Structural Stability of Planar Bimodal Linear Systems

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Abstract. We consider bimodal linear dynamical systems consisting of two linear dynamics acting on each side of a given hyperplane, assuming continuity along the separating hyperplane. Focusing in the planar case, we describe which of these systems are structurally stable.

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INTRODUCTION

Piecewise linear control systems have attracted the interest of the researchers in recent years by their wide range of applications, as well as by the possible theoretical approaches. See, for example, [1], [2], [3], [4], [5], [6], [7] and [8].

Bimodal linear systems consist of two subsystems acting on each side of a given hyperplane, assuming continuity along the separating hyperplane. We focus in the planar case. Indeed, it is very commonly found in applications (see the above references).

We adapt the conditions stated in [9] for piecewise-linear vector fields to the particular class of planar bimodal linear systems, describing which of these systems are structurally stable.

Throughout the paper, $\mathbb{R}$ will denote the set of real numbers, $M_{n\times m}(\mathbb{R})$ the set of matrices having n rows and m columns and entries in $\mathbb{R}$ (in the case where $n=m$, we will simply write $M_n(\mathbb{R})$) and $Gl_n(\mathbb{R})$ the group of non-singular matrices in $M_n(\mathbb{R})$. Finally, we will denote by $e_1,\dots,e_n$ the natural basis of the Euclidean space $\mathbb{R}^n$.

PRELIMINARIES

Let us consider a bimodal linear dynamical system given by

$$\begin{cases}\dot{x}(t)=A_1x(t)+B_1,\\ y(t)=Cx(t),\end{cases} \quad \text{if } y(t)\le 0, \quad \begin{cases}\dot{x}(t)=A_2x(t)+B_2,\\ y(t)=Cx(t),\end{cases} \quad \text{if } y(t)\ge 0$$

where $A_1,A_2\in M_n(\mathbb{R})$; $B_1,B_2\in M_{n\times 1}(\mathbb{R})$; $C\in M_{1\times n}(\mathbb{R})$. We assume that the dynamics is continuous along the separating hyperplane $H=\{x\in\mathbb{R}^n : Cx=0\}$; that is to say, that both subsystems coincide for $y(t)=0$.
By means of a linear change in the state variable $x(t)$, we can consider $C=(1\,0\dots0)\in M_{1\times n}(\mathbb{R})$. Hence $H=\{x\in\mathbb{R}^n : x_1=0\}$ and continuity along H is equivalent to:

$$B_2=B_1, \qquad A_2e_i=A_1e_i, \quad 2\le i\le n.$$

We will write from now on $B=B_1=B_2$.

Definition 1. *In the above conditions, we say that the triple of matrices (A_1,A_2,B) defines a bimodal linear dynamical system.*

A natural tool is simplifying the matrices A_1,A_2,B by means of changes in the variables $x(t)$ which preserve the qualitative behavior of the system. So, we consider linear changes in the state variables space preserving the hyperplanes $x_1(t)=k$, which will be called *admissible basis changes*. Thus, they are basis changes given by a matrix $S\in Gl_n(\mathbb{R})$,

$$S=\begin{pmatrix}1 & 0\\ U & T\end{pmatrix}, \quad T\in Gl_{n-1}(\mathbb{R}), \quad U\in M_{n-1\times 1}(\mathbb{R}).$$

Also, translations parallel to the hyperplane H are allowed.

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By adapting the definition in [9] to our case we have:

Definition 2. *A triple of matrices (A_1, A_2, B) defining a bimodal linear dynamical system is said to be (regularly) structurally stable if it has a neighborhood $V(A_1, A_2, B)$ such that for every $(A_1^{'}, A_2^{'}, B^{'}) \in V(A_1, A_2, B)$ there is a homeomorphism of $\mathbb{R}^2$ preserving the hyperplane H which maps the oriented orbits of $(A_1^{'}, A_2^{'}, B^{'})$ into those of (A_1, A_2, B) and it is differentiable when restricted to finite periodic orbits.*

Our aim is to adapt the necessary and sufficient conditions in [9] to our triples:

$$A_1 = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix}, A_2 = \begin{pmatrix} \gamma_1 & a_3 \\ \gamma_2 & a_4 \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Firstly, the singularities at infinity must be disjoint from the hyperplane H. Hence:

Lemma 1. *Given a bimodal linear system defined by a triple (A_1, A_2, B) as above, a necessary condition for it being structurally stable is $a_3 \neq 0$.*

Therefore, by means of a suitable admissible basis change (see [6]) and the translation $(0, b_1)$, we have:

Corollary 1. *The triples representing a structurally stable bimodal linear system can be reduced to the form:*

$$\begin{pmatrix} a_1 & 1 \\ a_2 & 0 \end{pmatrix}, \begin{pmatrix} \gamma_1 & 1 \\ \gamma_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b_2 \end{pmatrix} \quad (*)$$

Remark 1. *The observable bimodal linear systems have a unique tangency point, which one has translated to the origin $(0,0)$. Notice also that foci (i.e., symmetrical nodes) are excluded.*

Secondly, the conditions in [9] concerning finite singularities give:

Lemma 2. *Given a bimodal linear system as in (*), necessary conditions for it being structurally stable are:*

1. $b_2 \neq 0$
2. $a_2 > 0$, *or* $a_2 < 0$ *and* $a_1 \neq 0$
3. $\gamma_2 > 0$, *or* $\gamma_2 < 0$ *and* $\gamma_1 \neq 0$

Remark 2. *The above lemma excludes the centers.*

In particular, each subsystem has a unique equilibrium point (i.e., finite singularity). We say that is *real* if it is located in the semiplane corresponding to the considered subsystem. Otherwise, we say that the equilibrium point is *virtual*. The above lemma gives the following possible planar bimodal linear systems:

Corollary 2. *The only bimodal linear systems verifying the necessary conditions in Lemma 1 and Lemma 2 are those in Table 1.*

TABLE 1.

Subsystem 1 \ Subsystem 2	Virtual saddle	Real node	Real spiral	Real improper node
Real saddle	1 ($b_2 > 0$)	2 ($b_2 > 0$)	3 ($b_2 > 0$)	4 ($b_2 > 0$)
Virtual node	5 ($b_2 < 0$)	6 ($b_2 > 0$)	7 ($b_2 > 0$)	8 ($b_2 > 0$)
Virtual spiral	9 ($b_2 < 0$)	10 ($b_2 < 0$)	11 ($b_2 > 0$)	12 ($b_2 > 0$)
Virtual improper node	13 ($b_2 < 0$)	14 ($b_2 < 0$)	15 ($b_2 < 0$)	16 ($b_2 > 0$)

Finally, we must consider the conditions in [9] concerning finite orbits:

(a) the finite periodic orbits are hyperbolic and disjoint from the tangency points
(b) there are not finite orbits connecting two saddles
(c) there are not finite orbits connecting a saddle and a tangency point

It is quite clear that finite periodic orbits can appear only in cases 3, 7, 11 and 15, whereas (b)-(c) only in case 3. Hence:

Proposition 1. *The cases 1, 2, 4, 5, 6, 8, 9, 10, 12, 13, 14 and 16 are structurally stable.*

Due to the limited space, we focus on case 3, leaving the remainder ones for future works.

Theorem 1. *Let us consider the bimodal linear systems 3 in Corollary 2 with $\gamma_1 > 0$, that is to say:*

$$a_2 > 0, \gamma_1 > 0, 0 < \gamma_1^2 < -4\gamma_2, b_2 > 0$$

1. *The only saddel-loop orbit appears for γ_1 verifying*

$$\exp(\alpha t)\sin(\beta t - \varphi) + \frac{\beta}{M} = 0, \quad \pi + \frac{\varphi}{\beta} \leq t \leq \frac{3\pi}{2} + \frac{\varphi}{\beta}$$

being

$$t = \frac{1}{2\alpha}\ln\left(\frac{\alpha^2 + \beta^2 + \lambda_1^2 - 2\lambda_1\alpha}{\lambda_1^2(1 - 2\frac{\alpha}{\lambda_2} + \frac{\alpha^2+\beta^2}{\lambda_2^2})}\right)$$

where
$\lambda_1 > 0$, $\lambda_2 < 0$ *are the eigenvalues of the saddle,*
$\alpha \pm i\beta$, $\alpha, \beta > 0$ *are the eigenvalues of the spiral,*
$M\cos(\varphi) = \alpha - \frac{\alpha^2+\beta^2}{\lambda_2}, \quad M\sin(\varphi) = \beta.$

2. *For minor values of γ_1, a finite periodic orbit appears, which is hyperbolic and disjoint from the tangency points.*
3. *For these values of γ_1, no saddle-tangency orbits appear.*

Example 1. *We show the case $a_1 = -1, a_2 = 1, \gamma_1 = 0.1, \gamma_2 = -5, b_2 = 1$ in Figure 1. We plot the phase portrait corresponding to the Poincaré map on the section $x = 0$ for different initial points: for each of them the orbits are integrated until the next oriented cut. The continuous lines correspond to inward spiraling orbits and the discontinuous lines to outward spiraling ones. An hyperbolic finite periodic orbit exists between them.*

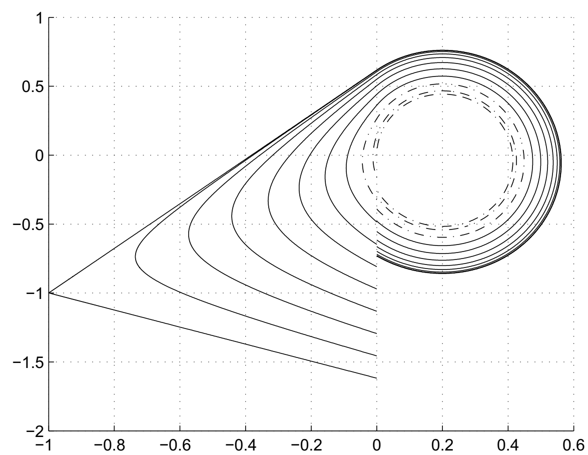

FIGURE 1.

The proof is based on the following lemmas:

Lemma 3. *Let us assume that a finite periodic orbit exists. Then*

$$\frac{A^+}{A^-} = -\frac{a_1}{\gamma_1}$$

where A^+, A^- are the enclosed areas in the right and the left side respectively.

Lemma 4. *Let us consider the saddle-spiral orbit passing through* $(0, -\frac{b_2}{\lambda_2})$. *Then its first intersection with the separating hyperplane (if exists) is determined by*

$$\exp(\alpha t)\sin(\beta t - \varphi) + \frac{\beta}{M} = 0, \quad \pi + \frac{\varphi}{\beta} \leq t \leq \frac{3\pi}{2} + \frac{\varphi}{\beta}$$

Corollary 3. *The systems in Theorem 1 with* γ_1 *as in 2./3. are structurally stable.*

Remark 3. *A homoclinic (i.e., saddle-loop) orbit bifurcation appears when 1. in Theorem 1 occurs.*

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