

F33: [Lo77, Gy78] A graph $G$ with $n \geq k + 1$ vertices is $k$-connected if and only if, for any distinct vertices $u_1, u_2, \ldots, u_k$ and any positive integers $n_1, n_2, \ldots, n_k$ such that $n_1 + n_2 + \cdots + n_k = n$, there is a partition $V_1, V_2, \ldots, V_k$ of $V(G)$ such that $u_i \in V_i, |V_i| = n_i$, and the induced subgraph $G(V_i)$ is connected, $1 \leq i \leq n$.

F34: [Su97] A digraph $G$ with at least $k$ edges is $k$-edge-connected if and only if, for any $k$ distinct arcs $e_i = (u_i, v_i), 1 \leq i \leq k$, the digraph $G - \{e_1, e_2, \ldots, e_k\}$ contains $k$ edge-disjoint spanning arborescences (rooted trees) $T_1, T_2, \ldots, T_k$ such that $T_i$ is rooted at $v_i$, $1 \leq i \leq n$.

4.1.3 Structural Connectivity

Here our purpose is to give results about certain configurations that must be present in a $k$-connected or $k$-edge-connected graph.

Cycles Containing Prescribed Vertices

The first is a classical result by Dirac, which generalizes Fact F15.

FACTS

F35: [Di60] Let $G$ be a $k$-connected graph, $k \geq 2$. Then $G$ contains a cycle through any given $k$ vertices.

F36: [WaMe67] Let $G$ be a $k$-connected graph with $k \geq 3$. Then $G$ has a cycle containing a given set $H$ with $k + 1$ vertices if and only if there is no set $T \subseteq V - H$ with $|T| = k$ vertices whose removal separates the vertices of $H$ from each other.

© 2014 by Taylor & Francis Group, LLC
The Lovász–Woodall Conjecture

Lovász [Lo74] and Woodall [Wo77] independently conjectured that every $k$-connected graph has a cycle containing a given set $F$ of $k$ independent edges (that is, no two edges have a vertex in common), if and only if $F$ is not an edge-disconnecting set of odd cardinality. Partial results on this conjecture are given in Facts F37 → F39.

FACTS

F37: [Lo74, Lo77, ErGy85, Lo90, Sa96] The Lovász–Woodall Conjecture is true for $k = 3, 4, 5$.

F38: [HaTh82] The Lovász–Woodall Conjecture is true assuming that $G$ is $(k + 1)$-connected (without restriction on the edge set $F$).

F39: [Ka02] Under the same assumptions of the conjecture, $F$ is either contained in a cycle or in two disjoint cycles.

Terminology: A subset of independent edges is also called a matching. Matchings are discussed in Section 11.3 of this Handbook.

Paths with Prescribed Initial and Final Vertices

Given any two subsets $A, B \subset V$ of $k$ vertices of a $k$-connected graph, the existence of $k$ disjoint paths $P_i$ ($1 \leq i \leq k$) connecting $A$ and $B$ is guaranteed by Menger’s theorem. Menger’s theorem does not, however, ensure that each of these paths can be so chosen to join a fixed $u_i, v_i$ pair of vertices, $u_i \in A, v_i \in B$, ($1 \leq i \leq k$). Now we consider the existence of paths with prescribed end-vertices.

Definitions

D26: A graph $G$ is called $k$-linked if it has at least $2k$ vertices, and for every sequence $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$ of $2k$ different vertices, there exists a $u_i - v_i$ path $P_i, i = 1, 2, \ldots, k$, such that the $k$ paths are vertex-disjoint.

D27: A graph is weakly $k$-linked if it has at least $2k$ vertices, and for every $k$ pairs of vertices $(u_i, v_i)$, there exists a $u_i - v_i$ path $P_i, 1 \leq i \leq k$, such that the $k$ paths are edge-disjoint.

D28: A graph is said to be $k$-parity-linked if one can find $k$ disjoint paths with prescribed end-vertices and prescribed parities of the lengths.

D29: The bipartite index of a graph is the smallest number of vertices whose deletion creates a bipartite graph.

FACTS

F40: A $k$-linked graph is always $(2k - 1)$-connected, but the converse is not true.

F41: [Ju70], [LaMa70] (independently) For each $k$, there exists an integer $f(k)$ such that if $\kappa \geq f(k)$ then $G$ is $k$-linked.
Chapter 4. Connectivity and Traversability

F42: Thomassen [Th80a] and Seymour independently characterized the graphs that are not 2-linked. This is the first problem in the so-called $k$-paths problem that has been solved using the Robertson–Seymour theory [RoSe85].

NOTATION: For $k \geq 1$, $g(k)$ denotes the smallest integer such that every $g(k)$-edge-connected graph $G$ is weakly $k$-linked.

CONJECTURE [Th80a] For every integer $k \geq 1$, $g(2k + 1) = g(2k) = 2k + 1$.

FACTS

F43: [Ok84, Ok85, Ok87] If $k \geq 3$ is odd, $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$ are (not necessarily distinct) vertices from a set $T$ with $|T| \leq 6$, and $\lambda(u_i, v_i) \geq k$ ($1 \leq i \leq k$), then there exists a $u_i - v_i$ path for $1 \leq i \leq k$ such that the $k$ paths are edge-disjoint.

F44: [Hu91] For every integer $k \geq 1$, $g(2k + 1) \leq 2k + 2$ and $g(2k) \leq 2k + 2$.

F45: [Ok88, Ok90a] For every integer $k \geq 1$,
(a) $g(2k + 1) \leq 3k$ and $g(2k + 2) \leq 3k + 2$,
(b) $g(3k) \leq 4k$ and $g(3k + 2) \leq 4k + 2$.

F46: [Th01] Every $f(k)$-connected graph (defined in Fact F41) with bipartite index at least $4k - 3$ is $k$-parity-linked.

F47: [Su97] Let $G$ be a $k$-edge-connected digraph, and let $(u_1, f_1, v_1)$, $(u_2, f_2, v_2)$, $\ldots$, $(u_k, f_k, v_k)$ be any $k$ triples, where $u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k$ are not necessarily distinct vertices, and $f_1, f_2, \ldots, f_k$ are $k$ distinct arcs, either of the form $f_i = (u_i, v_i)$, $i = 1, \ldots, k$, or $f_i = (t_i, v_i)$, $i = 1, \ldots, k$. Then there exist $k$ edge-disjoint $u_i - v_i$ paths $P_i$ in $G$ such that $f_i \in E(P_i)$, $i = 1, \ldots, k$.

Subgraphs

High connectivity implies a large minimum degree (Fact F7). Conversely, a large minimum degree does not guarantee high connectivity (Fact F8). However, it does ensure the existence of a highly connected subgraph.

FACT

F48: [Ma72a] Every graph of minimum degree at least $4k$ contains a $k$-connected subgraph.

REMARK

R6: In fact, Mader [Ma72a] proved that if the average of the degrees of the vertices of $G$ is at least $4k$, then $G$ contains a $k$-connected subgraph. Concerning the proof of Fact F48, see also Thomassen [Th88].
4.1.4 Analysis and Synthesis

An interesting question in the study of graph connectivity is to describe how to obtain every \(k\)-(edge-)connected graph from a given “simple” one by a succession of elementary operations preserving \(k\)-connectedness. A classical result on this topic is Tutte’s theorem, which states how to construct all 3-connected graphs, starting with a wheel graph. We also consider some relevant results dealing with deletion of edges or vertices. Finally, some facts concerning minimally and critically \(k\)-connected graphs, as well as a reference to connectivity augmentation problems, are considered.

Contractions and Splittings

DEFINITIONS

**D30:** The contraction of an edge \(uv\) consists of the identification of its endpoints \(u\) and \(v\) (keeping the old adjacencies but removing the self-loop from \(u = v\) to itself). Let \(G\) be a \(k\)-connected graph. An edge of \(G\) is said to be \(k\)-contractible if its contraction results in a \(k\)-connected graph.

**D31:** The converse operation is called splitting: A vertex \(w\) with degree \(\delta\) is replaced by an edge \(uv\) in such a way that some of the vertices adjacent to \(w\) are now adjacent to \(u\) and the rest are adjacent to \(v\). Moreover, if the new vertices \(u, v\) have degrees at least \(k = \delta/2 + 1\) we speak about a \(k\)-vertex-splitting.

**D32:** For any integer \(n \geq 4\), the wheel graph \(W_n\) is the \(n\)-vertex graph obtained by joining a vertex to each of the \(n - 1\) vertices of the cycle graph \(C_{n-1}\).

FACTS

**F49:** If \(G\) is a \(k\)-connected graph, the operations of \(k\)-vertex splitting and edge addition always produce a graph that is also (at least) \(k\)-connected. In fact, as shown below, for \(k = 3\) these operations suffice to derive all 3-connected graphs.

**F50:** [Th80b] Every 3-connected graph distinct from \(K_4\) has a 3-contractible edge.

**F51:** [Th81] Every triangle-free (no 3-cycles) \(k\)-connected graph has a \(k\)-contractible edge.

**F52:** [Tu61] Every 3-connected graph can be obtained from a wheel by a finite sequence of 3-vertex-splittings and edge additions.

REMARK

**R7:** In general, \(k\)-connectedness does not ensure the existence of \(k\)-contractible edges.
Chapter 4. Connectivity and Traversability

EXAMPLE

E2: In Figure 4.1.2, the cube graph $Q_3$ is synthesized from the wheel graph $W_5$ in four steps. All but the second step are 3-vertex-splittings.

![Figure 4.1.2: A 4-step Tutte synthesis of the cube graph $Q_3$.](image)

REMARKS

R8: Thomassen used Fact F50 to give a short proof of Kuratowski’s theorem on planarity. Fact F50 can also be derived from Tutte’s theorem (Fact F52).

R9: Since Tutte’s paper, the distribution of contractible edges in graphs of given connectivity has been extensively studied. For a comprehensive survey of this subject, we refer the reader to Kriesell [Kr02], where the author also considers subgraph contractions (see below).

R10: Fact F52 is a reformulation of the following proposition [Tu61]: a 3-connected graph is either a wheel, or it contains an edge whose removal leaves a 3-connected subgraph, or it contains a 3-contractible edge that is not in a cycle of length 3.

R11: Slater [Sl74] gave a similar result for constructing all 4-connected graphs starting from $K_5$, but in this case three more operations are required. For $k \geq 5$ the problem is still open. However, Lovász [Lo74] and Mader [Ma78a] managed to construct all $k$-edge-connected pseudographs (loops and multiple edges allowed) for every $k$ even and odd, respectively.

Subgraph Contraction

The contraction of a subgraph is a natural generalization of edge contraction.

DEFINITION

D33: A connected subgraph $H$ of a $k$-connected graph $G$ is said to be $k$-contractible if the contraction of $H$ into a single vertex results in a $k$-connected graph.
Section 4.1. Connectivity: Properties and Structure

FACTS

F53: [McOt94] Every 3-connected graph on \( n \geq 9 \) vertices has a 3-contractible path of length two.

F54: [ThTo81] Every 3-connected graph with minimum degree at least four contains a 3-contractible cycle.

F55: [Kr00] Every 3-connected graph of order at least eight has a 3-contractible subgraph of order four.

CONJECTURE

[McOt94] For every \( n \), a 3-connected graph of sufficiently large order has a 3-contractible subgraph of order \( n \).

Edge Deletion

DEFINITION

D34: A subgraph \( H \) of a \( k \)-edge-connected graph \( G \) is said to be \( \rho \)-reducible if the graph obtained from \( G \) by removing the edges of \( H \) is \( (k - \rho) \)-connected.

FACTS

F56: [Ma74] Every \( k \)-connected graph \( G \) with minimum degree at least \( k + 2 \) contains a cycle \( C \) such that \( G - E(C) \) is \( k \)-connected.

F57: [Ok88] Let \( G \) be a \( k \)-edge-connected graph with \( k \geq 4 \) even. Let \( \{u, v\} \subseteq V \) and \( \{e_1, e_2, f\} \subseteq E \), \( e_i \neq f \) (\( i = 1, 2 \)). Then,

(a) There exists a 2-reducible cycle containing \( e_1 \) and \( e_2 \), but not \( f \).

(b) There exists a 2-reducible \( u-v \) path containing \( e_1 \), but not \( f \).

F58: [Ok90b] Let \( G \) be a \( k \)-edge-connected graph with \( k \geq 2 \) even. If \( \{u_1, v_1, u_2, v_2\} \) are distinct vertices, with edges \( e_0 = v_1v_2 \), \( e_i = u_iv_i \) (\( i = 1, 2 \)), and there is no edge-cut with \( k \) or \( k + 1 \) elements containing \( \{e_0, e_1, e_2\} \), then there exists a 2-reducible cycle containing \( \{e_0, e_1, e_2\} \).

F59: [HuOk92] For each odd \( k \geq 3 \), there exists a \( k \)-edge-connected graph containing two vertices \( u \) and \( v \) such that every cycle passing through \( u, v \) is \( \rho \)-reducible with \( \rho \geq 3 \).

REMARK

R12: For the case of three consecutive edges \( e_1, e_2, e_3 \) of a \( k \)-connected graph, Okamura [Ok95] also found a nontrivial equivalent reformulation of the condition that no cycle of \( G \) containing \( e_1, e_2, \) and \( e_3 \) is 2-reducible.
Vertex Deletion

FACTS

F60: [ChKaLi72] Every 3-connected graph of minimum degree at least 4 has a vertex $v$ such that $G - v$ is 3-connected.

F61: [Th81] Every $(k + 3)$-connected graph has an induced (chordless) cycle whose deletion results in a $k$-connected graph.

F62: [Eg87] Every $(k + 2)$-connected triangle-free graph has an induced cycle whose deletion results in a $k$-connected graph.

REMARK

R13: Fact F61 was conjectured by Lovász, and Thomassen used Fact F51 to prove it.

Products of Graphs

DEFINITIONS

D35: Recall that the cartesian product of two graphs $G_i = (V_i, E_i)$, $i = 1, 2$, is the graph $G_1 \square G_2$ with vertex set $V_1 \times V_2$, and for which vertices $(x_1, x_2)$ and $(y_1, y_2)$ are adjacent if $x_1 = y_1$ and $x_2 y_2 \in E_2$, or $x_1 y_1 \in E_1$ and $x_2 = y_2$.

D36: The Kronecker product of two graphs $G_i = (V_i, E_i)$, $i = 1, 2$, is the graph $G_1 \times G_2$ with vertex set $V_1 \times V_2$, and for which vertices $(x_1, x_2)$ and $(y_1, y_2)$ are adjacent if $x_1 y_1 \in E_1$ and $x_2 y_2 \in E_2$.

D37: [BeDeFa84] Let $G_i = (V_i, E_i)$, $i = 1, 2$, be two graphs with the edges of $G_1$ arbitrarily oriented, in such a way that an oriented edge from $x_1$ to $y_1$ is denoted by $e_{x_1 y_1}$. For each arc $e_{x_1 y_1}$, let $\pi_{e_{x_1 y_1}}$ be a permutation of $V_2$. Then the twisted product $G_1 \times G_2$ has $V_1 \times V_2$ as vertex set, with two vertices $(x_1, x_2)$, $(y_1, y_2)$ being adjacent if and only if either $x_1 = y_1$ and $x_2 y_2 \in E_2$ or $x_1 y_1 \in E_1$ and $y_2 = \pi_{e_{x_1 y_1}}(x_2)$.

D38: [BaDaFiMi09] Given two graphs $G_i = (V_i, E_i)$, $i = 1, 2$, and a non-empty vertex subset $U_1 \subseteq V_1$, the generalized hierarchical product $G_1(U_1) \cap G_2$ is the graph with vertex set $V_1 \times V_2$, and for which vertices $(x_1, x_2)$ and $(y_1, y_2)$ are adjacent if $x_1 y_1 \in E_1$ and $x_2 = y_2$, or $x_1 = y_1 \in U_1$ and $x_2 y_2 \in E_2$.
Section 4.1. Connectivity: Properties and Structure

FACTS

F63: [XuYa06] For any nontrivial graphs $G_1$ and $G_2$,

$$\kappa(G_1 \square G_2) \geq \min\{\kappa(G_1) + \delta(G_2), \kappa(G_2) + \delta(G_1)\}$$

and

$$\lambda(G_1 \square G_2) \geq \min\{\lambda(G_1)|V_2|, \lambda(G_2)|V_1|, \delta(G_1) + \delta(G_2)\}.$$  

F64: [Sp08] For any nontrivial graphs $G_1$ and $G_2$,

$$\kappa(G_1 \square G_2) = \min\{\kappa(G_1)|V_2|, \kappa(G_2)|V_1|, \delta(G_1) + \delta(G_2)\}.$$  

F65: [We62] If $G_1$ and $G_2$ are two connected graphs, then $G_1 \times G_2$ is connected if and only if $G_1$ and $G_2$ are not both bipartite graphs.

F66:

(a) [MaVu08] $\kappa(K_n \times K_m) = (n-1)(m-1)$ for any $n \geq m \geq 2$ and $n \geq 3$.

(b) [WaWu11] $\kappa(G \times K_n) = \min\{n\kappa(G), (n-1)\delta(G)\}$ for any nontrivial graph $G$ and $n \geq 3$.

F67: [BaGVMa06, BaCeDiGVMa07]

(a) For any nontrivial graphs $G_1$ and $G_2$,

$$\min\{\kappa(G_1)|V_2|,(\delta_1 + 1)\kappa(G_2),\delta_1 + \delta_2\} \leq \kappa(G_1 \ast G_2) \leq \delta_1 + \delta_2;$$

and

$$\min\{\lambda(G_1)|V_2|,(\delta_1 + 1)\lambda(G_2),\delta_1 + \delta_2\} \leq \lambda(G_1 \ast G_2) \leq \delta_1 + \delta_2,$$

where $\delta_1 + \delta_2$ is the minimum degree of $G_1 \ast G_2$.

(b) If $G_1$ and $G_2$ are maximally connected, then $G_1 \ast G_2$ is also maximally connected.

(c) For every connected graph $G$, the graph $G \ast G$ is maximally connected.

F68: [BaDaFiMi09] The connectivity of the generalized hierarchical product satisfies

$$\kappa(G_1(U_1) \cap G_2) \leq \min\{\kappa(G_1)|V_2|, \kappa(U_1'|U_1), \delta(G_1(U_1) \cap G_2)\},$$

where $U_1' \subset V_1 - U_1$ and $\delta(G_1(U_1) \cap G_2) = \min\{\delta(G_1 - U_1), \delta(G_1(U_1)) + \delta_2\}.$

REMARKS

R14: The graph $G_1 \ast G_2$ can be viewed as formed by $|V_1|$ disjoint copies of $G_2$, each oriented edge $x_1y_1$ indicating that some perfect matching between the copies $G_1^{x_1}, G_2^{y_1}$ (respectively generated by the vertices $x_1$ and $y_1$ of $G_1$) is added. Moreover, $K_2 \ast G$ is a permutation graph [ChHa67].

R15: If in Definition D37, $\pi_{x_1,y_1}$ is the identity permutation for any oriented edge $e_{x_1,y_1}$, the twisted product $G_1 \ast G_2$ is the cartesian product $G_1 \square G_2$.

R16: If $U_1$ is consists of only one vertex, then $G_1(U_1) \cap G_2$ is the standard hierarchical product [BaCoDaF09], whereas if $U_1 = V_1$ we obtain the cartesian product $G_1 \square G_2$.

R17: Fact F66(b) was previously proved for $G$ bipartite in [GuVu09].

R18: Regarding Fact F66, the connectivity of Kronecker products by $K_2$ has been recently studied in [WaYa12].
Minimality and Criticality

A standard technique used to study a certain property $P$ is to consider those graphs that are edge-minimal or vertex-minimal (critical) with respect to $P$, in the sense that the removal of any vertex or edge produces a graph for which $P$ does not hold.

**DEFINITIONS**

**D39:** A graph or digraph $G$ is said to be **minimally $k$-connected** if $\kappa(G) \geq k$ but, for each edge $e \in E$, $\kappa(G - e) < k$. Analogously, $G$ is **minimally $k$-edge-connected** if $\lambda(G) \geq k$, but for each edge $e \in E$, $\lambda(G - e) < k$.

**D40:** A vertex $u$ of a digraph has **half degree** $k$ if either $\deg^+(u) = k$ or $\deg^-(u) = k$.

**FACTS**

**F69:** [Ma71, Ma72b] Every minimally $k$-connected (or $k$-edge-connected) graph contains at least $k + 1$ vertices of degree $k$.

**F70:** [Ma72b] Every cycle of a minimally $k$-connected graph contains a vertex of degree $k$.

**F71:** Every cycle in a $k$-connected graph $G$ contains either a vertex of degree $k$ or an edge whose removal does not lower the connectivity of $G$.

**F72:** [Ha81] Every minimally $k$-connected digraph contains at least $k + 1$ vertices of half degree $k$.

**F73:** [Ma02] Every minimally $k$-connected digraph contains at least $k + 1$ vertices of outdegree $k$ and at least $k + 1$ vertices of indegree $k$.

**REMARKS**

**R19:** Halin [Ha69, Ha00] proved the existence of a vertex of degree $k$ in every minimally $k$-connected graph, and the corresponding theorem for minimally $k$-edge-connected graphs was proved by Lick [Li72]. Both results were then improved by Mader (Fact F69).

**R20:** Fact F72, a consequence of Mader’s result Fact F73, is due to Hamidoune and is the digraph analogue of (and implies) Mader’s theorem (Fact F69) about the existence of vertices of degree $k$. The existence of at least one vertex of half degree $k$ had been previously asserted by Kameda [Ka74].

**Vertex-Minimal Connectivity – Criticality**

Maurer and Slater [MaSl77] introduced the general concept of **critically connected** and **critically edge-connected graphs**, graphs whose connectivity decreases when one or more vertices are removed.
DEFINITION

D41: A graph $G$ is called $k$-critically $n$-connected, or an $(n, k)$-graph, if, for each vertex subset $U$ with $|U| \leq k$, we have $\kappa(G - U) = n - |U|$. When $k = 1$, we simply refer to the graph as critically $n$-connected.

FACTS

F74: [MaSl77] The only $(n, n)$-graph is the complete graph $K_{n+1}$.

F75: The “cocktail party graph” (obtained from $K_{2n+2}$ by removing a 1-factor [perfect matching]) is a $(2n, n)$-graph but not a $(2n, n + 1)$-graph.

F76: [Su88] The complete graph on $k+1$ vertices is the unique $k$-critically $n$-connected graph with $n < 2k$.

F77: [Ma77] If $G$ is a $(n, 3)$-graph, then its order is at most $6n^2$. Thus, for each $n$, there are only finitely many of $(n, 3)$-critical graphs.

REMARKS

R21: An early survey about $(n, k)$-graphs can be found in [Ma84].

R22: Fact F75 led Slater to conjecture that, apart from $K_{n+1}$, there is no $(n, k)$-graph with $k > n/2$, which, after some partial results, was finally proved by Su (Fact F76).

R23: Fact F77 was generalized by Mader to the class of all finite $n$-connected graphs.

Connectivity Augmentation

We conclude the section by referring the reader to Frank [Fr94] for an in-depth discussion of connectivity augmentation. In the edge-connectivity augmentation problem, we are given a graph $G = (V, E)$ and a positive integer $k$, and the goal is to find the smallest set of edges $F$ that we can add to $G$ such that $G' = (V, E \cup F)$ is $k$-connected. Due to its applicability to the design of fault-tolerant networks, connectivity augmentation has also been widely investigated from an algorithmic point of view. Watanabe and Nakamura [WaNa87] gave the first polynomial-time algorithm solving the edge-connectivity augmentation problem. In the same paper, the authors formulated a necessary and sufficient condition to decide if a given graph $G$ can be made $k$-connected by adding at most a certain number of edges. The same question for digraphs was solved in [Fr92].

References


Chapter 4. Connectivity and Traversability


Section 4.1. Connectivity: Properties and Structure


© 2014 by Taylor & Francis Group, LLC
Chapter 4. Connectivity and Traversability


Section 4.1. Connectivity: Properties and Structure


[Ok90a] H. Okamura, Every 4k-edge-connected graph is weakly 3k-linked. Graphs Combin. 6 (1990), 179–185.
Chapter 4. Connectivity and Traversability


Section 4.1. Connectivity: Properties and Structure


