Differentiable Families of Stabilizers for Planar Bimodal Linear Control Systems

Josep Ferrer, M. Dolors Magret and Marta Peña

Departament de Matemàtica Aplicada I, ETSEIB, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain

Abstract. We consider bimodal linear control systems consisting of two subsystems acting on each side of a given hyperplane, assuming continuity along the separating hyperplane. For a differentiable family of controllable planar ones, we construct a differentiable family of feedbacks which pointwise stabilizes both subsystems.

INTRODUCTION

Piecewise linear control systems (in particular, the bimodal ones: see, for example, [1], [2], [3]) have attracted the interest of the researchers in recent years by their wide range of applications, as well as by the possible theoretical approaches, even in the planar case (see, for example, [4]).

Bimodal linear control systems (BLCS) consist of two subsystems acting on each side of a given hyperplane, assuming continuity along the separating hyperplane. In the space of triples of matrices defining those systems, we consider the natural equivalence relation defined by changes of bases in the state space which preserves the hyperplanes parallel to the separating one.

Here we consider parameterized families of controllable planar BLCS, and we tackle the construction of parameterized families of stabilizers. In both cases, we say differentiable family when the involved matrices depend differentially on a parameter. Given a differentiable family of controllable BLCS, for each value of the parameter there is a feedback which stabilizes the corresponding system (see [3]). However, this parameterized family of pointwise stabilizers may not be differentiable (not even continuous). Our results allow to construct a differentiable family of feedbacks which stabilizes the corresponding system for each value of the parameter.

We point out that when dealing with parameterized families of BLCS, the non-generic case of unobservable ones appears in a natural way (see Example 1).

Throughout the paper, $\mathbb{R}$ will denote the set of real numbers, $M_{n \times m}(\mathbb{R})$ the set of matrices having n rows and m columns and entries in $\mathbb{R}$ (in the case where $n = m$, we will simply write $M_n(\mathbb{R})$).

PRELIMINARIES

A bimodal linear control system (BLCS) is given by two subsystems

$$\begin{cases} \dot{x}(t) = A_1 x(t) + B_1 u(t), \\ y(t) = Cx(t), \end{cases} \quad \text{if } y(t) \leq 0, \quad \begin{cases} \dot{x}(t) = A_2 x(t) + B_2 u(t), \\ y(t) = Cx(t), \end{cases} \quad \text{if } y(t) \geq 0$$

where $A_1, A_2 \in M_n(\mathbb{R})$; $B_1, B_2 \in M_{n \times 1}(\mathbb{R})$; $C \in M_{1 \times n}(\mathbb{R})$. One assumes that the dynamics is continuous along the separating hyperplane $H = \{x \in \mathbb{R}^n : Cx = 0\}$ and one can consider $C = (1\, 0 \ldots 0) \in M_{1 \times n}(\mathbb{R})$. Hence $H = \{x \in \mathbb{R}^n : x_1 = 0\}$ and continuity along H is equivalent to:

$$B_2 = B_1, \qquad A_2 e_i = A_1 e_i, \quad 2 \leq i \leq n.$$

We will write from now on $B = B_1 = B_2$.

We recall some definitions related to BLCS.

Definition 1. *The above BLCS is called* observable *if*

$$\text{rank} \begin{pmatrix} C \\ CA_i \\ \dots \\ CA_i^{n-1} \end{pmatrix} = n, \quad i = 1, 2.$$

The above ranks are equal for both subsytems due to the continuity property of the system ([5]).

Definition 2. *A BLCS is* (completely) controllable *if for any pair of states* (x_0, x_f) *there exists a locally integrable input u such that the solution* $x^{x_0,u}$ *passes through* x_f, *i.e.* $x^{x_0,u}(T) = x_f$ *for some* $T > 0$.

A well-known remarkable fact is that a single linear system $\dot{x} = Ax + Bu$ is controllable if and only if its 'controllability matrix' $\begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix}$ has maximal rank. For planar BLCS we recall the characterization of their controllability obtained in [1] for observable systems and generalized in [6] to unobservable ones:

Proposition 3. *Let us consider a planar BLCS defined by* (A_1, A_2, B). *We write* C_1, C_2 *the controllability matrices of both subsystems*

$$C_1 = (B \quad A_1 B), \quad C_2 = (B \quad A_2 B).$$

Then, it is controllable if and only if

$$\det C_1 \det C_2 > 0$$

Remark 4. *(1) Notice that, in particular, both subsystems must be controllable, but it is not a sufficient condition.*

(2) Whereas for single systems the subset of controllable ones is open and dense, the above proposition shows that it is not for bimodal planar systems: controllability is an open, but not generic, property.

FAMILIES OF STABILIZERS

If the control function is a so-called 'feedback' of the type $u(t) = f(x(t))$, one obtains a dynamical system ('in closed loop'). In the linear case $\dot{x} = Ax + Bu$, a feedback $u = Fx$ gives $\dot{x} = (A + BF)x$. A remarkable fact is that it is stable for some suitable F, provided that the initial control system be controllable.

As a natural generalization, in [3] any controllable planar BLCS is proved to be feedback stabilizable. Hence, if a differentiable parameterized family $(A_1(s), A_2(s), B(s))$ is pointwise controllable (observable or not) then it is also pointwise stabilizable, that is to say, for any $s \in \mathbb{R}$ there is a common feedback $F(s)$ such that both closed-loop systems $A_1(s) + B(s)F(s)$, $A_2(s) + B(s)F(s)$ are stable. However, the family $F(s)$ may not be differentiable (not even continuous). Here we prove that differentiable families of stabilizer feedbacks exist for $n = 2$.

As we have pointed out in the Introduction, the unobservable case appears generically in parameterized families of BLCS. A typical case is considered in the following example. As an application of the above Proposition, we characterize when this family is pointwise controllable.

Example 1. *Let us consider the parameterized family of planar BLCS*

$$A_1(s) = \begin{pmatrix} a_1 & s \\ a_2 & a_4 \end{pmatrix}, A_2(s) = \begin{pmatrix} \gamma_1 & s \\ \gamma_2 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

where $s \in \mathbb{R}$. *Obviously, the systems defined by these matrices are observable except for* $s = 0$. *Let us see that the family is pointwise controllable (i.e., for any* $s \in \mathbb{R}$ *the corresponding system is controllable) if and only if* $b_1 \neq 0$ *and*

(i) $a_2 \gamma_2 > 0$, *if* $b_2 = 0$

(ii) $\det \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \det \begin{pmatrix} \gamma_1 & b_1 \\ \gamma_2 & b_2 \end{pmatrix}$, *otherwise.*

From Proposition 3 and the analogous result in [1], for any $s \in \mathbb{R}$ (including the case where $s = 0$) the corresponding system is controllable if and only if

$$(b_1^2 a_2 + b_1 b_2 a_4 - b_1 b_2 a_1 - b_2^2 s)(b_1^2 \gamma_2 + b_1 b_2 a_4 - b_1 b_2 \gamma_1 - b_2^2 s) > 0$$

In particular $b_1 \neq 0$ (it suffices to take $s = 0$).
If $b_2 = 0$, the above inequality is

$$(b_1^2 a_2)(b_1^2 \gamma_2) > 0$$

that is to say

$$a_2 \gamma_2 > 0$$

Assume now $b_2 \neq 0$. In general, two polynomials of degree 1 have the same sign at any point if and only if they have the same root and the slopes have the same sign. In our case both slopes are $-b_2^2$, so that the above inequality holds if and only if

$$\frac{b_1^2 a_2 + b_1 b_2 a_4 - b_1 b_2 a_1}{b_2^2} = \frac{b_1^2 \gamma_2 + b_1 b_2 a_4 - b_1 b_2 \gamma_1}{b_2^2}$$

which is equivalent (recall $b_1 \neq 0$) to

$$b_1 a_2 - b_2 a_1 = b_1 \gamma_2 - b_2 \gamma_1$$

Finally, we prove the existence of differentiable families of stabilizers for differentiable families of planar controllable bimodal systems.

Proposition 5. *Let*

$$(A_1(s), A_2(s), B(s)), \quad s \in \mathbb{R}$$

be a differentiable family of planar BLCS. If it is pointwise controllable, then there is a differentiable family of feedbacks $F(s)$, $s \in \mathbb{R}$, such that

$$A_1(s) + B(s)F(s), \quad A_2(s) + B(s)F(s)$$

are stable for any $s \in \mathbb{R}$.

Proof 1. *Let us write*

$$A_1(s) = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}, A_2(s) = \begin{pmatrix} \gamma_1 & a_3 \\ \gamma_2 & a_4 \end{pmatrix}, B(s) = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$C_1 = (B(s) \quad A_1(s)B(s)), C_2 = (B(s) \quad A_2(s)B(s))$$

where all the coefficients are assumed differentiably depending on $s \in \mathbb{R}$. By hypothesis, we assume

$$\det C_1 \det C_2 > 0$$

for any $s \in \mathbb{R}$.

We look for $F(s) = (f_1 \quad f_2)$ where again we assume the coefficients depending on $s \in \mathbb{R}$, such that the eigenvalues of

$$\begin{pmatrix} a_1 + b_1 f_1 & a_3 + b_1 f_2 \\ a_2 + b_2 f_1 & a_4 + b_2 f_2 \end{pmatrix}, \begin{pmatrix} \gamma_1 + b_1 f_1 & a_3 + b_1 f_2 \\ \gamma_2 + b_2 f_1 & a_4 + b_2 f_2 \end{pmatrix}$$

have negative real part for any $s \in \mathbb{R}$ or, equivalently, the matrices have negative trace and positive determinant, that is to say:

$$b_1 f_1 + b_2 f_2 < -a_1 - a_4,$$
$$b_1 f_1 + b_2 f_2 < -\gamma_1 - a_4,$$
$$f_1(a_3 b_2 - a_4 b_1) + f_2(a_2 b_1 - a_1 b_2) < a_1 a_4 - a_2 a_3,$$
$$f_1(a_3 b_2 - a_4 b_1) + f_2(\gamma_2 b_1 - \gamma_1 b_2) < \gamma_1 a_4 - \gamma_2 a_3.$$

We change the variables (f_1, f_2) by (x, y) defined by

$$x = b_1 f_1 + b_2 f_2,$$
$$y = (b_2 a_3 - b_1 a_4)f_1 + (b_1 a_2 - b_2 a_1)f_2,$$

which is a change of variables, because (by hypothesis):

$$\det \begin{pmatrix} b_1 & b_2 a_3 - b_1 a_4 \\ b_2 & b_1 a_2 - b_2 a_1 \end{pmatrix} = \det C_1 \neq 0.$$

Then:

$$f_1 = \frac{(b_1a_2 - b_2a_1)x - b_2y}{\det C_1}, \qquad f_2 = -\frac{(b_2a_3 - b_1a_4)x - b_1y}{\det C_1}.$$

With this change of variables, the desired inequalities become:

$$x < -a_1 - a_4,$$
$$x < -\gamma_1 - a_4,$$
$$y < a_1a_4 - a_2a_3,$$
$$(a_3b_2 - a_4b_1)\frac{(b_1a_2 - b_2a_1)x - b_2y}{\det C_1} - (\gamma_2b_1 - \gamma_1b_2)\frac{(b_2a_3 - b_1a_4)x - b_1y}{\det C_1} < \gamma_1a_4 - \gamma_2a_3.$$

In order to see that there exist solutions (x,y), it is sufficient that some of the coefficients of the variables x, y are positive. For the last one we have:

$$-\frac{b_2(a_3b_2 - a_4b_1)}{\det C_1} + \frac{b_1(\gamma_2b_1 - \gamma_1b_2)}{\det C_1} = \frac{-a_3b_2^2 + a_4b_1b_2 + \gamma_2b_1^2 - \gamma_1b_1b_2}{\det C_1} = \frac{\det C_2}{\det C_1} > 0$$

again by the hypothesis.

REFERENCES

1. CAMLIBEL, K., HEEMELS, M. & SCHUMACHER, H., *Stability and controllability of planar bimodal linear complementarity systems,* Proceedings of the 42nd IEEE Conference on Decision and Control, 1651–1656, 2003.
2. CAMLIBEL, K., HEEMELS, M. & SCHUMACHER, H., *On the controllability of bimodal piecewise linear systems,* LNCS **2993**, 250–264, 2004.
3. CAMLIBEL, K., HEEMELS, M. & SCHUMACHER, H., *A full characterization of stabilizability of bimodal piecewise linear systems with scalar inputs,* Automatica, **44**, 1261–1267, 2008.
4. DI BERNARDO, M., PAGANO, D. J. & PONCE, E., *Nonhyperbolic boundary equilibrium bifurcations in planar Filippov systems: a case study approach*, J. Bifur. Chaos Appl. Sci. Engin., **18**, 1377–1392, 2008.
5. FERRER, J., MAGRET, M. & PEÑA, M., *Bimodal piecewise linear systems. Reduced forms*, International Journal of Bifurcation and Chaos, **20**, 2795–2808, 2010.
6. FERRER, J., PACHA, J. R. & PEÑA, M., *Controllability of continuous bimodal linear systems,* Mathematical Problems in Engineering (special issue Mathematical Modeling, Analysis, and Control of Hybrid Dynamical Systems), Volume 2013 (2013), Article ID 342548, 14 pages.