ON THE UNIQUENESS AND ANALYTICITY OF SOLUTIONS IN MICROPOLAR THERMOVISCOELASTICITY

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Abstract: This paper deals with the linear theory of isotropic micropolar thermoviscoelastic materials. When the dissipation is positive definite, we present two uniqueness theorems. The first one requieres the extra assumption that some coupling terms vanish; in this case, the instability of solutions is also proved. When the internal energy and the dissipation are both positive definite, we prove the well-posedness of the problem and the analyticity of the solutions. Exponential decay and impossibility of localization are corollaries of the analyticity.

Keywords: micropolar thermoviscoelasticity, uniqueness, analyticity, exponential decay.

1. Introduction

A great effort has been made in the last years to understand the behavior of the so-called "non-classical materials". Solids with voids, mixtures of materials or non-simple materials are examples of them. Some mathematical and mechanical studies about these materials can be found in the book of Ieşan [11].

It can be said that the first analysis concerning the time decay properties of these alternative solids was proposed by Quintanilla [26]. The author proved the slow decay of the solutions with respect to the time for elastic-porous materials when the only dissipation mechanism is the porous dissipation. After that, a lot of works were intended to clarify the behavior of the solutions (exponential decay, slow decay, impossibility of localization and/or analyticity) for solids with voids [4, 5, 9, 10, 13, 14, 16, 17, 18, 20, 19, 21, 23, 24, 29], for non-simple materials [8, 25] or for mixtures of elastic solids [1, 2, 3, 27, 28]. Nevertheless, any attention has been paid up to now to micropolar elastic solids. We believe that these kind of properties are a relevant issue to clarify in order to understand better the thermomechanial behavior ot these materials.

The origin of the rational theories about polar continua is attributed to E. and F. Cosserat (see [11] or [7]) at the beginning of the twentieth century. In the sixties, other contributions on this field were done: we want to highlight the work of Eringen [6], among others. Nowadays, these materials are a subclass of the micromorphic materials. Metals, polymers, rocks, wood, ceramics, soils, biological materials or pressed powders are typical examples of them.

In this work we focuss on the analysis of the qualitative properties of the isotropic micropolar thermoviscoelastic materials. That is, materials that, apart from the usual macroscopic movements, allow its material points to rotate. We consider thermal effects as well as viscosity effects at the macroscopic and microscopic levels. We have two main purposes. First, we will suppose that the dissipation is positive definite. In this case we will see the uniqueness of the solutions

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and its instability when the internal energy is not necessarily positive definite. Second, if the dissipation and the internal energy are both positive definite, we will obtain the analyticity of the solutions. This is an important fact: analyticity implies that the solutions are very regular. That means that the orbits are analytic functions in the time variable and that the solution can be obtained from the derivatives in any point. Two consequences of this fact are the exponential decay and the impossibility of localization of the solutions. In particular, the exponential stability tells us that the perturbations are damped in a very fast way and, from an empirical point of view, they will be imperceptible after a small period of time.

As we said above, the study of the qualitative properties of the micropolar elastic solids has got little attention. Our aim is to improve this circumstance. We take the situation where the more number of dissipation mechanisms can appear. We think that this could be a good beginning because this situation determines when the solutions will be the more regular possible. The analysis of the solutions when less dissipation mechanisms are present could be the aim of future works.

The structure of the paper is the following. In Section 2 we recall the basic equations with the assumptions that we need to set down the problem. In Section 3, we present uniqueness and instability results for the solutions using the logarithmic convexity argument for a particular (but quite general) case. In Section 4, uniqueness is proved working with the general system of equations. The existence of solution is proved in Section 5 making use of the linear operators semigroup theory. Finally, in Section 6, we prove the analyticity of the solutions, that, among other properties, show that the solutions are exponentially stable and also the impossibility of localization.

2. Basic equations

Let us consider an homogeneous isotropic tridimensional micropolar viscoelastic body which occupies a three-dimensional domain Γ with a boundary, $\partial\Gamma$, smooth enough to apply the Divergence theorem. We consider the strain measures e_{ij} and κ_{ij} which are defined by

$$e_{ij} = u_{j,i} + \epsilon_{jik}\phi_k, \quad \kappa_{ij} = \phi_{j,i}, \tag{2.1}$$

where u_i are the components of the displacement vector, ϕ_i are the components of the microrotation and ϵ_{jik} is the alternating symbol. In this paper we assume that the stress tensor t_{ij}, the microstress m_{ij}, the entropy η and the heat flux vector q_i are related to the strain measures e_{ij} and κ_{ij} and also to the temperature T and the gradient of the temperature by means of the constitutive equations

$$\begin{aligned}
t_{ij} &= \lambda e_{rr}\delta_{ij} + (\mu+\sigma)e_{ij} + \mu e_{ji} + \lambda_v \dot{e}_{rr}\delta_{ij} + (\mu_v+\sigma_v)\dot{e}_{ij} + \mu_v \dot{e}_{ji} - bT\delta_{ij},\\
m_{ij} &= \alpha\kappa_{rr}\delta_{ij} + \beta\kappa_{ji} + \gamma\kappa_{ij} + \alpha_v\dot{\kappa}_{rr} + \beta_v\dot{\kappa}_{ji} + \gamma_v\dot{\kappa}_{ij} + b^*\epsilon_{ijr}T_{,r},\\
\rho\eta &= be_{rr} + aT,\\
q_i &= kT_{,i} + k^*\epsilon_{irs}\dot{\kappa}_{rs}.
\end{aligned}$$

Here $\lambda, \mu, \sigma, \lambda_v, \mu_v, \sigma_v, b, b^*, \alpha, \beta, \gamma, \alpha_v, \beta_v, \gamma_v, \rho, a, k$ and k^* are the constitutive coefficients and δ_{ij} the Kronecker delta.

The evolution equations are

$$\begin{aligned}
&t_{ji,j} + \rho F_i^{(1)} = \rho\ddot{u}_i,\\
&\epsilon_{ijk}t_{jk} + m_{ji,j} + JF_i^{(2)} = J\ddot{\phi}_i,\\
&\rho T_0\dot{\eta} = q_{i,i} + \rho r,
\end{aligned}$$

where ρ and J are positive constants whose physical meaning is well known and $F_i^{(k)}$ and r are the supply terms.

To have a well determined problem we need to impose boundary and initial conditions. Therefore, we will assume Dirichlet boundary conditions

$$u_i(\boldsymbol{x},t)=\overline{u}_i,\ \ \phi_i(\boldsymbol{x},t)=\overline{\phi}_i,\ \ T(\boldsymbol{x},t)=\overline{T},\ \ \boldsymbol{x}\in\partial\Gamma. \tag{2.2}$$

and the following initial conditions

$$\begin{aligned} u_i(\boldsymbol{x},0)=u_i^0(\boldsymbol{x}),\ \ \dot{u}_i(\boldsymbol{x},0)=v_i^0(\boldsymbol{x}),\phi_i(\boldsymbol{x},0)=\phi_i^0(\boldsymbol{x}),\ \ \dot{\phi}_i(\boldsymbol{x},0)=\varphi_i^0(\boldsymbol{x}),\\ T(\boldsymbol{x},0)=T^0(\boldsymbol{x}),\ \ \boldsymbol{x}\in\Gamma. \end{aligned} \tag{2.3}$$

As the material is isotropic the system of the field equations is given by

$$\begin{aligned} (\mu+\sigma)\Delta u_i &+ (\lambda+\mu)u_{r,ri}+\sigma\epsilon_{irs}\phi_{s,r}+(\mu_v+\sigma_v)\Delta\dot{u}_i\\ &+ (\lambda_v+\mu_v)\dot{u}_{r,ri}+\sigma_v\epsilon_{irs}\dot{\phi}_{s,r}-bT_{,i}+\rho F_i^{(1)}=\rho\ddot{u}_i\\ \gamma\Delta\phi_i+b^*\epsilon_{ijk}T_{,kj} &+ (\alpha+\beta)\phi_{r,ri}+\sigma\epsilon_{irs}u_{s,r}-2\sigma\phi_i+\gamma_v\Delta\dot{\phi}_i\\ &+ (\alpha_v+\beta_v)\dot{\phi}_{r,ri}+\sigma_v\epsilon_{irs}\dot{u}_{s,r}-2\sigma_v\dot{\phi}_i+JF_i^{(2)}=J\ddot{\phi}_i\\ & T_0(b\dot{u}_{i,i}+a\dot{T})=k\Delta T+k^*\epsilon_{irs}\dot{\kappa}_{rs,i}+\rho r. \end{aligned} \tag{2.4}$$

Here Δ means the Laplace operator.

As usual in this paper we assume that the coefficients satisfy $\rho>0$, $J>0$, $a>0$ and $k>0$.

It is worth noting that the mechanical internal energy is given by

$$2W=\lambda e_{rr}e_{ss}+(\mu+\sigma)e_{ij}e_{ij}+\mu e_{ij}e_{ji}+\alpha\kappa_{rr}\kappa_{ss}+\beta\kappa_{ij}\kappa_{ji}+\gamma\kappa_{ij}\kappa_{ij}. \tag{2.5}$$

If we assume that W is a positive definite quadratic form, then we find that the constitutive coefficients of an isotropic body satisfy the inequalities

$$\begin{aligned} 3\lambda+2\mu+\sigma>0,2\mu+\sigma>0,\sigma>0,\\ 3\alpha+\beta+\gamma>0,\gamma+\beta>0,\gamma-\beta>0. \end{aligned} \tag{2.6}$$

The dissipation in this case is given by

$$D=\lambda_v\dot{e}_{rr}\dot{e}_{ss}+(\mu_v+\sigma_v)\dot{e}_{ij}\dot{e}_{ij}+\mu_v\dot{e}_{ij}\dot{e}_{ji}+\alpha_v\dot{\kappa}_{rr}\dot{\kappa}_{ss}+\beta_v\dot{\kappa}_{ij}\dot{\kappa}_{ji}+\gamma_v\dot{\kappa}_{ij}\dot{\kappa}_{ij}+\frac{k}{T_0}T_{,i}T_{,i}+\left(b^*+\frac{k^*}{T_0}\right)\epsilon_{jir}\dot{\kappa}_{ji}T_{,r}. \tag{2.7}$$

In this paper we will assume that D is a positive definite quadratic form, that is, we assume the existence of a positive constant C such that the inequality

$$D\geq C(\dot{e}_{ij}\dot{e}_{ij}+\dot{\kappa}_{ij}\dot{\kappa}_{ij}+T_{,i}T_{,i}) \tag{2.8}$$

is satisfied.

3. Uniqueness and instability

We restrict our attention to the particular case obtained when $k^*=b^*=0$ and, for this case, we propose the logarithmic convexity argument to show the uniqueness and instability of the solutions. For this purpose, we assume that the supply terms vanish.

If D is a positive definite quadratic form, then we find that the constitutive coefficients of an isotropic body satisfy the inequalities

$$\begin{aligned} 3\lambda_v + 2\mu_v + \sigma_v > 0, 2\mu_v + \sigma_v > 0, \sigma_v > 0, \\ 3\alpha_v + \beta_v + \gamma_v > 0, \gamma_v + \beta_v > 0, \gamma_v - \beta_v > 0. \end{aligned} \tag{3.1}$$

In this section we suppose that ρ, J, a and k are positive, and that (3.1) holds. To prove the uniqueness of the solutions it is sufficient to see that the only solution of the homogeneous boundary-value problem is the null solution. Thus, from now on, we will work with the null boundary conditions:

$$u_i = 0,\ \ \phi_i = 0,\ \ T = 0,\ \text{ on } \partial\Gamma \times (0,t). \tag{3.2}$$

The logarithmic convexity argument is strongly based on the choice of a relevant measure function. To define this function it is convenient to consider several preliminaries. Let us denote by

$$\theta(t) = \int_0^t T(s)ds,$$

and integrate the last equation of (2.4) with respect to the time. We obtain

$$T_0(bu_{i,i} + aT) = k\theta_{,ii} + T_0(bu^0_{i,i} + aT^0). \tag{3.3}$$

We denote by $P(\mathbf{x})$ the solution of the boundary value problem defined by the equation

$$kP_{,ii} = -T_0(bu^0_{i,i} + aT^0) \tag{3.4}$$

and the homogeneous Dirichlet conditions $P = 0$ on $\partial\Gamma$.

It is worth noting that the existence and uniqueness of P is guaranteed by the theory of elliptic equations. If we define

$$z(\mathbf{x},t) = \theta(\mathbf{x},t) - P(\mathbf{x}),$$

then equation (3.3) becomes

$$T_0(bu_{i,i} + aT) = kz_{,ii}. \tag{3.5}$$

The energy equality can be written as

$$\begin{aligned} E(t) = & \int_\Gamma \big(\rho\dot{u}_i\dot{u}_i + J\dot{\phi}_i\dot{\phi}_i + aT^2\big)dV \\ & + \int_\Gamma \big(\lambda e_{rr}e_{ss} + (\mu+\sigma)e_{ij}e_{ij} + \mu e_{ij}e_{ji} + \alpha\kappa_{rr}\kappa_{ss} + \beta\kappa_{ij}\kappa_{ji} + \gamma\kappa_{ij}\kappa_{ij}\big)dV \\ & + 2\int_0^t \int_\Gamma D\,dV\,ds = E(0), \end{aligned} \tag{3.6}$$

where D has been defined previously.

Now we define the function $F_{h,t_0}(t)$ in the following way:

(3.7)

$$\begin{aligned} F_{h,t_0}(t) = & \int_\Gamma \big(\rho u_i u_i + J\phi_i\phi_i\big)dV \\ & + \int_0^t \int_\Gamma \big(\lambda_v e_{rr}e_{ss} + (\mu_v+\sigma_v)e_{ij}e_{ij} + \mu_v e_{ij}e_{ji} + \alpha_v\kappa_{rr}\kappa_{ss} + \beta_v\kappa_{ij}\kappa_{ji} + \gamma_v\kappa_{ij}\kappa_{ij} + \frac{k}{T_0}z_{,i}z_{,i}\big)dV\,ds \\ & + h(t+t_0)^2. \end{aligned}$$

Here h and t_0 are two positive constants to be determined later (see [12]).

If we compute the first and second derivatives of $F_{h,t_0}(t)$ we have

(3.8)
$$\begin{aligned}
\dot{F}_{h,t_0}(t) &= 2\int_\Gamma \big(\rho u_i\dot{u}_i + J\phi_i\dot{\phi}_i\big)dV \\
&+ 2\int_0^t\int_\Gamma \big(\lambda_v e_{rr}\dot{e}_{ss} + (\mu_v+\sigma_v)e_{ij}\dot{e}_{ij} + \mu_v e_{ij}\dot{e}_{ji} + \alpha_v\kappa_{rr}\dot{\kappa}_{ss} + \beta_v\kappa_{ij}\dot{\kappa}_{ji} + \gamma_v\kappa_{ij}\dot{\kappa}_{ij} + \frac{k}{T_0}z_{,i}\dot{z}_{,i}\big)dV ds \\
&+ \int_\Gamma \big(\lambda_v e_{rr}(0)e_{ss}(0) + (\mu_v+\sigma_v)e_{ij}(0)e_{ij}(0) + \mu_v e_{ij}(0)e_{ji}(0) + \alpha_v\kappa_{rr}(0)\kappa_{ss}(0) + \beta_v\kappa_{ij}(0)\kappa_{ji}(0) \\
&+ \gamma_v\kappa_{ij}(0)\kappa_{ij}(0) + \frac{k}{T_0}P_{,i}P_{,i}\big)dV + 2h(t+t_0),
\end{aligned}$$

and

(3.9)
$$\begin{aligned}
\ddot{F}_{h,t_0}(t) &= 2\int_\Gamma \big(\rho\dot{u}_i\dot{u}_i + J\dot{\phi}_i\dot{\phi}_i + \rho u_i\ddot{u}_i + J\phi_i\ddot{\phi}_i\big)dV \\
&+ 2\int_\Gamma \big(\lambda_v e_{rr}\dot{e}_{ss} + (\mu_v+\sigma_v)e_{ij}\dot{e}_{ij} + \mu_v e_{ij}\dot{e}_{ji} + \alpha_v\kappa_{rr}\dot{\kappa}_{ss} + \beta_v\kappa_{ij}\dot{\kappa}_{ji} + \gamma_v\kappa_{ij}\dot{\kappa}_{ij} + \frac{k}{T_0}z_{,i}\dot{z}_{,i}\big)dV + 2h.
\end{aligned}$$

In view of the evolution equations and using the divergence theorem we obtain

$$\begin{aligned}
\ddot{F}_{h,t_0}(t) &= 2\int_\Gamma \big(\rho\dot{u}_i\dot{u}_i + J\dot{\phi}_i\dot{\phi}_i\big)dV \\
&- 2\int_\Gamma \big(\lambda e_{rr}e_{ss} + (\mu+\sigma)e_{ij}e_{ij} + \mu e_{ij}e_{ji} + \alpha\kappa_{rr}\kappa_{ss} + \beta\kappa_{ij}\kappa_{ji} + \gamma\kappa_{ij}\kappa_{ij}\big)dV \\
&+ 2\int_\Gamma \frac{k}{T_0}z_{,i}\dot{z}_{,i}dV + 2\int_\Gamma bTu_{i,i}dV + 2h.
\end{aligned} \tag{3.10}$$

Therefore, taking into account that

$$\int_\Gamma bu_{i,i}TdV = -\int_\Gamma \left(\frac{k}{T_0}z_{,i}T_{,i} - aT^2\right)dV,$$

we see that

$$\begin{aligned}
\ddot{F}_{h,t_0}(t) &= 2\int_\Gamma \big(\rho\dot{u}_i\dot{u}_i + J\dot{\phi}_i\dot{\phi}_i\big)dV \\
&- 2\int_\Gamma \big(\lambda e_{rr}e_{ss} + (\mu+\sigma)e_{ij}e_{ij} + \mu e_{ij}e_{ji} + \alpha\kappa_{rr}\kappa_{ss} + \beta\kappa_{ij}\kappa_{ji} + \gamma\kappa_{ij}\kappa_{ij}\big)dV \\
&- 2\int_\Gamma aT^2dV + 2h.
\end{aligned} \tag{3.11}$$

Using the conservation of the energy we find that

$$\ddot{F}_{h,t_0}(t) = 4\int_\Gamma \big(\rho\dot{u}_i\dot{u}_i + J\dot{\phi}_i\dot{\phi}_i\big)dV + 4\int_0^t\int_\Gamma DdVds - 2(E(0)-h). \tag{3.12}$$

Then, if we denote by

(3.13)

$$\nu = \int_\Gamma \big(\lambda_v e_{rr}(0)e_{ss}(0) + (\mu_v+\sigma_v)e_{ij}(0)e_{ij}(0) + \mu_v e_{ij}(0)e_{ji}(0) + \alpha_v \kappa_{rr}(0)\kappa_{ss}(0) + \beta_v \kappa_{ij}(0)\kappa_{ji}(0) + \gamma_v \kappa_{ij}(0)\kappa_{ij}(0) + \frac{k}{T_0}P_{,i}P_{,i}\big)dV,$$

then we obtain the following inequality:

$$F_{h,t_0}\ddot{F}_{h,t_0} - (\dot{F}_{h,t_0} - \nu)^2 \geq 2(h+E(0))F_{h,t_0}. \tag{3.14}$$

If we assume that the initial data vanish, then we have $\nu = 0$, and the above inequality, for $h = t_0 = 0$ becomes

$$F\ddot{F} - \dot{F}^2 \geq 0.$$

(To simplify, we have denoted $F = F_{0,0}$.) From this last inequality we find

$$F(t) \leq F(0)^{1-t/t_1}F(t_1)^{t/t_1}, \quad 0 \leq t \leq t_1,$$

and we conclude that $F(t) = 0$ for $0 \leq t \leq t_1$. This gives us the uniqueness result.

In the general case, assuming that $E(0) < 0$, we can always take t_0 large enough to guarantee that $\dot{F}_{h,t_0} > \nu$. Then we get

$$F_{h,t_0}(t) \geq \frac{F_{h,t_0}(0)\dot{F}_{h,t_0}(0)}{\dot{F}_{h,t_0}(0)-\nu}\exp\left(\frac{\dot{F}_{h,t_0}(0)-\nu}{F_{h,t_0}(0)}t\right) - \frac{\nu F_{h,t_0}(0)}{\dot{F}_{h,t_0}(0)-\nu}.$$

This inequality gives the exponential growth of the solutions.

Therefore, we have proved the following result.

Theorem 3.1. *Let us assume that $\rho > 0$, $J > 0$, $a > 0$, $k > 0$ and that (3.1) hold. Then:*

(1) *The Dirichlet boundary value problem has at most one solution.*
(2) *If $E(0) < 0$, then the solution becomes unbounded in an exponential way.*

4. Uniqueness for the general case

In this section we will study the problem in the general case, that is, when b^* or k^* can be different from zero, and we will show the uniqueness of the solutions. In fact, we will prove that the only solution with null initial conditions is the null solution. This fact will prove the uniqueness. In this section we assume again that ρ, J, a and k are positive and D is positive definite.

In this case, the energy equation (see (3.6)) gives

$$\begin{aligned} E(t) = &\int_\Gamma \big(\rho\dot{u}_i\dot{u}_i + J\dot{\phi}_i\dot{\phi}_i + aT^2\big)dV \\ &+ \int_\Gamma \big(\lambda e_{rr}e_{ss} + (\mu+\sigma)e_{ij}e_{ij} + \mu e_{ij}e_{ji} + \alpha\kappa_{rr}\kappa_{ss} + \beta\kappa_{ij}\kappa_{ji} + \gamma\kappa_{ij}\kappa_{ij}\big)dV \\ &+ 2\int_0^t\int_\Gamma D\,dV\,ds = 0. \end{aligned} \tag{4.1}$$

From the above expression, we introduce the following notation:

$$
\begin{aligned}
I_1 &= \int_\Gamma \left(\rho \dot{u}_i \dot{u}_i + J \dot{\phi}_i \dot{\phi}_i + a T^2\right) dV \\
I_2 &= \int_\Gamma \left(\lambda e_{rr} e_{ss} + (\mu+\sigma) e_{ij} e_{ij} + \mu e_{ij} e_{ji} + \alpha \kappa_{rr} \kappa_{ss} + \beta \kappa_{ij} \kappa_{ji} + \gamma \kappa_{ij} \kappa_{ij}\right) dV \\
I_3 &= 2 \int_0^t \int_\Gamma D dV ds.
\end{aligned}
$$

And we define the function $J(t) = I_1 + I_3$. Obviously, $J(t) = -I_2$. Therefore,

$$
J(t) \leq C \left(\int_0^t I_2 \right)^{\frac{1}{2}} I_3^{\frac{1}{2}},
$$

where C is a calculable positive constant. In view of a Poincaré type inequality, there exists a positive constant C_1 such that

$$
\int_0^t I_2 ds \leq \frac{4t^2}{\pi^2} C_1 \int_0^t (\dot{e}_{ij} \dot{e}_{ij} + \dot{\kappa}_{ij} \dot{\kappa}_{ij}) ds.
$$

Using the positivity of the dissipation function, it can be seen that

$$
J(t) \leq C^* t I_3 \leq C^* t J(t), \tag{4.2}
$$

where C^* can be calculated in terms of the constitutive coefficients. From (4.2), it follows that $(1 - C^* t) J(t) \leq 0$. If we consider $t_0 = (C^*)^{-1}$, then we find that $J(t)$ vanishes in the interval $[0, t_0]$. From the definition of $J(t)$, it follows that $\dot{u}_i = 0$ and $T = 0$ for every $t \leq t_0$. Thus, we have proved that the problem has only the null solution in the interval $[0, t_0]$. Applying the same argument to the problem determined by the field equations, the same boundary conditions and the null initial data for the initial instant t_0, that is, $u_i(\boldsymbol{x}, t_0) = 0$, $\dot{T}(\boldsymbol{x}, t_0) = 0$, we conclude that $u_i = 0$ and $T = 0$ for every $t \leq 2t_0$. The theorem is proved applying recurrently this argument.

Theorem 4.1. *Let us assume that $\rho > 0$, $J > 0$, $a > 0$, $k > 0$ and that D is positive definite. Then the Dirichlet boundary value problem has at most one solution.*

5. Existence of solution

In this section we use the results of the semigroup of linear operators theory to obtain an existence theorem. Though other boundary conditions could be proposed, we restrict our attention to the boundary conditions proposed at (3.2).

In the remains of the paper we assume that ρ, J, a and k are positive, D is positive definite and that the internal energy density W is a positive definite quadratic form.

We now transform the boundary initial value problem defined by system (2.4), initial conditions (2.3) and boundary conditions (2.2) into an abstract problem on a suitable Hilbert space. We denote

$$
\mathcal{Z} = \{\mathbf{U} = (\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}, \boldsymbol{\varphi}, T); u_i, \phi_i, \in W_0^{1,2}(\Gamma), v_i, \varphi_i, T \in L^2(\Gamma)\}, \tag{5.1}
$$

where $W_0^{1,2}(\Gamma)$ and $L^2(\Gamma)$ are the usual Sobolev spaces, which take values at the complex field.

Let us consider the operators

$$\begin{aligned}
A_i(\mathbf{u}) =& \frac{1}{\rho}\left[(\mu+\sigma)u_{i,jj} + (\lambda+\mu)u_{j,ji}\right],\\
B_i(\boldsymbol{\phi}) =& \frac{1}{\rho}\left[\sigma\epsilon_{irs}\phi_{s,r}\right],\\
C_i(T) =& \frac{1}{\rho}\left[-bT_{,i}\right],\\
D_i(\mathbf{v}) =& \frac{1}{\rho}\left[(\mu_v+\sigma_v)v_{i,jj} + (\lambda_v+\mu_v)v_{j,ji}\right],\\
E_i(\boldsymbol{\varphi}) =& \frac{1}{\rho}\left[\sigma_v\epsilon_{irs}\varphi_{r,s}\right],
\end{aligned}$$

and

$$\begin{aligned}
K_i(\mathbf{v}) =& \frac{1}{J}\left[\sigma_v\epsilon_{irs}v_{s,r}\right],\\
Z_i(\mathbf{u}) =& \frac{1}{J}\left[\sigma\epsilon_{irs}u_{s,r}\right],\\
M_i(\boldsymbol{\phi}) =& \frac{1}{J}\left[\gamma\phi_{i,jj} + (\alpha+\beta)\phi_{j,ji} - 2\sigma\phi_i\right],\\
F_i(\boldsymbol{\varphi}) =& \frac{1}{J}\left[\gamma_v\varphi_{i,jj} + (\alpha_v+\beta_v)\varphi_{j,ji} - 2\sigma_v\varphi_i\right],\\
N_i(T) =& \frac{b^*}{J}\epsilon_{ijk}T_{,jk},\\
U_1(T) =& \frac{1}{aT_0}\left[kT_{,jj}\right],\\
V_1(\mathbf{v}) =& \frac{1}{a}\left[-bv_{i,i}\right],\\
X_1(\boldsymbol{\varphi}) =& \frac{k^*}{aT_0}\epsilon_{irs}\varphi_{s,ri}.
\end{aligned}$$

We denote

$$\mathcal{D} = \left(\mathbf{W}_0^{1,2}\cap\mathbf{W}^{2,2}\right)\times\left(\mathbf{W}_0^{1,2}\cap\mathbf{W}^{2,2}\right)\times\left(\mathbf{W}_0^{1,2}\cap\mathbf{W}^{2,2}\right)\times\mathbf{W}_0^{1,2}\times\left(W^{2,2}\cap W_0^{1,2}\right). \tag{5.2}$$

Let $\mathcal{A}$ be the matrix operator defined on $\mathcal{D}$ by

$$\mathcal{A} = \begin{pmatrix} \mathbf{0} & \mathbf{Id} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{A} & \mathbf{D} & \mathbf{B} & \mathbf{E} & \mathbf{C}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{Id} & \mathbf{0}\\ \mathbf{Z} & \mathbf{K} & \mathbf{M} & \mathbf{F} & \mathbf{N}\\ \mathbf{0} & V_1 & \mathbf{0} & X_1 & U_1 \end{pmatrix}, \tag{5.3}$$

where $\mathbf{A}=(A_i), \mathbf{B}=(B_i), \mathbf{C}=(C_i), \mathbf{D}=(D_i), \mathbf{E}=(E_i), \mathbf{F}=(F_i), \mathbf{Z}=(Z_i), \mathbf{M}=(M_i), \mathbf{N}=(N_i), \mathbf{K}=(K_i)$ and $\mathbf{Id}$ represent the identity in the respective space. We note that the domain of $\mathcal{A}$ contains $\mathcal{D}$ which is dense in $\mathcal{Z}$.

The initial boundary value problem (2.4), (2.3), (2.2) can be transformed into the following abstract equation in the Hilbert space $\mathcal{Z}$,

$$\frac{d\mathbf{U}}{dt} = \mathcal{A}\mathbf{U}(t) + \mathbf{\Gamma}(t), \quad \mathbf{U}(0) = \mathbf{U}_0, \tag{5.4}$$

where

$$\mathbf{\Gamma} = \left(\mathbf{0}, \mathbf{F}^{(1)}, \mathbf{0}, \mathbf{F}^{(2)}, \frac{\rho r}{aT_0}\right), \quad \mathbf{U}_0 = \left(u_i^0, v_i^0, \phi_i^0, \varphi_i^0, T^0\right). \tag{5.5}$$

We introduce in $\mathcal{Z}$ the inner product

$$< \mathbf{U}, \mathbf{V} >= \int_\Gamma \left(\rho v_i \overline{v_i}^\star + J\varphi_i \overline{\varphi_i}^\star + aT\overline{T}^\star + \mathcal{M}\big[\mathbf{U}^0, \mathbf{V}^0\big]\right) dV, \tag{5.6}$$

where

$$\mathbf{U} = (\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}, \boldsymbol{\varphi}, T), \ \ \mathbf{V} = (\mathbf{u}^\star, \mathbf{v}^\star, \boldsymbol{\phi}^\star, \boldsymbol{\varphi}^\star, T^\star),$$

$$\mathbf{U}^0 = (\mathbf{u}, \boldsymbol{\phi}), \ \ \mathbf{V}^0 = (\mathbf{u}^\star, \boldsymbol{\phi}^\star)$$

and

$$\mathcal{M}\big[\mathbf{U}^0, \mathbf{V}^0\big] = \lambda e_{rr}\overline{e_{ss}^\star} + (\mu+\sigma)e_{ij}\overline{e_{ij}^\star} + \mu e_{ij}\overline{e_{ji}^\star} + \alpha\kappa_{rr}\overline{\kappa_{ss}^\star} + \beta\kappa_{ij}\overline{\kappa_{ji}^\star} + \gamma\kappa_{ij}\overline{\kappa_{ij}^\star}.$$

As before, we use relations (2.1).

We note that (5.6) defines a norm which is given by

$$\begin{aligned}\|(\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}, \boldsymbol{\varphi}, T)\|^2 = \int_{\mathcal{Z}} \big(&\rho v_i \overline{v_i} + J\varphi_i\overline{\varphi_i} + aT\overline{T} + \lambda e_{rr}\overline{e_{ss}} + (\mu+\sigma)e_{ij}\overline{e_{ij}} + \\ &\mu e_{ij}\overline{e_{ji}} + \alpha\kappa_{rr}\overline{\kappa_{ss}} + \beta\kappa_{ij}\overline{\kappa_{ji}} + \gamma\kappa_{ij}\overline{\kappa_{ij}}\big)\, dV\end{aligned} \tag{5.7}$$

In view of the assumptions (2.6) and also of the equality

$$e_{ij}e_{ij} = u_{(i,j)}u_{(i,j)} + \epsilon_{ijk}\epsilon_{ijk}(\gamma_k - \phi_k)^2,$$

where $\gamma_k = \frac{1}{2}\epsilon_{krs}u_{s,r}$ and $u_{(i,j)} = \frac{1}{2}(u_{i,j} + u_{j,i})$, we can assure that our inner product defines a norm which is equivalent to the usual norm in $\mathcal{Z}$.

Lemma 5.1. *The operator $\mathcal{A}$ has the property*

$$\Re < \mathcal{A}\mathbf{U}, \mathbf{U} > \leq 0, \tag{5.8}$$

for any $\mathbf{U} \in \mathcal{D}$, where the inner product $< .,. >$ is defined at (5.6).

Proof. Let $\mathbf{U} = (\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}, \boldsymbol{\varphi}, T) \in \mathcal{D}$. We denote $e_{ij}^* = v_{j,i} + \epsilon_{jik}\varphi_k$ and $\kappa_{ij}^* = \varphi_{j,i}$.

Using the divergence theorem and the boundary conditions we have

$$\begin{aligned}< \mathcal{A}\mathbf{U}, \mathbf{U} >= -\int_\Gamma \Big(&\lambda_v e_{rr}^*\overline{e^*}_{ss} + (\mu_v+\sigma_v)e_{ij}^*\overline{e^*}_{ij} + \mu_v e_{ij}^*\overline{e^*}_{ji} + \alpha_v\kappa_{rr}^*\overline{\kappa^*}_{ss} + \beta_v\kappa_{ij}^*\overline{\kappa^*}_{ji} \\ &+\gamma_v\kappa_{ij}^*\overline{\kappa^*}_{ij} + \frac{k^*}{T_0}\epsilon_{jir}\kappa_{ji}^*\overline{T}_{,r} + b^*\epsilon_{jir}\overline{\kappa^*}_{ji}T_{,r} + \frac{k}{T_0}T_{,i}\overline{T}_{,i}\Big)\, dV\end{aligned} \tag{5.9}$$

In view of (2.8), it is clear that $\Re < \mathcal{A}\mathbf{U}, \mathbf{U} > \leq 0$.

Lemma 5.2. *The operator $\mathcal{A}$ satisfies the condition $0 \in \varrho(\mathcal{A})$.*

Proof. Let $\mathbf{U}^* = (\mathbf{u}^*, \mathbf{v}^*, \boldsymbol{\phi}^*, \boldsymbol{\varphi}^*, T^*) \in \mathcal{Z}$. We must show that the equation

$$\mathcal{A}\mathbf{U} = \mathbf{U}^*, \tag{5.10}$$

has a solution $\mathbf{U} = (\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}, \boldsymbol{\varphi}, T) \in \mathcal{D}$. If we take into account the operator $\mathcal{A}$ described by (5.3), then we find the system

$$\begin{aligned} &\mathbf{v} = \mathbf{u}^* \\ &\mathbf{A}\mathbf{u} + \mathbf{D}\mathbf{v} + \mathbf{B}\boldsymbol{\phi} + \mathbf{E}\boldsymbol{\varphi} + \mathbf{C}T = \mathbf{v}^* \\ &\boldsymbol{\varphi} = \boldsymbol{\phi}^* \\ &\mathbf{Z}\mathbf{u} + \mathbf{K}\mathbf{v} + \mathbf{M}\boldsymbol{\phi} + \mathbf{F}\boldsymbol{\varphi} + \mathbf{N}T = \boldsymbol{\varphi}^* \\ &V_1\mathbf{v} + U_1 T + X_1\boldsymbol{\phi}^* = T^*. \end{aligned} \tag{5.11}$$

Substituting the first and the third equations into the others, we obtain the following system with unknowns $\mathbf{u}$, $\boldsymbol{\phi}$ and T.

$$\begin{aligned} &\mathbf{A}\mathbf{u} + \mathbf{B}\boldsymbol{\phi} + \mathbf{C}T = \mathbf{v}^* - \mathbf{D}\mathbf{u}^* - \mathbf{E}\boldsymbol{\phi}^* \\ &\mathbf{Z}\mathbf{u} + \mathbf{M}\boldsymbol{\phi} + \mathbf{N}T = \boldsymbol{\varphi}^* - \mathbf{K}\mathbf{u}^* - \mathbf{F}\boldsymbol{\phi}^* \\ &U_1 T = T^* - V_1\mathbf{u}^* - X_1\boldsymbol{\varphi}. \end{aligned} \tag{5.12}$$

From the last equation we obtain the solution $T(x)$, which can be substituted in the first and second equations of the above system. Hence, we get

$$\begin{aligned} &\mathbf{A}\mathbf{u} + \mathbf{B}\boldsymbol{\phi} = \mathbf{v}^* - \mathbf{D}\mathbf{u}^* - \mathbf{E}\boldsymbol{\phi}^* - \mathbf{C}T \\ &\mathbf{Z}\mathbf{u} + \mathbf{M}\boldsymbol{\phi} = \boldsymbol{\varphi}^* - \mathbf{K}\mathbf{u}^* - \mathbf{F}\boldsymbol{\phi}^* - \mathbf{N}T. \end{aligned} \tag{5.13}$$

Notice that

$$(\mathbf{v}^* - \mathbf{D}\mathbf{u}^* - \mathbf{E}\boldsymbol{\phi}^* - \mathbf{C}T\,,\, \boldsymbol{\varphi}^* - \mathbf{K}\mathbf{u}^* - \mathbf{F}\boldsymbol{\phi}^* - \mathbf{N}T) \in W^{-1,2} \times W^{-1,2}.$$

On the other side,

$$\mathcal{B}\big((\mathbf{u}, \boldsymbol{\phi}), (\mathbf{u}^*, \boldsymbol{\phi}^*)\big) = \langle (\mathbf{A}\mathbf{u} + \mathbf{B}\boldsymbol{\phi}, \mathbf{Z}\mathbf{u} + \mathbf{M}\boldsymbol{\phi}), (\rho\mathbf{u}^*, J\boldsymbol{\phi}^*)\rangle$$

defines a coercive and bounded bilinear form on $W_0^{1,2} \times W_0^{1,2}$. Hence, in $W_0^{1,2} \times W_0^{1,2}$, the Lax-Milgram theorem implies the existence of a solution to the system of equations (5.13).Thus, equation (5.10) has also a solution.

Theorem 5.3. *The operator $\mathcal{A}$ generates a semigroup of contractions in $\mathcal{Z}$.*

The proof follows from the above lemmas and the Lumer-Phillips corollary to the Hille-Yosida theorem.

Theorem 5.4. *Assume that $F_i^\alpha, r \in C^1([0,\infty), L^2)$ and $\mathbf{U}_0 \in \mathcal{D}$. Then, there exists a unique solution $\mathbf{U}(t) \in C^1([0,\infty), \mathcal{Z}) \cap C^0([0,\infty), \mathcal{D})$ to the problem (5.4).*

Since the semigroup defined by the operator $\mathcal{A}$ is contractive, we obtain the estimate

$$||\mathbf{U}(t)||_{\mathcal{Z}} \leq ||\mathbf{U}_0||_{\mathcal{Z}} + \int_0^t \Big(||\mathbf{F}^{(1)}(\tau)||_{L^2} + ||\mathbf{F}^{(2)}(\tau)||_{L^2} + ||r(\tau)||_{L^2} \Big)\, d\tau, \tag{5.14}$$

which proves the continuous dependence of the solutions upon initial data and body loads. Thus, the problem is well posed.

6. Analyticity of solutions

In this section we prove the analyticity of the solutions to the problem (5.4) supposing that the supply terms are absent.

In order to prove the main result of this section we will use a theorem that can be found in Liu and Zheng [15].

Theorem 6.1. *Let us consider* $S(t) = e^{\mathcal{A}t}$ *a* C_0*-semigroup of contractions generated by the operator* $\mathcal{A}$ *in the Hilbert space* $\mathcal{Z}$*. Suppose that* $\varrho(\mathcal{A}) \supseteq \{i\beta; \beta \in \mathbb{R}\} \equiv i\mathbb{R}$*. Then* $S(t)$ *is analytic if and only if*

$$\overline{\lim}_{|\beta|\to\infty} \|\beta(i\beta\mathcal{I} - \mathcal{A})^{-1}\| < \infty, \quad \beta \in \mathbb{R}$$

holds.

To apply this theorem to our situation, we need to consider the resolvent equation

$$\lambda \mathbf{U} - \mathcal{A}\mathbf{U} = \mathbf{F},$$

where $\mathbf{U} = (\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}, \boldsymbol{\varphi}, T)$ and $\mathbf{F} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4, f_5)$. We shall take $\lambda = i\alpha$, with $\alpha \in \mathbb{R}$. Therefore, our equation becomes

$$\begin{array}{l} i\alpha\mathbf{u} - \mathbf{v} = \mathbf{f}_1 \\ i\alpha\mathbf{v} - \mathbf{A}\mathbf{u} - \mathbf{D}\mathbf{v} - \mathbf{B}\boldsymbol{\phi} - \mathbf{E}\boldsymbol{\varphi} - \mathbf{C}T = \mathbf{f}_2 \\ i\alpha\boldsymbol{\phi} - \boldsymbol{\varphi} = \mathbf{f}_3 \\ i\alpha\boldsymbol{\varphi} - \mathbf{Z}\mathbf{u} - \mathbf{K}\mathbf{v} - \mathbf{M}\boldsymbol{\phi} - \mathbf{F}\boldsymbol{\varphi} - \mathbf{N}T = \mathbf{f}_4 \\ i\alpha T - V_1\mathbf{v} - U_1 T - X_1\boldsymbol{\varphi} = f_5. \end{array} \tag{6.1}$$

Lemma 6.2. *For any* $\mathbf{F} \in \mathcal{Z}$ *there exists a positive constant* C *such that*

$$\int_\Gamma (e^*_{ij}\overline{e}^*_{ij} + \kappa^*_{ij}\overline{\kappa}^*_{ij} + T_{,i}T_{,i})dV \leq C\|\mathbf{F}\|_{\mathcal{Z}}\|\mathbf{U}\|_{\mathcal{Z}},$$

where, as before,

$$e^*_{ij} = v_{j,i} + \epsilon_{jik}\varphi_k \text{ and } \kappa^*_{ij} = \varphi_{j,i}.$$

Proof. If we multiply the first equation of (6.1) by $-\mathbf{A}\overline{\mathbf{u}} - \mathbf{B}\overline{\boldsymbol{\phi}}$, the second by $\overline{\mathbf{v}}$, the third by $-\mathbf{M}\overline{\boldsymbol{\phi}} - \mathbf{Z}\overline{\mathbf{u}}$, the fourth by $\overline{\boldsymbol{\varphi}}$, the last one by $\overline{T}$ and we add all the results we obtain at the left hand side the following expression:

$$\begin{aligned} & i\alpha\left(-\langle \mathbf{u}, \mathbf{A}\mathbf{u} + \mathbf{B}\boldsymbol{\phi}\rangle - \langle \boldsymbol{\phi}, \mathbf{Z}\mathbf{u} + \mathbf{M}\boldsymbol{\phi}\rangle + \langle \mathbf{v}, \mathbf{v}\rangle + \langle \boldsymbol{\varphi}, \boldsymbol{\varphi}\rangle + \langle T, T\rangle\right) \\ & +\langle \mathbf{v}, \mathbf{A}\mathbf{u} + \mathbf{B}\boldsymbol{\phi}\rangle - \langle \mathbf{A}\mathbf{u} + \mathbf{B}\boldsymbol{\phi}, \mathbf{v}\rangle + \langle \boldsymbol{\varphi}, \mathbf{Z}\mathbf{u} + \mathbf{M}\boldsymbol{\phi}\rangle - \langle \mathbf{Z}\mathbf{u} + \mathbf{M}\boldsymbol{\phi}, \boldsymbol{\varphi}\rangle \\ & -\langle \mathbf{D}\mathbf{v} + \mathbf{E}\boldsymbol{\varphi}, \mathbf{v}\rangle - \langle \mathbf{K}\mathbf{v} + \mathbf{F}\boldsymbol{\varphi}, \boldsymbol{\varphi}\rangle - \langle U_1 T, T\rangle \\ & -\langle NT, \boldsymbol{\varphi}\rangle - \langle X_1\boldsymbol{\varphi}, T\rangle \\ & -\langle \mathbf{C}T, \mathbf{v}\rangle - \langle V_1\mathbf{v}, T\rangle. \end{aligned} \tag{6.2}$$

And the norm of the right hand side is bounded by $C\|\mathbf{F}\|\|\mathbf{U}\|$, for a positive constant C.

Notice that the first line of (6.2) becomes

$$i\alpha \int_\Gamma \left[\lambda e_{rr}\overline{e}_{ss} + (\mu + \sigma)e_{ij}\overline{e}_{ij} + \mu e_{ij}\overline{e}_{ji} + \alpha\kappa_{rr}\overline{\kappa}_{ss} + \beta\kappa_{ij}\overline{\kappa}_{ji} + \gamma\kappa_{ij}\overline{\kappa}_{ij} + \rho v_i\overline{v}_i + J\varphi_i\overline{\varphi}_i + aT\overline{T}\right] dV \tag{6.3}$$

and, therefore, it is clear that this number is imaginary.

The second line of (6.2) is also imaginary, as it obtained from the difference of conjugate complex numbers.

On the other hand, the third line becomes

$$\int_\Gamma \left[\lambda_v e^*_{rr}\overline{e}^*_{ss} + (\mu_v + \sigma_v)e^*_{ij}\overline{e}^*_{ij} + \mu_v e^*_{ij}\overline{e}^*_{ji} + \alpha_v\kappa^*_{rr}\overline{\kappa}^*_{ss} + \beta_v\kappa^*_{ij}\overline{\kappa}^*_{ji} + \gamma_v\kappa^*_{ij}\overline{\kappa}^*_{ij} + \frac{k}{T_0}T_{,j}\overline{T}_{,j}\right] dV, \tag{6.4}$$

and, hence, it is real.

The expression in the fourth line of (6.2) has real and imaginary parts. Applying the operators we get

$$\int_\Gamma \left[\big(b^* + \frac{k^*}{T_0}\big)\epsilon_{ijk}\Re\kappa^*_{ki}\overline{T}_{,j} + i\big(b^* - \frac{k^*}{T_0}\big)\epsilon_{ijk}\Im\kappa^*_{ki}\overline{T}_{,j}\right] dV \tag{6.5}$$

Finally, recalling the operators $\mathbf{C}$ and V_1, from the fifth line of (6.2) we obtain

$$\int_\Gamma \left(bT_{,i}\overline{v}_i - bv_i\overline{T}_{,i}\right) dV,$$

which is imaginary.

Therefore, taking only the real parts and recalling the positivity of the dissipation, we obtain the desired result.

Lemma 6.3. *For any* $\mathbf{F} \in \mathcal{Z}$ *there exists a positive constant* C *such that*

$$|\alpha| \|\mathbf{U}\|_{\mathcal{Z}} \leq C\|\mathbf{F}\|_{\mathcal{Z}} \quad \forall \alpha \in \mathbb{R},$$

where $\mathbf{U}$ *is the solution of (6.1).*

Proof. Now we multiply the first equation of (6.1) by $-i(\mathbf{A}\overline{\mathbf{u}} + \mathbf{B}\overline{\boldsymbol{\phi}})$, the second by $i\overline{\mathbf{v}}$, the third by $-i(\mathbf{M}\overline{\boldsymbol{\phi}} + \mathbf{Z}\overline{\mathbf{u}})$, the fourth by $i\overline{\boldsymbol{\varphi}}$, the last one by $i\overline{T}$ and we add all the results. Notice that if $\langle x, y\rangle = \Re\langle x, y\rangle + i\Im\langle x, y\rangle$, then $\langle x, iy\rangle = \Im\langle x, y\rangle - i\Re\langle x, y\rangle$. That means, that now the real part of the sum of the left hand side is just the imaginary part of (6.2).

The norm of the right hand side is bounded by $C\|\mathbf{F}\|\|\mathbf{U}\|$, for a positive constant C.

Notice that

$$-i\left(\langle \mathbf{v}, \mathbf{A}\mathbf{u} + \mathbf{B}\boldsymbol{\phi}\rangle - \langle \mathbf{A}\mathbf{u} + \mathbf{B}\boldsymbol{\phi}, \mathbf{v}\rangle + \langle \boldsymbol{\varphi}, \mathbf{Z}\mathbf{u} + \mathbf{M}\boldsymbol{\phi}\rangle - \langle \mathbf{Z}\mathbf{u} + \mathbf{M}\boldsymbol{\phi}, \boldsymbol{\varphi}\rangle\right) = 2\Im\mathcal{M}[(\mathbf{v}, \boldsymbol{\varphi}), (\mathbf{u}, \boldsymbol{\phi})], \tag{6.6}$$

and, hence, it is real.

Notice also that now the first line of (6.2) is now real:

$$\alpha \int_\Gamma \left[\lambda e_{rr}\overline{e}_{ss} + (\mu + \sigma)e_{ij}\overline{e}_{ij} + \mu e_{ij}\overline{e}_{ji} + \alpha\kappa_{rr}\overline{\kappa}_{ss} + \beta\kappa_{ij}\overline{\kappa}_{ji} + \gamma\kappa_{ij}\overline{\kappa}_{ij} + \rho v_i\overline{v}_i + J\varphi_i\overline{\varphi}_i + aT\overline{T}\right] dV \tag{6.7}$$

Moreover,

$$-i\int_\Gamma \left(bT_{,i}\overline{v}_i - bv_i\overline{T}_{,i}\right) dV = 2\Im\int_b T_{,i}\overline{v}_i dV \tag{6.8}$$

is also real.

We know that there exists positive constants C_1 and C_2 such that

$$\begin{array}{l} 2\Im\mathcal{M}[(\mathbf{v}, \boldsymbol{\varphi}), (\mathbf{u}, \boldsymbol{\phi})] \leq C_1\|\mathbf{U}\|^{1/2}\|\mathbf{F}\|^{1/2}\|\mathbf{U}\| \\ 2\Im\int_\Gamma bT_{,i}\overline{v}_i dV \leq C_2\|\mathbf{U}\|\|\mathbf{F}\| \end{array} \tag{6.9}$$

Therefore, we obtain

$$\alpha \int_\Gamma \left[\lambda e_{rr}\overline{e}_{ss} + (\mu + \sigma)e_{ij}\overline{e}_{ij} + \mu e_{ij}\overline{e}_{ji} + \alpha\kappa_{rr}\overline{\kappa}_{ss} + \beta\kappa_{ij}\overline{\kappa}_{ji} + \gamma\kappa_{ij}\overline{\kappa}_{ij} + \rho v_i\overline{v}_i + J\varphi_i\overline{\varphi}_i + aT\overline{T}\right] dV \leq \int_\Gamma \left[\big(b^* - \frac{k^*}{T_0}\big)\epsilon_{ijk}\Im\kappa^*_{ki}\overline{T}_{,j}\right] dV + 2\Im\mathcal{M}[(\mathbf{v}, \boldsymbol{\varphi}), (\mathbf{u}, \boldsymbol{\phi})] + 2\Im\int_\Gamma bT_{,i}\overline{v}_i dV + C\|\mathbf{F}\|\|\mathbf{U}\|, \tag{6.10}$$

and, hence,

$$|\alpha|\|\mathbf{U}\|^2 \leq C_1\|\mathbf{U}\|^{3/2}\|\mathbf{F}\|^{1/2} + C_2^*\|\mathbf{U}\|\|\mathbf{F}\|. \tag{6.11}$$

From this inequality we get that

$$|\alpha|\|\mathbf{U}\| \leq C\|\mathbf{F}\|,$$

where $C > 0$ and α is sufficiently greater.

Theorem 6.4. *The semigroup generated by the operator $\mathcal{A}$ is analytic.*

Proof. Since $\mathcal{A}$ is the infinitesimal generator of a strongly continuous semigroup, $\mathbb{R}_+ \in \varrho(\mathcal{A})$ and as $0 \in \varrho(\mathcal{A})$, we have $i\mathbb{R} \subset \varrho(\mathcal{A})$. From Lemma (6.3) we have

$$\|\alpha(i\alpha\mathcal{I} - \mathcal{A})^{-1}\mathbf{F}\|_{\mathcal{Z}} = |\alpha|\|\mathbf{U}\|_{\mathcal{Z}} \leq C\|\mathbf{F}\|_{\mathcal{Z}}.$$

Then,

$$\overline{\lim}_{|\alpha|\to\infty}\|\alpha(i\alpha\mathcal{I} - \mathcal{A})^{-1})\| < \infty.$$

Corollary 6.5. *As a consequence of the analyticity, the system (2.4) is exponentially stable. Moreover, the system has a regularity effect in the sense that the solution $\mathbf{U} = (\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}, \boldsymbol{\varphi}, T)$ satisfies $\mathbf{U} \in C^\infty((0,t); \mathcal{D}(\mathcal{A}^\infty))$.*

However, $\mathcal{D}(\mathcal{A})$ is not necessarily a regular space. This implies that $\mathbf{U}$ is not in $C^\infty\big((0,t) \times \Gamma\big)$ when the initial data are not regular.

Another consequence of the analyticity of the solutions is the impossibility of localization. That means that the only solution that can be identically zero after a finite period of time is the null solution.

Corollary 6.6. *Let $\mathbf{U} = (\mathbf{u}, \mathbf{v}, \boldsymbol{\phi}, \boldsymbol{\varphi}, T)$ be a solution of the system (2.4) with initial conditions (2.3) and boundary conditions (2.2) such that $\mathbf{u} = \boldsymbol{\phi} = T \equiv 0$ after a finite time $t_0 > 0$. Then, $\mathbf{u} = \boldsymbol{\phi} = T \equiv 0$ for every $t \geq 0$.*

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