Communications in Statistics - Simulation and Computation

Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lssp20

Reliability Bounds for Fault-Tolerant Systems with Deferred Repair using Bounding Split Regenerative Randomization

Jamal Temsamani [a] & Juan A. Carrasco [b]

[a] Manipulados Maroc , Tanger , Morocco

[b] Departament d'Enginyeria Electrònica , Universitat Politècnica de Catalunya , Barcelona , Spain

Accepted author version posted online: 16 May 2013.Published online: 13 Aug 2013.

To cite this article: Jamal Temsamani & Juan A. Carrasco (2014) Reliability Bounds for Fault-Tolerant Systems with Deferred Repair using Bounding Split Regenerative Randomization, Communications in Statistics - Simulation and Computation, 43:1, 183-208, DOI: 10.1080/03610918.2012.700360

To link to this article: http://dx.doi.org/10.1080/03610918.2012.700360

Communications in Statistics—Simulation and Computation®, 43: 183–208, 2014

ISSN: 0361-0918 print / 1532-4141 online
DOI: 10.1080/03610918.2012.700360

Reliability Bounds for Fault-Tolerant Systems with Deferred Repair using Bounding Split Regenerative Randomization

JAMAL TEMSAMANI[1] AND JUAN A. CARRASCO[2]

[1]Manipulados Maroc, Tanger, Morocco
[2]Departament d'Enginyeria Electrònica, Universitat Politècnica de Catalunya, Barcelona, Spain

A numerically stable method is developed which computes seemingly tight bounds at a small computational cost relative to the model size, when that model size is large, for the unreliability and bounds for the unreliability using, respectively, exact and bounding failure/repair continuous-time Markov chain models of fault-tolerant systems with exponential failure and repair time distributions, in which repair is deferred until some condition on the collection of failed components is satisfied, and, then, proceeds until reaching the state without failed components, with failure rates much smaller than repair rates and not too different output rates from states with deferred repair.

Keywords Bounds; Continuous-time Markov chains; Deferred repair; Fault-tolerant systems; Randomization.

Mathematics Subject Classification 68U01; 60J25; 60J22.

1. Introduction

Repair deferment is an interesting approach for some fault-tolerant systems, particularly, fault-tolerant systems for which repair actions have high costs either because the system is located at a remote site or because some components are accessible with difficulty. In those systems, in order to reduce the number of repair actions, repair may be deferred until some condition on the collection of failed components is satisfied, for instance, until the number of failed components becomes greater than or equal to two, and, then, it may proceed until reaching the state without failed components. The key property of fault-tolerant systems with deferred repair is the presence of states with failed components in which no repair action is underway.

The unreliability (probability that the system has been failed in the time interval $[0, t]$) is an appropriate dependability measure for many fault-tolerant systems, specifically, mission-oriented fault-tolerant systems with applications requiring continuous operation. Assuming that the fault-tolerant system is modeled by an (homogeneous) continuous-time

Received September 3, 2010; Accepted May 28, 2012
Address correspondence to Juan Antonio Carrasco, Universitat Politècnica de Catalunya, Diagonal 647, plta. 9, Barcelona, 08028 Spain; E-mail: juan.a.carrasco@upc.edu

Markov chain (CTMC), computation of the reliability of the system requires the computation of the transient probability vector of the CTMC model. Available methods to perform such transient analysis when the CTMC model is large include ODE (ordinary differential equation) solvers and randomization methods. Good recent reviews of those methods with new developments can be found in Malhotra (1995), Malhotra et al. (1994), Reibman and Trivedi (1988), and Stewart (1994). Randomization methods (also called uniformization methods or Jensen's methods) are attractive because the absolute truncation error can be bounded from above and because they are numerically stable. Here, a method is considered to be numerically stable when the impact of round-off errors in the computed solution if the computed solution is a scalar, or, in each component of the computed solution, if the computed solution is a vector, can be expected to be small in relative value. The basic idea of randomization methods is due to Jensen (1953), and a randomization method was independently proposed by Grassmann (1977). The method proposed by Grassmann and randomization methods described in Malhotra et al. (1994), Reibman and Trivedi (1988), and Stewart (1994) are applicable to finite CTMC models with infinitesimal generator and compute the transient probability vector of the CTMC model.

Randomization methods and its proposed variants can be derived from the following results. Let $X = \{X(t); t \geq 0\}$ be a, not necessarily finite, uniformizable CTMC with state space Ω, let $\mathbf{A} = (a_{i,j})_{i,j\in\Omega}$ denote the infinitesimal generator of X, with $a_{i,j} = \lambda_{i,j}$, $i, j \in \Omega, i \neq j$, $\lambda_{i,j}$ denoting the transition rate of X from state i to state j, and with $a_{i,i} = -\lambda_i = -\sum_{j\in\Omega, j\neq i} \lambda_{i,j}$, λ_i denoting the output rate from state i. Consider any $\Lambda \geq \sup_{i\in\Omega}$ and the (homogeneous) discrete-time Markov chain (DTMC) $\widehat{X} = \{\widehat{X}_n; n = 0, 1, 2, \ldots\}$ with same state space and initial probability distribution as X and transition matrix $\mathbf{P} = \mathbf{I} + \Lambda^{-1}\mathbf{A}$, $\mathbf{I}$ denoting the identity matrix. Let $Q = \{Q(t); t \geq 0\}$ be a Poisson process with arrival rate Λ independent of $\widehat{X}$. Then, $X = \{X(t); t \geq 0\}$ is probabilistically identical to $\{\widehat{X}_{Q(t)}; t \geq 0\}$. The result allows to compute anything depending solely on the probabilistic behavior of X using $\{\widehat{X}_{Q(t)}; t \geq 0\}$ instead, and can be found in Çinlar (1975, pp. 259–260), Feller (1971, pp. 321–332), Heyman and Sobel (1982, pp. 310–311), and Kijima (1997, Theorem 4.19). The DTMC $\widehat{X}$ is said to be randomized with rate Λ when $\{\widehat{X}_{Q(t)}; t \geq 0\}$ is considered. It can also be said that $\widehat{X}$ is subordinated to the Poisson process Q. $\widehat{X}$ can also be subordinated to a jump process with exponentially distributed jump times with non-decreasing rates equal to the supremum of the output rates from the states at which $\widehat{X}$ can be at every step (Van Moorsel and Sanders, 1994).

In this article, we will mostly consider finite CTMCs $X = \{X(t); t \geq 0\}$ with infinitesimal generator and state space $\Omega = S \cup \{f_1, \ldots, f_A\}$, $|S| \geq 1$, $A \geq 1$, where f_i are absorbing states, satisfying some conditions, and will consider the problem of computing bounds for $m(t) = \sum_{i=1}^{A} r_{f_i} P[X(t) = f_i]$, where $r_{f_i} \geq 0$ and the r_{f_i} are different. The measure $m(t)$ can be considered an special case of the more general expected transient reward rate measure, $E[r_{X(t)}]$, if the r_{f_i} are interpreted as reward rates associated with states f_i and null reward rates are associated with states in S. The unreliability, $ur(t)$, of a fault-tolerant system is a particular instance of the $m(t)$ measure. In that case, S includes operational states, $A = 1$, $r_{f_1} = 1$, and stay of X in f_1 models the fact that the system has been failed.

Both lower and upper bounds for $ur(t)$ can also be formulated by using bounding CTMC models and particular instances of the $m(t)$ measure. Those bounding CTMC models are useful when an exact CTMC model would have an state space of unmanageable size. In a lower bounding CTMC model, X^{lb}, subset S, S^{lb}, would be a subset of the subset S of the exact CTMC model, X, A would be 2, r_{f_1} would be 1, r_{f_2} would be 0, transition rates among states in S^{lb} would be as the transition rates among those states in X, transition rates from states in S^{lb} to f_1 would be as the transition rates from those states to f_1 in X, transition rates from states in S^{lb} to f_2 would be as the transition rates

from those states to $S - S^{\text{lb}}$ in X, the initial probabilities of states in S^{lb} would be as the initial probabilities of those states in X, $P[X^{\text{lb}}(0) = f_1]$ would be $P[X(0) = f_1]$, and $P[X^{\text{lb}}(0) = f_2]$ would be $P[X(0) \in S - S^{\text{lb}}]$. Then, the $m(t)$ measure of X^{lb} would be a lower bound for $ur(t)$. In an upper bounding CTMC model, X^{ub}, subset S, S^{ub}, would be a subset of the subset S of the exact CTMC model, X, A would be 1, r_{f_1} would be 1, transition rates among states in S^{ub} would be as the transition rates among those states in X, transition rates from states in S^{ub} to f_1 would be as the transition rates from those states to $(S - S^{\text{ub}}) \cup \{f_1\}$ in X, and $P[X^{\text{ub}}(0) = f_1]$ would be $P[X(0) \in (S - S^{\text{ub}}) \cup \{f_1\}]$. Then, the $m(t)$ measure of X^{ub} would be an upper bound for $ur(t)$.

We will review next what we will call the standard randomization (SR) version of the randomization method for the computation of $m(t)$. That version can be trivially extended to cover the computation of $E[r_{X(t)}]$ for the more general case $A \geq 0$ and is numerically stable. Using the facts that X and $\{\widehat{X}_{Q(t)}; t \geq 0\}$ are probabilistically identical and that Q is independent of $\widehat{X}$, we can express $m(t)$ as

$$m(t) = \sum_{i=1}^{A} r_{f_i} P[X(t) = f_i] = \sum_{i=1}^{A} r_{f_i} \sum_{n=0}^{\infty} P[\widehat{X}_n = f_i \mid Q(t) = n] P[Q(t) = n]$$

$$= \sum_{n=0}^{\infty} \sum_{i=1}^{A} r_{f_i} P[\widehat{X}_n = f_i] \, e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} = \sum_{n=0}^{\infty} d(n) \, e^{-\Lambda t} \frac{(\Lambda t)^n}{n!},$$

with $d(n) = \sum_{i=1}^{A} r_{f_i} P[\widehat{X}_n = f_i]$. In the method, an approximate value for $m(t)$, $m_N^a(t)$, is obtained by truncating the infinite sum:

$$m_N^a(t) = \sum_{n=0}^{N} d(n) \, e^{-\Lambda t} \frac{(\Lambda t)^n}{n!}.$$

Then, ε being an error control parameter and with $r_{\max} = \max_{1 \leq i \leq A} r_{f_i}$, N can be chosen as

$$N = \min \left\{ m \geq 0 : r_{\max} \sum_{n=m+1}^{\infty} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} \leq \varepsilon \right\},$$

guaranteeing a non-negative absolute truncation error $\leq \varepsilon$ in $m(t)$. Let $\mathbf{q}(n)$ be the row vector $(P[\widehat{X}_n = i])_{i \in \Omega}$. Computation of $m_N^a(t)$ can be done from $\mathbf{q}(n)$, $0 \leq n \leq N$. Vector $\mathbf{q}(0)$ is known. Vectors $\mathbf{q}(n)$, $0 < n \leq N$ can be computed from $\mathbf{q}(0)$ using, for increasing n,

$$\mathbf{q}(n+1) = \mathbf{q}(n)\mathbf{P}. \tag{1}$$

For large CTMC models, the computational cost in terms of CPU time of the method is roughly due to the N vector-matrix multiplications (1). The truncation parameter N increases with Λt, and, for that reason, Λ is taken equal to $\max_{i \in \Omega} \lambda_i$. Using the well-known result (Ross, 1983, Theorem 3.3.5) that $Q(t)$ has, for $\Lambda t \to \infty$, an asymptotic normal distribution with mean and variance Λt, it is easy to realize that, for large Λt and $\varepsilon \ll 1$, the required N will be $\approx \Lambda t$. Then, we can conclude that for large CTMC models the method will be expensive if Λt is large. That problem is common to all randomization methods.

Several variants of randomization methods have been proposed to improve their efficiency or to widen to, not necessarily finite, uniformizable CTMC models the CTMC models they can handle. All those variants are, or potentially are, numerically stable, the

latter depending on using specific procedures for computing Poisson probabilities, and compute either the transient probability vector of the CTMC or $E[r_{X(t)}]$. Gross and Miller (1984) have proposed a variant applicable to uniformizable CTMC models with finite support for the initial probability vector of X that bounds from above the 1-norm of the absolute truncation error in the transient probability vector. Reibman and Trivedi (1988) have proposed an approach based on the multi-step concept. The idea is to compute $\mathbf{P}^M$ explicitly by squaring, where M is the length of the multi-step, and use the recurrence $\mathbf{q}(n + M) = \mathbf{q}(n)\mathbf{P}^M$ to advance $\widehat{X}$ faster for steps that have negligible contributions to the transient probability vector of X at time t. However, significant fill-in can occur when computing $\mathbf{P}^M$. The variant bounds from above the 1-norm of the absolute truncation error in the transient probability vector. Stewart (1994) has proposed a variant (uniformized powering) very similar to the approach proposed by Reibman an Trivedi (1988) and another variant in which the time interval $[0, t]$ is divided into subintervals of identical length, randomization is used to analyze in terms of transient probability vectors the intervals, and the common matrix relating the initial and final transient probability vectors of the subintervals is computed and stored. In both those variants, the 1-norm of the absolute truncation error can be bounded from above, but fill-in is a problem. Adaptive uniformization layered uniformization (Van Moorsel and Sanders, 1994) is applicable to uniformizable CTMC models with finite support for the initial probability vector and bounds from above the 1-norm of the absolute truncation error in the transient probability vector. Its numerical stability has been argued by Diener and Sanders (1995) using interval arithmetic. It has been proposed to combine another version of adaptive uniformization with randomization to obtain a variant, called adaptive uniformization/standard uniformization, with smaller computational cost for most finite CTMC models that bounds from above the 1-norm of the absolute approximation error in the transient probability vector (Van Moorsel and Sanders, 1997). A steady-state detection based variant that requires the knowledge of the steady-state probability vector of $\widehat{X}$, is applicable to some, not necessarily finite, uniformizable CTMC models and to any finite CTMC model, and bounds from above the 1-norm of the absolute truncation error in the transient probability vector, has been proposed by Van den Hout (1996, Chapter 3). Another steady-state detection based variant that does not need the knowledge of the steady-state probability vector of $\widehat{X}$, is applicable to finite CTMC models with a single recurrent class of states, bounds from above the absolute approximation error in $E[r_{X(t)}]$, and is numerically stable if reward rates are ≥ 0, has been developed by Sericola (1999). Steady-state detection based variants are useful for CTMC models in which $\widehat{X}$ reaches its steady-state probability vector long before the largest step of $\widehat{X}$ which would have to be considered in randomization methods.

Regenerative randomization (RR) (Carrasco, 2003) is another variant targeted at a class of CTMC models, class C′, including both exact and bounding failure/repair CTMC models of fault-tolerant systems with exponential failure and repair time distributions, repair in every state with failed components, and failure rates much smaller than repair rates, computing $E[r_{X(t)}]$ with reward rates ≥ 0 with bounded from above absolute truncation error, and having, for class C′ models, a computational cost in terms of CPU time that can be smaller than that of SR. Based on RR, bounding regenerative randomization (Carrasco, 2002) is targeted at a class of CTMC models, class C″, slightly less general than class C′, but also including both exact and bounding failure/repair CTMC models of fault-tolerant systems with exponential failure and repair time distributions, repair in every state with failed components, and failure rates much smaller than repair rates, and computing seemingly tight bounds at a computational cost in terms of CPU time that should be small relative to the model size when that model size is large.

The randomization with quasistationarity detection method (RQD) (Carrasco, 2004) covers CTMC models with finite state space $\Omega = S \cup \{f_1, \ldots, f_A\}$, $|S| \geq 1$, $A \geq 1$, where S is a transient class of states and states f_i are absorbing, computes $E[r_{X(t)}]$ with reward rates ≥ 0 bounding from above the absolute approximation error in $E[r_{X(t)}]$, and has a computational cost in terms of CPU time that can be smaller than that of SR. An efficient implementation of the randomization method, covering CTMC models with finite state space and computing $E[r_{X(t)}]$ with reward rates ≥ 0 with control of the absolute relative truncation error, has been proposed by Suñé and Carrasco (2005).

The split regenerative randomization method (SRR) (Temsamani and Carrasco, 2005) is a variant covering CTMC models with finite state space $S \cup \{f_1, f_2, \ldots, f_A\}$, $|S| \geq 3$, $A \geq 0$, where f_i are absorbing states with different reward rates, satisfying some conditions; computing $E[r_{X(t)}]$ with reward rates ≥ 0 with bounded from above absolute truncation error; targeted at a class of CTMC models, class C_2', including both exact and bounding failure/repair CTMC models of fault-tolerant systems with exponential failure and repair time distributions, in which repair is deferred until some condition on the collection of failed components is satisfied, and, then, proceeds until reaching the state without failed components, with failure rates much smaller than repair rates; and having a computational cost in terms of CPU time that can be smaller than those of SR, RR, a version of adaptive uniformization layered uniformization that bounds from above the absolute truncation error in $E[r_{X(t)}]$ and is numerically stable, and RQD. In this article, we take as starting point SRR applied to CTMC models with $A \geq 1$ for computing the measure $m(t) = \sum_{i=1}^{A} r_{f_i} P[X(t) = r_{f_i}]$ (Temsamani and Carrasco, 2006), and develop a method called bounding split regenerative randomization (BSRR) for computing bounds for $m(t)$. The rest of the article is organized as follows. Section 2 reviews SRR with $A \geq 1$ for computing the $m(t)$ measure at the detail required by the developments to follow. Section 3 develops and describes BSRR, including a particular, more efficient, implementation for an important case. Section 4 analyzes the behavior of BSRR using a representative large CTMC model and compares using that CTMC model the computational cost in terms of CPU time of the method with those of SRR, SR, RR, a version of adaptive uniformization layered uniformization (AU) that bounds from above the absolute truncation error in $m(t)$ and is a particular case of the version considered in Temsamani and Carrasco (2005), and RQD. Finally, Section 5 presents the conclusions.

2. Preliminaries

The SRR method with $A \geq 1$ applied to the computation of $m(t) = \sum_{i=1}^{A} r_{f_i} P[X(t) = f_i]$, where $r_{f_i} \geq 0$ and the r_{f_i} are different, requires the selection of a subset of states E and a regenerative state $r \in E$. With $E' = E - \{r\}$ and $\bar{E} = S - E$, the method covers CTMC models X with finite state space Ω for which there exist selections for the subset E and the regenerative state r such that the following conditions are satisfied:

C1. $\Omega = S \cup \{f_1, \ldots, f_A\}$, $|S| \geq 3$, $A \geq 1$, where the states f_i are absorbing and either all states in S are transient or S includes a single recurrent class of states $C \subset S$.
C2. All states are reachable from some state with non-null initial probability.
C3. $E \subset S$.
C4. if X includes a single recurrent class of states $C \subset S$, $r \in C$.
C5. $|E| \geq 2$.
C6. $|\bar{E}| \geq 1$.
C7. r can only be entered from $\bar{E}$, i.e., $\lambda_{i,r} = 0$, $i \in E'$.

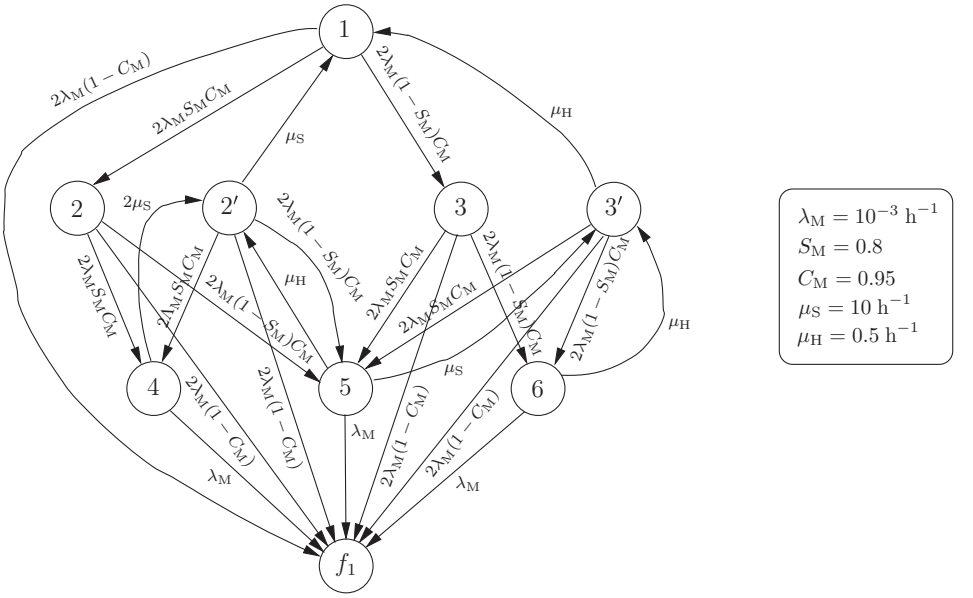

Figure 1. State diagram of a small CTMC model of a repairable fault-tolerant system with deferred repair using the pair-and-spare technique.

C8. r is the only entry point in E, i.e. $\lambda_{i,j} = 0, i \in \bar{E}, j \in E'$.
C9. $\lambda_{r,j} > 0$ for some $j \in E'$.

That all r_{f_i} have to be different can be fulfilled by merging absorbing states f_i with same r_{f_i}. Condition C9 can be fulfilled by adding, in case $\lambda_{r,j} = 0$ for all $j \in E'$, a tiny transition rate $\lambda \leq 10^{-10}\varepsilon/(2r_{\max}t_{\max})$ from r to some state in E', where ε is the allowed truncation error, $r_{\max} = \max_{1\leq i\leq A} r_{f_i}$, and $t_{\max}$ is the largest time at which the measure has to be computed, introducing a negligible absolute error $\leq 10^{-10}\varepsilon$ in $m(t)$, $t \leq t_{\max}$ (see Carrasco (2005b)). If X has a single recurrent class of states $C \subset S$, by conditions C4 and C9, $|C| \geq 2$, since $|C| = 1$ would imply through condition C4 that r would be absorbing, in contradiction with condition C9. Therefore, when the method is applicable, $f_1, f_2, \ldots, f_A$ have to be the only absorbing states. This makes it easy to identify the set of states S for which the conditions have to be checked to determine whether the method is applicable to a given finite CTMC with given selections for E and $r \in E$. Finally, condition C2 can be fulfilled by deleting unreachable states.

To illustrate the CTMC models covered by the method, Fig. 1 depicts the state diagram of a small CTMC model of a fault-tolerant system with deferred repair using the pair-and-spare technique (Johnson, 1989), in which active modules have failure rate λ_M, the spare module does not fail, the failure of an active module is "soft" with probability S_M and "hard" with probability $1 - S_M$, and whether soft or hard, the failure of an active module is covered with probability C_M. Modules in soft failure mode are independently recovered at rate μ_S and modules in hard failure mode are repaired by a single repairman at rate μ_H. Repair is deferred until two modules are failed and, then, proceeds until reaching the state 1 without failed components. The states with deferred repair are states 2 and 3. It is assumed that the CTMC model has some initial probability distribution in the states 1, 2, 3, 4, 5, 6,

$2'$, and $3'$. The subset of states S would be $S = \{1, 2, 3, 4, 5, 6, 2', 3'\}$. The measure $m(t)$ with $r_{f_1} = 1$ would be the unreliability of the system at time t. Possible selections for the subset of states E and the regenerative state r are $E = \{1, 2, 3\}$ and $r = 1$.

Although SRR applied to the computation of $m(t) = \sum_{i=1}^{A} r_{f_i} P[X(t) = f_i]$ is per se applicable to any CTMC model with finite state space satisfying, with appropriate selections for E and r, conditions C1–C9, the method is targeted at CTMC models in a certain CTMC model class, class C_2, with natural selections for E and r. Class C_2 includes all CTMC models X with finite state space Ω satisfying conditions C1–C2 and the condition

C10. There exists a partition $S_0 \cup S_1 \cup \cdots \cup S_{N_C} \cup \bar{S}_1 \cup \bar{S}_2 \cup \cdots \cup \bar{S}_{\bar{N}_C}$ for S satisfying the following properties:

P1. $S_0 = \{o\}$, i.e., $|S_0| = 1$.
P2. If X has a single recurrent class of states $C \subset S$, then $o \in C$.
P3. $|S_0 \cup S_1 \cup \cdots \cup S_{N_C}| \geq 2$ and $|\bar{S}_1 \cup \bar{S}_2 \cup \cdots \cup \bar{S}_{\bar{N}_C}| \geq 1$.
P4. $\lambda_{o, S_1 \cup \cdots \cup S_{N_C}} > 0$.
P5. for each $i \in S_k$, $0 < k \leq N_C$, $\lambda_{i, S_0 \cup \cdots \cup S_{k-1} \cup S_k - \{i\}} = 0$.
P6. for each $i \in \bar{S}_k$, $1 \leq k \leq \bar{N}_C$, $\lambda_{i, S_1 \cup \cdots \cup S_{N_C}} = 0$.
P7. $\max_{1 \leq k \leq \bar{N}_C} \max_{i \in \bar{S}_k} \lambda_{i, \bar{S}_k - \{i\} \cup \bar{S}_{k+1} \cup \cdots \cup \bar{S}_{\bar{N}_C}} \ll \min_{1 \leq k \leq \bar{N}_C} \min_{i \in \bar{S}_k} \lambda_{i, S_0 \cup \bar{S}_1 \cup \cdots \cup \bar{S}_{k-1} \cup \{f_1, f_2, \ldots, f_A\}} > 0$.

Natural selections for E and r are $E = S_0 \cup S_1 \cup \cdots \cup S_{N_C}$ and $r = o$. Obviously, the natural selection for E implies condition C3. In addition, with the natural selections for E and r, properties P2, P3, P4, P5, and P6 of the partition for S imply the fulfillment of, respectively, conditions C4, C5 and C6, C9, C7, and C8. Model class C_2 includes both exact and bounding failure/repair CTMC models of fault-tolerant systems with exponential failure and repair time distributions, in which repair is deferred until some condition on the collections of failed components is satisfied, and, then, proceeds until reaching the state without failed components, with failure rates much smaller than repair rates. For those CTMC models, the partition for S for which properties P1—P7 would be satisfied is the partition in which S_k includes the states without repair and the same number of failed components, with the subsets S_k ordered following increasing number of failed components, and $\bar{S}_k$ includes the states with repair and the same number of failed components, with the subsets $\bar{S}_k$ similarly ordered following increasing number of failed components. The small CTMC model with the state diagram of Fig. 1 illustrates class C_2. A partition for $S = \{1, 2, 3, 2', 3', 4, 5, 6\}$ showing that that CTMC is in class C_2 is $S_0 = \{1\}$, $S_1 = \{2, 3\}$, $S'_1 = \{2', 3'\}$, and $S'_2 = \{4, 5, 6\}$. Natural selections for E and r are $E = \{1, 2, 3\}$ and $r = 1$.

SRR with $A \geq 1$ applied to the computation of $m(t) = \sum_{i=1}^{A} r_{f_i} P[X(t) = f_i]$ has two phases. In the first phase, a truncated transformed CTMC model, V_T, is built that has the same $m(t)$ measure as the original CTMC model X with absolute model truncation error $\leq \varepsilon/2$. In the second phase, the $m(t)$ measure of V_T is computed with absolute truncation error $\leq \varepsilon/2$ by using SR. This results in the computation of $m(t)$ for X with an absolute truncation error $\leq \varepsilon$. The truncated transformed CTMC model V_T is obtained by characterizing the behavior of X from state r until either next hit of state r or hit of a state f_i and from $S' = S - \{r\}$ until either hit of state r or hit of a state f_i. The reader is referred to Temsamani and Carrasco (2006) for details.

Let $\Lambda_E = (1 + \theta) \max_{i \in E} \lambda_i$ and $\Lambda_{\bar{E}} = (1 + \theta) \max_{i \in \bar{E}} \lambda_i$, where θ is a small quantity > 0, say $\theta = 10^{-4}$. Let $\widehat{X} = \{\widehat{X}_n; n = 0, 1, 2, \ldots\}$ be the DTMC with same state space and initial probability distribution as X and transition matrix $\mathbf{P} = (P_{i,j})_{i,j \in \Omega}$, where

$P_{i,j} = \lambda_{i,j}/\Lambda_E,\ i \in E,\ j \neq i,\ P_{i,i} = 1 - \lambda_i/\Lambda_E,\ i \in E,\ P_{i,j} = \lambda_{i,j}/\Lambda_{\bar{E}},\ i \in \bar{E} \cup \{f_1, f_2, \ldots, f_A\}, j \neq i, P_{i,i} = 1-\lambda_i/\Lambda_{\bar{E}}, i \in \bar{E} \cup \{f_1, f_2, \ldots, f_A\}$, and let $\widehat{X}' = \{\widehat{X}'_n; n = 0, 1, 2, \ldots\}$ denote a version of $\widehat{X}$ with initial probability distribution concentrated in state r. Given a DTMC $Y = \{Y_n; n = 0, 1, 2, \ldots\}$, let $Y_{m_1:m_2}c$ be the predicate which is satisfied when Y_n, $m_1 \leq n \leq m_2$ satisfies condition c (by convention $Y_{m_1:m_2}c$ is satisfied for $m_2 < m_1$), and consider the row vectors $\boldsymbol{\pi}(n) = (\pi_i(n))_{i\in E}$, $n \geq 0$, $\boldsymbol{\pi}(n,k) = (\pi_i(n,k))_{i\in\bar{E}}$, $n \geq 0$, $k \geq 1$, $\boldsymbol{\pi}'(n) = (\pi'_i(n))_{i\in E'}$, $n \geq 0$, $\boldsymbol{\pi}'(n,k) = (\pi'_i(n,k))_{i\in\bar{E}}$, $n \geq 0$, $k \geq 1$, $\boldsymbol{\pi}''(n) = (\pi''_i(n))_{i\in\bar{E}}$, $n \geq 0$, where $\pi_i(n) = P[\widehat{X}'_{1:n} \in E' \wedge \widehat{X}'_n = i]$, $\pi_i(n,k) = P[\widehat{X}'_{1:n} \in E' \wedge \widehat{X}'_{n+1:n+k} \in \bar{E} \wedge \widehat{X}'_{n+k} = i]$, $\pi'_i(n) = P[\widehat{X}_{0:n} \in E' \wedge \widehat{X}_n = i]$, $\pi'_i(n,k) = P[\widehat{X}_{0:n} \in E' \wedge \widehat{X}_{n+1:n+k} \in \bar{E} \wedge \widehat{X}_{n+k} = i]$, and $\pi''_i(n) = P[\widehat{X}_{0:n} \in \bar{E} \wedge \widehat{X}_n = i]$. Let $a(n) = \sum_{i\in E}\pi_i(n)$, $n \geq 0$, $a(n,k) = \sum_{i\in\bar{E}}\pi_i(n,k)$, $n \geq 0$, $k \geq 1$, $a'(n) = \sum_{i\in E'}\pi'_i(n)$, $n \geq 0$, $a'(n,k) = \sum_{i\in\bar{E}}\pi'_i(n,k)$, $n \geq 0$, $k \geq 1$, and $a''(n) = \sum_{i\in\bar{E}}\pi''_i(n)$, $n \geq 0$. With $\alpha_B = \sum_{i\in B}\alpha_i$, $B \subset \Omega$, $\alpha_i = P[X(0) = i]$, $i \in \Omega$, V_T is defined by the truncation parameters: M, if $\alpha_{\bar{E}} > 0$; K, K_n, $n \in \gamma_K$, where $\gamma_K = \{n : 0 \leq n \leq K-1 \wedge a(n,1) > 0\}$; and L, L_n, $n \in \gamma'_L$, where $\gamma'_L = \{n : 0 \leq n \leq L-1 \wedge a'(n,1) > 0\}$, if $\alpha_{E'} > 0$. With $t_{\max}$ denoting the maximum value of t at which $m(t)$ has to be computed and remembering that $r_{\max} = \max_{1\leq i\leq A} r_{f_i}$, the values of the truncation parameters are

$$M = \min\left\{n \geq 1 : r_{\max}a''(n)\sum_{m=n+1}^{\infty} e^{-\Lambda_{\bar{E}}t_{\max}}\frac{(\Lambda_{\bar{E}}t_{\max})^m}{m!} \leq \varepsilon_1\right\}, \tag{2}$$

where $\varepsilon_1 = \varepsilon/6$ if $\alpha_{E'} > 0$ and $\varepsilon_1 = \varepsilon/4$ if $\alpha_{E'} = 0$;

$$K = \min\Bigg\{n \geq 1 : r_{\max}(\alpha_S - \mathbf{1}_{\alpha_{\bar{E}}>0}a''(M))\,a(n) \sum_{m=n+1}^{\infty}(m-n)e^{-\Lambda_E t_{\max}}\frac{(\Lambda_E t_{\max})^m}{m!} \leq \varepsilon_2\Bigg\}, \tag{3}$$

$$K_n = \min\Bigg\{k \geq 1 : r_{\max}(\alpha_S - \mathbf{1}_{\alpha_{\bar{E}}>0}a''(M))\,a(n,k) \sum_{m=n+1}^{\infty}(m-n)\,e^{-\Lambda_E t_{\max}}\frac{(\Lambda_E t_{\max})^m}{m!} \leq \frac{\varepsilon_2}{|\gamma_K|}\Bigg\}, \tag{4}$$

where $\mathbf{1}_c$ is the indicator function returning value 1 if condition c is satisfied and value 0 otherwise, $\varepsilon_2 = \varepsilon/12$ if $\alpha_{E'} > 0$ and $\alpha_{\bar{E}} > 0$, $\varepsilon_2 = \varepsilon/8$ if $\alpha_{E'} > 0$ and $\alpha_{\bar{E}} = 0$ or $\alpha_{E'} = 0$ and $\alpha_{\bar{E}} > 0$, and $\varepsilon_2 = \varepsilon/4$ if $\alpha_{E'} = 0$ and $\alpha_{\bar{E}} = 0$; and

$$L = \min\left\{n \geq 1 : r_{\max}a'(n)\sum_{m=n+1}^{\infty} e^{-\Lambda_E t_{\max}}\frac{(\Lambda_E t_{\max})^m}{m!} \leq \varepsilon_3\right\}, \tag{5}$$

$$L_n = \min\left\{k \geq 1 : r_{\max}a'(n,k)\sum_{m=n+1}^{\infty} e^{-\Lambda_E t_{\max}}\frac{(\Lambda_E t_{\max})^m}{m!} \leq \frac{\varepsilon_3}{|\gamma'_L|}\right\}, \tag{6}$$

where $\varepsilon_3 = \varepsilon/12$ if $\alpha_{\bar{E}} > 0$ and $\varepsilon_3 = \varepsilon/8$ if $\alpha_{\bar{E}} = 0$.

The truncated transformed CTMC model V_T has, for the case $\alpha_{E'} > 0$ and $\alpha_{\bar{E}} > 0$, the state space $E_V^T \cup \bar{E}_V^T \cup \{f_1, f_2, \ldots, f_A, a\}$, $E_V^T = \{s_n, 0 \leq n \leq K\} \cup \{s'_n, 0 \leq n \leq L\}$, $\bar{E}_V^T = \{s_{n,k} : n \in \gamma_K \wedge 1 \leq k \leq K_n\} \cup \{s'_{n,k} : n \in \gamma'_L \wedge 1 \leq k \leq L_n\} \cup \{s''_n, 0 \leq n \leq M\}$

and initial probability distribution $P[\widehat{V}_0 = s_0] = \alpha_r$, $P[\widehat{V}_0 = s_0'] = \alpha_{E'}$, $P[\widehat{V}_0 = s_0''] = \alpha_{\bar{E}}$, $P[\widehat{V}_0 = f_i] = \alpha_{f_i}$, $1 \le i \le A$, $P[\widehat{V}_0 = i] = 0$, $i \notin \{s_0, s_0', s_0'', f_1, f_2, \ldots, f_A\}$. If $\alpha_{E'} = 0$, the states s_n' and $s_{n,k}'$ disappear. If $\alpha_{\bar{E}} = 0$, the states s_n'' disappear.

With $P_{i,B} = \sum_{j \in B} P_{i,j}$, $B \subset \Omega$, let

$$w_n = \frac{\sum_{i \in E} \pi_i(n) P_{i,E'}}{a(n)}, \tag{7}$$

$$v_n^i = \frac{\sum_{j \in E} \pi_j(n) P_{j,f_i}}{a(n)}, \tag{8}$$

$$h_n = \frac{\sum_{i \in E} \pi_i(n) P_{i,\bar{E}}}{a(n)}, \tag{9}$$

$$w_{n,k} = \frac{\sum_{i \in \bar{E}} \pi_i(n,k) P_{i,\bar{E}}}{a(n,k)}, \tag{10}$$

$$q_{n,k} = \frac{\sum_{i \in \bar{E}} \pi_i(n,k) P_{i,r}}{a(n,k)}, \tag{11}$$

$$v_{n,k}^i = \frac{\sum_{j \in \bar{E}} \pi_j(n,k) P_{j,f_i}}{a(n,k)}, \tag{12}$$

$$w_n' = \frac{\sum_{i \in E'} \pi_i'(n) P_{i,E'}}{a'(n)}, \tag{13}$$

$$v_n'^i = \frac{\sum_{j \in E'} \pi_j'(n) P_{j,f_i}}{a'(n)}, \tag{14}$$

$$h_n' = \frac{\sum_{i \in E'} \pi_i'(n) P_{i,\bar{E}}}{a'(n)}, \tag{15}$$

$$w_{n,k}' = \frac{\sum_{i \in \bar{E}} \pi_i'(n,k) P_{i,\bar{E}}}{a'(n,k)}, \tag{16}$$

$$q_{n,k}' = \frac{\sum_{i \in \bar{E}} \pi_i'(n,k) P_{i,r}}{a'(n,k)}, \tag{17}$$

$$v_{n,k}'^i = \frac{\sum_{j \in \bar{E}} \pi_j'(n,k) P_{j,f_i}}{a'(n,k)}, \tag{18}$$

$$w_n'' = \frac{\sum_{i \in \bar{E}} \pi_i''(n) P_{i,\bar{E}}}{a''(n)}, \tag{19}$$

$$q_n'' = \frac{\sum_{i \in \bar{E}} \pi_i''(n) P_{i,r}}{a''(n)}, \tag{20}$$

$$v_n''^i = \frac{\sum_{j \in \bar{E}} \pi_j''(n) P_{j,f_i}}{a''(n)}. \tag{21}$$

Then, the transition rates in V_T are as follows:

- Each state s_n, $0 \le n < K$, has a transition rate $w_n \Lambda_E$ to state s_{n+1}, a transition rate $v_n^i \Lambda_E$ to each state f_i, and a transition rate $h_n \Lambda_E$ to state $s_{n,1}$ if $a(n, 1) > 0$.
- State s_K has a transition rate Λ_E to state a.

- Each state $s_{n,k}$, $n \in \gamma_K$, $1 \leq k < K_n$, has a transition rate $w_{n,k}\Lambda_{\bar{E}}$ to state $s_{n,k+1}$, a transition rate $q_{n,k}\Lambda_{\bar{E}}$ to state s_0, and a transition rate $v^i_{n,k}\Lambda_{\bar{E}}$ to each state f_i.
- Each state s_{n,K_n}, $n \in \gamma_K$, has a transition rate $\Lambda_{\bar{E}}$ to state a.
- Each state s'_n, $0 \leq n < L$, has a transition rate $w'_n\Lambda_E$ to state s'_{n+1}, a transition rate $v'^i_n\Lambda_E$ to each state f_i, and a transition rate $h'_n\Lambda_E$ to state $s'_{n,1}$ if $a'(n,1) > 0$.
- State s'_L has a transition rate Λ_E to state a.
- Each state $s'_{n,k}$, $n \in \gamma'_L$, $1 \leq k < L_n$, has a transition rate $w'_{n,k}\Lambda_{\bar{E}}$ to state $s'_{n,k+1}$, a transition rate $q'_{n,k}\Lambda_{\bar{E}}$ to state s_0, and a transition rate $v'^i_{n,k}\Lambda_{\bar{E}}$ to each state f_i.
- Each state s'_{n,L_n}, $n \in \gamma'_L$, has a transition rate $\Lambda_{\bar{E}}$ to state a.
- Each state s''_n, $0 \leq n < M$, has a transition rate $w''_n\Lambda_{\bar{E}}$ to state s''_{n+1}, a transition rate $q''_n\Lambda_{\bar{E}}$ to state s_0, and a transition rate $v''^i_n\Lambda_{\bar{E}}$ to each state f_i.
- State s''_M has a transition rate $\Lambda_{\bar{E}}$ to state a.

Figure 2 illustrates the state diagram of V_T for the case $\alpha_{E'} > 0$, $\alpha_{\bar{E}} > 0$ and $A = 1$.

The CTMC V_T is built as follows (see Temsamani and Carrasco (2006) for details.). First, if $\alpha_{\bar{E}} > 0$, the vectors $\boldsymbol{\pi}''(n)$ and, if $n \geq 1$, the sums $\sum_{m=n+1}^{\infty} e^{-\Lambda_{\bar{E}} t_{\max}}((\Lambda_{\bar{E}} t_{\max})^m/m!)$ are computed for increasing $n \geq 0$. This allows the determination of the truncation parameter M (2), noting that $a''(n) = \sum_{i\in\bar{E}} \pi''_i(n)$, the determination of the transition rates $w''_n\Lambda_{\bar{E}}$, $q''_n\Lambda_{\bar{E}}$, and $v''^i_n\Lambda_{\bar{E}}$, $1 \leq i \leq A$, $0 \leq n \leq M-1$, using (19)–(21), and the building up of the part of V_T hanging on the strip of states $s''_0, s''_1, \ldots, s''_M$. Second, the vectors $\boldsymbol{\pi}(n)$ and, if $n \geq 1$, the sums $\sum_{m=n+1}^{\infty}(m-n)e^{-\Lambda_E t_{\max}}((\Lambda_E t_{\max})^m/m!)$ are computed for increasing $n \geq 0$. This allows the determination of the truncation parameter K (3), noting that $a(n) = \sum_{i\in E} \pi_i(n)$, the determination of the transition rates $w_n\Lambda_E$ and $v^i_n\Lambda_E$, $1 \leq i \leq A$, $0 \leq n \leq K-1$, using (7), (8), the determination of h_n, $0 \leq n \leq K-1$, using (9), and the building up, except for the transition rates $h_n\Lambda_E$, of the part of V_T hanging on the strip of states $s_0, s_1, \ldots, s_K$. Third, vectors $\boldsymbol{\pi}(n)$, $0 \leq n \leq K-1$ are recomputed for increasing n. For each such n, noting that $a(n,1) > 0$ if and only if $h_n > 0$, if $h_n > 0$, the sum $\sum_{m=n+1}^{\infty}(m-n)e^{-\Lambda_E t_{\max}}((\Lambda_E t_{\max})^m/m!)$ is computed and the vectors $\boldsymbol{\pi}(n,k)$ are computed for increasing $k \geq 1$. This allows the determination of the truncation parameter K_n (4), noting that $a(n,k) = \sum_{i\in\bar{E}} \pi_i(n,k)$, the determination of the transition rates $w_{n,k}\Lambda_{\bar{E}}$, $q_{n,k}\Lambda_{\bar{E}}$ and $v^i_{n,k}\Lambda_{\bar{E}}$, $1 \leq i \leq A$, $1 \leq k \leq K_n - 1$, using (10)–(12), and the building up of the part of V_T hanging on the strip of states $s_{n,1}, s_{n,2}, \ldots, s_{n,K_n}$. The transition rate $h_n\Lambda_E$ from s_n to $s_{n,1}$ is also added, if $h_n > 0$. Fourth, if $\alpha_{E'} > 0$, the vectors $\boldsymbol{\pi}'(n)$ and, if $n \geq 1$, the sums $\sum_{m=n+1}^{\infty} e^{-\Lambda_E t_{\max}}((\Lambda_E t_{\max})^m/m!)$ are computed for increasing $n \geq 0$. This allows the determination of the truncation parameter L (5), noting that $a'(n) = \sum_{i\in E'} \pi'_i(n)$, the determination of the transition rates $w'_n\Lambda_E$ and $v'^i_n\Lambda_E$, $1 \leq i \leq A$, $0 \leq n \leq L-1$, using (13), (14), the determination of h'_n, $0 \leq n \leq L-1$, using (15), and the building up, except for the transition rates $h'_n\Lambda_E$, of the part of V_T hanging on the strip of states $s'_0, s'_1, \ldots, s'_L$. Fifth and lastly, if $\alpha_{E'} > 0$, vectors $\boldsymbol{\pi}'(n)$, $0 \leq n \leq L-1$ are recomputed for increasing n. For each such n, noting that $a'(n,1) > 0$ if and only if $h'_n > 0$, if $h'_n > 0$, the sum $\sum_{m=n+1}^{\infty} e^{-\Lambda_E t_{\max}}((\Lambda_E t_{\max})^m/m!)$ is computed and the vectors $\boldsymbol{\pi}'(n,k)$ are computed for increasing $k \geq 1$. This allows the determination of the truncation parameter L_n (6), noting that $a'(n,k) = \sum_{i\in\bar{E}} \pi'_i(n,k)$, the determination of the transition rates $w'_{n,k}\Lambda_{\bar{E}}$, $q'_{n,k}\Lambda_{\bar{E}}$ and $v'^i_{n,k}\Lambda_{\bar{E}}$, $1 \leq i \leq A$, $1 \leq k \leq L_n - 1$, using (16)–(18), and the building up of the part of V_T hanging on the strip of states $s'_{n,1}, s'_{n,2}, \ldots, s'_{n,L_n}$. The transition rate $h'_n\Lambda_E$ from s'_n to $s'_{n,1}$ is also added, if $h'_n > 0$. The sums $\sum_{m=n+1}^{\infty} e^{-\Lambda_E t_{\max}}((\Lambda_E t_{\max})^m/m!)$,

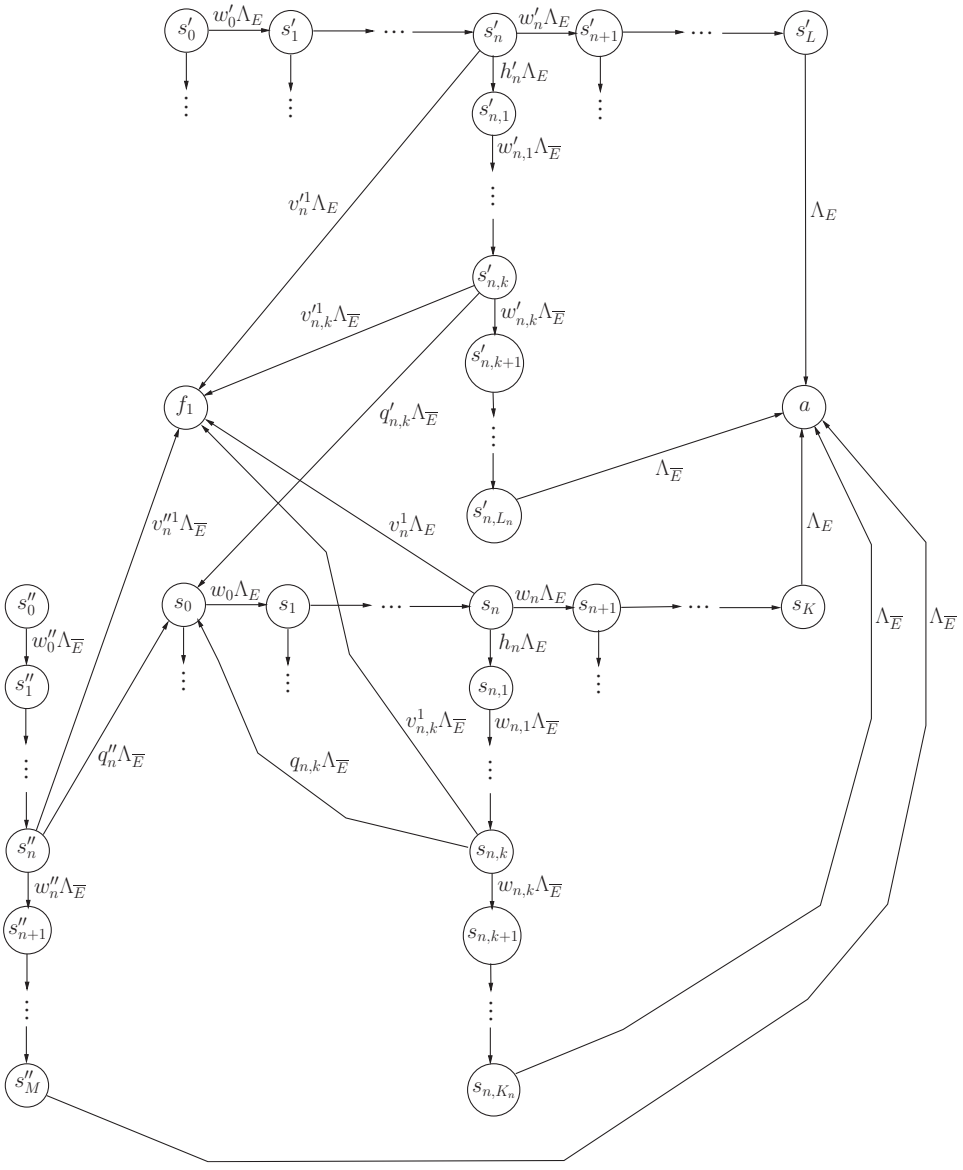

Figure 2. Illustration of the state diagram of V_T.

$\sum_{m=n+1}^{\infty} e^{-\Lambda_{\bar{E}} t_{\max}}((\Lambda_{\bar{E}} t_{\max})^m/m!)$ and $\sum_{m=n+1}^{\infty}(m-n)e^{-\Lambda_E t_{\max}}((\Lambda_E t_{\max})^m/m!)$ are computed for increasing n with numerical stability and efficiently using the procedures described in Carrasco (2005a).

The vectors $\boldsymbol{\pi}(n)$, $\boldsymbol{\pi}(n,k)$, $\boldsymbol{\pi}'(n)$, $\boldsymbol{\pi}'(n,k)$, and $\boldsymbol{\pi}''(n)$ can be obtained in the required ordering using the following recurrences, where $\mathbf{P}_{B,C}$, $B, C \subset \Omega$ denotes $(P_{i,j})_{i\in B, j\in C}$, $\mathbf{P}'_{E,E}$ denotes $\mathbf{P}_{E,E}$ with the column associated with state r set to a column of all zeroes,

and state r is numbered first in $\boldsymbol{\pi}(n)$:

$$\boldsymbol{\pi}(0) = (1\ 0\ 0\ \cdots\ 0)\,, \tag{22}$$

$$\boldsymbol{\pi}(n+1) = \boldsymbol{\pi}(n)\mathbf{P}'_{E,E},\ n \geq 0\,, \tag{23}$$

$$\boldsymbol{\pi}(n,1) = \boldsymbol{\pi}(n)\mathbf{P}_{E,\bar{E}},\ n \geq 0\,, \tag{24}$$

$$\boldsymbol{\pi}(n,k+1) = \boldsymbol{\pi}(n,k)\mathbf{P}_{\bar{E},\bar{E}},\ n \geq 0, k \geq 1\,, \tag{25}$$

$$\boldsymbol{\pi}'(0) = (\alpha_i)_{i\in E'}\,, \tag{26}$$

$$\boldsymbol{\pi}'(n+1) = \boldsymbol{\pi}'(n)\mathbf{P}_{E',E'},\ n \geq 0\,, \tag{27}$$

$$\boldsymbol{\pi}'(n,1) = \boldsymbol{\pi}'(n)\mathbf{P}_{E',\bar{E}},\ n \geq 0\,, \tag{28}$$

$$\boldsymbol{\pi}'(n,k+1) = \boldsymbol{\pi}'(n,k)\mathbf{P}_{\bar{E},\bar{E}},\ n \geq 0, k \geq 1\,, \tag{29}$$

$$\boldsymbol{\pi}''(0) = (\alpha_i)_{i\in\bar{E}}\,, \tag{30}$$

$$\boldsymbol{\pi}''(n+1) = \boldsymbol{\pi}''(n)\mathbf{P}_{\bar{E},\bar{E}},\ n \geq 0\,. \tag{31}$$

For class C_2 models we have the following result (see Temsamani and Carrasco, 2006), where $f(n) \sim g(n)$ denotes $\lim_{n\to\infty} f(n)/g(n) = 1$:

Theorem 2.1 *For class* C_2 *models and the selections* $E = S_0 \cup S_1 \cup \cdots \cup S_{N_C}$ *and* $r = o$, $a(n) \leq h_E(n)$ *and* $a'(n) \leq \alpha_{E'} h_{E'}(n)$, *where, for* $n \to \infty$, $h_E(n) \sim C\binom{n}{p-1} q_E^n$, $C > 0$, p *integer* ≥ 1, $h_{E'}(n) \sim C'\binom{n}{p'-1} q_E^n$, $C' > 0$, p' *integer* ≥ 1, *and* $q_E \approx 1 - 1/R_E$, *with*

$$R_E = \frac{\max_{0\leq k\leq N_C} \max_{i\in S_k} \lambda_i}{\min_{0<k\leq N_C} \min_{i\in S_k} \lambda_i}\,.$$

Furthermore, $a(n,k) \leq h_{\bar{E}}(k-1)$, $a'(n,k) \leq \alpha_{E'} h_{\bar{E}}(k-1)$ *and* $a''(n) \leq \alpha_{\bar{E}} h_{\bar{E}}(n)$, *where, for* $l \to \infty$, $h_{\bar{E}}(l) \sim C''\binom{l}{p''-1} q_{\bar{E}}^l$, $C'' \leq 0$, p'' *integer* ≥ 1, *and* $q_{\bar{E}} \approx 1 - 1/R_{\bar{E}}$, *with*

$$R_{\bar{E}} = \frac{\max_{1\leq k\leq \bar{N}_C} \max_{i\in \bar{S}_k} \lambda_i}{\min_{1\leq k\leq \bar{N}_C} \min_{i\in \bar{S}_k} \lambda_i}\,.$$

According to the theorem, for class C_2 models with natural selections for E and r, the closer R_E to 1, the faster asymptotically $h_E(n)$ and $h_{E'}(n)$ will decrease with n, and the smaller the truncation parameters K (3) and L (5) should be. Similarly, the closer $R_{\bar{E}}$ to 1, the faster asymptotically $h_{\bar{E}}(l)$ will decrease with l, and the smaller the truncation parameters M (2), K_n (4), and L_n (6) should be. Then, as R_E and $R_{\bar{E}}$ get closer to 1, the computational cost in terms of CPU time of the method should decrease, because the computational cost in terms of CPU time of the generation of V_T should decrease and the size of V_T should decrease.

3. The Method

3.1. *Motivation and General Case*

As SRR, BSRR requires the selection of a subset of states E and a regenerative state $r \in E$. The method covers the same CTMC models X with finite state space Ω as SRR with $A \geq 1$

applied to the computation of $m(t) = \sum_{i=1}^{A} r_{f_i} P[X(t) = f_i]$, where $r_{f_i} \geq 0$ and the r_{f_i} are different, reviewed in the previous section, and allows the computation of a lower bound for $m(t)$, an upper bound for $m(t)$ or both. The method is also targeted at CTMC models in class C_2 with same natural selections for E and r as SRR, is motivated by Theorem 2.1 and the discussion following it, and is based in the following intuitive result:

Theorem 3.1 *Let $X = \{X(t); t \geq 0\}$ be a CTMC model with state space $\Omega = S \cup \{f_1, f_2, \ldots, f_A\}$, $A \geq 1$, where f_i are absorbing states, transition rates $\lambda_{i,j}$, $i, j \in \Omega$, $j \neq i$, and let $m(t) = \sum_{1 \leq i \leq A} r_{f_i} P[X(t) = f_i]$, where $r_{f_i} \geq 0$ and the r_{f_i} are different. Let another CTMC model $X' = \{X'(t); t \geq 0\}$ with same state space and initial probability distribution as X and transition rates $\lambda'_{i,j} = \beta_i \lambda_{i,j}$, $i, j \in \Omega$, $0 < \beta_i \leq 1$, $i \in S$, and let $m'(t) = \sum_{1 \leq i \leq A} r_{f_i} P[X'(t) = f_i]$. Then, $m'(t) \leq m(t)$. Furthermore, it is enough that $\beta_k < 1$ for some reachable state $k \in S$ from which some state f_i with $r_{f_i} > 0$ can be reached to have $m'(t) < m(t)$.*

Proof. The result $m'(t) \leq m(t)$ follows immediately from developments in Carrasco (2002). The result $m'(t) < m(t)$ follows immediately by considering that strict inequality holds in Proposition 1 of Carrasco (2002) if $F_{U_j}(u) < F_{U'_j}(u)$ for some j and by noting that, then, following the developments of Proposition 2 in Carrasco (2002), we have $P[X'(t) = f_i] < P[X(t) = f_i]$ for some i such that $r_{f_i} > 0$. □

According to Theorem 3.1, scaling down the transition rates from some states in S of the given CTMC model X, will result in a CTMC model that will bound from below $m(t)$ and scaling up the transition rates from some states in S will result in a CTMC model that will bound from above $m(t)$. BSRR performs such scalings in states in $\bar{E}$. The scalings are performed using a control parameter D which is required to satisfy $1 \leq D < \lambda_{\max}/\lambda_{\min}$, where $\lambda_{\min} = \min_{i \in \bar{E}} \lambda_i$ and $\lambda_{\max} = \max_{i \in \bar{E}} \lambda_i$. The scalings are performed so that $\max_{i \in \bar{E}} \lambda_i / \min_{i \in \bar{E}} \lambda_i$ is reduced. Note that no selection for D is possible in the case $\lambda_{\min} = \lambda_{\max}$. However, in that case $\max_{i \in \bar{E}} \lambda_i / \min_{i \in \bar{E}} \lambda_i = 1$ and cannot be reduced further.

More precisely, in BSRR, the lower bound, $m^{\text{lb}}(t)$, for $m(t)$ is computed with absolute truncation error $\leq \varepsilon$ by generating a lower bounding CTMC model, X^{lb}, from the given CTMC model X, and computing the $m(t)$ measure of X^{lb} by using SRR with subset of states E, regenerative state r and absolute truncation error $\leq \varepsilon$. The lower bounding CTMC model X^{lb} is obtained from the given CTMC model X by scaling down transition rates from states $i \in \bar{E}$ using $\lambda^{\text{lb}}_{i,j} = \lambda_{i,j}(\lambda^{\text{lb}}_i/\lambda_i)$, $\lambda^{\text{lb}}_i = \min\{\lambda_i, D\lambda_{\min}\}$, $i \in \bar{E}$, where the superscript $^{\text{lb}}$ makes reference to quantities defining X^{lb}. The upper bound, $m^{\text{ub}}(t)$, for $m(t)$ is computed with absolute truncation error $\leq \varepsilon$ by generating an upper bounding CTMC model, X^{ub}, from the given CTMC model X, and computing the $m(t)$ measure of X^{ub} by using SRR with subset of states E, regenerative state r and absolute truncation error $\leq \varepsilon$. The upper bounding CTMC model X^{ub} is obtained from the given CTMC model X by scaling up transition rates from states $i \in \bar{E}$ using $\lambda^{\text{ub}}_{i,j} = \lambda_{i,j}(\lambda^{\text{ub}}_i/\lambda_i)$, $\lambda^{\text{ub}}_i = \max\{\lambda_i, \lambda_{\max}/D\}$, $i \in \bar{E}$, where the superscript $^{\text{ub}}$ makes reference to quantities defining X^{ub}. Figure 3 presents an schematic representation of BSRR in terms of SRR, detailing the two phases of the latter. In the figure, the truncated transformed CTMC model obtained in the first phase of SRR applied to X^{lb} is denoted by V^{lb}_T and the truncated transformed CTMC model obtained in the first phase of SRR applied to X^{ub} is denoted by V^{ub}_T. We will also use the superscripts $^{\text{lb}}$ and $^{\text{ub}}$ to denote quantities and objects associated with the generation of, respectively, V^{lb}_T and V^{ub}_T in SRR.

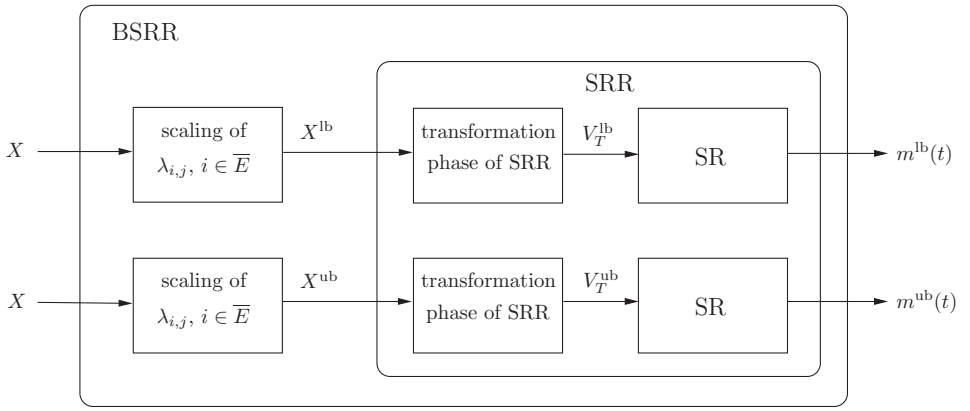

Figure 3. Schematic representation of BSRR.

Note that larger values of D potentially give larger values of the output rates from some states $i \in \bar{E}$ in X^{lb} and, therefore, according to Theorem 3.1, give potentially larger values of $m^{\mathrm{lb}}(t)$. Similarly, larger values of D potentially give smaller values of the output rates from some states $i \in \bar{E}$ in X^{ub} and, therefore, according to Theorem 3.1, give potentially smaller values of $m^{\mathrm{ub}}(t)$. Then, the parameter D controls the tightness of the bounds: the larger D, the tighter $m^{\mathrm{lb}}(t)$ and $m^{\mathrm{ub}}(t)$ can be. In fact, for $D \to \lambda_{\max}/\lambda_{\min}$, no transition rates are scaled and $m^{\mathrm{lb}}(t)$ and $m^{\mathrm{ub}}(t)$ become $m(t)$.

For class C_2 models X, with natural selections for E and r, X^{lb} and X^{ub} still belong to class C_2 and, with those natural selections for E and r, have an $R_{\bar{E}}$ parameter equal to D:

Theorem 3.2 *Let X be a class C_2 model. Then, with natural selections for E and r, the CTMC models X^{lb} and X^{ub} generated in bounding split regenerative randomization belong to class C_2 and, with those natural selections for E and r, have an $R_{\bar{E}}$ parameter equal to D.*

Proof. To show that, with natural selections for E and r, X^{lb} and X^{ub} belong to class C_2, it suffices to check that X^{lb} and X^{ub} with the same partition as X for S showing that X is in class C_2 satisfy property P7. To that end, note that P7 of X with that partition for S implies $\min_{1\le k\le \bar{N}_C} \min_{i\in\bar{S}_k} \lambda_{i,S_0\cup\bar{S}_1\cup\cdots\cup\bar{S}_{k-1}\cup\{f_1,f_2,\ldots,f_A\}} \approx \min_{1\le k\le \bar{N}_C} \min_{i\in\bar{S}_k} \lambda_i = \min_{i\in\bar{E}} \lambda_i = \lambda_{\min}$. Using it, we have, for X^{lb},

$$
\begin{aligned}
&\max_{1\le k\le \bar{N}_C} \max_{i\in\bar{S}_k} \lambda^{\mathrm{lb}}_{i,\bar{S}_k-\{i\}\cup\bar{S}_{k+1}\cup\cdots\cup\bar{S}_{\bar{N}_C}} \\
&\quad = \max_{1\le k\le \bar{N}_C} \max_{i\in\bar{S}_k} \frac{\lambda^{\mathrm{lb}}_i}{\lambda_i} \lambda_{i,\bar{S}_k-\{i\}\cup\bar{S}_{k+1}\cup\cdots\cup\bar{S}_{\bar{N}_C}} \\
&\quad \le \max_{1\le k\le \bar{N}_C} \max_{i\in\bar{S}_k} \lambda_{i,\bar{S}_k-\{i\}\cup\bar{S}_{k+1}\cup\cdots\cup\bar{S}_{\bar{N}_C}} \\
&\quad \ll \min_{1\le k\le \bar{N}_C} \min_{i\in\bar{S}_k} \lambda_{i,S_0\cup\bar{S}_1\cup\cdots\cup\bar{S}_{k-1}\cup\{f_1,f_2,\ldots,f_A\}} \\
&\quad \approx \lambda_{\min} = \lambda^{\mathrm{lb}}_{\min} = \min_{i\in\bar{E}} \lambda^{\mathrm{lb}}_i ,
\end{aligned}
$$

implying

$$\max_{1\le k\le \bar{N}_C}\ \max_{i\in\bar{S}_k} \lambda^{\mathrm{lb}}_{i,\bar{S}_k-\{i\}\cup\bar{S}_{k+1}\cup\cdots\cup\bar{S}_{\bar{N}_C}} \ll \min_{1\le k\le \bar{N}_C}\ \min_{i\in\bar{S}_k} \lambda^{\mathrm{lb}}_{i,S_0\cup\bar{S}_1\cup\cdots\cup\bar{S}_{k-1}\cup\{f_1,f_2,\ldots,f_A\}} .$$

Similarly, for X^{ub}, noting that $\lambda_i^{\mathrm{ub}}/\lambda_i \le \lambda_{\max}/(D\lambda_{\min})$, because the maximum of $\lambda_i^{\mathrm{ub}}/\lambda_i$, $i \in \bar{E}$ is obtained for $\lambda_i = \lambda_{\min}$,

$$\begin{aligned}
&\max_{1\le k\le \bar{N}_C}\ \max_{i\in\bar{S}_k} \lambda^{\mathrm{ub}}_{i,\bar{S}_k-\{i\}\cup\bar{S}_{k+1}\cup\cdots\cup\bar{S}_{\bar{N}_C}} \\
&\quad= \max_{1\le k\le \bar{N}_C}\ \max_{i\in\bar{S}_k} \frac{\lambda_i^{\mathrm{ub}}}{\lambda_i}\, \lambda_{i,\bar{S}_k-\{i\}\cup\bar{S}_{k+1}\cup\cdots\cup\bar{S}_{\bar{N}_C}} \\
&\quad\le \frac{\lambda_{\max}}{D\lambda_{\min}} \max_{1\le k\le \bar{N}_C}\ \max_{i\in\bar{S}_k} \lambda_{i,\bar{S}_k-\{i\}\cup\bar{S}_{k+1}\cup\cdots\cup\bar{S}_{\bar{N}_C}} \\
&\quad\ll \frac{\lambda_{\max}}{D\lambda_{\min}} \min_{1\le k\le \bar{N}_C}\ \min_{i\in\bar{S}_k} \lambda_{i,S_0\cup\bar{S}_1\cup\cdots\cup\bar{S}_{k-1}\cup\{f_1,f_2,\ldots,f_A\}} \\
&\quad\approx \frac{\lambda_{\max}}{D\lambda_{\min}}\lambda_{\min} = \frac{\lambda_{\max}}{D} = \lambda^{\mathrm{ub}}_{\min} = \min_{i\in\bar{E}} \lambda_i^{\mathrm{ub}} ,
\end{aligned}$$

implying

$$\max_{1\le k\le \bar{N}_C}\ \max_{i\in\bar{S}_k} \lambda^{\mathrm{ub}}_{i,\bar{S}_k-\{i\}\cup\bar{S}_{k+1}\cup\cdots\cup\bar{S}_{\bar{N}_C}} \ll \min_{1\le k\le \bar{N}_C}\ \min_{i\in\bar{S}_k} \lambda^{\mathrm{ub}}_{i,S_0\cup\bar{S}_1\cup\cdots\cup\bar{S}_{k-1}\cup\{f_1,f_2,\ldots,f_A\}} .$$

Finally,

$$R^{\mathrm{lb}}_{\bar{E}} = \frac{\max_{1\le k\le \bar{N}_C} \max_{i\in\bar{S}_k} \lambda_i^{\mathrm{lb}}}{\min_{1\le k\le \bar{N}_C} \min_{i\in\bar{S}_k} \lambda_i^{\mathrm{lb}}} = \frac{\max_{i\in\bar{E}} \lambda_i^{\mathrm{lb}}}{\min_{i\in\bar{E}} \lambda_i^{\mathrm{lb}}} = \frac{D\lambda_{\min}}{\lambda_{\min}} = D ,$$

and

$$R^{\mathrm{ub}}_{\bar{E}} = \frac{\max_{1\le k\le \bar{N}_C} \max_{i\in\bar{S}_k} \lambda_i^{\mathrm{ub}}}{\min_{1\le k\le \bar{N}_C} \min_{i\in\bar{S}_k} \lambda_i^{\mathrm{ub}}} = \frac{\max_{i\in\bar{E}} \lambda_i^{\mathrm{ub}}}{\min_{i\in\bar{E}} \lambda_i^{\mathrm{ub}}} = \frac{\lambda_{\max}}{\lambda_{\max}/D} = D ,$$

showing that X^{lb} and X^{ub} with the natural selections for E and r have an $R_{\bar{E}}$ parameter equal to D. □

Then, according to the discussion following Theorem 2.1, for class C_2 CTMC models with natural selections for E and r, the smaller D, the smaller the computational cost in terms of CPU time of BSRR should be, and the method can have computational cost in terms of CPU time smaller than SRR even if both bounds are computed. Further, since, as discussed, larger values of D potentially give tighter bounds $m^{\mathrm{lb}}(t)$ and $m^{\mathrm{ub}}(t)$, for class C_2 models with natural selections for E and r, the parameter D should trade off computational cost in terms of CPU time with bounds tightness.

For class C_2 models X with natural selections for E and r and $D = 1$, we have, $R^{\mathrm{lb}}_{\bar{E}} = R^{\mathrm{ub}}_{\bar{E}} = 1$. Then, according to discussion following Theorem 2.1, M^{lb}, K_n^{lb}, L_n^{lb}, M^{ub}, K_n^{ub}, and L_n^{ub} should be small. If, in addition, R_E has a moderate value ≥ 1, since $R^{\mathrm{lb}}_E = R^{\mathrm{ub}}_E = R_E$, because the transition rates from states $i \in E$ are not scaled, K^{lb}, L^{lb}, K^{ub}, and L^{ub} should be small. Then, if X is large, the computational cost in terms of CPU time of the first phase of SRR applied to both X^{lb} and X^{ub} should be small relative to the size of X. The sizes of V_T^{lb} and V_T^{ub} should also be small and much smaller than the size of X,

implying that the computational cost in terms of CPU time of the second phase of SRR applied to both X^{lb} and X^{ub} should be small relative to the size of X. In summary, for large class C_2 CTMC models with natural selections for E and r and $D = 1$, the computational cost in terms of CPU time of the method relative to the model size should be small if the model size is large and R_E has not too a large value ≥ 1. For both exact and bounding failure/repair CTMC models of fault-tolerant systems with exponential failure and repair time distributions, in which repair is deferred until some condition on the collection of failed components is satisfied, and, then, proceeds until reaching the state without failed components, with failure rates much smaller than repair rates, that R_E has not too a large value ≥ 1 means that output rates from states with deferred repair are not too different.

Finally, since SRR is numerically stable, BSRR is also numerically stable.

3.2. *Particular Case*

The particular case in which both bounds are to be computed and $D = 1$ allows a particular, more efficient, implementation of the method than that described in the previous subsection. That more efficient implementation is based on the fact that V_T^{lb} can be obtained from quantities and objects associated with the generation of V_T^{ub} during the first phase of SRR applied to X^{ub}. The particular case is important, since it is in that case that, for class C_2 models with natural selections for E and r, BSRR should have the smallest computational cost, and, often, both bounds would be sought to bracket $m(t)$.

The quantities and objects associated with the generation of V_T^{ub} which have to be saved are Λ_E^{ub}, $\Lambda_{\bar{E}}^{\mathrm{ub}}$, K^{ub}, $a^{\mathrm{ub}}(n)$, $1 \leq n \leq K^{\mathrm{ub}}$, γ_K^{ub}, K_n^{ub}, $n \in \gamma_K^{\mathrm{ub}}$, and $a^{\mathrm{ub}}(n,k)$, $n \in \gamma_K^{\mathrm{ub}}$, $1 \leq k \leq K_n^{\mathrm{ub}}$; if $\alpha_{E'} > 0$, L^{ub}, $\gamma_L^{\prime\,\mathrm{ub}}$ and L_n^{ub}, $n \in \gamma_L^{\prime\,\mathrm{ub}}$; if $\alpha_{\bar{E}} > 0$, M^{ub} and $a''^{\mathrm{ub}}(n)$, $1 \leq n \leq M^{\mathrm{ub}}$; w_n^{ub}, $0 \leq n \leq K^{\mathrm{ub}} - 1$, $v_n^{i\,\mathrm{ub}}$, $0 \leq n \leq K^{\mathrm{ub}} - 1$, $1 \leq i \leq A$, h_n^{ub}, $n \in \gamma_K^{\mathrm{ub}}$, $w_{n,k}^{\mathrm{ub}}$, $n \in \gamma_K^{\mathrm{ub}}$, $1 \leq k \leq K_n^{\mathrm{ub}} - 1$, $q_{n,k}^{\mathrm{ub}}$, $n \in \gamma_K^{\mathrm{ub}}$, $1 \leq k \leq K_n^{\mathrm{ub}} - 1$, and $v_{n,k}^{i\,\mathrm{ub}}$, $n \in \gamma_K^{\mathrm{ub}}$, $1 \leq k \leq K_n^{\mathrm{ub}} - 1$, $1 \leq i \leq A$; if $\alpha_{E'} > 0$, $w_n'^{\mathrm{ub}}$, $0 \leq n \leq L^{\mathrm{ub}} - 1$, $v_n'^{i\,\mathrm{ub}}$, $0 \leq n \leq L^{\mathrm{ub}} - 1$, $1 \leq i \leq A$, $h_n'^{\mathrm{ub}}$, $n \in \gamma_L^{\prime\,\mathrm{ub}}$, $w_{n,k}'^{\mathrm{ub}}$, $n \in \gamma_L^{\prime\,\mathrm{ub}}$, $1 \leq k \leq L_n^{\mathrm{ub}} - 1$, $q_{n,k}'^{\mathrm{ub}}$, $n \in \gamma_L^{\prime\,\mathrm{ub}}$, $1 \leq k \leq L_n^{\mathrm{ub}} - 1$, and $v_{n,k}'^{i\,\mathrm{ub}}$, $n \in \gamma_L^{\prime\,\mathrm{ub}}$, $1 \leq k \leq L_n^{\mathrm{ub}} - 1$, $1 \leq i \leq A$; and, if $\alpha_{\bar{E}} > 0$, $w_n''^{\mathrm{ub}}$, $0 \leq n \leq M^{\mathrm{ub}} - 1$, $q_n''^{\mathrm{ub}}$, $0 \leq n \leq M^{\mathrm{ub}} - 1$ and $v_n''^{i\,\mathrm{ub}}$, $0 \leq n \leq M^{\mathrm{ub}} - 1$, $1 \leq i \leq A$.

Construction of V_T^{lb} from those quantities and objects is possible because, as we shall show: (1) $K^{\mathrm{lb}} \leq K^{\mathrm{ub}}$, $n \in \gamma_K^{\mathrm{lb}}$ if and only if $0 \leq n \leq K^{\mathrm{lb}} - 1$ and $n \in \gamma_K^{\mathrm{ub}}$, and $K_n^{\mathrm{lb}} \leq K_n^{\mathrm{ub}}$, $n \in \gamma_K^{\mathrm{lb}}$; if $\alpha_{E'} > 0$, $L^{\mathrm{lb}} = L^{\mathrm{ub}}$, $\gamma_L'^{\mathrm{lb}} = \gamma_L'^{\mathrm{ub}}$ and $L_n^{\mathrm{lb}} = L_n^{\mathrm{ub}}$, $n \in \gamma_L'^{\mathrm{lb}}$; and, if $\alpha_{\bar{E}} > 0$, $M^{\mathrm{lb}} \leq M^{\mathrm{ub}}$; and, (2) there exist simple relationships between Λ_E^{lb}, $\Lambda_{\bar{E}}^{\mathrm{lb}}$, $a^{\mathrm{lb}}(n)$, $n \geq 0$, $a^{\mathrm{lb}}(n,k)$, $n \in \gamma_K^{\mathrm{lb}}$, $k \geq 1$; if $\alpha_{\bar{E}} > 0$, $a''^{\mathrm{lb}}(n)$, $n \geq 0$; w_n^{lb}, $n \geq 0$, $v_n^{i\,\mathrm{lb}}$, $n \geq 0$, $1 \leq i \leq A$, h_n^{lb}, $n \in \gamma_K^{\mathrm{lb}}$, $w_{n,k}^{\mathrm{lb}}$, $n \in \gamma_K^{\mathrm{lb}}$, $k \geq 1$, $q_{n,k}^{\mathrm{lb}}$, $n \in \gamma_K^{\mathrm{lb}}$, $k \geq 1$, and $v_{n,k}^{i\,\mathrm{lb}}$, $n \in \gamma_K^{\mathrm{lb}}$, $k \geq 1$, $1 \leq i \leq A$; if $\alpha_{E'} > 0$, $w_n'^{\mathrm{lb}}$, $n \geq 0$, $v_n'^{i\,\mathrm{lb}}$, $n \geq 0$, $1 \leq i \leq A$, $h_n'^{\mathrm{lb}}$, $n \in \gamma_L'^{\mathrm{lb}}$, $w_{n,k}'^{\mathrm{lb}}$, $n \in \gamma_L'^{\mathrm{lb}}$, $k \geq 1$, $q_{n,k}'^{\mathrm{lb}}$, $n \in \gamma_L'^{\mathrm{lb}}$, $k \geq 1$, and $v_{n,k}'^{i\,\mathrm{lb}}$, $n \in \gamma_L'^{\mathrm{lb}}$, $k \geq 1$, $1 \leq i \leq A$; if $\alpha_{\bar{E}} > 0$, $w_n''^{\mathrm{lb}}$, $n \geq 0$, $q_n''^{\mathrm{lb}}$, $n \geq 0$, and $v_n''^{i\,\mathrm{lb}}$, $n \geq 0$, $1 \leq i \leq A$ and the corresponding quantities for V_T^{ub}. Using those relationships, it is possible to determine (2)–(4) M^{lb}, K^{lb} and K_n^{lb}, $n \in \gamma_K^{\mathrm{lb}}$, and, taking into account that V_T^{lb} and V_T^{ub} have the same initial probability distribution and the structure of the state diagram of V_T^{lb} illustrated by Fig. 2, to build V_T^{lb}. In the remaining of this section, we will prove (1) and will obtain the mentioned relationships. Towards that end, we will consider the parameter $R'_{\bar{E}} = \lambda_{\max}/\lambda_{\min} > 1$, with, we remember, $\lambda_{\max} = \max_{i \in \bar{E}} \lambda_i$ and $\lambda_{\min} = \min_{i \in \bar{E}} \lambda_i$.

We start by relating Λ_E, $\Lambda_{\bar{E}}$ and the transition probabilities $P_{i,j}$ of the DTMC $\widehat{X}$:

Theorem 3.3 *Assume $D = 1$. Then, $\Lambda_E^{\mathrm{lb}} = \Lambda_E^{\mathrm{ub}}$, $\Lambda_{\bar{E}}^{\mathrm{lb}} = \Lambda_{\bar{E}}^{\mathrm{ub}}/R'_{\bar{E}}$ and $P_{i,j}^{\mathrm{lb}} = P_{i,j}^{\mathrm{ub}}$, $i \in S$, $j \in \Omega$.*

Proof. Since transition rates are only scaled in states $i \in \bar{E}$, $\Lambda_E^{\text{lb}} = (1+\theta)\max_{i\in E}\lambda_i^{\text{lb}} = (1+\theta)\max_{i\in E}\lambda_i = (1+\theta)\max_{i\in E}\lambda_i^{\text{ub}} = \Lambda_E^{\text{ub}}$. With $D=1$, the scalings give $\lambda_i^{\text{lb}} = \lambda_{\min}$, $i \in \bar{E}$ and $\lambda_i^{\text{ub}} = \lambda_{\max}$, $i \in \bar{E}$. Then, $\Lambda_{\bar{E}}^{\text{lb}} = (1+\theta)\max_{i\in\bar{E}}\lambda_i^{\text{lb}} = (1+\theta)\lambda_{\min} = (1+\theta)\lambda_{\max}/R'_{\bar{E}} = (1+\theta)\max_{i\in\bar{E}}\lambda_i^{\text{ub}}/R'_{\bar{E}} = \Lambda_{\bar{E}}^{\text{ub}}/R'_{\bar{E}}$. For $i \in E$, $j \neq i$,

$$P_{i,j}^{\text{lb}} = \frac{\lambda_{i,j}^{\text{lb}}}{\Lambda_E^{\text{lb}}} = \frac{\lambda_{i,j}}{\Lambda_E} = \frac{\lambda_{i,j}^{\text{ub}}}{\Lambda_E^{\text{ub}}} = P_{i,j}^{\text{ub}},$$

which combined with $\sum_{j\in\Omega} P_{i,j}^{\text{lb}} = \sum_{j\in\Omega} P_{i,j}^{\text{ub}} = 1$, gives $P_{i,i}^{\text{lb}} = P_{i,i}^{\text{ub}}$, $i \in E$. For $i \in \bar{E}$, $j \neq i$,

$$P_{i,j}^{\text{lb}} = \frac{\lambda_{i,j}^{\text{lb}}}{\Lambda_{\bar{E}}^{\text{lb}}} = \frac{\lambda_{i,j}(\lambda_i^{\text{lb}}/\lambda_i)}{(1+\theta)\max_{i\in\bar{E}}\lambda_i^{\text{lb}}} = \frac{\lambda_{i,j}(\lambda_{\min}/\lambda_i)}{(1+\theta)\lambda_{\min}} = \frac{\lambda_{i,j}}{(1+\theta)\lambda_i}$$

$$= \frac{\lambda_{i,j}(\lambda_{\max}/\lambda_i)}{(1+\theta)\lambda_{\max}} = \frac{\lambda_{i,j}(\lambda_i^{\text{ub}}/\lambda_i)}{(1+\theta)\max_{i\in\bar{E}}\lambda_i^{\text{ub}}} = \frac{\lambda_{i,j}^{\text{ub}}}{\Lambda_{\bar{E}}^{\text{ub}}} = P_{i,j}^{\text{ub}},$$

which combined with $\sum_{j\in\Omega} P_{i,j}^{\text{lb}} = \sum_{j\in\Omega} P_{i,j}^{\text{ub}} = 1$, gives $P_{i,i}^{\text{lb}} = P_{i,i}^{\text{ub}}$, $i \in \bar{E}$. □

Using Theorem 3.3, it is immediate to relate the vectors $\boldsymbol{\pi}(n)$, $\boldsymbol{\pi}(n,k)$, $\boldsymbol{\pi}'(n)$, $\boldsymbol{\pi}'(n,k)$, and $\boldsymbol{\pi}''(n)$:

Proposition 3.1 *Assume $D=1$. Then; $\boldsymbol{\pi}^{\text{lb}}(n) = \boldsymbol{\pi}^{\text{ub}}(n)$, $n \geq 0$ and $\boldsymbol{\pi}^{\text{lb}}(n,k) = \boldsymbol{\pi}^{\text{ub}}(n,k)$, $n \in \gamma_K^{\text{lb}}$, $k \geq 1$; if $\alpha_{E'} > 0$, $\boldsymbol{\pi}'^{\text{lb}}(n) = \boldsymbol{\pi}'^{\text{ub}}(n)$, $n \geq 0$ and $\boldsymbol{\pi}'^{\text{lb}}(n,k) = \boldsymbol{\pi}'^{\text{ub}}(n,k)$, $n \in \gamma_L'^{\text{lb}}$, $k \geq 1$; and, if $\alpha_{\bar{E}} > 0$, $\boldsymbol{\pi}''^{\text{lb}}(n) = \boldsymbol{\pi}''^{\text{ub}}(n)$, $n \geq 0$.*

Proof. An immediate consequence of the last part of Theorem 3.3 and the recurrences (22)–(31), taking into account that X^{lb} and X^{ub} have the same initial probability distribution. □

The following theorem, relating the quantities $a(n)$, $a(n,k)$, $a'(n)$, $a'(n,k)$, and $a''(n)$ is an immediate consequence of Proposition 3.1, taking into account $a(n) = \sum_{i\in E}\pi_i(n)$, $a(n,k) = \sum_{i\in\bar{E}}\pi_i(n,k)$, $a'(n) = \sum_{i\in E'}\pi'_i(n)$, $a'(n,k) = \sum_{i\in\bar{E}}\pi'(n,k)$, and $a''(n) = \sum_{i\in\bar{E}}\pi''_i(n)$.

Theorem 3.4 *Assume $D=1$. Then; $a^{\text{lb}}(n) = a^{\text{ub}}(n)$, $n \geq 0$ and $a^{\text{lb}}(n,k) = a^{\text{ub}}(n,k)$, $n \in \gamma_K^{\text{lb}}$, $k \geq 1$; if $\alpha_{E'} > 0$, $a'^{\text{lb}}(n) = a'^{\text{ub}}(n)$, $n \geq 0$ and $a'^{\text{lb}}(n,k) = a'^{\text{ub}}(n,k)$, $n \in \gamma_L'^{\text{lb}}$, $k \geq 1$; and, if $\alpha_{\bar{E}} > 0$, $a''^{\text{lb}}(n) = a''^{\text{ub}}(n)$, $n \geq 0$.*

The following theorem relates the truncation parameters K, K_n, L, L_n, M, and the sets γ_K and γ'_L.

Theorem 3.5 *Assume $D=1$. Then; $K^{\text{lb}} \leq K^{\text{ub}}$, $n \in \gamma_K^{\text{lb}}$ if and only if $0 \leq n \leq K^{\text{lb}}-1$ and $n \in \gamma_K^{\text{ub}}$, and $K_n^{\text{lb}} \leq K_n^{\text{ub}}$, $n \in \gamma_K^{\text{lb}}$; if $\alpha_{E'} > 0$, $L^{\text{lb}} = L^{\text{ub}}$, $\gamma_L'^{\text{lb}} = \gamma_L'^{\text{ub}}$ and $L_n^{\text{lb}} = L_n^{\text{ub}}$, $n \in \gamma_L'^{\text{lb}}$; and, if $\alpha_{\bar{E}} > 0$, $M^{\text{lb}} \leq M^{\text{ub}}$.*

Proof. We start by showing $M^{\text{lb}} \leq M^{\text{ub}}$, assuming $\alpha_{\bar{E}} > 0$. By Theorem 3.4, $a''^{\text{lb}}(n) = a''^{\text{ub}}(n)$, $n \geq 0$, and, then, $r_{\max}a''^{\text{lb}}(n)\sum_{m=n+1}^{\infty} e^{-\Lambda_{\bar{E}}^{\text{lb}}t_{\max}}((\Lambda_{\bar{E}}^{\text{lb}}t_{\max})^m/m!) = r_{\max}a''^{\text{ub}}(n)\sum_{m=n+1}^{\infty} e^{-\Lambda_{\bar{E}}^{\text{lb}}t_{\max}}((\Lambda_{\bar{E}}^{\text{lb}}t_{\max})^m/m!)$, $n \geq 1$. By Theorem 3.3, taking into account $R'_{\bar{E}} > 1$,

we have $\Lambda^{\text{lb}}_{\bar{E}} < \Lambda^{\text{ub}}_{\bar{E}}$. Then, since $\sum_{m=n+1}^{\infty} e^{-\Lambda t_{\max}}((\Lambda t_{\max})^m/m!)$, $n \geq 1$ is the probability that the number of arrivals in a Poisson process with arrival rate $t_{\max}$ in the time interval $[0, \Lambda]$ is $\geq n+1$, which is increasing with Λ, we have $\sum_{m=n+1}^{\infty} e^{-\Lambda^{\text{lb}}_{\bar{E}} t_{\max}}((\Lambda^{\text{lb}}_{\bar{E}} t_{\max})^m/m!) \leq \sum_{m=n+1}^{\infty} e^{-\Lambda^{\text{ub}}_{\bar{E}} t_{\max}}((\Lambda^{\text{ub}}_{\bar{E}} t_{\max})^m/m!)$, $n \geq 1$ and $r_{\max} a''^{\text{lb}}(n) \sum_{m=n+1}^{\infty} e^{-\Lambda^{\text{lb}}_{\bar{E}} t_{\max}}((\Lambda^{\text{lb}}_{\bar{E}} t_{\max})^m/m!) \leq r_{\max} a''^{\text{ub}}(n) \sum_{m=n+1}^{\infty} e^{-\Lambda^{\text{ub}}_{\bar{E}} t_{\max}}((\Lambda^{\text{ub}}_{\bar{E}} t_{\max})^m/m!)$, $n \geq 1$, which, using (2), implies $M^{\text{lb}} \leq M^{\text{ub}}$.

We next show $K^{\text{lb}} \leq K^{\text{ub}}$. Assuming $\alpha_{\bar{E}} > 0$, since $a''^{\text{lb}}(n) = \|\boldsymbol{\pi}''^{\text{lb}}(n)\|_\infty$ is nonincreasing with n, because of (31) and $\|\mathbf{P}_{\bar{E},\bar{E}}\|_\infty \leq 1$, and $M^{\text{lb}} \leq M^{\text{ub}}$, we have $a''^{\text{lb}}(M^{\text{lb}}) \geq a''^{\text{lb}}(M^{\text{ub}})$. By Theorem 3.4, assuming $\alpha_{\bar{E}} > 0$, $a''^{\text{lb}}(n) = a''^{\text{ub}}(n)$, $n \geq 0$. Then, assuming $\alpha_{\bar{E}} > 0$, $a''^{\text{lb}}(M^{\text{ub}}) = a''^{\text{ub}}(M^{\text{ub}})$ and $a''^{\text{lb}}(M^{\text{lb}}) \geq a''^{\text{ub}}(M^{\text{ub}})$. Then, $\alpha_S - \mathbf{1}_{\alpha_{\bar{E}}>0} a''^{\text{lb}}(M^{\text{lb}}) \leq \alpha_S - \mathbf{1}_{\alpha_{\bar{E}}>0} a''^{\text{ub}}(M^{\text{ub}})$. By Theorem 3.4, $a^{\text{lb}}(n) = a^{\text{ub}}(n)$, $n \geq 0$. Then, $r_{\max}(\alpha_S - \mathbf{1}_{\alpha_{\bar{E}}>0} a''^{\text{lb}}(M^{\text{lb}})) a^{\text{lb}}(n) \sum_{m=n+1}^{\infty}(m-n) e^{-\Lambda^{\text{lb}}_E t_{\max}}((\Lambda^{\text{lb}}_E t_{\max})^m/m!) \leq r_{\max}(\alpha_S - \mathbf{1}_{\alpha_{\bar{E}}>0} a''^{\text{ub}}(M^{\text{ub}})) a^{\text{ub}}(n) \sum_{m=n+1}^{\infty}(m-n) e^{-\Lambda^{\text{lb}}_E t_{\max}}((\Lambda^{\text{lb}}_E t_{\max})^m/m!)$, $n \geq 1$. But, by Theorem 3.3, $\Lambda^{\text{lb}}_E = \Lambda^{\text{ub}}_E$, and, then, $\sum_{m=n+1}^{\infty}(m-n) e^{-\Lambda^{\text{lb}}_E t_{\max}}((\Lambda^{\text{lb}}_E t_{\max})^m/m!) = \sum_{m=n+1}^{\infty}(m-n) e^{-\Lambda^{\text{ub}}_E t_{\max}}((\Lambda^{\text{ub}}_E t_{\max})^m/m!)$, $n \geq 1$ and $r_{\max}(\alpha_S - \mathbf{1}_{\alpha_{\bar{E}}>0} a''^{\text{lb}}(M^{\text{lb}})) a^{\text{lb}}(n) \sum_{m=n+1}^{\infty}(m-n) e^{-\Lambda^{\text{lb}}_E t_{\max}}((\Lambda^{\text{lb}}_E t_{\max})^m/m!) \leq r_{\max}(\alpha_S - \mathbf{1}_{\alpha_{\bar{E}}>0} a''^{\text{ub}}(M^{\text{ub}})) a^{\text{ub}}(n) \sum_{m=n+1}^{\infty}(m-n) e^{-\Lambda^{\text{ub}}_E t_{\max}}((\Lambda^{\text{ub}}_E t_{\max})^m/m!)$, $n \geq 1$, which, using (3), implies $K^{\text{lb}} \leq K^{\text{ub}}$.

That $n \in \gamma^{\text{lb}}_K$ if and only if $0 \leq n \leq K^{\text{lb}} - 1$ and $n \in \gamma^{\text{ub}}_K$ follows immediately from the facts that $K^{\text{lb}} \leq K^{\text{ub}}$ and that, according to Theorem 3.4, $a^{\text{lb}}(n, 1) = a^{\text{ub}}(n, 1)$, $n \in \gamma^{\text{lb}}_K$. Note that the result implies $|\gamma^{\text{lb}}_K| \leq |\gamma^{\text{ub}}_K|$.

We next show $K^{\text{lb}}_n \leq K^{\text{ub}}_n$, $n \in \gamma^{\text{lb}}_K$. It can be shown similarly as it was previously shown $r_{\max}(\alpha_S - \mathbf{1}_{\alpha_{\bar{E}}>0} a''^{\text{lb}}(M^{\text{lb}})) a^{\text{lb}}(n) \sum_{m=n+1}^{\infty}(m-n) e^{-\Lambda^{\text{lb}}_E t_{\max}}((\Lambda^{\text{lb}}_E t_{\max})^m/m!) \leq r_{\max}(\alpha_S - \mathbf{1}_{\alpha_{\bar{E}}>0} a''^{\text{ub}}(M^{\text{ub}})) a^{\text{ub}}(n) \sum_{m=n+1}^{\infty}(m-n) e^{-\Lambda^{\text{ub}}_E t_{\max}}((\Lambda^{\text{ub}}_E t_{\max})^m/m!)$, $n \geq 1$ that $r_{\max}(\alpha_S - \mathbf{1}_{\alpha_{\bar{E}}>0} a''^{\text{lb}}(M^{\text{lb}})) a^{\text{lb}}(n, k) \sum_{m=n+1}^{\infty}(m-n) e^{-\Lambda^{\text{lb}}_E t_{\max}}((\Lambda^{\text{lb}}_E t_{\max})^m/m!) \leq r_{\max}(\alpha_S - \mathbf{1}_{\alpha_{\bar{E}}>0} a''^{\text{ub}}(M^{\text{ub}})) a^{\text{ub}}(n, k) \sum_{m=n+1}^{\infty}(m-n) e^{-\Lambda^{\text{ub}}_E t_{\max}}((\Lambda^{\text{ub}}_E t_{\max})^m/m!)$, $n \in \gamma^{\text{lb}}_K$, $k \geq 1$. Then, that $K^{\text{lb}}_n \leq K^{\text{ub}}_n$, $n \in \gamma^{\text{lb}}_K$ follows using (4), taking into account $|\gamma^{\text{lb}}_K| \leq |\gamma^{\text{ub}}_K|$.

That, if $\alpha_{E'} > 0$, $L^{\text{lb}} = L^{\text{ub}}$ follows immediately using (5), taking into account that, according to Theorem 3.3, $\Lambda^{\text{lb}}_E = \Lambda^{\text{ub}}_E$ and, according to Theorem 3.4, $a'^{\text{lb}}(n) = a'^{\text{lb}}(n)$, $n \geq 1$.

That, if $\alpha_{E'} > 0$, $\gamma'^{\text{lb}}_L = \gamma'^{\text{ub}}_L$ follows immediately taking into account $L^{\text{lb}} = L^{\text{ub}}$ and, according to Theorem 3.4, $a'^{\text{lb}}(n, 1) = a'^{\text{ub}}(n, 1)$, $n \in \gamma'^{\text{lb}}_L$. Note that the result implies $|\gamma'^{\text{lb}}_L| = |\gamma'^{\text{ub}}_L|$.

Finally, that, if $\alpha_{E'} > 0$, $L^{\text{lb}}_n = L^{\text{ub}}_n$, $n \in \gamma'^{\text{lb}}_L$ follows immediately using (6), taking into account that, according to Theorem 3.3, $\Lambda^{\text{lb}}_E = \Lambda^{\text{ub}}_E$, according to Theorem 3.4, $a'^{\text{lb}}(n, k) = a'^{\text{ub}}(n, k)$, $n \in \gamma'^{\text{lb}}_L$, $k \geq 1$, and, as just shown, $|\gamma'^{\text{lb}}_L| = |\gamma'^{\text{ub}}_L|$. □

Finally, the following theorem relates w_n, v^i_n, h_n, $w_{n,k}$, $q_{n,k}$, $v^i_{n,k}$, w'_n, v'^i_n, h'_n, $w'_{n,k}$, $q'_{n,k}$, $v'^i_{n,k}$, w''_n, q''_n, and v''^i_n.

Theorem 3.6 *Assume $D = 1$. Then; $w^{\text{lb}}_n = w^{\text{ub}}_n$, $0 \leq n \leq K^{\text{lb}} - 1$, $v^{i\,\text{lb}}_n = v^{i\,\text{ub}}_n$, $0 \leq n \leq K^{\text{lb}} - 1$, $1 \leq i \leq A$, $h^{\text{lb}}_n = h^{\text{ub}}_n$, $n \in \gamma^{\text{lb}}_K$, $w^{\text{lb}}_{n,k} = w^{\text{ub}}_{n,k}$, $n \in \gamma^{\text{lb}}_K$, $1 \leq k \leq K^{\text{lb}}_n - 1$, $q^{\text{lb}}_{n,k} = q^{\text{ub}}_{n,k}$, $n \in \gamma^{\text{lb}}_K$, $1 \leq k \leq K^{\text{lb}}_n - 1$, and $v^{i\,\text{lb}}_{n,k} = w^{i\,\text{ub}}_{n,k}$, $n \in \gamma^{\text{lb}}_K$, $1 \leq k \leq K^{\text{lb}}_n - 1$, $1 \leq i \leq A$; if $\alpha_{E'} > 0$, $w'^{\text{lb}}_n = w'^{\text{ub}}_n$, $0 \leq n \leq L^{\text{lb}} - 1$, $v'^{i\,\text{lb}}_n = v'^{i\,\text{ub}}_n$, $0 \leq n \leq L^{\text{lb}} - 1$, $1 \leq i \leq A$, $h'^{\text{lb}}_n = h'^{\text{ub}}_n$, $n \in \gamma'^{\text{lb}}_L$, $w'^{\text{lb}}_{n,k} = w'^{\text{ub}}_{n,k}$, $n \in \gamma'^{\text{lb}}_L$, $1 \leq k \leq L^{\text{lb}}_n - 1$, $q'^{\text{lb}}_{n,k} = q'^{\text{ub}}_{n,k}$, $n \in \gamma'^{\text{lb}}_L$, $1 \leq k \leq L^{\text{lb}}_n - 1$, and $v'^{i\,\text{lb}}_{n,k} = v'^{i\,\text{ub}}_{n,k}$, $n \in \gamma'^{\text{lb}}_L$, $1 \leq k \leq L^{\text{lb}}_n - 1$, $1 \leq i \leq A$; and, if $\alpha_{\bar{E}} > 0$, $w''^{\text{lb}}_n = w''^{\text{ub}}_n$,*

$0 \le n \le M^{\mathrm{lb}} - 1$, $q_n''^{\mathrm{lb}} = q_n''^{\mathrm{ub}}$, $0 \le n \le M^{\mathrm{lb}} - 1$, *and* $v_n''^{i\,\mathrm{lb}} = v_n''^{i\,\mathrm{ub}}$, $0 \le n \le M^{\mathrm{lb}} - 1$, $1 \le i \le A$.

Proof. An immediate combination of (7)–(21) with the last part of Theorem 3.3, Proposition 3.1 and Theorem 3.4. □

4. Analysis

In this section, we will illustrate the behavior of BSRR and will compare its computational cost in terms of CPU time with those of SRR, SR, RR, AU, and RQD. In AU, a truncated jump process with an absorbing state with reward rates associated with states is considered to be the result of the first truncation performed in adaptive uniformization layered uniformization, and that rewarded truncated jump process is solved by SR.

We will consider a CTMC model of a 16×16 ASEN-MAX interconnection network (Kumar and Reddy, 1987) appropriate to compute its unreliability, $ur(t)$. The architecture of the interconnection network is depicted in Fig. 4. The interconnection network includes a stage of multiplexers, three stages of switches, and a stage of demultiplexers. Links are assumed perfect (they do not fail). The network is assumed to be operational if the set of unfailed components is such that, in the presence of a single request from any source to any destination, the nonbacktracking routing protocol used by the interconnection network succeeds in routing the request. That assumption is guaranteed to be pessimistic (Rincu et al., 2002). In addition, we consider that faults in multiplexers, switches and demultiplexers may not be covered and that an uncovered fault leads to system failure. The structure of the interconnection network is such that, assuming perfect fault coverage, the interconnection network tolerates the failure of any single component. Multiplexers and demultiplexers fail at rate λ_{M}; switches fail at rate λ_{S}. Multiplexer and demultiplexer faults are covered with probability C_{M}; switch faults are covered with probability C_{S}. There is a single repairman. Repair starts when the number of failed components gets ≥ 2 and, then, proceeds until reaching the state without failed components. Multiplexers and demultiplexers are repaired at rate μ_{M}; switches are repaired at rate μ_{S}. The repair of switches has priority over the repair of multiplexers and demultiplexers. Failed components with same repair priority are chosen at random by the repairman. There is repair preemption. Thus, if a switch fails when a multiplexer or demultiplexer is being repaired, the repair of the multiplexer or demultiplexer is interrupted and the repairman starts immediately the repair of the failed switch. We will use the model parameter values $\lambda_{\mathrm{M}} = 4 \times 10^{-6}\ \mathrm{h}^{-1}$, $\lambda_{\mathrm{S}} = 1.2 \times 10^{-5}\ \mathrm{h}^{-1}$, $C_{\mathrm{M}} = 0.995$, $C_{\mathrm{S}} = 0.99$, $\mu_{\mathrm{M}} = 2\ \mathrm{h}^{-1}$, and, for μ_{S}, two values: $\mu_{\mathrm{S}} = 0.08\ \mathrm{h}^{-1}$ and $\mu_{\mathrm{S}} = 0.4\ \mathrm{h}^{-1}$. Regarding the initial probability distribution, we will consider two cases: case 1—the initial state is the state without failed components; case 2—with probability 0.5 the initial state is the state in which the only failed component is switch 0 of stage 0 and, with probability 0.5, the initial state is the state in which the failed components are switch 0 of stage 0 and the top multiplexer.

An exact CTMC model of the network has an unmanageably large number of states. The problem can be circumvented by using bounding CTMC models such as those described in Sec. 1. Including in S the operational states with up to 4 failed components gives very tight bounds for $ur(t)$. With that selection, $|S| = 315{,}045$ and both bounding CTMC models have manageable sizes. We will illustrate the performance of the method by using the bounding CTMC model yielding the upper bound for $ur(t)$, $ur_{\mathrm{ub}}(t)$. That upper bounding CTMC model belongs to class C_2, and, for BSRR and SRR, we took the natural selections in which E includes the operational states without repair and state r is the single state without failed

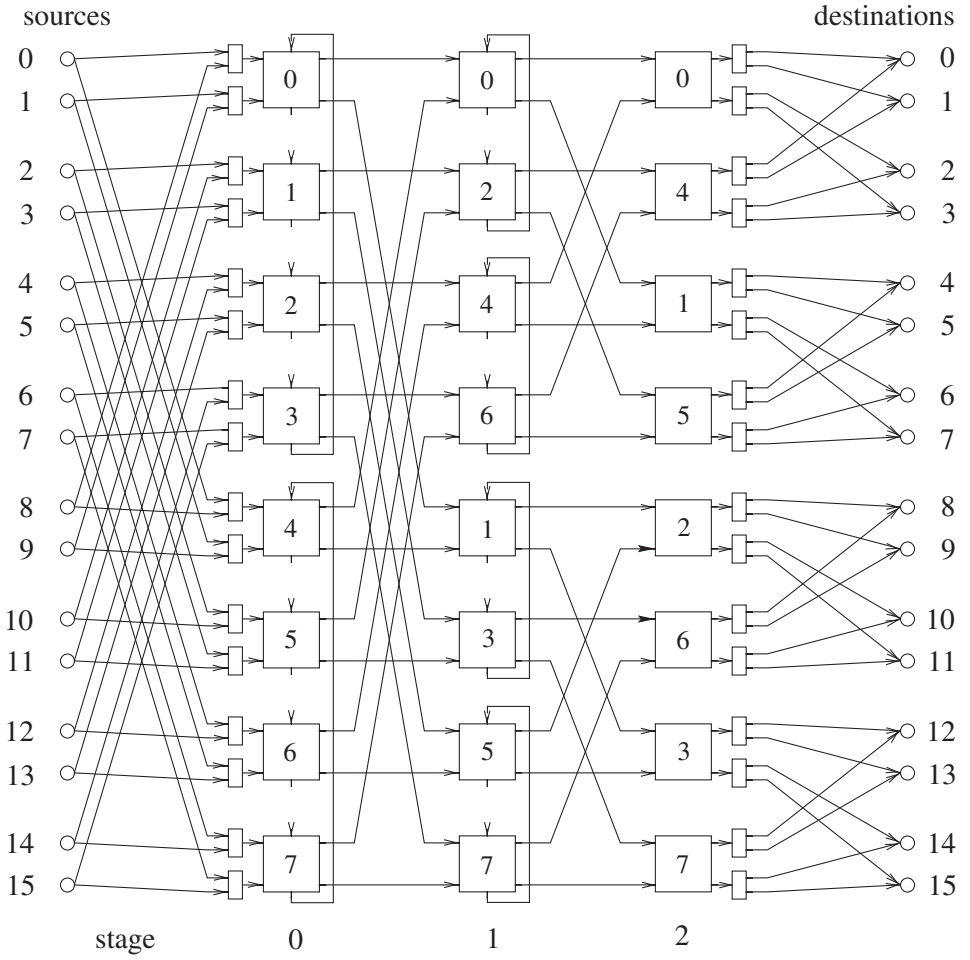

Figure 4. Architecture of the 16 × 16 ASEN-MAX interconnection network.

components. With those selections, case 1 illustrates the case $\alpha_{E'} = \alpha_{\bar{E}} = 0$ and case 2 illustrates the case $\alpha_{E'} > 0$, $\alpha_{\bar{E}} > 0$. For the RR method we took as regenerative state r the single state without failed components. All methods were run with a truncation error target $\varepsilon = 10^{-10}$ and CPU times were measured on a workstation with a SunBlade 1000 processor. The randomization rates considered in SRR, RR, and RQD are larger than the minimum required by a factor $1 + \theta$, and we took $\theta = 10^{-4}$.

Table 1 gives the bounds for $ur_{\mathrm{ub}}(t)$ obtained by BSRR with $D = 1$ for case 1 and several values of t. We can note that the bounds are tight for all values of t. Intuitively, this is because X^{lb} and X^{ub} spend most of the time up to absorption in the subset E, and X^{lb} and X^{ub} only differ in the values of the transition rates from states in $\bar{E}$. For large t, this will be the case for class C_2 models with natural selections for E and r, provided that the partition $S_0 \cup S_1 \cup \cdots \cup S_{N_C} \cup \bar{S}_1 \cup \bar{S}_2 \cup \cdots \cup \bar{S}_{\bar{N}_C}$ for S satisfies the following two additional properties:

Table 1
Bounds for $ur_{\text{ub}}(t)$ obtained by bounding split regenerative randomization with $D = 1$ as a function of t for case 1 and $\mu_S = 0.4\ \text{h}^{-1}$ (top) / $\mu_S = 0.08\ \text{h}^{-1}$ (bottom)

t (h)	Lower bound	Upper bound
1	3.523×10^{-6}	3.523×10^{-6}
10	3.5487×10^{-5}	3.5487×10^{-5}
100	3.79994×10^{-4}	3.80008×10^{-4}
1,000	5.741090×10^{-3}	5.742394×10^{-3}
10,000	0.092991136	0.093051379
100,000	0.651968927	0.652259360
1	3.523×10^{-6}	3.523×10^{-6}
10	3.5487×10^{-5}	3.5490×10^{-5}
100	3.80162×10^{-4}	3.80302×10^{-4}
1,000	5.757722×10^{-3}	5.765937×10^{-3}
10,000	0.093287668	0.093656971
100,000	0.653129492	0.654870606

P8. $\max_{1\le k\le \bar{N}_C} \max_{i\in\bar{S}_k} \lambda_{i,\bar{S}_k-\{i\}\cup\bar{S}_{k+1}\cup\cdots\cup\bar{S}_{\bar{N}_C}\cup\{f_1,f_2,\ldots,f_A\}}$
$\ll \min_{1\le k\le \bar{N}_C} \min_{i\in\bar{S}_k} \lambda_{i,S_o\cup\bar{S}_1\cup\cdots\cup\bar{S}_{k-1}}$.
P9. $\max_{1\le k\le N_C} \max_{i\in S_k} \lambda_i \ll \min_{1\le k\le \bar{N}_C} \min_{i\in\bar{S}_k} \lambda_i$.

The reason is that P8 implies that the embedded DTMC of both X^{lb} and X^{ub} will go from $\bar{E}$ towards state o with almost one probability and P9 implies that state holding times in E will be much larger than state holding times in $\bar{E}$. The fact that the bounds are also tight for small t seems to have to do with the fact that the initial probability distribution of X in nonabsorbing states is concentrated in subset E. Table 2 gives the bounds obtained for case 2, a case in which X has a non-negligible initial probability in $\bar{E}$. In that case, the bounds

Table 2
Bounds for $ur_{ub}(t)$ obtained by BSRR with $D = 1$ as a function of t for case 2 and $\mu_S = 0.4\ \text{h}^{-1}$ (top) / $\mu_S = 0.08\ \text{h}^{-1}$ (bottom)

t (h)	Lower bound	Upper bound
1	2.1528×10^{-5}	4.0092×10^{-5}
10	1.58424×10^{-4}	1.66109×10^{-4}
100	1.358243×10^{-3}	1.366681×10^{-3}
1,000	0.012891783	0.012922037
10,000	0.107421165	0.107509980
100,000	0.657558500	0.657855188
1	2.2793×10^{-5}	1.40807×10^{-4}
10	1.98521×10^{-4}	2.84253×10^{-4}
100	1.443059×10^{-3}	1.494011×10^{-3}
1,000	0.013009002	0.013125405
10,000	0.107827753	0.108352430
100,000	0.658744807	0.660523036

Table 3
Values of the truncation parameters for BSRR and SRR for case 1 and $\mu_S = 0.4\,h^{-1}$ (top) $/\mu_S = 0.08\,h^{-1}$ (bottom)

	BSRR		SRR	
t (h)	K^{ub}	$K^{ub}_{\bar{E}}$	K	$K_{\bar{E}}$
1	2	3	2	35
10	3	8	3	69
100	3	12	3	128
1,000	5	24	5	262
10,000	7	40	7	474
100,000	9	54	9	641
1	2	5	2	181
10	3	10	3	359
100	3	14	3	686
1,000	5	28	5	1,414
10,000	7	48	7	2,572
100,000	9	68	9	3,499

are not tight for small values of t. Properties P8 and P9 for the partition for S are satisfied by both exact and bounding failure/repair CTMC models with exponential failure and repair time distributions, in which repair is deferred until some condition on the collection of failed components is satisfied, and, then, proceeds until reaching the state without failed components, with failure rates much smaller than repair rates.

Tables 3 and 4 give the truncation parameters of BSRR for $D = 1$ and SRR, for, respectively, case 1 and case 2. The truncation parameter N of SR ranged in both cases

Table 4
Values of the truncation parameters of BSRR for $D = 1$ and SRR for case 2 and $\mu_S = 0.4\,h^{-1}$ (top) $/\mu_S = 0.08\,h^{-1}$ (bottom)

	BSRR					SRR				
t (h)	K^{ub}	L^{ub}	M^{ub}	$K^{ub}_{\bar{E}}$	$L^{ub}_{\bar{E}}$	K	L	M	$K_{\bar{E}}$	$L_{\bar{E}}$
1	2	2	6	5	10	2	2	14	40	108
10	3	2	8	8	12	3	2	43	81	141
100	3	3	8	12	19	3	3	100	138	210
1,000	5	4	8	24	24	5	4	100	283	316
10,000	7	6	8	42	40	7	6	100	505	481
100,000	9	7	8	57	45	9	7	100	692	524
1	2	2	7	5	12	2	2	15	204	616
10	3	2	10	10	14	3	2	51	443	774
100	3	3	10	14	23	3	3	263	745	1,137
1,000	5	4	10	30	32	5	4	554	1,535	1,717
10,000	7	6	10	52	50	7	6	554	2,750	2,609
100,000	9	7	10	72	55	9	7	554	3,785	2,840

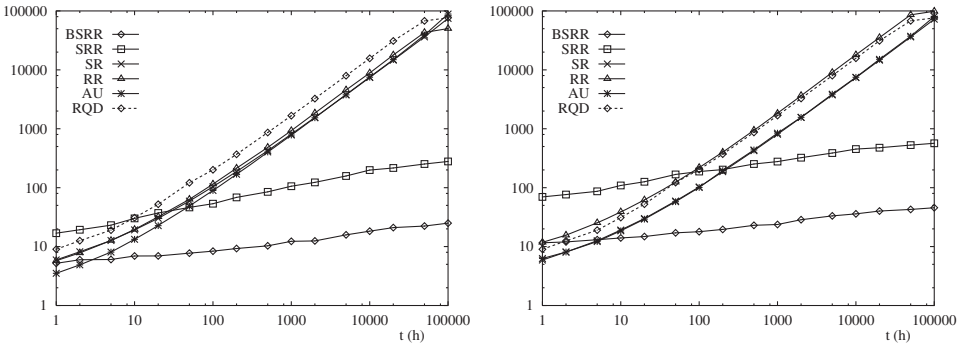

Figure 5. CPU times in seconds consumed by the methods as a function of t for $\mu_S = 0.4\ \mathrm{h}^{-1}$ for case 1 (left) and case 2 (right).

from $N = 15$ to 202,730 for t ranging from 1 to 100,000 h. In the tables, for SRR, $K_{\bar{E}}$ denotes $\sum_{n\in\gamma_K} K_n$ and $L_{\bar{E}}$ denotes $\sum_{n\in\gamma'_L} L_n$. A similar notation is used for BSRR. For BSRR, we only give the truncation parameters associated with the generation of V_T^{ub}, since the truncation parameters associated with the generation of V_T^{lb} are guaranteed to be nonlarger (see Theorem 3.5). We can note that, as suggested theoretically, D being equal to 1 and R_E being close to 1 ($R_E = 1.051$), the truncation parameters of BSRR are small. The truncation parameters of SRR are significantly larger ($R_{\bar{E}} \approx 5$ for $\mu_S = 0.4\ \mathrm{h}^{-1}$ and $R_{\bar{E}} \approx 25$ for $\mu_S = 0.08\ \mathrm{h}^{-1}$).

The CPU times consumed by BSRR with $D = 1$, SRR, SR, RR, AU, and RQD are given in Figs. 5 and 6. We can note that BSRR with $D = 1$ has small computational cost in terms of CPU time for all t. This is because the truncation parameters in BSRR are small. The computational cost in terms of CPU time of SRR is somewhat larger. This is because the truncation parameters of BSRR are larger. For large t, the CPU times for SR, RR, AU, and RQD are enormous. The CPU times for AU are almost identical to those of SR. Regarding the dependence of the CPU times of RR and RQD with t, there seems to exist a knee beyond which the increase gets smooth, but that knee is reached for very large values of t.

Finally, we will explore the tradeoff in BSRR between computational cost in terms of CPU time and bounds tightness controlled by the parameter D. For the bounding CTMC

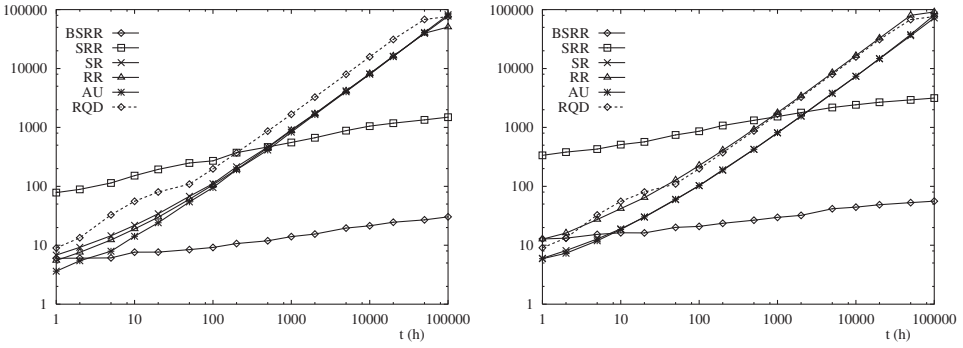

Figure 6. CPU times in seconds consumed by the methods as a function of t for $\mu_S = 0.08\ \mathrm{h}^{-1}$ for case 1 (left) and case 2 (right).

Table 5
Bounds for $ur_{\mathrm{ub}}(t)$ obtained by BSRR and consumed CPU times in seconds for $\mu_{\mathrm{S}} = 0.08\,\mathrm{h}^{-1}$, case 1, $t = 50{,}000$ h, and several values of D

D	Lower bound	Upper bound	CPU time (s)
1	0.408435010	0.409890092	278
2	0.408668614	0.409847831	444
5	0.408808680	0.409721102	842
10	0.408855393	0.409510071	1,487
20	0.408878753	0.409088694	2,860

model of the network, all states in S communicate and constitute a single transient class. Then, the conditions for k of the second part of Theorem 3.1 are satisfied for all states in $\bar{E}$, implying that as D increases the bounds obtained by BSRR will get strictly tighter. Table 5 gives the bounds for $ur_{\mathrm{ub}}(t)$ obtained by BSRR and the consumed CPU times for $\mu_{\mathrm{S}} = 0.08\,\mathrm{h}^{-1}$, case 1, $t = 50{,}000$ h, and several values of D, $1 \leq D < R'_{\bar{E}}$ ($R'_{\bar{E}} \approx 25$). We can note that the bounds get tighter as D increases, but the computational cost in terms of CPU time of the method increases significantly. Thus, the selection $D = 1$ seems to be the most interesting one.

5. Conclusions

Taking as starting point SRR, a method called BSRR has been developed. The method computes bounds for some measures, covers some CTMC models, and has an input parameter controlling the tightness of the bounds. The method requires the selection of a subset of states E and a regenerative state r and is targeted at a certain class of CTMC models, class C_2. For CTMC models in that class, natural selections for E and r exist, with those natural selections the input parameter should tradeoff computational cost in terms of CPU time with bounds tightness, the method can be computationally less costly in terms of CPU time than SRR, and the version of the method obtained with the value of the input parameter that should have smallest computational cost in terms of CPU time should have small computational cost in terms of CPU time relative to the model size when that model size is large, if output rates from states in E are not too different. Furthermore, under additional conditions, the bounds obtained by the version of the method that should have smallest computational cost in terms of CPU time seem to be tight for all times or not short times, depending on whether the initial probability of the CTMC model in nonabsorbing states is concentrated in E or not. Combinations of measures that can be computed by the method and CTMC models in class C_2 include, respectively, (1) the unreliability and exact failure/repair CTMC models of fault-tolerant systems with exponential failure and repair time distributions, in which repair is deferred until some condition on the collection of failed components is satisfied, and, then, proceeds until reaching the state without failed components, with failure rates much smaller than repair rates, and, (2) lower and upper bounds for the unreliability and bounding failure/repair CTMC models of fault-tolerant systems with exponential failure and repair time distributions, in which repair is deferred until some condition on the collection of failed components is satisfied, and, then, proceeds until reaching the state without failed components, with failure rates much smaller than repair rates. For those CTMC models, natural selections for E and r are, respectively, the

subset of states without repair and the state without failed components, and, with those natural selections, the version of the method that should have smallest computational cost in terms of CPU time should have small computational cost in terms of CPU time relative to the model size when that model size is large if output rates from states without repair are not too different and the additional conditions under which the bounds obtained with that version seem to be tight or not are satisfied.

Acknowledgments

This research work has been supported by the Ministry of Science and Technology of Spain and FEDER ("Fondo Europeo de Desarrollo Regional") under the research grant DPI2004-05077 and by the Ministry of Science and Innovation of Spain and FEDER under the research grant TIN2008–06735–C02–02. The authors are indebted to the associate editor and the anonymous reviewers, whose comments have been helpful in improving the quality of the presentation.

References

Carrasco, J. A. (2002). Computationally efficient and numerically stable reliability bounds for repairable fault-tolerant systems. *IEEE Transactions on Computers* 51:254–268.

Carrasco, J. A. (2003). Computation of bounds for transient measures of large rewarded Markov models using regenerative randomization. *Computers & Operations Research* 30:1005–1035.

Carrasco, J. A. (2004). Transient analysis of some rewarded Markov models using randomization with quasistationarity detection. *IEEE Transactions on Computers* 53:1106–1120.

Carrasco, J. A. (2005a). Transient analysis of large Markov models with absorbing states using regenerative randomization. Technical report DMSD_99_2. Available at: ftp://ftp-eel.upc.es/techreports.

Carrasco, J. A. (2005b). Transient analysis of large Markov models with absorbing states using regenerative randomization. *Communications in Statistics—Simulation and Computation* 34:1027–1052.

Çinlar, E. (1975). *Introduction to Stochastic Processes*. Englewood Cliffs, NJ: Prentice-Hall, Inc.

Diener, J. D., Sanders, W. H. (1995). Empirical comparison of uniformization methods for continuous-time Markov chains. In: *Computation with Markov Chains*. Boston: Kluwer Academic Publishers, pp. 547–570.

Feller, W. (1971). *An Introduction to Probability Theory and Its Applications*. Vol. 2. New York: John Wiley & Sons.

Grassmann, W. K. (1977). Transient solutions in Markovian queueing systems. *Computers & Operations Research* 4:47–53.

Gross, D., Miller, D. R. (1984). The randomization technique as a modelling tool and solution procedure for transient Markov processes. *Operations Research* 32:343–361.

Heyman, D. P., Sobel, M. J. (1984). *Stochastic Models in Operations Research*. Vol. 2. New York: McGraw-Hill.

Jensen, A. (1953). Markoff chains as an aid in the study of Markoff processes. *Skandinavisk Aktuarietidskriff* 36:87–91.

Johnson, B. W. (1989). *Design and Analysis of Fault Tolerant Digital Systems*. Reading, MA: Addison-Wesley.

Kijima, M. (1997). *Markov Processes for Stochastic Modeling*. London: Chapman & Hall.

Kumar, V. K., Reddy, S. M. (1987). Augmented shuffle-exchange multistage interconnection networks. *IEEE Computer* 20:30–40.

Malhotra, M. (1995). A computationally efficient technique for transient analysis of repairable Markovian systems. *Performance Evaluation* 24:311–331.

Malhotra, M., Muppala, J. K., Trivedi, K. S. (1994). Stiffness-tolerant methods for transient analysis of stiff Markov chains. *Microelectronics and Reliability* 34:1825–1841.

Reibman, A., Trivedi, K. S. (1988). Numerical transient analysis of Markov models. *Computers & Operations Research* 15:19–36.

Rincu, C. I., Carrasco, J. A., Suñé, V. (2002). Markovian reliability analysis of non-coherent non-repairable fault-tolerant interconnection networks. *Proceedings Communications 2002 International Conference*108–113.

Ross, S. M. (1983). *Stochastic Processes*. New York: John Wiley & Sons.

Sericola, B. (1999). Availability analysis of repairable computer systems and stationarity detection. *IEEE Transactions on Computers* 48:1166–1172.

Stewart, W. J. (1994). *Introduction to the Numerical Solution of Markov Chains*. Princeton: Princeton University Press.

Suñé, V., Carrasco, J. A. (2005). Efficient implementations of the randomization method with control of the relative error. *Computers & Operations Research* 32:1089–1114.

Temsamani, J., Carrasco, J. A. (2005). A generalized method for the transient analysis of Markov models of fault-tolerant systems with deferred repair. *Communications in Statistics—Simulation and Computation* 34:631–661.

Temsamani, J., Carrasco, J. A. (2006). Transient analysis of Markov models of fault-tolerant systems with deferred repair using split regenerative randomization. *Naval Research Logistics* 53:318–353.

Van den Hout, W. B. (1996). *The Power-series Algorithm. A Numerical Approach to Markov Processes*. Ph.D. dissertation. CenTER for Economic Research, Tilburg University, The Netherlands.

Van Moorsel, A. P. A., Sanders, W. H. (1994). Adaptive uniformization. *Communications in Statistics—Stochastic Models* 10:619–647.

Van Moorsel, A. P. A., Sanders, W. H. (1997). Transient solution of Markov models by combining adaptive & standard uniformization. *IEEE Transactions on Reliability* 46:430–440.