A Boolean Algebra Approach to the Construction of Snarks

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ABSTRACT

This work deals with the construction of snarks, that is, cubic graphs that cannot by 3–edge–colored. A natural generalization of the concept of "color", that describes in a simple way the coloring ("0" or "1") of any set of (semi)edges, is introduced. This approach allows us to apply the Boolean logic theory to find an ample family of snarks, which includes many of the previously known constructions and also some interesting new ones.

1. Introduction

Let G be a graph with maximum degree Δ. The *chromatic index* of G, denoted by $\chi'(G)$, is the minimum integer k such that G is k–edge–colorable. By the well–known theorem of Vizing [25],

$$\Delta \leq \chi'(G) \leq \Delta + 1.$$

When $\chi'(G) = \Delta$, the graph G is said to be of *class 1*. Otherwise, i.e. when $\chi'(G) = \Delta + 1$, G is said to be of *class* 2.

The term *Tait coloring* of G is used to mean a 3–edge–coloring of G when such a graph is cubic [24]. For a general textbook on edge–coloring we refer the reader to [11].

The main concern of this paper is the construction of snarks. Following [3], we define a *snark* as a cubic graph that cannot be Tait colored (i.e., with chromatic index 4). This name was proposed by M. Gardner [13] who borrowed it from the Lewis Carroll ballad *"The Hunting of the Snark"*. Usually, and in order to avoid "trivial cases", a class 2 cubic graph is called snark only if it is cyclically 4–edge–connected and with girth at least 5. See, for instance, [6], [18] or [26]. However, the decomposition results

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given in [3] and [15] showed that this notion of nontriviality may not be appropriate. Hence, we adopt the most simple definition given above. In the next section we will study this question in detail.

To the author's knowledge, the history of the hunt of (nontrivial) snarks may be summarized as follows. In 1973 only four snarks were known, the earliest one being the ubiquitous Petersen graph P [22]. The other three, on 18, 210, and 50 vertices, were found by D. Blanuša [2], B. Descartes [7] and G. Szekeres [23] respectively. Quoting A. Chetwynd and R. Wilson [6], "In 1975 the art of snark hunting underwent a dramatic change when R. Isaacs [18] described two infinite families of snarks." One of these families, called the *BDS class*, included all (three) snarks previously known. In fact, this family is based on a construction also discovered independently by G.M. Adelson–Velski and A. Titov in [1]. The members of the other family are the so–called *flower snarks*. They were also found independently by Grinberg in 1972, although he never published his work.

In [20], Jakobsen proposed a method, based on the well–known Hajós–union [16], to construct class 2 graphs. As it was pointed out by M.K. Goldberg in [15], some snarks of the BDS class can also be obtained by using this approach.

Later, R. Isaacs [19] described two new infinite sets of snarks found by F. Loupekine.

In [8], the author proposed a new method of generating snarks, based on Boolean algebra. This method led to a new characterization of the BDS class and also to a significant enlargement of it. For instance, Loupekhine's graphs [19] and most of the Goldberg's snarks [14], [15] can be viewed as members of this class. This paper is mainly devoted to the study of such a method.

In [8], infinitely many snarks of another family, called by R. Isaacs the *Q class*, were also given. Apart from the Petersen graph P and the flower snark J_5, in [18] Isaacs had given a further snark of this class: the *double star graph*. The graphs of this class are all cyclically 5–edge–connected. Recently, P. Cameron, A.G. Chetwynd and J.J. Watkins [3] gave a method to construct new snarks belonging to such a family.

Other constructions of snarks, most of them belonging to the BDS class, have been proposed by several authors. See, for instance, the papers of U.A. Celmins and E.R. Swart [5], J.L. Fouquet, J.L. Jolivet and M. Rivière [12], and J.J. Watkins [26].

2. Multipoles

In the study of snarks it is useful to think of them as made up by joining two or more graphs with "dangling edges". We call these graphs multipoles. More precisely, a *multipole* or *m–pole* $Gp = (V, E, \mathcal{X})$ consists of a (finite) set of vertices $V = V(Gp)$, a set of edges $E = E(Gp)$ or unordered pair of vertices, and a set $\mathcal{X} = \mathcal{X}(Gp)$, $|\mathcal{X}| = m$, whose elements χ_i are called *semiedges*. Each semiedge is associated either with one vertex or with another semiedge making up what we call an *isolated edge*. An example of multipole is depicted in Fig. 1a. Notice that, according to our definition, a multipole can be disconnected or even be "empty" in the sense that it can have no vertices. The diagram of a generic m–pole will be as shown in Fig. 1b.

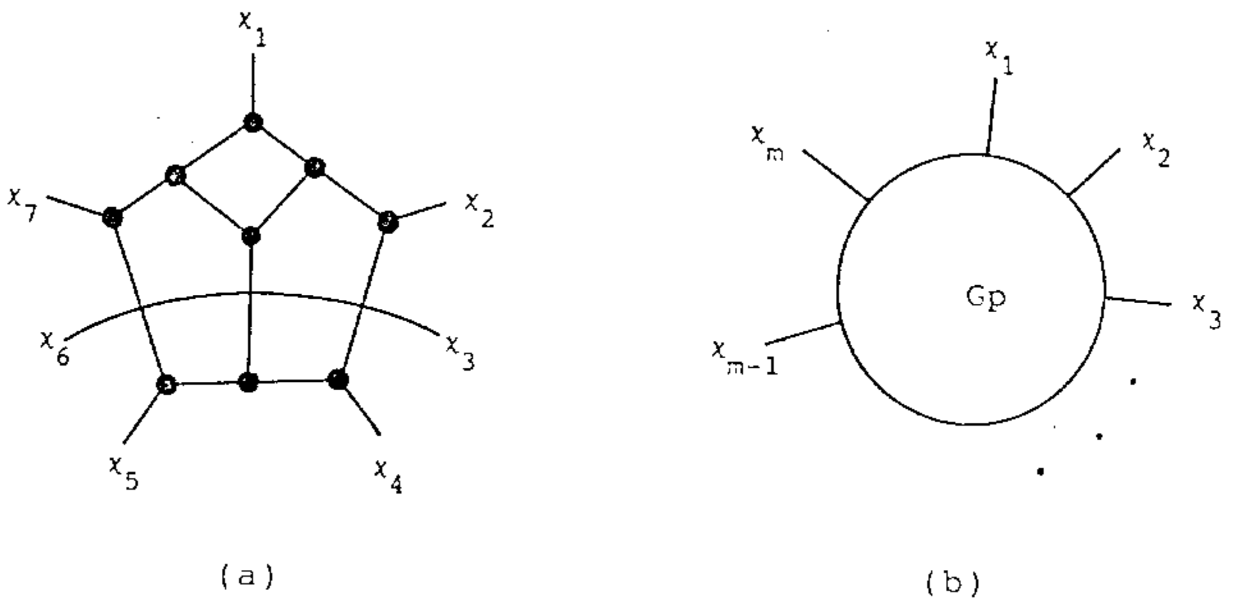


Figure 1

The behavior of the semiedges is as expected. For instance, if the semiedge χ is associated with vertex u, we say that χ is *incident* to u. Then we write $\chi = (u)$ following Goldberg's notation [15]. By joining the semiedges (u) and (v) we obtain the edge (u, v). The *degree* of u, d(u) is defined as the number of edges plus the number of semiedges incident to it. Throughout this paper, a multipole will be supposed to by cubic, i.e. $d(u) = 3$ for all $u \in V$.

Given a multipole Gp, we denote by Gp* the graph (with maximum degree 3) obtained from Gp by leaving out all its semiedges. Then, Gp is said to be *contained* in a cubic graph G if Gp* is a (proper) subdigraph of G. Notice that, in this case, Gp can be thought of as being obtained from G by cutting (in one or more points) some of its edges.

Let $C = \{1, 2, 3\}$ be a set of "colors". A *Tait coloring* of a m–pole $(V, E, \mathcal{X})$ is an assignment of colors to its edges and semiedges, i.e. a mapping $\phi : E \cup \mathcal{X} \to C$, such that in each vertex incide edges and/or semiedges with different color and each isolated edge has both semiedges of the same color. For example, Fig. 2 shows a Tait coloring of the 7–pole of Fig. 1a. Note that the numbers of semiedges with equal color have the same parity. The following basic lemma states that this is always the case.

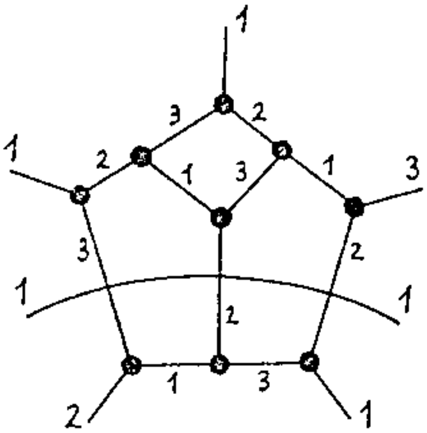


Figure 2

The Parity Lemma *Let m_i be the number of semiedges with color i, i = 1, 2, 3, in a Tait colored m–pole. Then*

$$m_1 \equiv m_2 \equiv m_3 \equiv m \pmod 2). \tag{1}$$

This result has been used extensively in the literature on the subject. See for instance [2], [7], [15] or [18]. Although in these references isolated edges are not allowed, the proof is basically the same and hence we refer the reader to them.

Given a m–pole Gp with semiedges $\chi_1, ..., \chi_m$, we define its *set C(Gp). of semiedge colorings* as

$$C(Gp) = \{(\phi(\chi_1), \phi(\chi_2), ... \phi(\chi_m)) : \phi \text{ is a Tait coloring of } Gp\}.$$

Note that C(Gp) depends on the order in which the semiedges are considered. Thus, when referring to such a set we will implicitly assume that this ordering is given.

Of course, $C(Gp) = \varnothing$ iff Gp is not Tait colorable. In this case it is trivial to obtain a class 2 graph from Gp. Indeed, we can either remove all its semiedges or join them properly in order to achieve regularity (using additional vertices if necessary). By the parity lemma, the simplest example of non–Tait–colorable m–pole is when m = 1, so that any cubic graph with a bridge is trivially of class 2.

In the other extreme, we will say that Gp is *c–complete* if C(Gp) has maximum cardinality. In other words, Gp is c–complete if it can be Tait colored so that its semiedges have any combination of colors satisfying the parity lemma. For instance, all Tait colorable 2–poles and 3–poles are c–complete because, according to (1), the only possibilities, up to permutation of the colors, are $(\phi(\chi_1), \phi(\chi_2)) = (a, a)$ and $(\phi(\chi_1), \phi(\chi_2), \phi(\chi_3)) = (a, b, c)$ respectively —here, and henceforth, the letters a, b, c stand for the colors 1, 2, 3 in any order. Clearly, the simplest c–complete 2–pole and 3–pole are respectively an isolated edge and a single vertex with 3 semiedges incident to it. They will be denoted by **e** and **v** respectively. On the other hand, a c–complete

4–pole has four different values of $(\phi(x_1), \phi(x_2), \phi(x_3), \phi(x_4))$— namely, (a, a, a, a), (a, a, b, b), (a, b, a, b) and (a, b, b, a). In general, a c–complete multipole will be denoted by **Z**.

Other useful definitions related with the semiedge coloring set follow. Let Gp_1 and Gp_2 be two m–poles. Then:

Gp_1 and Gp_2 are said to be *c–equivalent* if $C(Gp_1) = C(Gp_2)$;

Gp_1 is said to be *c–contained* in Gp_2 if $C(Gp_1) \subseteq C(Gp_2)$;

A multipole Gp is said to be *c–reducible* if there exists another multipole Gp', c-contained in Gp, such that $V(Gp') < V(Gp)$. In such a case we also say that Gp is *c–reducible to Gp'*.

Let us assume that a snark U contains the multipole Gp which c–contains the multipole Gp'. Then, it is clear that Gp can be replaced by Gp' —in the obvious way— without affecting the non–Tait–colorability of the resulting graph. Moreover, if Gp is c–reducible to Gp', such a graph will have fewer vertices than U.

By the above, the conditions of nontriviality for snarks, given in the Introduction, are a consequence of the following statements:

A1 Any 1–pole is not Tait colorable.

A2 Any 2–pole different from e is either not Tait colorable or c–reducible to e.

A3 Any 3–pole different from v is either not Tait colorable or c–reducible to v.

As A square, i.e. a 4–cycle with one semiedge incident to each vertex, is c–reducible to two parallel isolated edges (the cyclic orderings of the semiedges being induced by the drawings).

Let U denote a snark. M.K. Goldberg [15] and P.J. Cameron, A.G. Chetwynd and J.J. Watkins [3] implicitly proved the following result.

A4 Any 4–pole Gp contained in U with $V(Gp) > 2$ is either not Tait colorable or c–reducible.

In the latter paper the following result was also proved.

A5 Any 5–pole contained in U with $V(Gp) > 5$ is either not Tait colorable or c-reducible.

In general, since the number of possible semiedge colorings $(\phi(x_1), \phi(x_2), ..., \phi(x_m))$ is finite, there exists a positive integer–valued function $v(m)$ such that the following result holds.

Am Any m–pole Gp contained in U with $V(Gp) > v(m)$ is either not Tait colorable or c–reducible.

The exact value of $v(m)$ is unknown for $m \geq 6$. Even so, the above statement shows that any snark U with a cutset of m edges and $V(U) > 2v(m)$ can be "reduced" to another snark with fewer vertices. See [3] for the cases $m = 4, 5$.

Let Gp_1 and Gp_2 be two m–poles with semiedges x_i and y_i, $i = 1, 2, ..., m$, respectively, and assume that by joining x_i with y_i for all i we obtain the cubic graph G. Then we will say that Gp_1 and Gp_2 are *complementary (with respect to G)*, or that Gp_2 is the *complement of* Gp_1, written $Gp_2 = Gp'_1$.

On the other hand, the m–poles Gp_1 and Gp_2 are said to be *c–disjoint* if $C(Gp_1) \cap C(Gp_2) = \emptyset$. In particular that is the case when one of the m–poles is not Tait colorable.

The analysis and synthesis of snarks is based on the following straightforward result.

Proposition *Let Gp and Gp' be two complementary multipoles of graph G. Then G is a snark iff Gp and Gp' are c–disjoint.*

Thus, the problem of constructing snarks can be reduced to the problem of finding pairs of c–disjoint multipoles. The main problem to proceed in this way is that, when the number of semiedges increases, the characterization of the set C(Gp) becomes more and more difficult. To overcome this drawback the idea is to group the semiedges in different sets and give a proper characterization of the "global" coloring of their elements, as we do in the next section.

3. Multisets and Boole Colorings

A *mulitset* or *n–set* is simply an n'–pole Gp whose n' semiedges are grouped in $n \leq n'$ sets, say $\mathcal{X}_i$, $i = 1, 2, ..., n$, with $n_i = |\mathcal{X}_i|$. See Fig. 3.

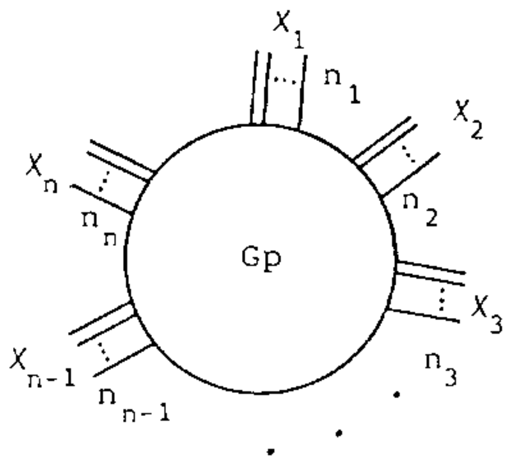


Figure 3

The main concern of this section is to characterize in a useful and simple way the coloring of the sets $\mathcal{X}_i$ for a given Tait coloring ϕ of Gp. To this end, let $\mathcal{X}$ be a generic set of m semiedges, m_i of which have color i, $i = 1, 2, 3$. Then, depending upon the parity of these numbers, we basically distinguish two cases:

Case 0 *They have the same parity:*

We say that the set $\mathcal{X}$ has *Boole coloring* 0, denoted by $\phi(\mathcal{X}) = X = 0$, iff

$$m_1 \equiv m_2 \equiv m_3 \equiv m \pmod 2). \tag{2}$$

Case 1 *They have different parity:*

We say that the set $\mathcal{X}$ has *Boole coloring* 1 (or, more specifically, 1_a), denoted by $\phi(\mathcal{X}) = X = 1(1_a)$, iff

$$m_a + 1 \equiv m_b \equiv m_c \equiv m + 1 \pmod 2). \tag{3}$$

This characterization (as well as a generalization of it involving more than 3 colors) was introduced in [8] to construct snarks. Other related applications are discussed in [9] and [10].

Note that if $\mathcal{X}$ coincides with the semiedge set of a Tait colored multipole (1–set), the parity lemma holds and hence $X = 0$.

Clearly, the above definitions can also be used to characterize the Boole coloring of any 3–colored set of edges.

By way of example, Table 1 shows the Boole coloring of $\mathcal{X}$ for different values of the coloring–vector (m_1, m_2, m_3) when $1 \leq m \leq 3$.

Boole coloring X	coloring-vector (m_1, m_2, m_3) $m = 1$	$m = 2$	$m=3$
0		(2, 0, 0) (0, 2, 0) (0, 0, 2)	(1, 1, 1)
1_1	(1,0,0)	(0,1,1)	(1,0,2) (1,2,0)
1_2	(0,1,0)	(1,0,1)	(0,1,2) (2,1,0)
1_3	(0,0,1)	(1,1,0)	(0,2,1) (2,0,1)

Table 1

It is interesting to note the following remarks:

B 1 When $m = 1$, the only possible Boole coloring of (the semiedge of) $\mathcal{X}$ is 1. Moreover, $X = 1_i$ iff such a semiedge has color i, $i = 1, 2, 3$.

B 2 When $m = 2$, we have a Boole coloring 0 (resp. 1) iff the two semiedges of $\mathcal{X}$ have the same (resp. different) color. This characterization was independently used by M. Goldberg [14], [15]; and by I. Holyer [17] whose values T ("true") and F ("false") correspond to our 0 and 1 respectively. More specifically, note that the Boole coloring is 1_a iff the missing color is a.

B 3 When $m = 3$, the Boole coloring of $\mathcal{X}$ is 0 iff the three colors of the semiedges are all different. Thus, we can say that a cubic graph is Tait colored iff the set of edges incident to each vertex has Boole coloring 0. Otherwise, if two semiedges have the same color, the Boole coloring of $\mathcal{X}$ is 1_a where a is the color of the third semiedge. An equivalent characterization, but without using the 1's, was used by B. Descartes in [7] to construct his graph.

A natural definition of the sum of Boole colorings is now the following. Let $\mathcal{X}$ and $\mathcal{Y}$ be two sets of semiedges with Boole colorings X and Y respectively. Then we define the sum X + Y as the Boole coloring that, according to (2) and (3), corresponds to the set $\mathcal{X} \cup \mathcal{Y}$. It is very easy to check that this definition leads to Table 2, so that we obtain the "Klein group of Boole colorings".

+	0	1_1	1_2	1_3
0	0	1_1	1_2	1_3
1_1	1_1	0	1_3	1_2
1_2	1_2	1_3	0	1_1
1_3	1_3	1_2	1_1	0

Table 2

Note that, as each element coincides with its inverse, $m1_i = 1_i + \ldots + 1_i$ is 0 when m is even and 1_i when m is odd. The following result, based on this simple fact, is of fundamental importance in our study. Because of **B1**, it can be seen as a generalization of the parity lemma.

Lemma 1 *Let Gp be a Tait colored n–set with $m_i \geq 0$ sets of semiedges having Boole coloring 1_i, $i = 1, 2, 3$, $m_1 + m_2 + m_3 = m \leq n$. Then,*

$$m_1 \equiv m_2 \equiv m_3 \equiv m \pmod 2.$$

Proof As stated before, the Boole coloring of the whole set of semiedges of Gp must be 0. So we have

$$\sum_{i=1}^{3} m_i 1_i + (n - m)0 = \sum_{i=1}^{3} m_i 1_i = 0.$$

But this equality only holds if either $m_i 1_i = 0$ or $m_i 1_i = 1_i$ for all i. Since $m_1 + m_2 + m_3 = m$, the lemma follows. ❑

Table 3 shows the feasible Boole colorings of (the semiedge sets of) a Tait colored n–set, $1 \leq n \leq 3$, according to the above result.

n	X_1	X_2	X_3
1	0		
2	0	0	
	1_a	1_a	
3	0	0	0
	0	1_a	1_a
	1_a	0	1_a
	1_a	1_a	0
	1_a	1_b	1_c

Table 3

Notice that, as pointed out before, if $n = 1$ it must be $X_1 = 0$, which is just a reformulation of the parity lemma.

Leaving out the subindexes of the Boole colorings 1, the entries in Table 3 can be thought of as being the possible values of the "logic variables" X_i. This suggests the possibility of using Boolean algebra in order to characterize the possible Boole colorings of a given multiset.

For any value of n we have the following corollary of Lemma 1.

Corollary 1 *A Tait colored n–set, $n \geq 1$, cannot have only one set with Boole coloring 1, i.e., $X_j = 1$ and $X_i = 0$ for all $i \neq j$.*

Given an n–set Gp with semiedge sets $\mathcal{X}_j$, $j = 1, 2, ..., n$, we define its *set B(Gp) of Boole coloring vectors*, or simply *Boole colorings*, as

$$B(Gp) = \{(\phi(\mathcal{X}_1), \phi(\mathcal{X}_2), ..., \phi(\mathcal{X}_n)) : \phi \text{ is a Tait coloring of } Gp\}.$$

Of course, this set depends upon the subindexes of the Boole colorings 1 begin considered or not, but that will be either immaterial or clear from the context. Hence the unified notation.

Most of the definitions and remarks involving the semiedge coloring set C(Gp) apply also to the set B(Gp) with minor and trivial changes. For instance, if B(Gp) has maximum cardinality, the multiset Gp is said to be c–complete. It is readily seen that if a multipole is c–complete any multiset obtained from it —with sets of at least two semiedges— is c–complete (considering subindexes or not). However, the converse does not hold in general.

Let us now consider a family of multisets which are joined in such a way that each set $\mathcal{X}_i$ of n_i semiedges is joined to exactly one set $\mathcal{X}_j$ of n_j semiedges, so making up a set of edges that we denote by $(\mathcal{X}_i, \mathcal{X}_j)$ —for simplicity we can now assume that $n_i = n_j$, but later we shall see that a junction can be easily done in general. If the pairs of semiedges to be joined are not specified, this structure, which will be called a *logic network*, represents a family of cubic graphs.

Let $B = \{0, 1_1, 1_2, 1_3\}$ or $\{0, 1\}$, depending upon the case, be a set of "Boole colors". Then, a *Boole coloring* of a logic network is defined to be an assignment of Boole colors to its edge sets $(\mathcal{X}_i, \mathcal{X}_j)$ such that the induced Boole coloring of each multiset Gp belongs to B(Gp).

Obviously, the same ideas above apply to the construction, from some multisets, of a *"logic multiset"*, as well as to its Boole colorability. Besides, note that from a non–Boole–colorable (logic) multiset we can readily obtain a non–Boole–colorable logic network.

In this context, it is clear that if a logic network is not Boole colorable, its "underlying" graphs are not Tait colorable. So, we will manage to construct snarks if we know how to construct such logic networks.

The most obvious way to construct a non–Boole–colorable logic network is by joining two c–disjoint (in terms of Boole colorings) multisets. Some methods to obtain them are explained in the last section, but first we need to consider some useful configurations which are the concern of the next section.

4. Multisets and Boolean Algebra

As said before, in our study we make ample use of a (2–valued) Boolean algebra. So, we have a set $B = \{0, 1\}$ jointly with two binary opreations, + and $\cdot$, called *logic sum* and *logic product* respectively, satisfying some well–known axioms. It will be clear from the context when "+" denotes logic sum or sum of Boole colorings. Given $x \in B$, we use $\overline{x}$ to mean the complement of x, i.e., $\overline{0} = 1$ and $\overline{1} = 0$. The remaining notation used below is, I hope, self–explanatory.

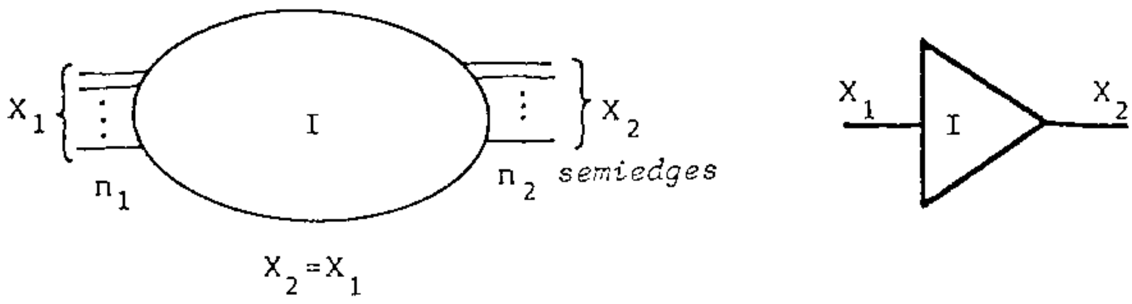
The Identity Operator

According to Table 3, the Boole coloring (X_1, X_2) of a Tait colorable 2–set must be $(0, 0)$ or $(1, 1)$. Therefore we can write

$$X_2 = X_1,$$

which corresponds to the *identity* logic function.

Then a 2–set will allow us to join two sets with any number, say n_1 and n_2, of semiedges without changing their Boole colorings, see Fig. 4a. The symbol used for this "operator" is shown in Fig. 4b, where bold lines represent semiedge sets.



(a) (b)

Figure 4

In particular, when one set of a Tait colored 2–set, say $\mathcal{X}_1$, has only one semiedge (with color a) we will have, by **B1**, $X_2 = X_1 = 1$ (1_a).

The Truth Operator

A method to obtain a 2–set with the only possible Boole coloring $(X_1, X_2) = (1, 1)$ —and sets with more than one semiedge— is the following. Assume that a snark U contains the multipole (2–set) shown in Fig. 5a jointly with its only possible Boole coloring, and denoted by $\{\mathbf{Z}_1, \mathbf{Z}_2\}$ or, simply, $\{\mathbf{Z}, \mathbf{Z}\}$. Then it is clear that its complement, $\{\mathbf{Z}, \mathbf{Z}\}'$, see Fig. 5b, cannot have the Boole coloring (0, 0). Otherwise, since $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are c–complete, the colorings of the semiedges of $\mathcal{X}_1$ and $\mathcal{X}_2$ would be elements of $C(\mathbf{Z}_1)$ and $C(\mathbf{Z}_2)$ repsectively, giving in this way a Tait coloring of U. Hence, the 2–set $\{\mathbf{Z}, \mathbf{Z}\}'$ can only have the Boole coloring

$$X_1 = X_2 = 1,$$

which corresponds to the *"truth"* logic function (with 2 variables). Fig. 5c shows the symbol used for such a 2–set.

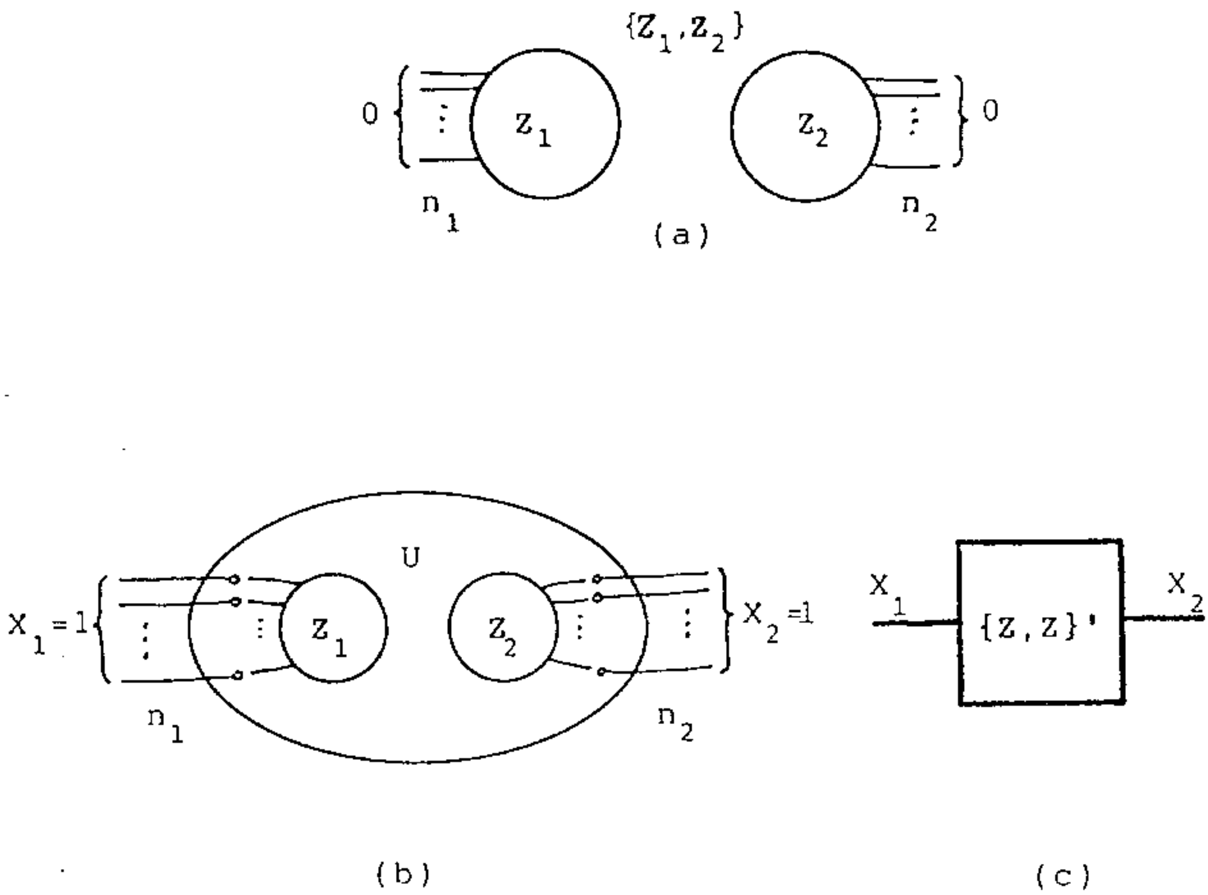


Figure 5

In particular, the simplest cases are obtained when the Z_i's are e ($n_i = 2$) or v ($n_i = 3$). For instance, the case $Z_1 = Z_2 = e$ was considered by R. Isaacs in [18] and by G.M. Adelson–Velski and A. Titov in [1]; and the case $Z_1 = Z_2 = v$, $U = P$, was used by B. Descartes in [7] to obtain his graph. In the next section we will discuss the case $n_i \geq 4$.

The Untruth Operator

Let us now consider a snark U containing the 2–set shown in Fig. 6a, that we denote by {ZeZ}. Then, since the Boole coloring of the edge e must be 1, its only possible Boole coloring is $(X_1, X_2) = (1, 1)$. Hence, reasoning as in the preceding subsection, we conclude that the 2–set {ZeZ}', shown in Fig. 6b, can only have the Boole coloring

$$X_1 = X_2 = 0,$$

which corresponds to the *"untruth"* logic function (with 2 variables). The symbol used for this 2–set is shown in Fig. 6c.

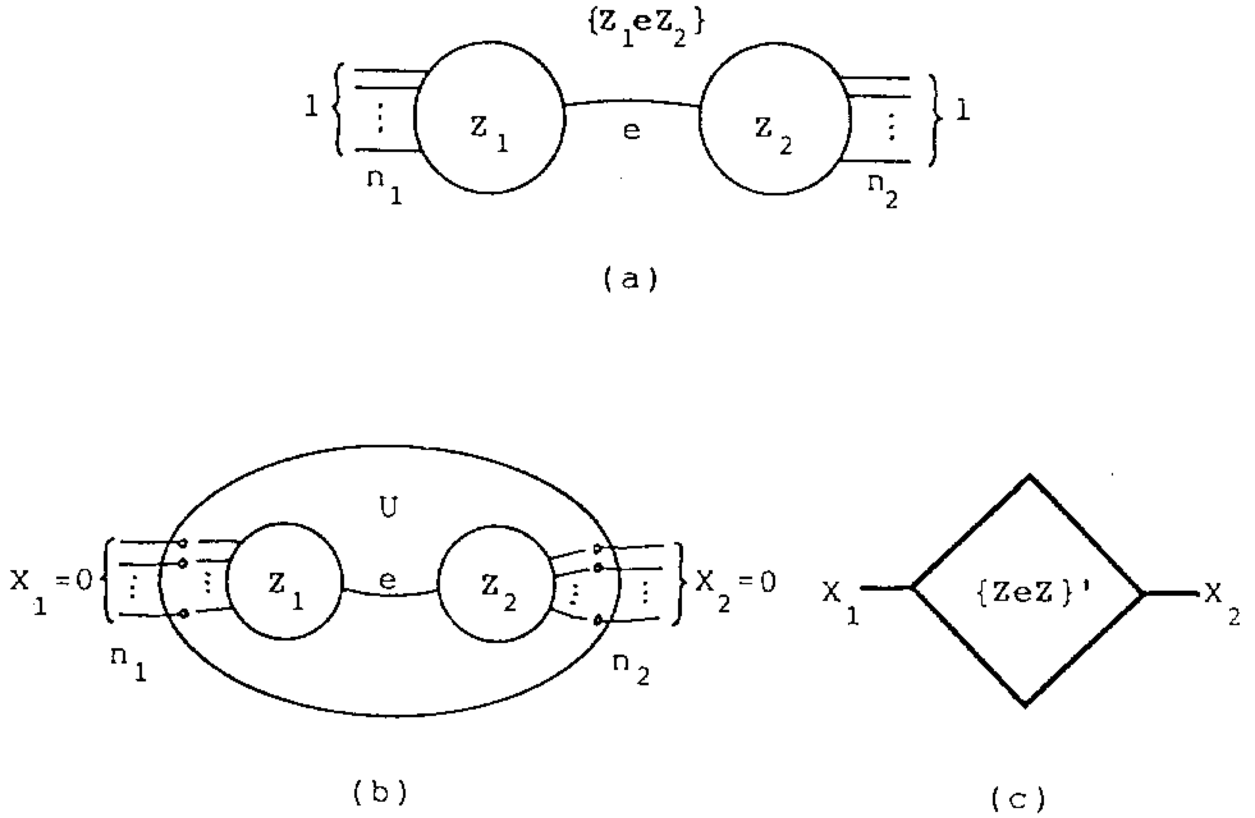


Figure 6

Apart from the trivial cases $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{e}$ and $\mathbf{Z}_1 = \mathbf{e}, \mathbf{Z}_2 = \mathbf{v}$, the most simple 2-set of this type is obtained when $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{v}$ $(n_1 = n_2 = 2)$, which was also dealt with in the above references [1] and [18].

The Or and Exclusive-Or Operators

Reasoning as above we can obtain 3–sets with only some of the possible Boole colorings shown in Table 3. As before, all these configurations are derived by taking the complement, with respect to a snark U, of a suitable multiset made up by some c-complete multipoles. The 3–sets thus obtained are shown in Table 4 together with their truth table, the corresponding logic function if any, and the symbol —usually borrowed from logic circuit theory— we use to represent them.

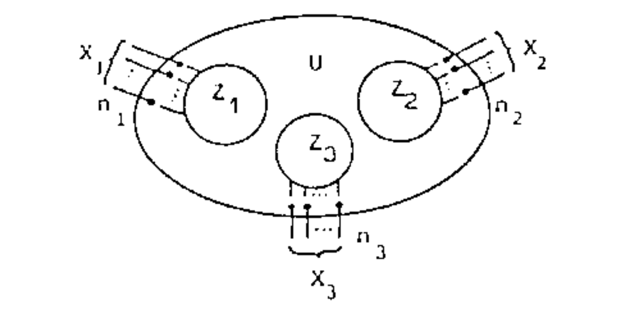
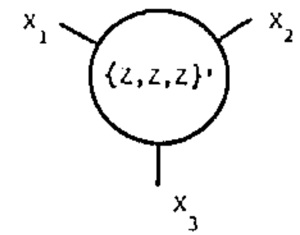
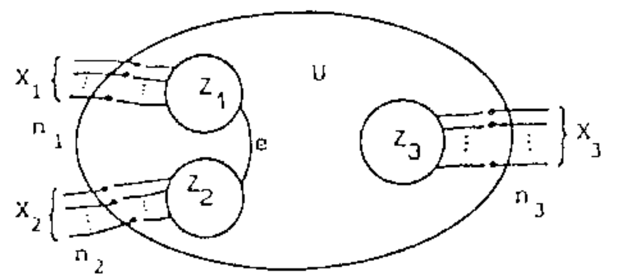
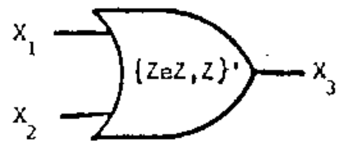
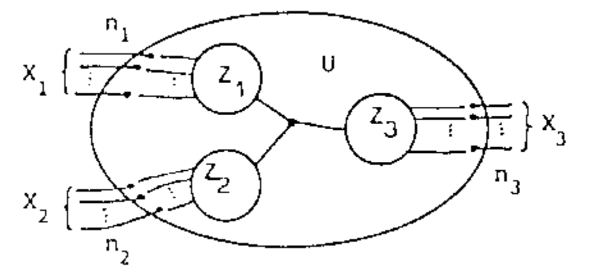
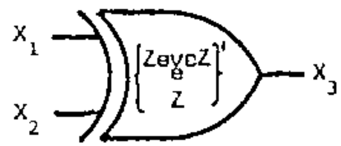
3-set	Truth table	Logic function	Symbol
	X_1 X_2 X_3 0 1 1 1 0 1 1 1 0 1 1 1	---	
	X_1 X_2 X_3 0 0 0 0 1 1 1 0 1 1 1 1	$X_3 = X_1 + X_2$ or	
	X_1 X_2 X_3 0 0 0 0 1 1 1 0 1 1 1 0	$X_3 = X_1\bar{X}_2 + \bar{X}_1 X_2 =$ $X_1 \oplus X_2$ exclusive-or	

Table 4

From these configurations, many particular cases may be derived. For instance, the *exclusive–or* 3–set obtained by taking $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathbf{Z}_3 = \mathrm{v}$ $(n_1 = n_2 = n_3 = 2)$ was independently considered in [15] and [5]. In [15], M.K. Goldberg proposed it as an example of *"even cell"*, i.e. a multiset with exactly two semiedges in each set and all Boole coloring vectors having an even number of 1's, see Section 5.

It should be pointed out that, for each general configuration, the truth table gives only its possible Boole colorings, Thus, in some particular cases the multiset we obtain may have not all of such colorings, or even it may have none of them (if it is not Tait colorable). For instance, when some $\mathcal{X}_i$ has only one element the value of X_i cannot be 0, making some Boole colorings in the truth table to be not possible. The two following particular cases of the *or* and *exclusive–or* operators are based on this fact.

The Not Operator

If in the 3–set $\left\{\begin{matrix}\mathrm{ZeveZ} \\ \mathbf{e} \\ \mathrm{Z}\end{matrix}\right\}'$ we consider a c–completé multipole, say Z_3, equal to e, we obtain the structure shown in Table 5 jointly with its truth table and symbol (*"not* gate" with an additional semiedge x). Notice that, disregarding X_3 (= 1), the other variables satisfy

$$X_2 = \overline{X}_1,$$

which corresponds to the *not* logic function.

By Corollary 1, we know that there is no 2–set with the Boole coloring (0, 1) or (1, 0). Hence, our *not* operator must have an additional semiedge —or semiedge set— whose Boole coloring is, in general, immaterial. For this reason it will be referred as a *neutral* semiedge.

The simplest particular case $\mathbf{Z}_1 = \mathbf{Z}_2 = \mathrm{v}$ $(n_1 = n_2 = 2)$, denoted by $\{\mathrm{vevev}\}'$, is the basic structure (5–pole) of Loupekine's snarks [19].

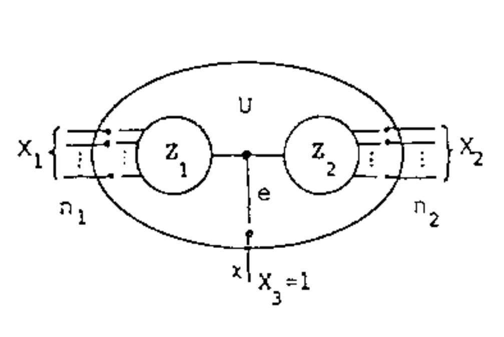
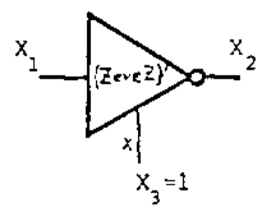
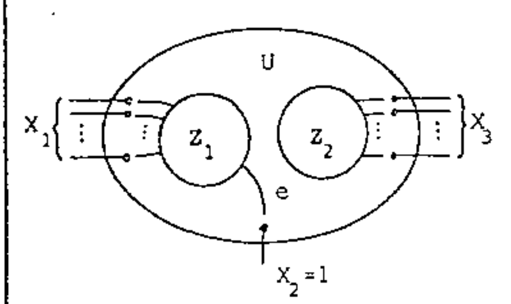
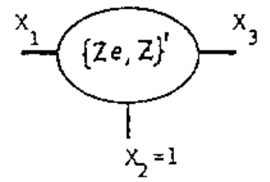
3-set	Truth table	Logic function	Symbol
	X_1 X_2 0 1 1 0	$X_2 = \overline{X}_1$ not	
	X_1 X_2 X_3 0 1_a 1_a 1_a 1_b 1_c	--- ("partial-or")	

Table 5

The Partial-Or Operator

This 3-set is obtained by considering the *or* operator $\{Z_1eZ_2, Z_3\}'$ with $Z_2 = e$. It is also shown in Table 5 together with its symbol and truth table where the subindexes corresponding to the 1's have been explicitly written.

As a summary, Table 6 shows some particular cases of the above multisets when $U = P$.

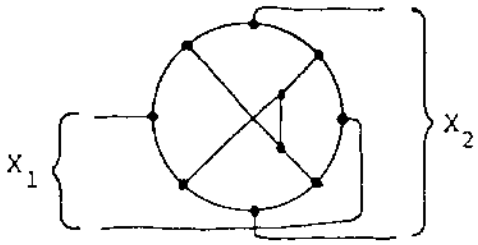
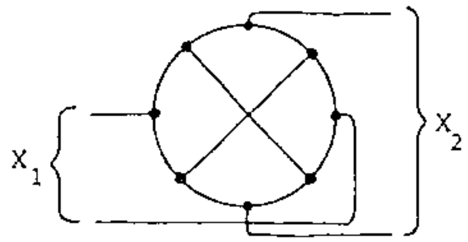
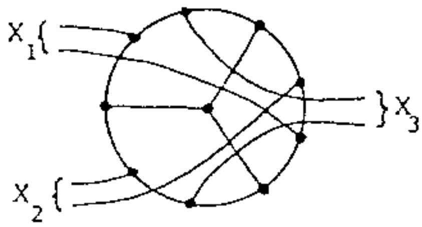
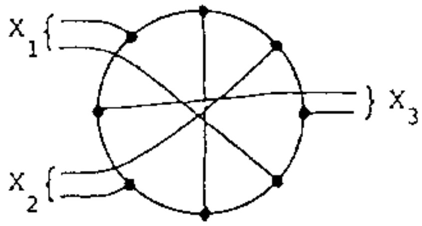
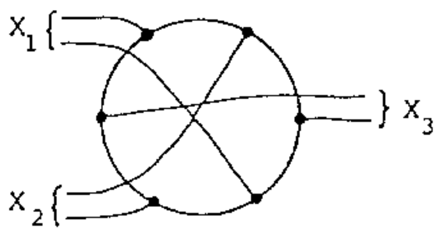
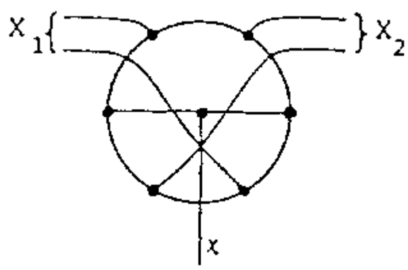
Type	Multiset	Function
$\{e,e\}'$		$x_1 = x_2 = 1$
$\{vev\}'$		$x_1 = x_2 = 0$
$\{e,e,e\}'$		—
$\{vev,e\}'$		$x_3 = x_1 + x_2$
$\left\{\begin{smallmatrix}vevev\\ e\\ v\end{smallmatrix}\right\}'$		$x_3 = x_1 \oplus x_2$
$\{vevev\}'$		$x_2 = \bar{x}_1$

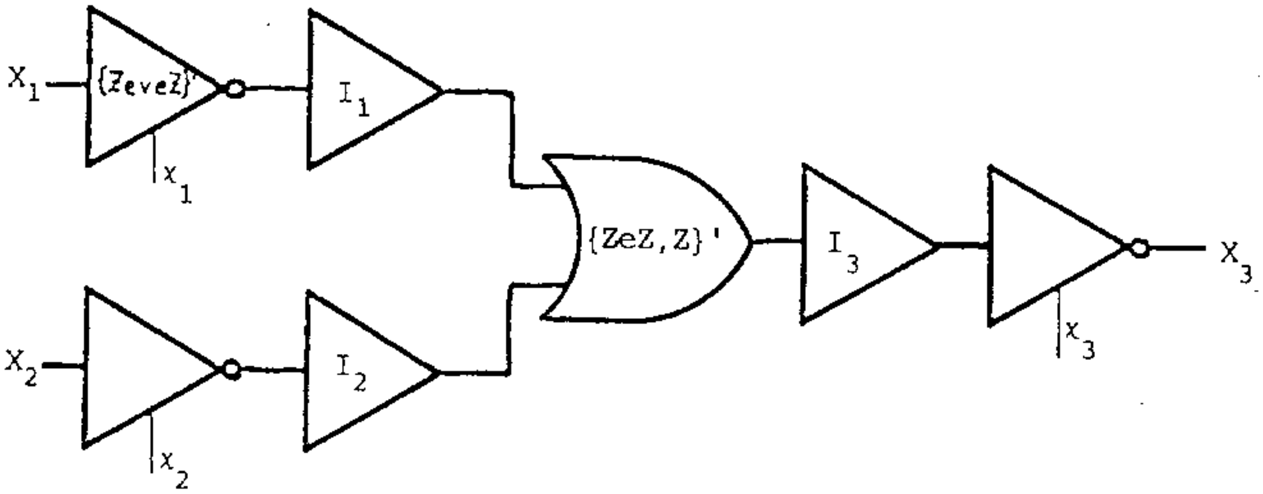
Table 6

Other Operators

From the logic operators obtained before it is now easy to construct multisets associated with other basic logic functions. For instance, by De Morgan's laws the *and* function can be written as

$$X_3 = X_1X_2 = \overline{\overline{X_1X_2}} = \overline{\overline{X}_1 + \overline{X}_2},$$

which lead us to the structure shown in Fig. 7 jointly with its truth table.



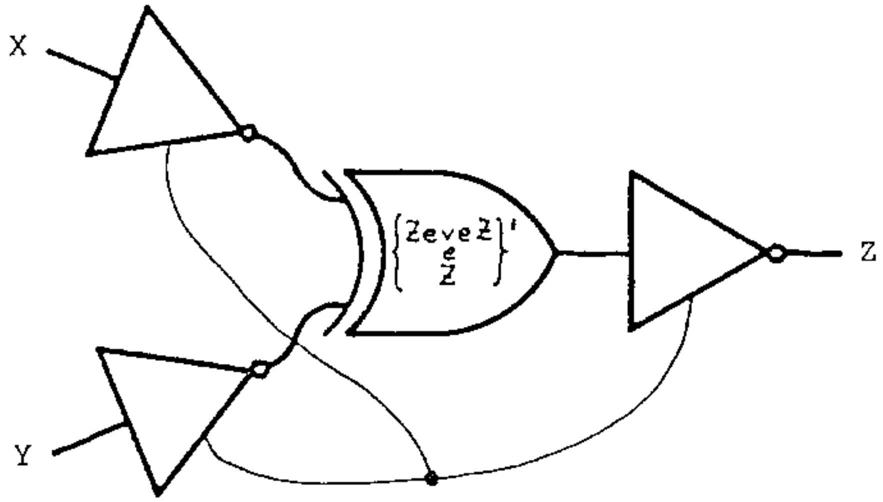
X_1	X_2	X_3
0	0	0
0	1	0
1	0	0
1	1	1

Figure 7

Notice the presence of the 2–sets I_i which, as said before, are used to join two sets with different numbers of semiedges, "transferring" the Boole coloring from one to the other. As the presence of such 2–sets is obvious, we will not draw them henceforth.

When the neutral semiedges x_1, x_2 and x_3 are joined to a common vertex, we obtain a 3–set which, by Corollary 1, cannot have the Boole colorings (0, 1, 0) and (1, 0, 0). Then, since the remaining Boole colorings satisfy $X_1 = X_2 = X_3$ (*identity* operator with 3 variables), we call this configuration a *meeting point*. Notice that it might also be obtained by using other 3–sets (e.g. a c–complete 3–set) instead of the *or* operator.

Besides, if in such a structure we replace the *or* operator by the *exclusive–or* operator, we obtain a "*truth* operator with 3 variables", see Fig. 8. Note the interesting "color isomorphism" $i \equiv 1_i$, $i = 1, 2, 3$, between this 3–set and the most simple 3-pole v.



X	Y	Z
1_a	1_b	1_c

Figure 8

Actually, because of Corollary 1, it is readily seen that our *not* operator is "functionally complete", that is, any other logic operator can be derived from it. For instance, Table 7 shows how we can obtain the *exclusive–or* and *equivalence* operators, as well as other "3–sets" (disregarding neutral semiedges) whose truth table does not correspond to any logic function. Note that in obtaining such an *exclusive–or* operator, A may be any c–complete 3–set. Then, when the most simple of them —made up by 3 isolated edges— is used, and the *not* operators are of type {vevev}', the 3–set of Fig. 9 appears.

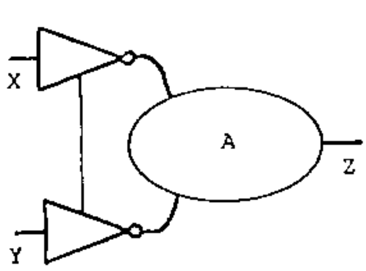
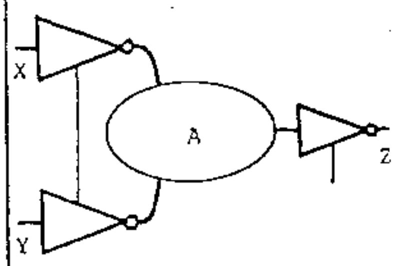
Diagram \ A	c-complete 3-set	$\{Z_1 e Z_2, Z_3\}'$ or operator	$\{Z_1, Z_2, Z_3\}'$
	X Y Z 0 0 0 0 1 1 1 0 1 1 1 0 exclusive-or	X Y Z 0 1 1 1 0 1 1 1 0 —	X Y Z 0 0 0 0 1 1 1 0 1 —
	X Y Z 0 0 1 0 1 0 1 0 1 1 1 1 equivalence	X Y Z 0 1 0 1 0 0 1 1 1	X Y Z 0 0 1 0 1 0 1 0 0

Table 7

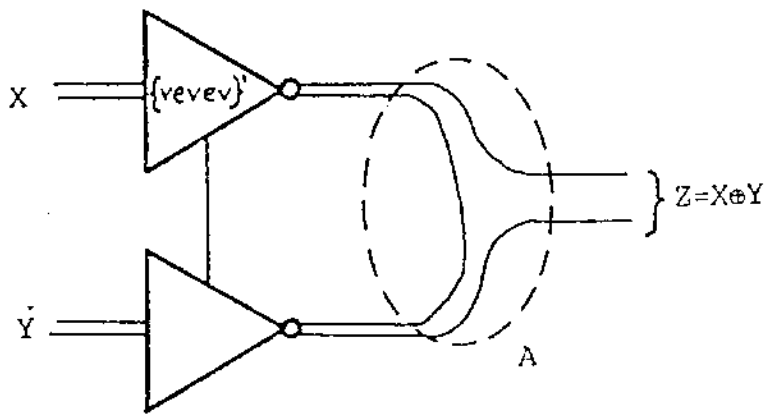


Figure 9

In order to construct a meeting point with more than 3 variables, we use a 3–set made up by joining "in series" an even number p of *not* operators and grouping all their neutral semiedges x_i, $1 \leq i \leq p$, into a set W, see Fig. 10a, its symbol being shown in Fig. 10b. From Corollary 1 and the parity of p it is trivial to check that $X = 0 \Rightarrow Y = W = 0$; $Y = 0 \Rightarrow X = W = 0$; and $W = 1 \Rightarrow X = Y = 1$. Hence it follows that the n–set shown in Fig. 10c has only the Boole colorings $(0, 0, ..., 0)$ or $(1, 1, ..., 1)$ as claimed.

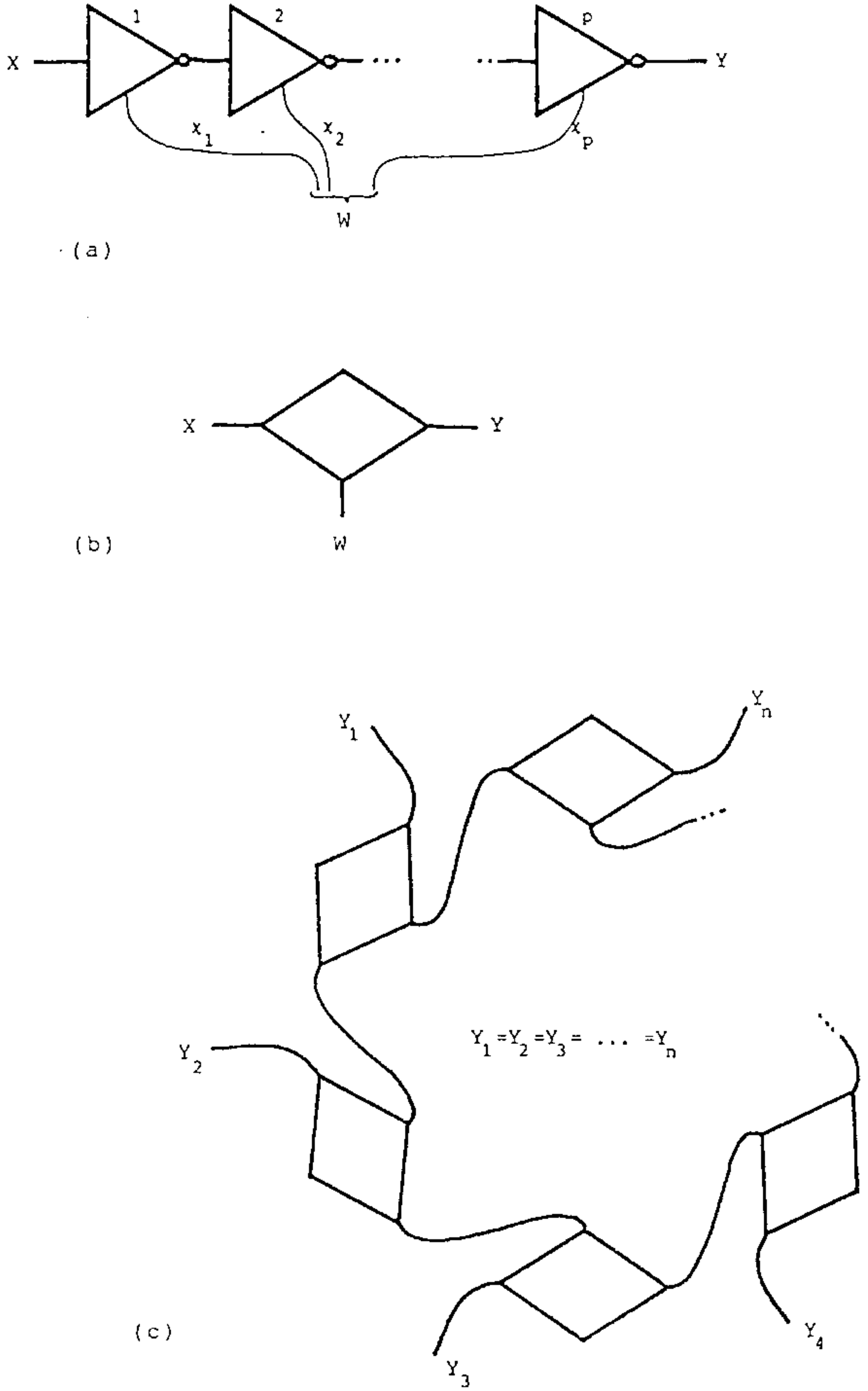
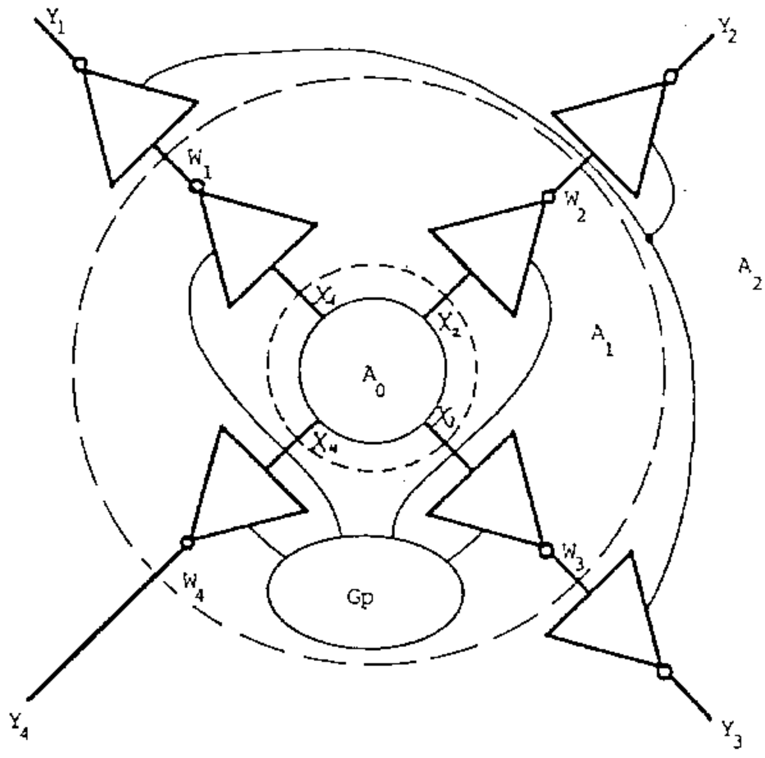


Figure 10

A particular case of this configuration, obtained from pairs of *not* operators $(p = 2)$ like the one shown in Table 6, was independently used by I. Holyer [17] to prove that the problem of finding the chromatic index of an arbitrary cubic graph is NP–complete.

By joining conveniently the logic operators and the meeting points studied throughout this section, we can now obtain multisets implementing any logic function $\mathbf{f}: B^n \to B^m$, $B = \{0, 1\}$. Also, by Corollary 1, many interesting configurations can be derived using only *not* operators. As an example, we have chosen the 4–set A_2 depicted in Fig. 11a, where A_0 represents a c–complete 4–set. Since A_1 and A_2, as well as A_0 itself, must satisfy Corollary 1, it is readily seen that the resulting 4–pole A_2 has the truth table shown in Fig. 11b, where subindexes have been included (just obtain the possible Boole coloring vectors of A_i from those of A_{i-1}, $i = 1, 2$, and, after each step, leave out the vectors with only one 1). Analogously to the case of the *truth* operator with 3 variables, it is worth noting the color isomorphism between this 4–set and a vertex with 4 semiedges (colored with, say, 0, 1, 2, 3).



(a)

Y_1	Y_2	Y_3	Y_4
0	1_a	1_b	1_c
1_a	0	1_b	1_c
1_a	1_b	0	1_c
1_a	1_b	1_c	0

(b)

Figure 11

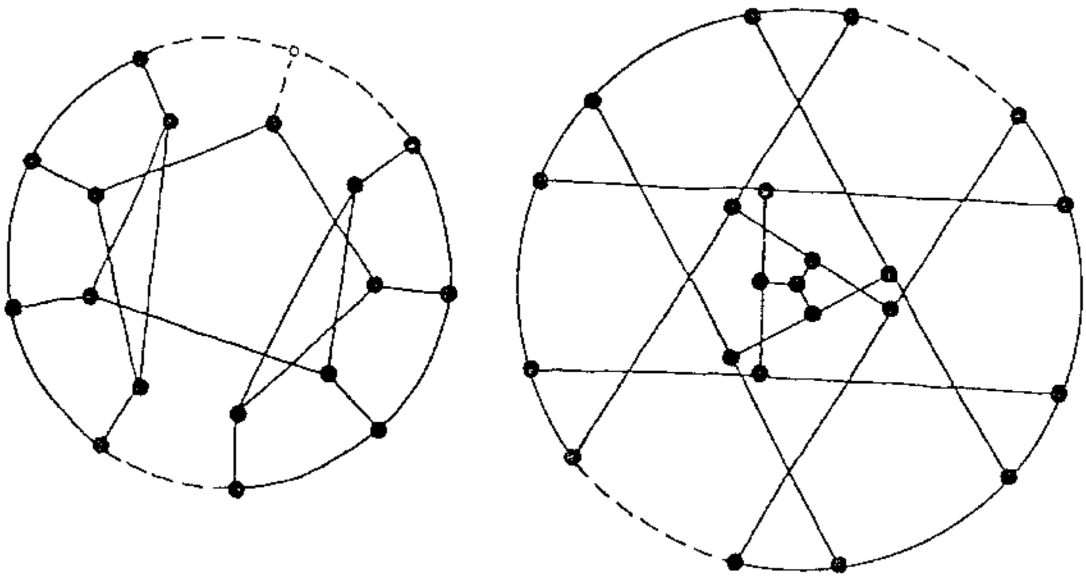
5. The BBDS Class of Snarks

We propose to slightly change the name of the BDS class of snarks and call it the *BBDS class*. The capital letters would, of course, stand for Boole and Blanuša & Descartes & Szekeres whose logic and graphs respectively inspired its construction.

Roughly speaking, we say that a snark U belongs to the BBDS class, denoted by U ∈ {BBDS}, if it derives from a non–Boole–colorable logic network. Hence, a basic characteristic of such a graph is that, when constructing it, we can insert an arbitrary 2-set (e.g. the *identity* operator) between every pair of semiedge sets to be joined. This fact, and some further minor considerations, led the author [8] to the following more precise definition.

The graph U belongs to {BBDS} iff it contains at least one m–pole Gp, $m > 3$, such that it can be replaced by a c–complete m–pole Z without affecting non–Tait–colorability; and there is at most one semiedge incident to each vertex of the complementary m–pole Gp'.

Note that, by the proposition in Section 2, Gp' must be non–Tait–colorable or, what is the same, the (non–regular) graph Gp'* is of class 2. Thus an alternative definition is to say that U ∈ {BBDS} iff it contains a subgraph of class 2 with $m > 3$ vertices of degree 2 (and none of degree 1). For instance, in the Blanuša graph and also in Loupekine's snark L_3 one can find such subgraphs with $m = 5$ and 4 respectively, see Figs. 12a and 12b (the corresponding m–poles Gp are drawn in dashed lines). The subdigraph of L_3 is a counter–example to the critical graph conjecture, see [4], [9], [15] and [21].



(a) (b)

Figure 12

The two snarks above are particular instances of two simple non–Boole–colorable logic networks. Namely, those shown in Figs. 13a and 13b (k odd), which correspond to the constructions of R. Isaacs [18], called the *dot product*, and F. Loupekine [19]. The non–Boole–colorability of both structures follows trivially from the truth tables of the operators involved.

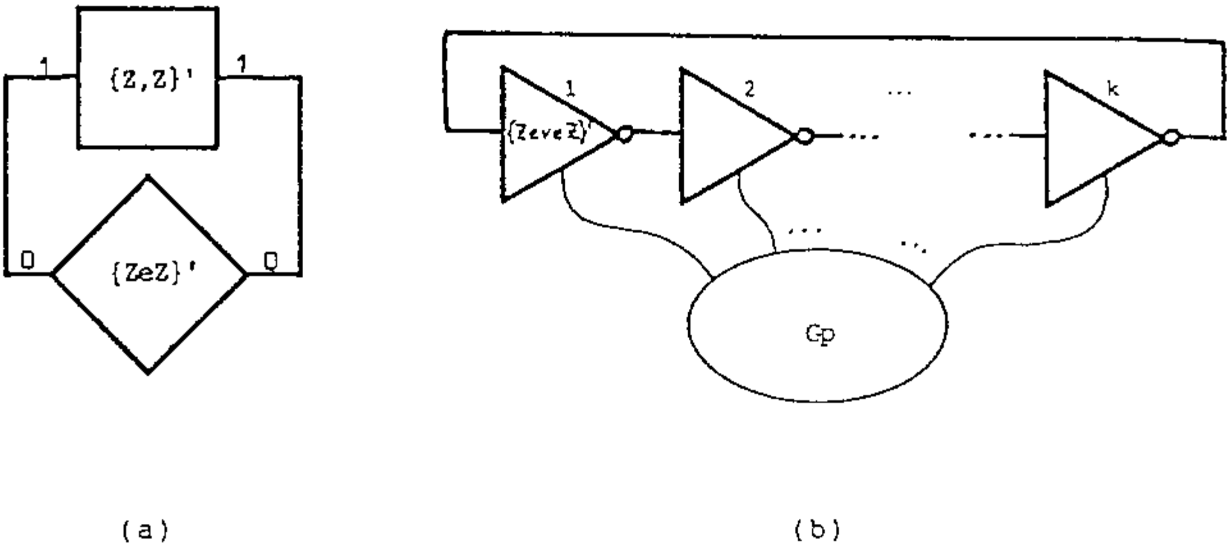


Figure 13

Let $U \in \{BBDS\}$. By the considerations above, it can be assumed that this graph contains some c–complete m–poles with $m \geq 4$. Then, applying the method of Section 4, we can obtain logic operators with larger semiedge sets than those obtained there. For example, if U is the "Star of David" of Fig. 12b (as we called L_3 before knowing of Loupekine's work), the *not* operator $\{Z_1 eve Z_2\}'$ with Z_1 a c–complete 4–pole ($n_1 = 3$) and $Z_2 = v$ ($n_2 = 2$) is shown in Fig. 14.

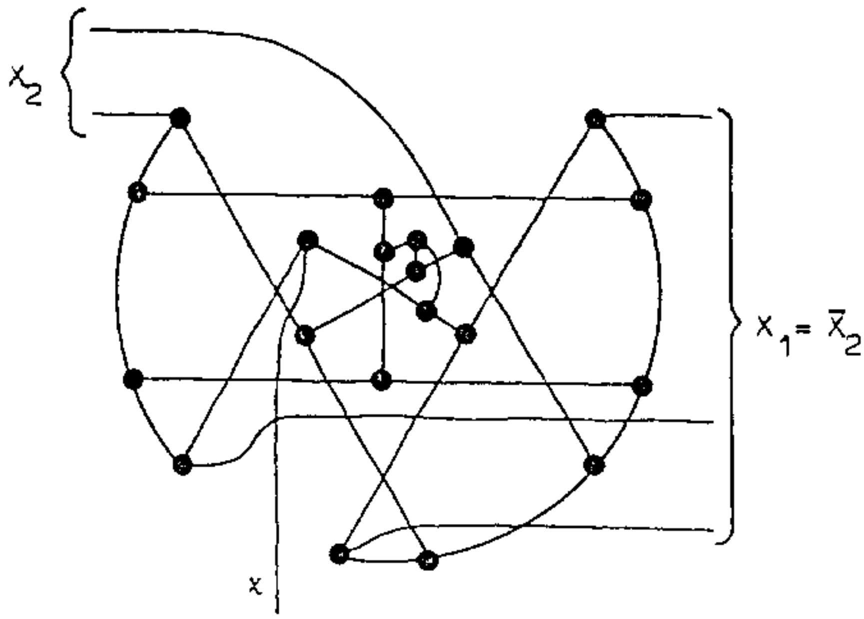


Figure 14

In what follows several methods of constructing snarks, based on the "Boole coloring theory", are given. As expected, most of the graphs obtained belong to the BBDS class.

Snarks from Logic Functions

Let us begin with an ample family of non–Boole–colorable logic networks, which include those given above. Let $f = (f_1, f_2, ..., f_m)$ and $\overline{f} = \overline{f}_1, \overline{f}_2, ..., \overline{f}_m)$ be two logic functions from B^n to B^m, $B = \{0, 1\}$, such that $y_i = f_i(x_1, x_2, ..., x_n)$, $1 \leq i \leq m$, $z_i = \overline{f}_i(x_1, x_2, ..., x_n) = \overline{y}_i$ if $i \in I \subset \{1, 2, ..., m\}$, $I \neq \emptyset$, and $z_i = y_i$ otherwise. Then, as the values of y_i and z_i, $i \in I$, mismatch for any given $(x_1, x_2, ..., x_n)$, it is clear that any pair of multisets implementing such functions are c–disjoint and, as said before, they can be used to derive a non–Boole–colorable logic network.

Snarks from Even and Odd Multisets

When the truth table of the considered multisets do not correspond to any logic function we can use other methods to find pairs of c–disjoint multisets. The method described here, based on the parity of the number of Boole colorings 1 (or 0), was proposed in [8]. The same method was independently developed in [15] for the above–mentioned cells.

A Tait colorable n–set Gp is said to be *even* (resp. *odd*) if all its Boole coloring vectors have an even (resp. odd) number of 1's. Note that some of the multisets studied in the preceding section belong to one of these categories. For instance the meeting point with an even number of variables and the $(n + 1)$–set corresponding to the *exclusive–or* function $y = x_1 \oplus x_2 \oplus \dots \oplus x_n$ are even, whereas the 4–set of Fig. 11a is odd. Other interesting examples can easily be found applying the proposed methods.

Of course, any logic network obtained by joining two n–sets with different parity is not Boole colorable. More general results are obtained from the following statement, which is easily proved using a simple parity argument.

C *Let Gp a (logic) n–set obtained by joining n_E even multisets and n_O odd mulitsets. Then the "parity" of Gp coincides with the parity of n_O.*

Considering a logic network as a (even) 0–set and applying Corollary 1, we have the following corollaries.

C1 *Any logic network made up by joining an odd number of odd multisets and any number of even mulisets is not Boole colorable.*

C2 *Any 1–set or 2–set made up as in* **C1** *is not Boole colorable.*

C3 *Any 3–set constructed as in* **C1** *has the only possible Boole coloring (1, 1, 1) —truth operator with 3 variables.*

C4 *Any 4–set constructed as in* **C1** *has only the possible Boole colorings shown in Fig. 11b.*

Snarks from Corollary 1

The method described at the end of Section 4, which is based on the use of *not* operators and Corollary 1, applies also the the construction of non–Boole–colorable logic networks or multisets. For instance, if each semiedge set $\mathcal{Y}_i$ of the 4–set of Fig. 11a is joined to a *not* operator (the neutral terminals being joined to any multipole), we obtain a 4–set with semiedge sets $\mathcal{Z}_i$ such that $Z_i = \overline{Y_i}$, $1 \le i \le 4$. Therefore, from the truth table of Fig. 11b, its possible Boole colorings would have only one 1, which is impossible.

Snarks from Color Isomorphisms

Let us now consider the subindexes of the Boole colorings 1. Then, as said before, some interesting color isomorphisms between multisets and multipoles appear. The first of them is based on the equivalence $i \equiv 1_i$, $i = 1, 2, 3$, between colors and Boole colorings (see **B1**). Thus, considering an m–pole as a special case of m–set, we can say that the *truth* operator of 2 (resp. 3) variables is c–equivalent to the 2–set **e** (resp. 3–set **v**), i.e., they have the same set of Boole coloring vectors. This fact allows us to construct easily a non–Boole–colorable logic network from a snark U. For instance, we can replace all the edges and vertices of U by *truth* operators of 2 variables and arbitrary 3–sets respectively —using *identity* operators if necessary. An example of this construction is the graph of B. Descartes [7]. Another possibility is to replace all the vertices of U by *truth* operators of 3 variables. Many other variations can also be considered.

Let us now consider the "c–equivalence" which exists between the 4–set of Fig. 11a or **C4** and a vertex with 4 semiedges, being "colored" with $\{0, 1_1, 1_2, 1_3\}$ and $\{0, 1, 2, 3\}$ respectively. In this case, we can derive, in the obvious way, a non–Boole–colorable logic network from a non–edge–colorable 4–regular graph. Examples of such 4–regular graphs are the line graphs of snarks and those graphs having an odd number of vertices (note that, in this latter case, the proposed construction consists in joining an odd number of even 4–sets). Analogously, using also *not* operators and *truth* operators of 3 variables to replace vertices of degree 2 and 3 respectively, non–Boole–colorable logic networks can be obtained from class 2 (non–regular) graphs with maximum degree 4.

Snarks from c–Equivalent Multipoles

As said in Section 2, a multipole Gp contained in a snark U can be replaced by a multipole Gp', c–contained in Gp, giving rise to another snark U'. In this subsection we focus on the case when Gp and Gp' are c–equivalent.

The two following statements show that some of the configurations studied before are c–equivalent to some simple multipoles contained in any snark.

D1 The "untruth cell" of n variables, seen as a 2n–set, is c–equivalent to n isolated edges, see Fig. 15a.

D2 The truth cells of n = 2 and 3 variables are c–equivalent to the multipoles $\{vev\}$ and $\left\{\begin{smallmatrix} vevev \\ e \\ v \end{smallmatrix}\right\}$ respectively, see Figs. 15b and 15c.

The proofs are simple consequences of remark B2, Lemma 1 and standard Kempe–chain arguments. The cases n = 2 are well–known since the above–mentioned dot product [1], [18] is based on them. The case n = 3 in D2 was implicitly considered by M.K. Goldberg in [15] to construct his (even) "hooking–cells".

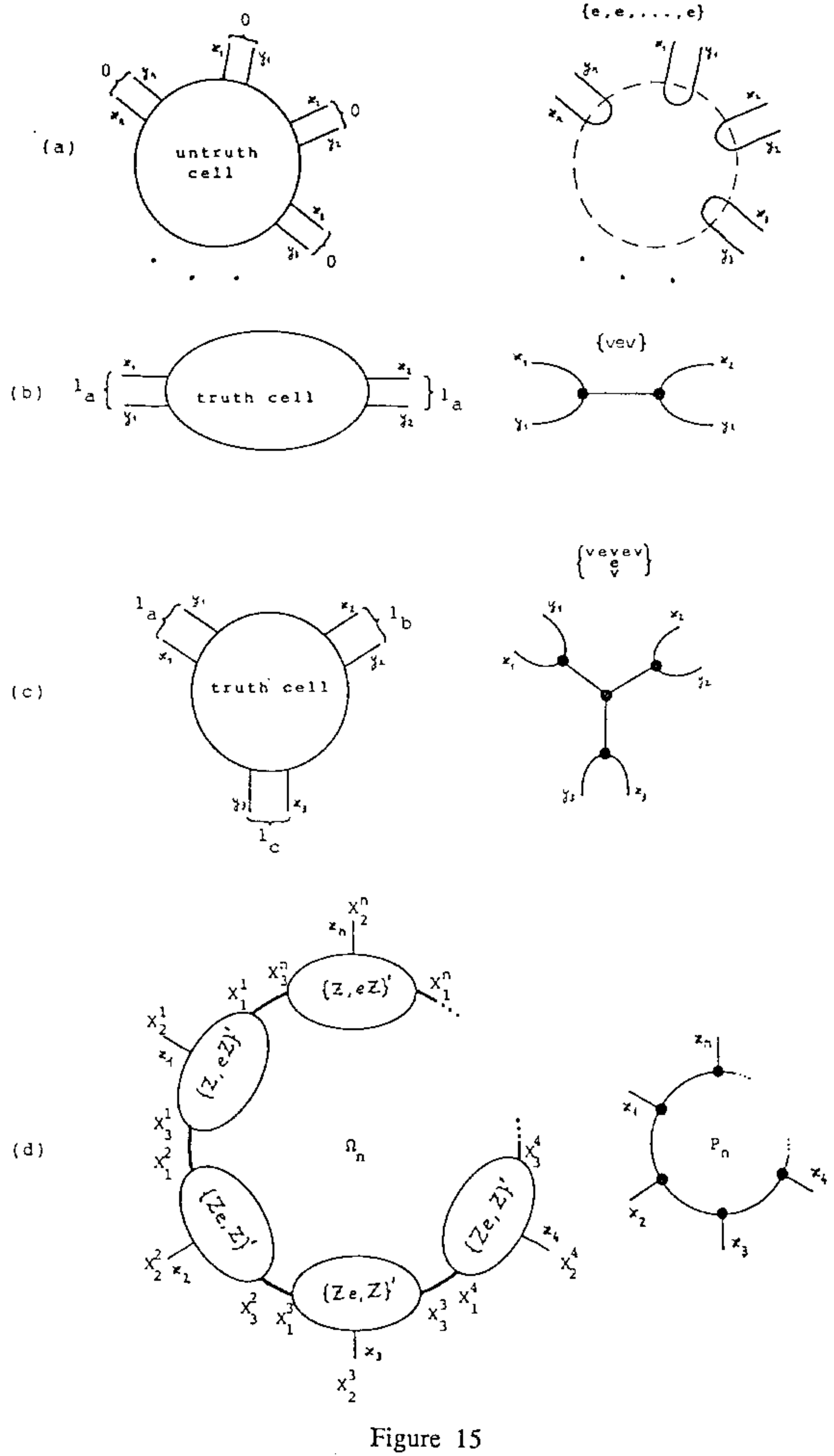

(a)
untruth cell
0
{e, e, . . . , e}
(b)
truth cell
1_a
{vev}
(c)
truth cell
1_a
1_b
1_c
(d)
Ω_n
{Z, eZ}'
{Ze, Z}'
P_n

Figure 15

Another interesting construction of c–equivalent multipoles is as follows. Let Ω_n be the n–pole obtained by joining *n partial–or* operators with semiedge sets X_1^i, X_2^i, X_3^i, $i \in Z_n$ —see Table 5— in such a way that $X_3^i = X_1^{i+1}$, as shown in Fig. 15d. Then,

D3 The multipole Ω_n is c–equivalent to an "n–gon", i.e., an n–cycle with one semiedge incident to each vertex. See the above figure.

Indeed, as $X_1^{i+1} = X_3^i = 1$, the only possible Boole coloring of each 3–set $\{Ze, Z\}'$ is $(X_1^i, X_2^i, X_3^i) = (1_a, 1_b, 1_c)$.

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