A probabilistic concept of accessibility for access structures *

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Abstract

In this paper we introduce the concept of weighted accessibility for access structures. In some sense, it represents a measure of how difficult or how easy is to recover the secret. We give also a numerical measure of accessibility for each participant depending on his position in the access structure. Both concepts, the accessibility of the access structure and the accessibility of the participants are closely related. We also provide an axiomatic characterization of the weighted accessibility for access structures based on four simple properties.

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1 Introduction and preliminaries

Access structures appear in cryptography, in the context of secret sharing schemes. These are methods of distributing a secret among a set of participants. Each participant receives a piece of the secret, its *share*, in such a way that only specified coalitions of participants can reconstruct the secret by pooling the shares of their members. The coalitions of participants which are able to reconstruct the secret are the *authorized coalitions* and, given a set of participants, the set of all authorized coalitions is the access structure.

It is known that for any access structure defined on a given set of participants, there is a secret sharing scheme realizing it [2]. Therefore, access structures can be studied independently of secret sharing schemes.

The aim of this paper is that of introducing new concepts to better understand and analyze access structures. Carreras *et al.* [1] introduce the notion of *accessibility*: it measures how many ways there exist for accessing to the secret and, hence, how easy or how difficult it is to recover. This concept depends only on the access structure and not on the particular scheme used for realizing it. In the work of Carreras *et al.*, the accessibility index is defined for every access structure as the number of authorized coalitions divided by 2^n, where n is the number of participants. It can be interpreted as the probability of a random coalition to be authorized when each participant has a probability of $1/2$ of belonging to it.

Here we propose a *weighted accessibility* for every access structure based on the assumption that the probability to form a random coalition with s members is $\alpha^s(1-\alpha)^{n-s}$ when each participant has a probability of α of belonging to it.

The plan of the paper is as follows. First, we briefly recall basic concepts on access structures. Section 2 is devoted to define the notion of weighted accessibility for an access structure and to study some of its properties. In Section 3 we turn our attention to the accessibility of the participants and we derive an interesting relation between this concept and the one defined before. Finally, in Section 4 we try to reconstruct the structure from the weighted accessibility of its participants and we prove that it is possible for structures with four or less participants but not for larger ones.

Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a set of n participants. An *access structure* on $\mathcal{P}$ is a set Γ of subsets of $\mathcal{P}$ ($\Gamma \subseteq 2^P$). The subsets in Γ are called *authorized coalitions* and they should be able to compute the shared secrets. We denote by $\mathcal{A}_\mathcal{P}$ the set of all access structures defined on $\mathcal{P}$. From now on, we suppose that the access structures are *monotone*, i.e., $A \subseteq B \subseteq \mathcal{P}$ with $A \in \Gamma$ implies $B \in \Gamma$. A monotone structure Γ on $\mathcal{P}$ is completely determined by means of its set Γ_0 of minimal authorized coalitions; Γ_0 is called the *basis* of Γ.

A participant $P_i \in \mathcal{P}$ that does not belong to any authorized coalition of the basis Γ_0 will be called *null* participant in the access structure Γ. Given two participants, P_i, P_j, we say that P_i is *over* P_j when $S \cup \{P_j\} \in \Gamma$ implies $S \cup \{P_i\} \in \Gamma$ for every $S \subseteq \mathcal{P} \setminus \{P_i, P_j\}$. In the particular case in which both P_i, P_j are simultaneously over the other, we say that P_i and P_j are *equivalent* participants.

Given two access structures Γ and Γ' on $\mathcal{P}$, the access structures *union* and *intersection* are defined as $\Gamma \cup \Gamma' = \{S \subseteq \mathcal{P} : S \in \Gamma \text{ or } S \in \Gamma'\}$ and $\Gamma \cap \Gamma' = \{S \subseteq \mathcal{P} : S \in \Gamma \text{ and } S \in \Gamma'\}$.

We say that two access structures Γ and Γ' on $\mathcal{P}$ are *isomorphic* if there exists a permutation $\pi : \mathcal{P} \to \mathcal{P}$ where $\pi(S)$ is an authorized coalition of Γ' if and only if S is an authorized coalition of Γ, for every subset $S \subseteq \mathcal{P}$.

2 The weighted accessibility of an access structure

Let us generalize the idea of accessibility. Suppose that each participant has a probability of α of belonging to a coalition.

Definition 2.1 *Let us consider a real number $\alpha \in (0, 1)$. The α-accessibility index on $\mathcal{P}$ is the map $\Omega^\alpha : \mathcal{A}_\mathcal{P} \to \mathbb{R}$ given by*

$$\Omega^\alpha(\Gamma) = \sum_{s=1}^{n} \alpha^s (1-\alpha)^{n-s} |\Gamma[s]| \tag{1}$$

where $\Gamma[s]$ denotes the subset of Γ formed by authorized coalitions with s participants.

The coalition size s takes values from $s = 1$ to $s = n = |\mathcal{P}|$, the total number of participants. As we said in the Introduction, the coefficient $\alpha^s(1-\alpha)^{n-s}$ can be interpreted as the probability that each random coalition with s members forms, when each participant has probability α to belong to it.

The 1/2-accessibility agrees with the *accessibilty index* [1] on $\mathcal{P}$, $\delta_\mathcal{P}$:

$$\Omega^{1/2}(\Gamma) = \sum_{s=1}^{n} \frac{1}{2^s} \frac{1}{2^{n-s}} |\Gamma[s]| = \frac{1}{2^n} \sum_{s=1}^{n} |\Gamma[s]| = \frac{|\Gamma|}{2^n} = \delta_\mathcal{P}(\Gamma) \quad \forall\, \Gamma \in \mathcal{A}_\mathcal{P}.$$

Since the number of authorized coalitions with s participants satisfies $0 \leq |\Gamma[s]| \leq \binom{n}{s}$, we have, $\forall\, \alpha \in (0, 1)$ and $\forall\, \Gamma \in \mathcal{A}_\mathcal{P}$,

$$0 \leq \Omega^\alpha(\Gamma) = \sum_{s=1}^{n} \alpha^s (1-\alpha)^{n-s} |\Gamma[s]| \leq \sum_{s=1}^{n} \alpha^s (1-\alpha)^{n-s} \binom{n}{s} = 1 - (1-\alpha)^n.$$

Thus, it is clear that, $0 \leq \Omega^\alpha(\Gamma) < 1$, and $\Omega^\alpha(\Gamma) = 0$ if, and only if, $\Gamma = \emptyset$.

We will say that Ω^α satisfies the Empty Structure property: $\Omega^\alpha(\emptyset) = 0$.

Example 2.2 Let $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ be a set of four participants. We consider the access structure Γ with basis $\Gamma_0 = \{\{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\}\}$. The cardinalities of authorized coalitions for Γ are $|\Gamma[1]| = 0$, $|\Gamma[2]| = 3$, $|\Gamma[3]| = 3$ and $|\Gamma[4]| = 1$, so that the α-accessibility index for Γ is given by

$$\Omega^\alpha(\Gamma) = \sum_{s=1}^{n} \alpha^s (1-\alpha)^{n-s} |\Gamma[s]| = 3\alpha^2(1-\alpha)^2 + 3\alpha^3(1-\alpha) + \alpha^4 = 3\alpha^2 - 3\alpha^3 + \alpha^4.$$

Following an analogous procedure, the α-accessibility index for the access structure Γ' on $\mathcal{P}$ with basis $\Gamma'_0 = \{\{P_1, P_2\}, \{P_3, P_4\}\}$ is

$$\Omega^\alpha(\Gamma') = \sum_{s=1}^{n} \alpha^s (1-\alpha)^{n-s} |\Gamma'[s]| = 2\alpha^2(1-\alpha)^2 + 4\alpha^3(1-\alpha) + \alpha^4 = 2\alpha^2 - \alpha^4.$$

In Figure 1 both weighted accessibilities are compared.

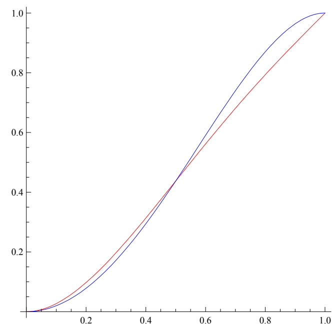

Figure 1: Comparing weighted accessibilities

For $0 < \alpha < 1/2$ the weighted accessibility of Γ is greater than the corresponding to Γ', for $\alpha = 1/2$ both coincide, and for $1/2 < \alpha < 1$, the first access structure has lower weighted accessibility than the second one.

This example shows that the α-accessibility index is not a simple ranking among access structures on $\mathcal{P}$. The values of α play an essential role so that an access structure can have higher or lower weighted accessibility than another one according to distinct values of α.

Let us state two properties of the α-accessibility.

Proposition 2.3 *Let Γ and Γ' be two access structures on $\mathcal{P}$ with $|\mathcal{P}| = n$.*

(i) If $|\Gamma[s]| \leq |\Gamma'[s]|$ for $s = 1, \ldots, n$, then $\Omega^\alpha(\Gamma) \leq \Omega^\alpha(\Gamma')$ $\forall \alpha \in (0, 1)$.

(ii) $\Omega^\alpha(\Gamma \cup \Gamma') = \Omega^\alpha(\Gamma) + \Omega^\alpha(\Gamma') - \Omega^\alpha(\Gamma \cap \Gamma')$ $\forall \alpha \in (0, 1)$. We will say that Ω^α satisfies the Transfer property.

Proof Part (i) directly follows from the definition of α-accessibility, whereas the relations $|(\Gamma \cup \Gamma')[s]| = |\Gamma[s]| + |\Gamma'[s]| - |(\Gamma \cap \Gamma')[s]|$ for $1 \leq s \leq n$ lead us to the expression in part (ii). $\square$

Definition 2.4 *An access structure Γ on $\mathcal{P}$ whose basis Γ_0 contains a unique authorized coalition $C \subseteq \mathcal{P}$ is called access structure expaned by C. Coalition C is the generator of structure Γ. If $\Gamma_0 = \{C\}$ with $C \subseteq \mathcal{P}$, we write $\Gamma =< C >$.*

Threshold schemes are important examples of access structures. On a set $\mathcal{P}$ of n participants, the basis of a (t, n)-threshold access structure is formed by all subsets of $\mathcal{P}$ with t participants ($t \leq n$). If we denote by $[t, n]$ a (t, n)-threshold access structure, we have $[t, n] = \bigcup_{|C|=t} < C >$.

Proposition 2.5 *Let $\mathcal{P}$ be a set of n participants.*

(i) If $C = \{P_{i_1}, \ldots, P_{i_k}\}$ with $|C| = k \leq n$, then $\Omega^\alpha(< C >) = \alpha^k \ \forall \alpha \in (0,1)$.

(ii) For a (t,n)-threshold access structure

$$\Omega^\alpha([t,n]) = \sum_{k=t}^{n} (-1)^{k-t} \binom{k-1}{k-t} \binom{n}{k} \alpha^k \quad \forall \alpha \in (0,1).$$

Proof (i) Let C be a subset of k participants ($k \leq n$). Then,

$$\Omega^\alpha(< C >) = \sum_{j=0}^{n-k} \binom{n-k}{j} \alpha^{k+j} (1-\alpha)^{n-k-j} = \alpha^k.$$

(ii) The α-accessibility index for a (t,n)-threshold access structure is

$$\begin{aligned}
\Omega^\alpha([t,n]) &= \sum_{s=t}^{n} \binom{n}{s} \alpha^s (1-\alpha)^{n-s} = \sum_{s=t}^{n} \binom{n}{s} \alpha^s \sum_{j=0}^{n-s} \binom{n-s}{j} (-\alpha)^{n-s-j} \\
&= \sum_{s=t}^{n} \binom{n}{s} \sum_{j=0}^{n-s} (-1)^{n-s-j} \binom{n-s}{j} \alpha^{n-j} \\
&= \sum_{j=0}^{n-t} \sum_{s=t}^{n-j} (-1)^{n-s-j} \binom{n-s}{j} \binom{n}{s} \alpha^{n-j}
\end{aligned}$$

If we write $n - j = k$, the last equality becomes

$$\begin{aligned}
\Omega^\alpha([t,n]) &= \sum_{k=t}^{n} \sum_{s=t}^{k} (-1)^{k-s} \binom{n-s}{n-k} \binom{n}{s} \alpha^k \\
&= \sum_{k=t}^{n} \binom{n}{k} \sum_{s=t}^{k} (-1)^{k-s} \binom{k}{s} \alpha^k \\
&= \sum_{k=t}^{n} \binom{n}{k} \left[\sum_{s=t}^{k-1} (-1)^{k-s} \left[\binom{k-1}{s-1} + \binom{k-1}{s} \right] + \binom{k}{k} \right] \alpha^k \\
&= \sum_{k=t}^{n} (-1)^{k-t} \binom{k-1}{t-1} \binom{n}{k} \alpha^k \qquad \square
\end{aligned}$$

Note that, in particular, for the $< \mathcal{P} >$ structure, we have $\Omega^\alpha(< \mathcal{P} >) = \alpha^n$. We will say that Ω^α satisfies the Unanimity property.

Remark 2.6 Expression in Proposition 2.3 (ii) can be recursively generalized for finite unions of access structures on a same set of participants $\mathcal{P}$ as follows:

$$\Omega^\alpha\Big(\bigcup_{i=1}^{k} \Gamma_i\Big) = \sum_{i=1}^{k} \Omega^\alpha(\Gamma_i) + \sum_{j=2}^{k} (-1)^{j-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq k} \Omega^\alpha(\Gamma_{i_1} \cap \Gamma i_2 \cap \cdots \cap \Gamma_{i_j}) \ \forall \alpha \in (0,1).$$

Given an access structure Γ whose basis $\Gamma_0 = \{C_1, \ldots, C_k\}$ contains two or more authorized coalitions, we can write its α-accessibility by means of α-accessibilities of expaned access structures according to Definition 2.4:

$$\Omega^\alpha(\Gamma) = \Omega^\alpha\Big(\bigcup_{i=1}^{k} < C_i >\Big)$$
$$= \sum_{i=1}^{k} \Omega^\alpha(< C_i >) + \sum_{j=2}^{k} (-1)^{j-1} \sum_{1 \le i_1 < \cdots < i_j \le k} \Omega^\alpha(< C_{i_1} > \cap \cdots \cap < C_{i_j} >)$$

Note that if C and C' are two authorized coalitions of Γ, then $< C > \cap < C' > = < C \cup C' >$. Therefore, the last sum can be computed according to Proposition 2.5 (i):

$$\Omega^\alpha(\Gamma) = \sum_{i=1}^{k} \Omega^\alpha(< C_i >) + \sum_{j=2}^{k} (-1)^{j-1} \sum_{1 \le i_1 < \cdots < i_j \le k} \Omega^\alpha(< C_{i_1} \cup \cdots \cup C_{i_j} >)$$
$$= \sum_{i=1}^{k} \alpha^{|C_i|} + \sum_{j=2}^{k} (-1)^{j-1} \sum_{1 \le i_1 < \cdots < i_j \le k} \alpha^{|C_{i_1} \cup \cdots \cup C_{i_j}|}$$

The above formula allows us to obtain the α-accessibility index of a monotone access structure as a polynomial in α from the authorized coalitions of its basis.

Example 2.7 We consider all possible access structures with three participants, up to isomorphisms. Table 1 contains in its first column the basis of each access structure, where, for simplicity, we write the authorized coalitions without braces. The second column shows the vector whose components are the number of authorized coalitions according to their size, and the third column the respective α-accessibility index.

Γ_0	$(\lvert\Gamma[1]\rvert, \lvert\Gamma[2]\rvert, \lvert\Gamma[3]\rvert)$	$\Omega^\alpha(\Gamma)$
$P_1P_2P_3$	$(0,0,1)$	α^3
P_1P_2	$(0,1,1)$	α^2
$P_1P_2,\ P_1P_3$	$(0,2,1)$	$2\alpha^2 - \alpha^3$
$P_1P_2,\ P_1P_3,\ P_2P_3$	$(0,3,1)$	$3\alpha^2 - 2\alpha^3$
P_1	$(1,2,1)$	α
$P_1,\ P_2P_3$	$(1,3,1)$	$\alpha + \alpha^2 - \alpha^3$
$P_1,\ P_2$	$(2,3,1)$	$2\alpha - \alpha^2$
$P_1,\ P_2,\ P_3$	$(3,3,1)$	$3\alpha - 3\alpha^2 + \alpha^3$

Table 1: α-accessibility of access structures with three participants

The access structures in Table 1 are ranked by increasing order of the vector $(|\Gamma[s]|)_{s=1}^{3}$. This example shows that, for three participants, non-isomorphic access structures have different α-accessibility index.

Definition 2.8 *Two distinct access structures* Γ *and* Γ' *defined on a set of participants* $\mathcal{P}$ *are strategically equivalent,* $\Gamma \backsim_{\mathcal{P}} \Gamma'$*, iff their vectors of cardinalities of authorized coalitions coincide:*

$$\Gamma \backsim_{\mathcal{P}} \Gamma' \Leftrightarrow |\Gamma[s]| = |\Gamma'[s]|,\ s = 1, \ldots, n = |\mathcal{P}|.$$

For each pair of access structures on a set of participants $\mathcal{P}$, strategically equivalent is weaker than isomorphic. Both concepts, isomorphism and strategically equivalent, agree on sets with two or three participants, but we can find two strategically equivalent access structures with four participants that are not isomorphic.

Example 2.9 Let Γ and Γ' be two access structures defined on $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ with respective basis

$$\Gamma_0 = \{P_1P_2,\ P_1P_3,\ P_1P_4,\ P_2P_3\} \quad \text{and} \quad \Gamma'_0 = \{P_1P_2,\ P_1P_3,\ P_2P_4,\ P_3P_4\}.$$

The access structures Γ and Γ' are not isomorphic, but $(|\Gamma[s]|)_{s=1}^4 = (|\Gamma'[s]|)_{s=1}^4 = (0, 4, 4, 1)$. Then $\Gamma \backsim_{\mathcal{P}} \Gamma'$ and the common α-accessibility is

$$\Omega^\alpha(\Gamma) = \Omega^\alpha(\Gamma') = 4\alpha^2(1-\alpha)^2 + 4\alpha^3(1-\alpha) + \alpha^4 = 4\alpha^2 - 4\alpha^3 + \alpha^4.$$

Theorem 2.10 *Two access structures* Γ *and* Γ' *defined on* $\mathcal{P}$ *are strategically equivalent if and only if their respective* α*-accessibilities agree for every* $\alpha \in (0, 1)$.

Proof It is obvious that the α-accessibilities agree for two strategically equivalent access structures. Conversely, we will prove that the α-accessibility for $\alpha \in (0, 1)$ univocally determines the vector of cardinalities of authorized coalitions. First, we write the α-accessibility index as a polynomial in α.

$$\begin{aligned}
\Omega^\alpha(\Gamma) &= \sum_{s=1}^{n} \alpha^s (1-\alpha)^{n-s} |\Gamma[s]| \\
&= \sum_{s=1}^{n} \alpha^s \sum_{j=0}^{n-s} \binom{n-s}{j} (-\alpha)^{n-s-j} |\Gamma[s]| \\
&= \sum_{s=1}^{n} \sum_{j=0}^{n-s} (-1)^{n-s-j} \binom{n-s}{j} \alpha^{n-j} |\Gamma[s]| \\
&= \sum_{s=1}^{n} \sum_{k=s}^{n} (-1)^{k-s} \binom{n-s}{n-k} \alpha^k |\Gamma[s]| \\
&= \sum_{k=1}^{n} \alpha^k \sum_{s=1}^{k} (-1)^{k-s} \binom{n-s}{n-k} |\Gamma[s]|.
\end{aligned}$$

This way, the α-accessibility for an access structure Γ can be written as

$$\Omega^\alpha(\Gamma) = \sum_{k=1}^{n} b_k \alpha^k \qquad \text{where} \qquad b_k = \sum_{s=1}^{k} (-1)^{k-s} \binom{n-s}{n-k} |\Gamma[s]|. \tag{2}$$

Now, if an α-accessibility is given, the numbers b_k, for $1 \le k \le n$, are known and the n equalities in the right of expression (2) form a linear system of equations with unknowns $|\Gamma[1]|, \dots, |\Gamma[n]|$. The matrix of the linear system is a lower triangular matrix whose entries in the main diagonal are all 1. This guarantees a unique solution for the unknowns $|\Gamma[s]|$, $1 \le s \le n$. $\square$

Example 2.11 On a set $\mathcal{P}$ with four participants, let us assume that the α-accessibility is given by $\Omega^\alpha(\Gamma) = 4\alpha^2 - 4\alpha^3 + \alpha^4$, that is, $b_1 = 0$, $b_2 = 4$, $b_3 = -4$ and $b_4 = 1$. The linear system with unknowns $|\Gamma[s]|$, $1 \le s \le 4$, becomes

$$\left.\begin{array}{ll} |\Gamma[1]| & = 0 \\ -3|\Gamma[1]| + |\Gamma[2]| & = 4 \\ 3|\Gamma[1]| - 2|\Gamma[2]| + |\Gamma[3]| & = -4 \\ -|\Gamma[1]| + |\Gamma[2]| - |\Gamma[3]| + |\Gamma[4]| & = 1 \end{array}\right\}$$

and its unique solution is the vector $(|\Gamma[1]|, |\Gamma[2]|, |\Gamma[3]|, |\Gamma[4]|) = (0, 4, 4, 1)$.

We can provide an axiomatic characterization for the α-accessibility index, but we need first another notion. Let $\mathcal{P}$ and $\mathcal{R}$ be two sets of participants such that $\mathcal{P} \subset \mathcal{R}$, and suppose that Γ is an access structure defined on $\mathcal{P}$.

Up to now, we have worked with access structures Γ on a unique set of participants $\mathcal{P} = \{P_1, \dots, P_n\}$. As from now on we will consider diverse sets of participants, to avoid misunderstanding, we will denote the access structures on $\mathcal{P}$ by $\Gamma_\mathcal{P}$ ($\Gamma_\mathcal{P} \subseteq 2^\mathcal{P}$) and its α-accessibility index by $\Omega^\alpha(\Gamma_\mathcal{P})$, for each $\alpha \in (0, 1)$.

Definition 2.12 *The null extension of* Γ *to* $\mathcal{R}$ *is the access structure*

$$\Gamma_\mathcal{R} = \{T \subseteq R : T \cap \mathcal{P} \in \Gamma\}.$$

Let us show that neither the adjunction nor the suppression of null participants affect the α-accessibility.

Proposition 2.13 *Let* $\mathcal{P} \subset \mathcal{R}$ *be two sets of participants and* Γ *an access structure defined on* $\mathcal{P}$. *If* $\Gamma_\mathcal{R}$ *is the null extension of* Γ *to* $\mathcal{R}$, *then* $\Omega^\alpha(\Gamma_\mathcal{R}) = \Omega^\alpha(\Gamma)$ *for all* $\alpha \in (0, 1)$.

Proof It suffices to show the case $\mathcal{R} = \mathcal{P} \cup \{P_i\}$, for $P_i \notin \mathcal{P}$.

We denote by $\Gamma_\mathcal{P}[s; P_i] = \{S \in \Gamma_\mathcal{P} \mid |S| = s \text{ and } P_i \in S\}$. Then

$$\begin{aligned} \Omega^\alpha(\mathcal{P} \cup \{P_i\}) &= \sum_{s=1}^{n+1} \alpha^s (1-\alpha)^{n+1-s} |\Gamma_{\mathcal{P}\cup\{P_i\}}[s]| \\ &= \sum_{s=1}^{n+1} \alpha^s (1-\alpha)^{n+1-s} |\Gamma_{\mathcal{P}\cup\{P_i\}}[s; P_i]| + \sum_{s=1}^{n} \alpha^s (1-\alpha)^{n+1-s} |\Gamma[s]|. \end{aligned}$$

As participant P_i is null in $\mathcal{P} \cup \{P_i\}$, we have

$$|\Gamma_{\mathcal{P}\cup\{P_i\}}[1; P_i]| = 0$$

and

$$|\Gamma_{\mathcal{P}\cup\{P_i\}}[s;P_i]| = |\Gamma[s-1]| \text{ for } s = 2,\dots,n+1.$$

Therefore,

$$\begin{aligned}\Omega^\alpha(\mathcal{P}\cup\{P_i\}) &= \sum_{s=2}^{n+1}\alpha^s(1-\alpha)^{n+1-s}|\Gamma[s-1]| + \sum_{s=1}^{n}\alpha^s(1-\alpha)^{n+1-s}|\Gamma[s]| \\ &= \sum_{s=1}^{n}\alpha^{s+1}(1-\alpha)^{n-s}|\Gamma[s]| + \sum_{s=1}^{n}\alpha^s(1-\alpha)^{n+1-s}|\Gamma[s]| \\ &= \sum_{s=1}^{n}\alpha^s(1-\alpha)^{n-s}[\alpha+(1-\alpha)]|\Gamma[s]| = \Omega^\alpha(\Gamma). \qquad \square\end{aligned}$$

We will say that Ω^α satisfies the Null Participant property.

Theorem 2.14 *Let $\mathcal{A} = \cup_{\mathcal{P}} A_{\mathcal{P}}$. A function $F : \mathcal{A} \longrightarrow \mathbb{R}$ satisfies the properties of Empty Structure, Transfer, Unanimity and Null Participant iff it is the α-accessibility index.*

Proof So far, we have shown that Ω^α satisfies the four properties. Conversely, let F be a function satisfying them. We will prove that $F = \Omega^\alpha$.

According to Remark 2.6, if Γ is an access structure on $\mathcal{P}$, then Γ can be written as the union of expaned structures. Applying the Unanimity property, F coincides with Ω^α on expaned structures and, applying the Null Participant property, they also coincide on extensions of expaned structures. Therefore, $F = \Omega^\alpha$ on $\mathcal{A}$. $\square$

3 The weighted accessibility of the participants

Now we want to offer a measure of the *importance* of each participant in the access structure based on the α-accessibility index of the structure. It seems reasonable to say that the importance of each participant is related to the number of authorized coalitions that contain the participant. More precisely, we will compare the number of authorized coalitions that contain a given participant with respect to the number of authorized coalitions without this participant. In addition these comparisons will be weighted by means of coefficients related to the probability of each random coalition to form, when each participant has a probability of α of belonging to it.

Definition 3.1 *Let $\Gamma_{\mathcal{P}}$ be an access structure defined on a set of participants $\mathcal{P}$. Consider a subset $\mathcal{Q}$ of $\mathcal{P}$, $\emptyset \neq \mathcal{Q} \subseteq \mathcal{P}$. The access substructure $\Gamma_{\mathcal{Q}}$ is the restriction of the access structure $\Gamma_{\mathcal{P}}$ to the subset $\mathcal{Q}$. We denote it by $\Gamma_{\mathcal{Q}}$.*

$$\Gamma_{\mathcal{Q}} = \Gamma_{\mathcal{P}}|_{\mathcal{Q}} = \{S | S \in \Gamma_{\mathcal{P}} \text{ and } S \subseteq \mathcal{Q}\}$$

Definition 3.2 *Let $\Gamma_{\mathcal{P}}$ be an access structure defined on a set of participants $\mathcal{P} = \{P_1,\dots,P_n\}$. For each real number $\alpha \in (0,1)$, the α-accessibility of a participant*

$P_i \in \mathcal{P}$ *in the access structure* $\Gamma_{\mathcal{P}}$ *is given by*

$$I_\alpha(P_i, \Gamma_{\mathcal{P}}) = \sum_{s=1}^{n} \alpha^s (1-\alpha)^{n-s} \big\{ |\Gamma_{\mathcal{P}}[s; P_i]| - |\Gamma_{\mathcal{P}\setminus\{P_i\}}[s-1]| \big\}, \quad i = 1, \ldots, n, \quad (3)$$

where $\Gamma_{\mathcal{P}}[s] = \{S \in \Gamma_{\mathcal{P}} \mid |S| = s\}$ *and* $\Gamma_{\mathcal{P}}[s; P_i] = \{S \in \Gamma_{\mathcal{P}} \mid |S| = s$ *and* $P_i \in S\}$.

Fixed a set of participants $\mathcal{P} = \{P_1, \ldots, P_n\}$, the α-accessibility of the participants can be considered as a map $I_\alpha : \mathcal{A}_{\mathcal{P}} \to \mathbb{R}^N$. The ith component of vector $I_\alpha(\Gamma_{\mathcal{P}})$ is $I_\alpha(P_i, \Gamma_{\mathcal{P}})$ for $i = 1, \ldots, n$.

Note that for all $P_i \in \mathcal{P}$, $|\Gamma_{\mathcal{P}}[1; P_i]| = 1$ if $\{P_i\} \in \Gamma_{\mathcal{P}}$ and $|\Gamma_{\mathcal{P}}[1; P_i]| = 0$ otherwise. In addition, $\Gamma_{\mathcal{P}\setminus\{P_i\}}[0] = \emptyset$, for all $P_i \in \mathcal{P}$.

Example 3.3 Let $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ be a set of four participants. We consider the access structure $\Gamma_{\mathcal{P}}$ with basis $(\Gamma_{\mathcal{P}})_0 = \{\{P_1, P_2\}, \{P_1, P_3\}, \{P_1, P_4\}, \{P_2, P_3\}\}$.

We want to compute, for instance, the α-accessibility of participant P_1. To do so, we need the cardinalities of authorized coalitions containing participant P_1:

$$|\Gamma_{\mathcal{P}}[1; P_1]| = 0, \; |\Gamma_{\mathcal{P}}[2; P_1]| = 3, \; |\Gamma_{\mathcal{P}}[3; P_1]| = 3, \; |\Gamma_{\mathcal{P}}[4; P_1]| = 1.$$

In addition, the authorized coalitions in the access substructure $\Gamma_{\mathcal{P}\setminus\{P_1\}}$ are $\{P_2, P_3\}$ and $\{P_2, P_3, P_4\}$; thus,

$$|\Gamma_{\mathcal{P}\setminus\{P_1\}}[0]| = 0, \; |\Gamma_{\mathcal{P}\setminus\{P_1\}}[1]| = 0, \; |\Gamma_{\mathcal{P}\setminus\{P_1\}}[2]| = 1, \; |\Gamma_{\mathcal{P}\setminus\{P_1\}}[3]| = 1,$$

so that the α-accessibility of participant P_1 becomes

$$I_\alpha(P_1, \Gamma_{\mathcal{P}}) = 3\alpha^2(1-\alpha)^2 + 2\alpha^3(1-\alpha) = 3\alpha^2 - 4\alpha^3 + \alpha^4.$$

An analogous procedure for the remaining participants allows us to obtain the vector of α-accessibilities:

$$I_\alpha(\Gamma_{\mathcal{P}}) = (\; 3\alpha^2 - 4\alpha^3 + \alpha^4, \; 2\alpha^2 - 3\alpha^3 + \alpha^4, \; 2\alpha^2 - 3\alpha^3 + \alpha^4, \; \alpha^2 - 2\alpha^3 + \alpha^4 \;).$$

Table 2 shows some vectors of accessibilities according to several selected values of α.

α	$I_\alpha(\Gamma_{\mathcal{P}})$	%
1/3	(0.1975, 0.1235, 0.1235, 0.0494)	(40, 25, 25, 10)
1/2	(0.3125, 0.1875, 0.1875, 0.0625)	(41.67, 25, 25, 8.33)
2/3	(0.3457, 0.1975, 0.1975, 0.0494)	(43.75, 25, 25, 6.25)

Table 2: α-accessibilities of the participants of $\Gamma_{\mathcal{P}}$ for several values of α

Proposition below groups several properties that can help to see the appropriateness of the concept above introduced of α-accessibility for the participants.

Proposition 3.4 *Let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a set of n participants.*

(i) The α-accessibility of a participant P_i in an access structure $\Gamma_{\mathcal{P}}$ vanishes if and only if P_i is a null participant in $\Gamma_{\mathcal{P}}$, $\forall \alpha \in (0,1)$.

(ii) Given $\alpha \in (0,1)$, for every access structure, the maximum of the α-accessibility for the participants is α and this value is reached by a unique participant when the access structure is expaned by its unipersonal coalition.

(iii) Given $\alpha \in (0,1)$, for non-null participants and every access structure, the minimum of the α-accessibility is α^n reached by every participant when the unique authorized coalition is $\mathcal{P}$.

(iv) If P_i and P_j are two distinct participants and P_i is over P_j in access structure $\Gamma_{\mathcal{P}}$, then $I_\alpha(P_i, \Gamma_{\mathcal{P}}) \geq I_\alpha(P_j, \Gamma_{\mathcal{P}})$ $\forall \alpha \in (0,1)$. In particular, if P_i and P_j are equivalent participants, their respective α-accessibilities coincide.

Proof In all cases, according to expression (3), we must analyze the differences $|\Gamma_{\mathcal{P}}[s; P_i]| - |\Gamma_{\mathcal{P}\setminus\{P_i\}}[s-1]|$ for $s = 1, \dots, n$. As we work with monotonic access structures, $C \in \Gamma_{\mathcal{P}\setminus\{P_i\}}[s-1]$ implies $C \cup \{P_i\} \in \Gamma_{\mathcal{P}}[s; P_i]$. Thus $|\Gamma_{\mathcal{P}}[s; P_i]| \geq |\Gamma_{\mathcal{P}\setminus\{P_i\}}[s-1]|$.

(i) Since $\alpha^s(1-\alpha)^{n-s} > 0$ $\forall \alpha \in (0,1)$, $I_\alpha(P_i, \Gamma_{\mathcal{P}}) = 0$ implies $|\Gamma_{\mathcal{P}}[s; P_i]| = |\Gamma_{\mathcal{P}\setminus\{P_i\}}[s-1]|$ for $s = 1, \dots, n$, of where $C \cup \{P_i\} \in \Gamma_{\mathcal{P}}[s; P_i]$ leads us to $C \in \Gamma_{\mathcal{P}\setminus\{P_i\}}[s-1]$ and, therefore, P_i is a null participant in $\Gamma_{\mathcal{P}}$. The converse property follows easily.

(ii) The maximum of $I_\alpha(P_i, \Gamma_{\mathcal{P}})$ is obtained when, simultaneously, $|\Gamma_{\mathcal{P}\setminus\{P_i\}}[s-1]| = 0$ and $|\Gamma_{\mathcal{P}}[s; P_i]| = \binom{n-1}{s-1}$, for $s = 1, \dots, n$. That means that the access substructure $\Gamma_{\mathcal{P}\setminus\{P_i\}}$ is the empty access structure and all coalitions containing participant P_i are authorized coalitions: $\Gamma_{\mathcal{P}} =< \{P_i\} >$. Note that the remaining participants are null participants. Then

$$I_\alpha(P_i, \Gamma_{\mathcal{P}}) = \sum_{s=1}^{n} \binom{n-1}{s-1} \alpha^s (1-\alpha)^{n-s} = \alpha \sum_{s=1}^{n} \binom{n-1}{s-1} \alpha^{s-1} (1-\alpha)^{n-1-(s-1)} = \alpha.$$

(iii) For the minimum of $I_\alpha(P_i, \Gamma_{\mathcal{P}})$ over non-null participants, we impose $|\Gamma_{\mathcal{P}}[s; P_i]| = |\Gamma_{\mathcal{P}\setminus\{P_i\}}[s-1]|$ for $s = 1, \dots, n-1$ and $|\Gamma_{\mathcal{P}}[n; P_i]| = 1$ with $|\Gamma_{\mathcal{P}\setminus\{P_i\}}[n-1]| = 0$. Now, $\Gamma_{\mathcal{P}} =< \{P_1, \dots, P_n\} >$ and $I_\alpha(P_i, \Gamma_{\mathcal{P}}) = \alpha^n$ $\forall P_i \in \mathcal{P}$.

(iv) For every pair of distinct participants in $\Gamma_{\mathcal{P}}$, the difference between their α-accessibilities can be written as

$$I_\alpha(P_i, \Gamma_{\mathcal{P}}) - I_\alpha(P_j, \Gamma_{\mathcal{P}}) = \sum_{s=1}^{n} \alpha^s (1-\alpha)^{n-s} \times$$
$$\Big\{ |\Gamma_{\mathcal{P}\setminus\{P_j\}}[s; P_i]| - |\Gamma_{\mathcal{P}\setminus\{P_i\}}[s; P_j]| + |\Gamma_{\mathcal{P}\setminus\{P_j\}}[s-1; P_i]| - |\Gamma_{\mathcal{P}\setminus\{P_i\}}[s-1; P_j]| \Big\}.$$

If participant P_i is over participant P_j, $C \in \Gamma_{\mathcal{P}\setminus\{P_i\}}[s; P_j]$ implies $C \setminus \{P_j\} \cup \{P_i\} \in \Gamma_{\mathcal{P}\setminus\{P_j\}}[s; P_i]$ so that $|\Gamma_{\mathcal{P}\setminus\{P_j\}}[s; P_i]| \geq |\Gamma_{\mathcal{P}\setminus\{P_i\}}[s; P_j]|$.

In a similar way, $\Gamma_{\mathcal{P}\setminus\{P_j\}}[s-1;P_i]| \geq |\Gamma_{\mathcal{P}\setminus\{P_i\}}[s-1;P_j]|$. We conclude $I_\alpha(P_i,\Gamma_\mathcal{P}) \geq I_\alpha(P_j,\Gamma_\mathcal{P})$, if P_i is over P_j. $\square$

For all monotonic access structures, we have introduced two definitions of accessibility: one for the whole structure a another one for each participant. The following result links together both concepts.

Theorem 3.5 *Let $\mathcal{P} = \{P_1,\ldots,P_n\}$ be a set of n participants. Consider an access structure $\Gamma_\mathcal{P}$ on $\mathcal{P}$. For each participant $P_i \in \mathcal{P}$ and every real number $\alpha \in (0,1)$,*

$$I_\alpha(P_i,\Gamma_\mathcal{P}) = \Omega^\alpha(\Gamma_\mathcal{P}) - \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}}), \quad i = 1,\ldots,n,$$

where $\Omega^\alpha(\Gamma_\mathcal{P})$ and $\Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}})$ denote the α-accessibility indices of access structures $\Gamma_\mathcal{P}$ and $\Gamma_{\mathcal{P}\setminus\{P_i\}}$, respectively.

Proof We will denote by $\Delta^\alpha[\Gamma_\mathcal{P};P_i]$ the difference $\Omega^\alpha(\Gamma_\mathcal{P})-\Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}})$. According to the definition of the α-accessibility index,

$$\begin{aligned}
\Delta^\alpha[\Gamma_\mathcal{P};P_i] &= \sum_{s=1}^{n} \alpha^s(1-\alpha)^{n-s}|\Gamma_\mathcal{P}[s]| - \sum_{s=1}^{n-1} \alpha^s(1-\alpha)^{n-1-s}|\Gamma_{\mathcal{P}\setminus\{P_i\}}[s]| \\
&= \sum_{s=1}^{n} \alpha^s(1-\alpha)^{n-s}|\Gamma_\mathcal{P}[s;P_i]| + \sum_{s=1}^{n-1} \alpha^s(1-\alpha)^{n-s}|\Gamma_{\mathcal{P}\setminus\{P_i\}}[s]| - \\
&\qquad \sum_{s=1}^{n-1} \alpha^s(1-\alpha)^{n-1-s}|\Gamma_{\mathcal{P}\setminus\{P_i\}}[s]| \\
&= \sum_{s=1}^{n} \alpha^s(1-\alpha)^{n-s}|\Gamma_\mathcal{P}[s;P_i]| - \sum_{s=1}^{n-1} \alpha^{s+1}(1-\alpha)^{n-1-s}|\Gamma_{\mathcal{P}\setminus\{P_i\}}[s]| \\
&= \sum_{s=1}^{n} \alpha^s(1-\alpha)^{n-s}|\Gamma_\mathcal{P}[s;P_i]| - \sum_{s=2}^{n} \alpha^s(1-\alpha)^{n-s}|\Gamma_{\mathcal{P}\setminus\{P_i\}}[s-1]| \\
&= I_\alpha(P_i,\Gamma_\mathcal{P})
\end{aligned}$$

where, in the last step, we have used the fact that $|\Gamma_{\mathcal{P}\setminus\{P_i\}}[0]| = 0$. $\square$

The above Theorem offers an alternative method to compute the α-accessibility of a participant, especially when both α-accessibility indices of the respective structures are known, as it happens in the following situations.

Corollary 3.6 *Let $\mathcal{P}$ be a set of n participants.*

(i) If $C = \{P_{i_1},\ldots,P_{i_k}\}$ with $|C| = k \leq n$, then $I_\alpha(P_i, < C >) = \alpha^k\ \forall P_i \in C$, whereas $I_\alpha(P_j, < C >) = 0\ \forall P_j \in \mathcal{P}\setminus C$, $\forall\alpha \in (0,1)$.

(ii) For a (t,n)-threshold access structure,

$$I_\alpha(P_i,[t,n]) = \sum_{k=t}^{n}(-1)^{k-t}\binom{k-1}{k-t}\binom{n-1}{k-1}\alpha^k \quad \forall P_i \in \mathcal{P},\ \forall\alpha \in (0,1).$$

Proof (i) All participants belonging to subset C are equivalent in the access structure $< C >$ and, according to Proposition 3.4 (iv), their α-accessibilities coincide. In addition, if participant P_i belongs to C, the access substructure $< C > \setminus \{P_i\}$ becomes the empty structure and its α-accessibility index vanishes. This way, according to Proposition 2.3 (i), if participant $P_i \in C$, $I_\alpha(P_i, < C >) = \Omega^\alpha(< C >) = \alpha^k$.

Participants in $\mathcal{P} \setminus C$ are null participants and their α-accessibilities are 0.

(ii) Now, all participants in $\mathcal{P}$ are equivalent. If $\Gamma_\mathcal{P}$ is a (t, n)-threshold access structure, $\Gamma_{\mathcal{P}\setminus\{P_i\}}$ is a $(t, n-1)$-threshold access structure so that we can use Proposition 2.3 (ii) for these access structures:

$$\begin{aligned} I_\alpha(P_i, [t, n]) &= \Omega^\alpha([t, n]) - \Omega^\alpha([t, n-1]) \\ &= \sum_{k=t}^{n} (-1)^{k-t} \binom{k-1}{k-t} \binom{n}{k} \alpha^k - \sum_{k=t}^{n-1} (-1)^{k-t} \binom{k-1}{k-t} \binom{n-1}{k} \alpha^k \\ &= (-1)^{n-t} \binom{n-1}{n-t} \alpha^n + \sum_{k=t}^{n-1} (-1)^{k-t} \binom{k-1}{k-t} \binom{n-1}{k-1} \alpha^k \end{aligned}$$

and the expression in the statement directly follows for every $\alpha \in (0, 1)$. $\square$

In Section 2 we introduced the concept of strategic equivalence in order to compare access structures. Now we try to compare participants in a given access structure by means of the following concept.

Definition 3.7 *Let $\Gamma_\mathcal{P}$ be an access structure defined on a set of participants $\mathcal{P}$. Two distinct participants P_i and P_j are strategically equivalent in $\Gamma_\mathcal{P}$, $P_i \backsim_{\Gamma_\mathcal{P}} P_j$, iff*

$$|\Gamma_\mathcal{P}[s; P_i]| = |\Gamma_\mathcal{P}[s; P_j]|, \quad s = 1, \ldots, n = |\mathcal{P}|.$$

Remark 3.8 An alternative definition of strategically equivalent participants is given by

$$P_i \backsim_{\Gamma_\mathcal{P}} P_j \;\Leftrightarrow |\Gamma_{\mathcal{P}\setminus\{P_i\}}[s]| = |\Gamma_{\mathcal{P}\setminus\{P_j\}}[s]|, \quad s = 1, \ldots, n-1.$$

To see the equivalence between both definitions, it suffices to consider the following families of identities for $s = 1, \ldots, n-1$:

$$|\Gamma_\mathcal{P}[s]| = |\Gamma_\mathcal{P}[s; P_i]| + |\Gamma_{\mathcal{P}\setminus\{P_i\}}[s]| \quad \text{and} \quad |\Gamma_\mathcal{P}[s]| = |\Gamma_\mathcal{P}[s; P_j]| + |\Gamma_{\mathcal{P}\setminus\{P_j\}}[s]|.$$

Conditions $|\Gamma_\mathcal{P}[s; P_i]| = |\Gamma_\mathcal{P}[s; P_j]|$ for $s = 1, \ldots, n-1$ are equivalent to conditions $|\Gamma_{\mathcal{P}\setminus\{P_j\}}[s]| = |\Gamma_{\mathcal{P}\setminus\{P_i\}}[s]|$ for $s = 1, \ldots, n-1$.

The last condition in Definition 3.7 for $n = |\mathcal{P}|$ is always satisfied: $|\Gamma_\mathcal{P}[n; P_i]| = |\Gamma_\mathcal{P}[n; P_j]| = |\Gamma_\mathcal{P}[n]|$, $\forall P_i, P_j \in \mathcal{P}$, $\forall \Gamma_\mathcal{P}$ access structure on $\mathcal{P}$. Since we have considered monotonic access structures, this amount only takes value 0 for the empty structure and value 1 otherwise.

Lemma 3.9 *Let $\Gamma_\mathcal{P}$ be an access structure defined on a set of n participants $\mathcal{P} = \{P_1, \ldots, P_n\}$. From the vector of α-accessibilities $I_\alpha(\Gamma_\mathcal{P})$ with $\alpha \in (0, 1)$, we can determine the amounts $|\Gamma_\mathcal{P}[s; P_i]|$ for $s = 1, \ldots, n$ and $i = 1, \ldots, n = |\mathcal{P}|$.*

Proof We proceed as in the proof of Theorem 2.10, but now for the α-accessibility of each participant $P_i \in \mathcal{P}$:

$$\begin{aligned} I_\alpha(P_i, \Gamma_\mathcal{P}) &= \sum_{s=1}^{n} \alpha^s (1-\alpha)^{n-s} \big\{ |\Gamma_\mathcal{P}[s; P_i]| - |\Gamma_{\mathcal{P}\setminus\{P_i\}}[s-1]| \big\} \\ &= \sum_{k=1}^{n} \alpha^k \sum_{s=1}^{k} (-1)^{k-s} \binom{n-s}{n-k} \big\{ |\Gamma_\mathcal{P}[s; P_i]| - |\Gamma_{\mathcal{P}\setminus\{P_i\}}[s-1]| \big\} \end{aligned}$$

Given the α-accessibility of each participant $P_i \in \mathcal{P}$ as a polynomial in α, $I_\alpha(P_i, \Gamma_\mathcal{P}) = \sum_{k=1}^{n} c_{k,i} \alpha^k$, it is possible to determine the amounts

$$|\Gamma_\mathcal{P}[s; P_i]| - |\Gamma_{\mathcal{P}\setminus\{P_i\}}[s-1]| = d_{s,i} \quad s = 1, \ldots, n, \; i = 1, \ldots, n.$$

Here, $d_{s,i}$ are known and its value is obtained from $c_{1,i}, \ldots, c_{n,i}$, for each $i = 1, \ldots, n$.

For $s = 1$, $|\Gamma_\mathcal{P}[1; P_i]| - |\Gamma_{\mathcal{P}\setminus\{P_i\}}[0]| = |\Gamma_\mathcal{P}[1; P_i]| = d_{1,i}$, so that $|\Gamma_\mathcal{P}[1; P_i]|$ are determined for $i = 1, \ldots, n$.

Assume these amounts known for cardinality $s = k$ ($1 \leq k < n$), $|\Gamma_\mathcal{P}[k; P_i]|$, for $i = 1, \ldots, n$. Then,

$$|\Gamma_\mathcal{P}[k+1; P_i]| = d_{k+1,i} + |\Gamma_{\mathcal{P}\setminus\{P_i\}}[k]| = d_{k+1,i} + \frac{1}{k} \sum_{j=1}^{n} |\Gamma_\mathcal{P}[k; P_j]| - |\Gamma_\mathcal{P}[k; P_i]|,$$

since $|\Gamma_\mathcal{P}[k]| = |\Gamma_\mathcal{P}[k; P_i]| + |\Gamma_{\mathcal{P}\setminus\{P_i\}}[k]|$ and $\sum_{j=1}^{n} |\Gamma_\mathcal{P}[k; P_j]| = k \; |\Gamma_\mathcal{P}[k]|$.

For $i = 1, \ldots, n$, we have proved that each amount $|\Gamma_\mathcal{P}[k+1; P_i]|$, is determined by means of all numbers $|\Gamma_\mathcal{P}[k; P_i]|$ for $i = 1, \ldots, n$. This finishes the proof. □

Theorem 3.10 *Let $\mathcal{P} = \{P_1, \ldots, P_n\}$ be a set of n participants and $\Gamma_\mathcal{P}$ an access structure on $\mathcal{P}$. Two distinct participants P_i and P_j are strategically equivalent in $\Gamma_\mathcal{P}$ if and only if their α-accessibilities coincide for every $\alpha \in (0,1)$.*

Proof If participants P_i and P_j are strategically equivalent in $\Gamma_\mathcal{P}$, $|\Gamma_\mathcal{P}[s; P_i]| = |\Gamma_\mathcal{P}[s; P_j]|$, for $s = 1, \ldots, n = |\mathcal{P}|$ and $|\Gamma_{\mathcal{P}\setminus\{P_i\}}[s]| = |\Gamma_{\mathcal{P}\setminus\{P_j\}}[s]|$, for $s = 1, \ldots, n-1$; this guarantees the equality $I_\alpha(P_i, \Gamma_\mathcal{P}) = I_\alpha(P_j, \Gamma_\mathcal{P})$.

Conversely, according to Lemma 3.9, once the α-accessibility of a participant P_i is given for every $\alpha \in (0,1)$, the amounts $|\Gamma_\mathcal{P}[s; P_i]|$ are univocally determined for $s = 1, \ldots, n$: if $I_\alpha(P_i, \Gamma_\mathcal{P}) = I_\alpha(P_j, \Gamma_\mathcal{P})$, in a first step, $d_{s,i} = d_{s,j}$ for $s = 1, \ldots, n$, and, in a second step, $|\Gamma_\mathcal{P}[s; P_i]| = |\Gamma_\mathcal{P}[s; P_j]|$ for $s = 1, \ldots, n$, since these amounts depend, in addition, on values $|\Gamma_\mathcal{P}[\widetilde{s}; P_k]|$ for all $P_k \in \mathcal{P}$ and $1 \leq \widetilde{s} < s$. □

Example 3.11 Let $\Gamma_\mathcal{P}$ be an access structure defined on a set with four participants $\mathcal{P}$. We assume a vector of α-accessibilities given by

$$I_\alpha(\Gamma_\mathcal{P}) = (\; 3\alpha^2 - 4\alpha^3 + \alpha^4, \; 2\alpha^2 - 3\alpha^3 + \alpha^4, \; 2\alpha^2 - 3\alpha^3 + \alpha^4, \; \alpha^2 - 2\alpha^3 + \alpha^4).$$

To determine the amounts $|\Gamma_{\mathcal{P}}[s; P_i]|$ for sizes $s = 1,\ 2,\ 3,\ 4$, and participants P_1, P_2, P_3 and P_4, we will follow the notation and procedure introduced in Lemma 3.9:

$$d_{s,i} = |\Gamma_{\mathcal{P}}[s; P_i]| - |\Gamma_{\mathcal{P}\setminus\{P_i\}}[s-1]| \quad s = 1, \ldots, 4,\ i = 1, \ldots, 4.$$

For participant P_1:

$$\left.\begin{array}{ll} d_{1,1} & = 0 \\ -3d_{1,1} + d_{2,1} & = 3 \\ 3d_{1,1} - 2d_{2,1} + d_{3,1} & = -4 \\ -d_{1,1} + d_{2,1} - d_{3,1} + d_{4,1} & = 1 \end{array}\right\} \Rightarrow\ (d_{1,1}, d_{2,1}, d_{3,1}, d_{4,1}) = (0, 3, 2, 0).$$

An analogous procedure for participants P_2, P_3 and P_4 leads us to determine the remaining unknowns:

$$(d_{1,2}, d_{2,2}, d_{3,2}, d_{4,2}) = (d_{1,3}, d_{2,3}, d_{3,3}, d_{4,3}) = (0, 2, 1, 0),\ (d_{1,4}, d_{2,4}, d_{3,4}, d_{4,4}) = (0, 1, 0, 0).$$

Now, in a first level, the amounts $|\Gamma_{\mathcal{P}}[1; P_i]|$ for each participant in $\mathcal{P}$ directly appear:

$$(|\Gamma_{\mathcal{P}}[1; P_1]|, |\Gamma_{\mathcal{P}}[1; P_2]|, |\Gamma_{\mathcal{P}}[1; P_3]|, |\Gamma_{\mathcal{P}}[1; P_4]|) = (d_{1,1}, d_{1,2}, d_{1,3}, d_{1,4}) = (0, 0, 0, 0),$$

that is, no participant individually forms authorized coalition in $\Gamma_{\mathcal{P}}$. In a second level we obtain the amounts $|\Gamma_{\mathcal{P}}[2; P_i]|$ for each participant in $\mathcal{P}$. In this particular case, since $|\Gamma_{\mathcal{P}}[1; P_i]| = 0\ \forall P_i \in \mathcal{P}$, we have

$$(|\Gamma_{\mathcal{P}}[2; P_1]|, |\Gamma_{\mathcal{P}}[2; P_2]|, |\Gamma_{\mathcal{P}}[2; P_3]|, |\Gamma_{\mathcal{P}}[2; P_4]|) = (d_{2,1}, d_{2,2}, d_{2,3}, d_{2,4}) = (3, 2, 2, 1).$$

A third level allows us obtain the amounts $|\Gamma_{\mathcal{P}}[3; P_i]|$ for each participant in $\mathcal{P}$. We compute, for instance, $|\Gamma_{\mathcal{P}}[3; P_1]|$.

$$\begin{aligned} |\Gamma_{\mathcal{P}}[3; P_1]| = d_{3,1} + |\Gamma_{\mathcal{P}\setminus\{P_1\}}[2]| &= d_{3,1} + \frac{1}{2}\sum_{j=1}^{4} |\Gamma_{\mathcal{P}}[2; P_j]| - |\Gamma_{\mathcal{P}}[2; P_1]| \\ &= 2 + \frac{3+2+2+1}{2} - 3 = 3 \end{aligned}$$

Analogous computations lead us to obtain the remaining amounts in this level:

$$(|\Gamma_{\mathcal{P}}[3; P_1]|, |\Gamma_{\mathcal{P}}[3; P_2]|, |\Gamma_{\mathcal{P}}[3; P_3]|, |\Gamma_{\mathcal{P}}[3; P_4]|) = (3, 3, 3, 3).$$

Finally, in a fourth level, it is obtained $|\Gamma_{\mathcal{P}}[4; P_i]|$ for each participant in $\mathcal{P}$.

$$\begin{aligned} |\Gamma_{\mathcal{P}}[4; P_1]| = d_{4,1} + |\Gamma_{\mathcal{P}\setminus\{P_1\}}[3]| &= d_{4,1} + \frac{1}{3}\sum_{j=1}^{4} |\Gamma_{\mathcal{P}}[3; P_j]| - |\Gamma_{\mathcal{P}}[3; P_1]| \\ &= 0 + \frac{3+3+3+3}{3} - 3 = 1. \end{aligned}$$

It is an expected result. In the last level, when $s = n = |\mathcal{P}|$, all amounts $|\Gamma_{\mathcal{P}}[4; P_i]|$ coincide and their value equals $|\Gamma_{\mathcal{P}}[4]| = 1$ for every nonempty access structure.

Note that, according to Theorem 3.10, $|\Gamma_{\mathcal{P}}[s; P_2]| = |\Gamma_{\mathcal{P}}[s; P_3]|$ for $s = 1, 2, 3, 4$, since both participants P_2 and P_3 have equal α-accessibility for every $\alpha \in (0, 1)$.

Definition 3.12 *Let $\mathcal{P}$ and $\mathcal{P}'$ be two sets with a same number of participants, $|\mathcal{P}| = |\mathcal{P}'|$. The participants of access structures $\Gamma_{\mathcal{P}}$ and $\Gamma'_{\mathcal{P}'}$ have equal α-accessibilities if there exists a one-to-one map $\pi : \mathcal{P} \to \mathcal{P}'$ so that*

$$I_\alpha(P_i, \Gamma_{\mathcal{P}}) = I_\alpha(\pi(P_i), \Gamma'_{\mathcal{P}'}) \quad \forall P_i \in \mathcal{P},\ \forall \alpha \in (0,1).$$

Corollary 3.13 *If the participants of $\Gamma_{\mathcal{P}}$ and $\Gamma'_{\mathcal{P}'}$ have the same α-accessibilities, then $\Omega^\alpha(\Gamma_{\mathcal{P}}) = \Omega^\alpha(\Gamma'_{\mathcal{P}'})$ $\forall \alpha \in (0,1)$.*

Proof For each participant P_i in access structure $\Gamma_{\mathcal{P}}$, let us consider P'_j the participant in access structure $\Gamma'_{\mathcal{P}'}$ obtained from the one-to-one map π, i.e., $P'_j = \pi(P_i)$. In a similar way as in Theorem 3.10, the amounts $\Gamma_{\mathcal{P}}[s; P_i]$, $s = 1, \ldots, n$, can be obtained from the vector of α-accessibilities and then,

$$\Gamma_{\mathcal{P}}[s; P_i] = \Gamma'_{\mathcal{P}'}[s; \pi(P_i)] = \Gamma'_{\mathcal{P}'}[s; P'_j] \quad s = 1, \ldots, n.$$

In a wide sense, we can affirm that participants P_i and P'_j are strategically equivalent (by means of π), although they belong to different sets of participants where access structures $\Gamma_{\mathcal{P}}$ and $\Gamma'_{\mathcal{P}'}$ are respectively defined. Consequently, for both access structures, the number of authorized coalitions with a same cardinality is coincident:

$$\Gamma_{\mathcal{P}}[s] = \frac{1}{s}\sum_{i=1}^{n} \Gamma_{\mathcal{P}}[s; P_i] = \frac{1}{s}\sum_{j=1}^{n} \Gamma'_{\mathcal{P}'}[s; P'_j] = \Gamma'_{\mathcal{P}'}[s] \quad s = 1, \ldots, n = |\mathcal{P}| = |\mathcal{P}'|.$$

The above equalities guarantee the coincidence of α-accessibility indices of both structures. □

Situations provided by the real world lead us to compare two access structures defined on different sets of participants when both sets have the same size. Thus, it seems reasonable to extend the concept of isomorphic access structures to these situations. With an abuse of notation, we will use a similar terminology.

Definition 3.14 *Let $\mathcal{P}$ and $\mathcal{P}'$ be two sets of participants with $|\mathcal{P}| = |\mathcal{P}'|$. Two access structures $\Gamma_{\mathcal{P}}$ and $\Gamma'_{\mathcal{P}'}$ are isomorphic if there exists a one-to-one map $\pi : \mathcal{P} \to \mathcal{P}'$ verifying $\pi(S)$ is an authorized coalition of $\Gamma'_{\mathcal{P}'}$ if and only if S is an authorized coalition of $\Gamma_{\mathcal{P}}$, for every subset $S \subseteq \mathcal{P}$.*

Corollary 3.15 *If $\Gamma_{\mathcal{P}}$ and $\Gamma'_{\mathcal{P}'}$ are two isomorphic access structures then, the α-accessibilities of their participants coincide.* □

4 Do the α-accessibilities determine the access structure?

Given two access structures defined on sets with the same number of participants, we have considered three concepts: (i) isomorphic, (ii) with an equal α-accessibilities for the participants, and (iii) strategically equivalent. Theorem 3.10 shows that an equal α-accessibilities between two structures $\Gamma_{\mathcal{P}}$ and $\Gamma'_{\mathcal{P}'}$, with $|\mathcal{P}| = |\mathcal{P}'|$, is equivalent to strategically equivalence (in wide sense) between the corresponding participants related by means of the one-to-one map, whereas Theorem 2.10 affirms that two access structures have an equal α-accessibility index if and only if they are strategically equivalent. From the previous corollaries, (i) implies (ii) and (ii) implies (iii). From now on, we will try to analyze the converse properties.

According to the results in Section 2, all access structures with three participants are univocally determined (up to isomorphisms) by means of their α-accessibility indices (see Table 1 in Example 2.7). Nevertheless, for four participants, it is possible to find two non-isomorphic access structures strategically equivalent, i.e., with a same vector of cardinalities of authorized coalitions and, consequently, with an equal α-accessibility index.

Example 4.1 Let us retake Example 2.9. The non-isomorphic access structures $\Gamma_{\mathcal{P}}$ and $\Gamma'_{\mathcal{P}}$ defined on $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$ with basis $(\Gamma_{\mathcal{P}})_0 = \{P_1P_2,\ P_1P_3,\ P_1P_4,\ P_2P_3\}$ and $(\Gamma'_{\mathcal{P}})_0 = \{P_1P_2,\ P_1P_3,\ P_2P_4,\ P_3P_4\}$, respectively, have a common α-accessibility $\Omega^\alpha(\Gamma_{\mathcal{P}}) = \Omega^\alpha(\Gamma'_{\mathcal{P}}) = 4\alpha^2 - 4\alpha^3 + \alpha^4$, $\forall \alpha \in (0,1)$. But, if we compute the respective vectors of α-accessibilities for their participants, both access structures obtain different allocations.

For access structure $\Gamma_{\mathcal{P}}$ we have:

$$\begin{array}{lcl}
(\Gamma_{\mathcal{P}\setminus\{P_1\}})_0 = \{P_2P_3\} & \Rightarrow & \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_1\}}) = \alpha^2 \\
(\Gamma_{\mathcal{P}\setminus\{P_2\}})_0 = \{P_1P_3,\ P_1P_4\} & \Rightarrow & \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_2\}}) = 2\alpha^2 - \alpha^3 \\
(\Gamma_{\mathcal{P}\setminus\{P_3\}})_0 = \{P_1P_2,\ P_1P_4\} & \Rightarrow & \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_3\}}) = 2\alpha^2 - \alpha^3 \\
(\Gamma_{\mathcal{P}\setminus\{P_4\}})_0 = \{P_1P_2,\ P_1P_3,\ P_2P_3\} & \Rightarrow & \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_4\}}) = 3\alpha^2 - 2\alpha^3
\end{array}$$

Theorem 3.5 allows us to compute the vector of α-accessibilities for the participants of $\Gamma_{\mathcal{P}}$:

$$I_\alpha(\Gamma_{\mathcal{P}}) = (\ 3\alpha^2 - 4\alpha^3 + \alpha^4,\ 2\alpha^2 - 3\alpha^3 + \alpha^4,\ 2\alpha^2 - 3\alpha^3 + \alpha^4,\ \alpha^2 - 2\alpha^3 + \alpha^4).$$

In a similar way, for access structure $\Gamma'_{\mathcal{P}}$, $\Omega^\alpha(\Gamma'_{\mathcal{P}\setminus\{P_i\}}) = 2\alpha^2 - \alpha^3\ \forall P_i \in \mathcal{P}$ and the α-accessibility of each participant is $I_\alpha(P_i, \Gamma'_{\mathcal{P}}) = 2\alpha^2 - 3\alpha^3 + \alpha^4\ \forall P_i \in \mathcal{P}$, $\forall \alpha \in (0,1)$.

This example shows that, for access structures with four or more participants, the converse of Corollary 3.13 is not true.

Now, we would analyze a different aspect. All access structures with three participants can be reconstructed from the α-accessibility index, up to isomorphisms: it suffices to observe Table 1 in Example 2.7 from the right column towards the left column. We ask ourselves for a similar reconstruction from the vector of α-accessibilities of the participants.

Let us assume that a vector of α-accessibilities is given in an access structure with four participants. Take, for instance, the corresponding to structure $\Gamma_{\mathcal{P}}$ in Example 4.1:

$$I_\alpha(\Gamma_{\mathcal{P}}) = (\ 3\alpha^2 - 4\alpha^3 + \alpha^4,\ 2\alpha^2 - 3\alpha^3 + \alpha^4,\ 2\alpha^2 - 3\alpha^3 + \alpha^4,\ \alpha^2 - 2\alpha^3 + \alpha^4).$$

Note that this is the vector in Example 3.11. From it, the number of authorized coalitions for each participant and each coalition size was obtained:

$$\begin{aligned}
(|\Gamma_{\mathcal{P}}[1;P_1]|, |\Gamma_{\mathcal{P}}[2;P_1]|, |\Gamma_{\mathcal{P}}[3;P_1]|, |\Gamma_{\mathcal{P}}[4;P_1]|) &= (0,3,3,1),\\
(|\Gamma_{\mathcal{P}}[1;P_k]|, |\Gamma_{\mathcal{P}}[2;P_k]|, |\Gamma_{\mathcal{P}}[3;P_k]|, |\Gamma_{\mathcal{P}}[4;P_k]|) &= (0,2,3,1),\ k=2,3,\\
(|\Gamma_{\mathcal{P}}[1;P_4]|, |\Gamma_{\mathcal{P}}[2;P_4]|, |\Gamma_{\mathcal{P}}[3;P_4]|, |\Gamma_{\mathcal{P}}[4;P_4]|) &= (0,1,3,1).
\end{aligned}$$

For all participants $i = 1,2,3,4$: $|\Gamma_{\mathcal{P}}[1;P_i]| = 0$ implies no participant individually forms authorized coalition; $|\Gamma_{\mathcal{P}}[3;P_i]| = 3$ and $|\Gamma_{\mathcal{P}}[4;P_i]| = 1$ implies all possible coalitions with a given participant of sizes three and four are authorized coalitions. Authorized coalitions with size 2:

$$|\Gamma_{\mathcal{P}}[2;P_1]| = 3 \ \Rightarrow\ P_1P_2,\ P_1P_3,\ P_1P_4\ \in \Gamma_{\mathcal{P}} \ \Rightarrow\ P_1P_2,\ P_1P_3,\ P_1P_4\ \in (\Gamma_{\mathcal{P}})_0$$

Since $|\Gamma_{\mathcal{P}}[2;P_4]| = 1$, only one coalition with size 2 containing participant P_4 is an authorized coalition: P_1P_4. For participants P_2 and P_3, since $|\Gamma_{\mathcal{P}}[2;P_2]| = 2$ and $|\Gamma_{\mathcal{P}}[2;P_3]| = 2$, in addition to authorized coalitions P_1P_2 and P_1P_3, another authorized coalition of size 2 necessarily exists, this is P_2P_3. Thus, also P_2P_3 belongs to $(\Gamma_{\mathcal{P}})_0$. Finally, if P_1P_2, P_1P_3, P_1P_4 and P_2P_3 are authorized coalitions, all coalitions of size three and four also are authorized coalitions, so that these four coalitions suffice to form the basis of $\Gamma_{\mathcal{P}}$,

$$(\Gamma_{\mathcal{P}})_0 = \{P_1P_2,\ P_1P_3,\ P_1P_4,\ P_2P_3\},$$

and the access structure $\Gamma_{\mathcal{P}}$ has been reconstructed from the vector of α-accessibilities of the participants.

This process of reconstruction also can be followed for the access structure $\Gamma'_{\mathcal{P}}$. Here, as every participant has the same α-accessibility $I_\alpha(P_i, \Gamma'_{\mathcal{P}}) = 2\alpha^2 - 3\alpha^3 + \alpha^4$, there are also the same number of authorized coalitions for each one of them:

$$(|\Gamma'_{\mathcal{P}}[1;P_i]|, |\Gamma'_{\mathcal{P}}[2;P_i]|, |\Gamma'_{\mathcal{P}}[3;P_i]|, |\Gamma'_{\mathcal{P}}[4;P_i]|) = (0,2,3,1) \quad i = 1,2,3,4.$$

No participant individually forms authorized coalition, whereas every participant belongs to two authorized coalition of size 2. Diverse structures can be obtained. For instance, beginning by participant P_1: (i) P_1P_2, $P_1P_3 \in \Gamma'^1_{\mathcal{P}}$ (ii) P_1P_2, $P_1P_4 \in \Gamma'^2_{\mathcal{P}}$ or (iii) P_1P_3, $P_1P_4 \in \Gamma'^3_{\mathcal{P}}$. Next, necessarily, in case (i) P_2P_4, $P_3P_4 \in \Gamma'^1_{\mathcal{P}}$, in case (ii) P_2P_3, $P_3P_4 \in \Gamma'^2_{\mathcal{P}}$ and in case (iii) P_2P_3, $P_2P_4 \in \Gamma'^3_{\mathcal{P}}$. All cases assure $|\Gamma'_{\mathcal{P}}[3;P_i]| = 3$ and $|\Gamma'_{\mathcal{P}}[4;P_i]| = 1$ for $i = 1,2,3,4$.

For this vector of α-accessibilities we obtain three possible access structures with respective basis

$$\begin{aligned}
(\Gamma'^1_{\mathcal{P}})_0 &= \{P_1P_2,\ P_1P_3,\ P_2P_4,\ P_3P_4\},\\
(\Gamma'^2_{\mathcal{P}})_0 &= \{P_1P_2,\ P_1P_4,\ P_2P_3,\ P_3P_4\},\\
(\Gamma'^3_{\mathcal{P}})_0 &= \{P_1P_3,\ P_1P_4,\ P_2P_3,\ P_2P_4\}.
\end{aligned}$$

Access structures $\Gamma'^1_{\mathcal{P}}$, $\Gamma'^2_{\mathcal{P}}$ and $\Gamma'^3_{\mathcal{P}}$ are isomorphic and again the access structure has been determined from the vector of α-accessibilities of the participants, up to isomorphisms.

Theorem 4.2 *Up to isomorphisms, every access structure defined in sets with at most four participants can be reconstructed from the vector of α-accessibilities of the participants.*

Proof For two or three participants the conclusion easily follows. More precisely, in both cases, it suffices the α-accessibility index to reconstruct the access structures and, by Corollary 3.13, the vector of α-accessibilities of the participants allows us to determine the α-accessibility index of the structure.

In $\mathcal{P} = \{P_1, P_2\}$ we have three possible access structures, up to isomorphisms.

Table 3 shows that each access structure is univocally determined from its α-accessibility in case of two participants and Table 1 in Example 2.7 proves it for three participants.

$(\Gamma_\mathcal{P})_0$	$\Omega^\alpha(\Gamma_\mathcal{P})$	$I_\alpha(\Gamma_\mathcal{P})$
P_1P_2	α^2	$(\alpha^2,\ \alpha^2)$
P_1	α	$(\alpha,\ 0)$
$P_1,\ P_2$	$2\alpha - \alpha^2$	$(\alpha - \alpha^2,\ \alpha - \alpha^2)$

Table 3: Access structures with two participants

For structures of four participants we know that it is necessary to work with the α-accessibilities of the participants. We have checked (see Appendix) one by one all twenty [1] access structures without null participants, up to isomorphisms, as we did with $\Gamma_\mathcal{P}$ and $\Gamma'_\mathcal{P}$ in Example 4.1. In each case, the access structure can be reconstructed up to isomorphisms.

Access structures of four participants with one, two or three null participants have been studied as access structures without null participants of three, two or one participant, respectively. □

The conclusion in the previous Theorem is not possible when the number of participants increases. We propose an example in a set of five participants where an equal vector of α-accessibilities for the participants leads to two non-isomorphic access structures.

Example 4.3 Let $\mathcal{P} = \{P_1, P_2, P_3, P_4, P_5\}$ be a set of five participants whit α-accessibilities given by $I_\alpha(P_1, \Gamma_\mathcal{P}) = 3\alpha^2 - 5\alpha^3 + \alpha^4 + \alpha^5$, $I_\alpha(P_k, \Gamma_\mathcal{P}) = 2\alpha^2 - 3\alpha^3 + \alpha^5$, $k = 2, 3, 4$, and $I_\alpha(P_5, \Gamma_\mathcal{P}) = \alpha^2 - \alpha^3 - \alpha^4 + \alpha^5$.

Following the procedure explained in Lemma 3.9 we obtain the amounts

$$(d_{1,1}, d_{2,1}, d_{3,1}, d_{4,1}, d_{5,1}) = (0, 3, 4, 0, 0),$$
$$(d_{1,k}, d_{2,k}, d_{3,k}, d_{4,k}, d_{5,k}) = (0, 2, 3, 0, 0),\ k = 2, 3, 4,\ \text{and}$$
$$(d_{1,5}, d_{2,5}, d_{3,5}, d_{4,5}, d_{5,5}) = (0, 1, 2, 0, 0)$$

that allows us to compute the number of authorized coalitions for each participant and each coalition size:

$$(|\Gamma_{\mathcal{P}}[1;P_1]|, |\Gamma_{\mathcal{P}}[2;P_1]|, |\Gamma_{\mathcal{P}}[3;P_1]|, |\Gamma_{\mathcal{P}}[4;P_1]|, |\Gamma_{\mathcal{P}}[5;P_1]|) = (0,3,6,4,1),$$
$$(|\Gamma_{\mathcal{P}}[1;P_k]|, |\Gamma_{\mathcal{P}}[2;P_k]|, |\Gamma_{\mathcal{P}}[3;P_k]|, |\Gamma_{\mathcal{P}}[4;P_k]|, |\Gamma_{\mathcal{P}}[5;P_k]|) = (0,2,6,4,1), \; k = 2,3,4,$$
$$(|\Gamma_{\mathcal{P}}[1;P_4]|, |\Gamma_{\mathcal{P}}[2;P_4]|, |\Gamma_{\mathcal{P}}[3;P_4]|, |\Gamma_{\mathcal{P}}[4;P_4]|, |\Gamma_{\mathcal{P}}[5;P_5]|) = (0,1,6,4,1).$$

At least two non-isomorphic access structures $\Gamma^1_{\mathcal{P}}$ and $\Gamma^2_{\mathcal{P}}$ satisfy all these conditions. Their respective basis are

$$(\Gamma^1_{\mathcal{P}})_0 = \{P_1P_2,\; P_1P_3,\; P_1P_4,\; P_2P_3,\; P_4P_5\} \quad \text{and}$$
$$(\Gamma^2_{\mathcal{P}})_0 = \{P_1P_2,\; P_1P_4,\; P_1P_5,\; P_2P_3,\; P_3P_4,\; P_2P_4P_5\}.$$

References

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5 Appendix

Access structures in sets of four non-null participants $\mathcal{P} = \{P_1, P_2, P_3, P_4, \}$, up isomorphisms, with α-accessibility index, α-accessibilities of the participants and determination of structure from their respective α-accessibilities.

(Reference number) Basis
α-accessibility indices $\Omega^\alpha(\Gamma_\mathcal{P})$ and $\Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}})$, $i \in \{1,2,3,4\}$
Vector of α-accessibilities $I_\alpha(\Gamma_\mathcal{P})$
Amounts $(d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i})$, $i \in \{1,2,3,4\}$
Number of authorized coalitions $(\lvert\Gamma_\mathcal{P}[1;P_i]\rvert, \lvert\Gamma_\mathcal{P}[2;P_i]\rvert, \lvert\Gamma_\mathcal{P}[3;P_i]\rvert, \lvert\Gamma_\mathcal{P}[4;P_i]\rvert)$, $i \in \{1,2,3,4\}$
$\Rightarrow$ **Basis of possible access structures from the α-accessibilities**

(1) $(\mathbf{\Gamma}_\mathcal{P})_\mathbf{0} = \{\mathbf{P_1},\ \mathbf{P_2},\ \mathbf{P_3},\ \mathbf{P_4}\}$

$\Omega^\alpha(\Gamma_\mathcal{P}) = 4\alpha - 6\alpha^2 + 4\alpha^3 - \alpha^4$, $\Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = 3\alpha - 3\alpha^2 + \alpha^3$, $i = 1,2,3,4.$

$I_\alpha(P_i, \Gamma_\mathcal{P}) = \alpha - 3\alpha^2 + 3\alpha^3 - \alpha^4$, $i = 1,2,3,4.$

$(d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (1,0,0,0)$, $i = 1,2,3,4.$

$(|\Gamma_\mathcal{P}[1;P_i]|, |\Gamma_\mathcal{P}[2;P_i]|, |\Gamma_\mathcal{P}[3;P_i]|, |\Gamma_\mathcal{P}[4;P_i]|) = (1,3,3,1)$, $i = 1,2,3,4.$

$\Rightarrow$ $(\mathbf{\Gamma}_\mathcal{P})_\mathbf{0} = \{\mathbf{P_1},\ \mathbf{P_2},\ \mathbf{P_3},\ \mathbf{P_4}\}$

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(2) $(\mathbf{\Gamma}_\mathcal{P})_\mathbf{0} = \{\mathbf{P_1},\ \mathbf{P_2},\ \mathbf{P_3 P_4}\}$

$\Omega^\alpha(\Gamma_\mathcal{P}) = 2\alpha - 2\alpha^3 + \alpha^4$,
$\Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = \alpha + \alpha^2 - \alpha^3$, $i = 1,2$, $\Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_j\}}) = 2\alpha - \alpha^2$, $j = 3,4.$

$I_\alpha(P_i, \Gamma_\mathcal{P}) = \alpha - \alpha^2 - \alpha^3 + \alpha^4$, $i = 1,2$, $I_\alpha(P_j, \Gamma_\mathcal{P}) = \alpha^2 - 2\alpha^3 + \alpha^4$, $j = 3,4.$

$(d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (1,2,0,0)$, $i = 1,2$, $(d_{1,j}, d_{2,j}, d_{3,j}, d_{4,j}) = (0,1,0,0)$, $j = 3,4.$

$(|\Gamma_\mathcal{P}[1;P_i]|, |\Gamma_\mathcal{P}[2;P_i]|, |\Gamma_\mathcal{P}[3;P_i]|, |\Gamma_\mathcal{P}[4;P_i]|) = (1,3,3,1)$, $i = 1,2,$
$(|\Gamma_\mathcal{P}[1;P_j]|, |\Gamma_\mathcal{P}[2;P_j]|, |\Gamma_\mathcal{P}[3;P_j]|, |\Gamma_\mathcal{P}[4;P_j]|) = (0,3,3,1)$, $j = 3,4.$

$\Rightarrow$ $(\mathbf{\Gamma}_\mathcal{P})_\mathbf{0} = \{\mathbf{P_1},\ \mathbf{P_2},\ \mathbf{P_3 P_4}\}$

$(\mathbf{3})$ $\quad (\mathbf{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1},\ \mathbf{P_2P_3},\ \mathbf{P_2P4},\ \mathbf{P_3P_4}\}$

$\Omega^{\alpha}(\Gamma_{\mathcal{P}}) = \alpha + 3\alpha^2 - 5\alpha^3 + 2\alpha^4,$
$\Omega^{\alpha}(\Gamma_{\mathcal{P}\setminus\{P_1\}}) = 3\alpha^2 - 2\alpha^3,\ \ \Omega^{\alpha}(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = \alpha + 2\alpha^2 - \alpha^3,\ \ i = 2, 3, 4.$

$I_{\alpha}(P_1, \Gamma_{\mathcal{P}}) = \alpha - 3\alpha^3 + 2\alpha^4,\ \ I_{\alpha}(P_i, \Gamma_{\mathcal{P}}) = 2\alpha^2 - 4\alpha^3 + 2\alpha^4,\ \ i = 2, 3, 4.$

$(d_{1,1}, d_{2,1}, d_{3,1}, d_{4,1}) = (1, 3, 0, 0),\ \ (d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (0, 2, 0, 0),\ \ i = 2, 3, 4.$

$(|\Gamma_{\mathcal{P}}[1; P_1]|, |\Gamma_{\mathcal{P}}[2; P_1]|, |\Gamma_{\mathcal{P}}[3; P_1]|, |\Gamma_{\mathcal{P}}[4; P_1]|) = (1, 3, 3, 1),$
$(|\Gamma_{\mathcal{P}}[1; P_i]|, |\Gamma_{\mathcal{P}}[2; P_i]|, |\Gamma_{\mathcal{P}}[3; P_i]|, |\Gamma_{\mathcal{P}}[4; P_i]|) = (0, 3, 3, 1),\ \ i = 2, 3, 4.$

$\Rightarrow\ \ (\mathbf{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1},\ \mathbf{P_2P_3},\ \mathbf{P_2P4},\ \mathbf{P_3P_4}\}$

$(\mathbf{4})$ $\quad (\mathbf{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3},\ \mathbf{P_1P_4},\ \mathbf{P_2P_3},\ \mathbf{P_2P4},\ \mathbf{P_3P_4}\}$

$\Omega^{\alpha}(\Gamma_{\mathcal{P}}) = 6\alpha^2 - 8\alpha^3 + 3\alpha^4,\ \ \Omega^{\alpha}(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = 3\alpha^2 - 2\alpha^3,\ \ i = 1, 2, 3, 4.$

$I_{\alpha}(P_i, \Gamma_{\mathcal{P}}) = 3\alpha^2 - 6\alpha^3 + 3\alpha^4,\ \ i = 1, 2, 3, 4.$

$(d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (0, 3, 0, 0),\ \ i = 1, 2, 3, 4.$

$(|\Gamma_{\mathcal{P}}[1; P_i]|, |\Gamma_{\mathcal{P}}[2; P_i]|, |\Gamma_{\mathcal{P}}[3; P_i]|, |\Gamma_{\mathcal{P}}[4; P_i]|) = (0, 3, 3, 1),\ \ i = 1, 2, 3, 4.$

$\Rightarrow\ \ (\mathbf{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3},\ \mathbf{P_1P_4},\ \mathbf{P_2P_3},\ \mathbf{P_2P4},\ \mathbf{P_3P_4}\}$

$(\mathbf{5})$ $\quad (\mathbf{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1},\ \mathbf{P_2P_3},\ \mathbf{P_2P4}\}$

$\Omega^{\alpha}(\Gamma_{\mathcal{P}}) = \alpha + 2\alpha^2 - 3\alpha^3 + \alpha^4,$
$\Omega^{\alpha}(\Gamma_{\mathcal{P}\setminus\{P_1\}}) = 2\alpha^2 - 3\alpha^3,\ \ \Omega^{\alpha}(\Gamma_{\mathcal{P}\setminus\{P_2\}}) = \alpha,\ \ \Omega^{\alpha}(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = \alpha + \alpha^2 - \alpha^3,\ \ i = 3, 4.$

$I_{\alpha}(P_1, \Gamma_{\mathcal{P}}) = \alpha - 2\alpha^3 + \alpha^4,\ \ I_{\alpha}(P_2, \Gamma_{\mathcal{P}}) = 2\alpha - 3\alpha^3 + \alpha^4,$
$I_{\alpha}(P_i, \Gamma_{\mathcal{P}}) = \alpha^2 - 2\alpha^3 + \alpha^4,\ \ i = 3, 4.$

$(d_{1,1}, d_{2,1}, d_{3,1}, d_{4,1}) = (1, 3, 1, 0),\ \ (d_{1,2}, d_{2,2}, d_{3,2}, d_{4,2}) = (0, 2, 1, 0),$
$(d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (0, 1, 0, 0),\ \ i = 3, 4.$

$(|\Gamma_{\mathcal{P}}[1; P_1]|, |\Gamma_{\mathcal{P}}[2; P_1]|, |\Gamma_{\mathcal{P}}[3; P_1]|, |\Gamma_{\mathcal{P}}[4; P_1]|) = (1, 3, 3, 1),$
$(|\Gamma_{\mathcal{P}}[1; P_2]|, |\Gamma_{\mathcal{P}}[2; P_2]|, |\Gamma_{\mathcal{P}}[3; P_2]|, |\Gamma_{\mathcal{P}}[4; P_2]|) = (0, 3, 3, 1),$
$(|\Gamma_{\mathcal{P}}[1; P_i]|, |\Gamma_{\mathcal{P}}[2; P_i]|, |\Gamma_{\mathcal{P}}[3; P_i]|, |\Gamma_{\mathcal{P}}[4; P_i]|) = (0, 2, 3, 1),\ \ i = 3, 4.$

$\Rightarrow\ \ (\mathbf{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1},\ \mathbf{P_2P_3},\ \mathbf{P_2P4}\}$

$(\mathbf{6}) \qquad (\boldsymbol{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3},\ \mathbf{P_1P_4},\ \mathbf{P_2P_3},\ \mathbf{P_2P4}\}$

$\Omega^\alpha(\Gamma_{\mathcal{P}}) = 5\alpha^2 - 6\alpha^3 + 2\alpha^4,$
$\Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = 2\alpha^2 - \alpha^3, \ \ i = 1,2, \ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_j\}}) = 3\alpha^2 - 2\alpha^3, \ \ j = 3,4.$

$I_\alpha(P_i, \Gamma_{\mathcal{P}}) = 3\alpha^2 - 5\alpha^3 + 2\alpha^4, \ \ i = 1,2, \ \ I_\alpha(P_j, \Gamma_{\mathcal{P}}) = 2\alpha^2 - 4\alpha^3 + 2\alpha^4, \ \ j = 3,4.$

$(d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (0,3,1,0), \ \ i = 1,2, \ \ (d_{1,j}, d_{2,j}, d_{3,j}, d_{4,j}) = (0,2,0,0), \ \ j = 3,4.$

$(|\Gamma_{\mathcal{P}}[1;P_i]|, |\Gamma_{\mathcal{P}}[2;P_i]|, |\Gamma_{\mathcal{P}}[3;P_i]|, |\Gamma_{\mathcal{P}}[4;P_i]|) = (0,3,3,1), \ \ i = 1,2,$
$(|\Gamma_{\mathcal{P}}[1;P_j]|, |\Gamma_{\mathcal{P}}[2;P_j]|, |\Gamma_{\mathcal{P}}[3;P_j]|, |\Gamma_{\mathcal{P}}[4;P_j]|) = (0,2,3,1), \ \ j = 3,4.$

$\Rightarrow \ \ (\boldsymbol{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3},\ \mathbf{P_1P_4},\ \mathbf{P_2P_3},\ \mathbf{P_2P4}\}$

$(\mathbf{7}) \qquad (\boldsymbol{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3},\ \mathbf{P_1P_4},\ \mathbf{P_2P_3}\}$

$\Omega^\alpha(\Gamma_{\mathcal{P}}) = 4\alpha^2 - 4\alpha^3 + \alpha^4,$
$\Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_1\}}) = \alpha^2, \ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = 2\alpha^2 - \alpha^3, \ \ i = 2,3, \ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_4\}}) = 3\alpha^2 - 2\alpha^3.$

$I_\alpha(P_1, \Gamma_{\mathcal{P}}) = 3\alpha^2 - 4\alpha^3 + \alpha^4, \ \ I_\alpha(P_i, \Gamma_{\mathcal{P}}) = 2\alpha^2 - 3\alpha^3 + \alpha^4, \ \ i = 2,3,$
$I_\alpha(P_4, \Gamma_{\mathcal{P}}) = \alpha^2 - 2\alpha^3 + \alpha^4.$

$(d_{1,1}, d_{2,1}, d_{3,1}, d_{4,1}) = (0,3,2,0), \ \ (d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (0,2,1,0), \ \ i = 2,3,$
$(d_{1,4}, d_{2,4}, d_{3,4}, d_{4,4}) = (0,1,0,0).$

$(|\Gamma_{\mathcal{P}}[1;P_1]|, |\Gamma_{\mathcal{P}}[2;P_1]|, |\Gamma_{\mathcal{P}}[3;P_1]|, |\Gamma_{\mathcal{P}}[4;P_1]|) = (0,3,3,1),$
$(|\Gamma_{\mathcal{P}}[1;P_i]|, |\Gamma_{\mathcal{P}}[2;P_i]|, |\Gamma_{\mathcal{P}}[3;P_i]|, |\Gamma_{\mathcal{P}}[4;P_i]|) = (0,2,3,1), \ \ i = 2,3,$
$(|\Gamma_{\mathcal{P}}[1;P_4]|, |\Gamma_{\mathcal{P}}[2;P_4]|, |\Gamma_{\mathcal{P}}[3;P_4]|, |\Gamma_{\mathcal{P}}[4;P_4]|) = (0,1,3,1),$

$\Rightarrow \ \ (\boldsymbol{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3},\ \mathbf{P_1P_4},\ \mathbf{P_2P_3}\}$

$(\mathbf{8}) \qquad (\boldsymbol{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3},\ \mathbf{P_2P_4},\ \mathbf{P_3P_4}\}$

$\Omega^\alpha(\Gamma_{\mathcal{P}}) = 4\alpha^2 - 4\alpha^3 + \alpha^4, \ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = 2\alpha^2 - \alpha^3, \ \ i = 1,2,3,4.$

$I_\alpha(P_i, \Gamma_{\mathcal{P}}) = 2\alpha^2 - 3\alpha^3 + \alpha^4, \ \ i = 1,2,3,4.$

$(d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (0,2,1,0), \ \ i = 1,2,3,4.$

$(|\Gamma_{\mathcal{P}}[1;P_i]|, |\Gamma_{\mathcal{P}}[2;P_i]|, |\Gamma_{\mathcal{P}}[3;P_i]|, |\Gamma_{\mathcal{P}}[4;P_i]|) = (0,2,3,1), \ \ i = 1,2,3,4.$

$$\Rightarrow \quad \begin{array}{l} (\boldsymbol{\Gamma}^{\mathbf{1}}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3},\ \mathbf{P_2P_4},\ \mathbf{P_3P_4}\} \\ (\boldsymbol{\Gamma}^{\mathbf{2}}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_4},\ \mathbf{P_2P_3},\ \mathbf{P_3P_4}\} \\ (\boldsymbol{\Gamma}^{\mathbf{3}}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_3},\ \mathbf{P_1P_4},\ \mathbf{P_2P_3},\ \mathbf{P_2P_4}\} \end{array}$$

$(\mathbf{9})\qquad (\boldsymbol{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1},\ \mathbf{P_2P_3P_4}\}$

$\Omega^\alpha(\Gamma_{\mathcal{P}}) = \alpha + \alpha^3 - \alpha^4,\ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_1\}}) = \alpha^3,\ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = \alpha,\ \ i = 2,3,4.$

$I_\alpha(P_1,\Gamma_{\mathcal{P}}) = \alpha - \alpha^4,\ \ I_\alpha(P_i,\Gamma_{\mathcal{P}}) = \alpha^3 - \alpha^4,\ \ i = 2,3,4.$

$(d_{1,1}, d_{2,1}, d_{3,1}, d_{4,1}) = (1,3,3,0),\ \ (d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (0,0,1,0),\ \ i = 2,3,4.$

$(|\Gamma_{\mathcal{P}}[1;P_1]|, |\Gamma_{\mathcal{P}}[2;P_1]|, |\Gamma_{\mathcal{P}}[3;P_1]|, |\Gamma_{\mathcal{P}}[4;P_1]|) = (1,3,3,1),$
$(|\Gamma_{\mathcal{P}}[1;P_i]|, |\Gamma_{\mathcal{P}}[2;P_i]|, |\Gamma_{\mathcal{P}}[3;P_i]|, |\Gamma_{\mathcal{P}}[4;P_i]|) = (0,1,3,1),\ \ i = 2,3,4.$

$\Rightarrow\ \ (\boldsymbol{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1},\ \mathbf{P_2P_3P_4}\}$

$(\mathbf{10})\qquad (\boldsymbol{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3},\ \mathbf{P_2P_4}\}$

$\Omega^\alpha(\Gamma_{\mathcal{P}}) = 3\alpha^2 - 2\alpha^3,$
$\Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = \alpha^2,\ \ i = 1,2,\ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_j\}}) = 2\alpha^2 - \alpha^3,\ \ j = 3,4.$

$I_\alpha(P_i,\Gamma_{\mathcal{P}}) = 2\alpha^2 - 2\alpha^3,\ \ i = 1,2,\ \ I_\alpha(P_j,\Gamma_{\mathcal{P}}) = \alpha^2 - \alpha^3,\ \ j = 3,4.$

$(d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (0,2,2,0),\ \ i = 1,2,\ \ (d_{1,j}, d_{2,j}, d_{3,j}, d_{4,j}) = (0,1,1,0),\ \ j = 3,4.$

$(|\Gamma_{\mathcal{P}}[1;P_i]|, |\Gamma_{\mathcal{P}}[2;P_i]|, |\Gamma_{\mathcal{P}}[3;P_i]|, |\Gamma_{\mathcal{P}}[4;P_i]|) = (0,2,3,1),\ \ i = 1,2,$
$(|\Gamma_{\mathcal{P}}[1;P_j]|, |\Gamma_{\mathcal{P}}[2;P_j]|, |\Gamma_{\mathcal{P}}[3;P_j]|, |\Gamma_{\mathcal{P}}[4;P_j]|) = (0,1,3,1),\ \ j = 3,4.$

$$\Rightarrow\quad \begin{array}{l} (\boldsymbol{\Gamma}^{\mathbf{1}}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3},\ \mathbf{P_2P_4}\} \\ (\boldsymbol{\Gamma}^{\mathbf{2}}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_4},\ \mathbf{P_2P_3}\} \end{array}$$

$(\mathbf{11})\qquad (\boldsymbol{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3},\ \mathbf{P_1P_4},\ \mathbf{P_2P_3P_4}\}$

$\Omega^\alpha(\Gamma_{\mathcal{P}}) = 3\alpha^2 - 2\alpha^3,\ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_1\}}) = \alpha^3,\ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = 2\alpha^2 - \alpha^3,\ \ i = 2,3,4.$

$I_\alpha(P_1,\Gamma_{\mathcal{P}}) = 3\alpha^2 - 3\alpha^3,\ \ I_\alpha(P_i,\Gamma_{\mathcal{P}}) = \alpha^2 - \alpha^3,\ \ i = 2,3,4.$

$(d_{1,1}, d_{2,1}, d_{3,1}, d_{4,1}) = (0,3,3,0),\ \ (d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (0,1,1,0),\ \ i = 2,3,4.$

$(|\Gamma_{\mathcal{P}}[1;P_1]|, |\Gamma_{\mathcal{P}}[2;P_1]|, |\Gamma_{\mathcal{P}}[3;P_1]|, |\Gamma_{\mathcal{P}}[4;P_1]|) = (0,3,3,1),$
$(|\Gamma_{\mathcal{P}}[1;P_i]|, |\Gamma_{\mathcal{P}}[2;P_i]|, |\Gamma_{\mathcal{P}}[3;P_i]|, |\Gamma_{\mathcal{P}}[4;P_i]|) = (0,1,3,1),\ \ i = 2,3,4.$

$\Rightarrow\ \ (\boldsymbol{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3},\ \mathbf{P_1P_4},\ \mathbf{P_2P_3P_4}\}$

$(\mathbf{12}) \qquad (\mathbf{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3},\ \mathbf{P_1P_4}\}$

$\Omega^\alpha(\Gamma_{\mathcal{P}}) = 3\alpha^2 - 3\alpha^3 + \alpha^4,\ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_1\}}) = 0,\ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = 2\alpha^2 - \alpha^3,\ \ i = 2,3,4.$

$I_\alpha(P_1, \Gamma_{\mathcal{P}}) = 3\alpha^2 - 3\alpha^3 + \alpha^4,\ \ I_\alpha(P_i, \Gamma_{\mathcal{P}}) = \alpha^2 - 2\alpha^3 + \alpha^4,\ \ i = 2,3,4.$

$(d_{1,1}, d_{2,1}, d_{3,1}, d_{4,1}) = (0,3,3,1),\ \ (d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (0,1,0,0),\ \ i = 2,3,4.$

$(|\Gamma_{\mathcal{P}}[1;P_1]|, |\Gamma_{\mathcal{P}}[2;P_1]|, |\Gamma_{\mathcal{P}}[3;P_1]|, |\Gamma_{\mathcal{P}}[4;P_1]|) = (0,3,3,1),$
$(|\Gamma_{\mathcal{P}}[1;P_i]|, |\Gamma_{\mathcal{P}}[2;P_i]|, |\Gamma_{\mathcal{P}}[3;P_i]|, |\Gamma_{\mathcal{P}}[4;P_i]|) = (0,1,2,1),\ \ i = 2,3,4.$

$\Rightarrow\ \ (\mathbf{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3},\ \mathbf{P_1P_4}\}$

$(\mathbf{13}) \qquad (\mathbf{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_3P_4}\}$

$\Omega^\alpha(\Gamma_{\mathcal{P}}) = 2\alpha^2 - \alpha^4,\ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = \alpha^2,\ \ i = 2,3,4.$

$I_\alpha(P_i, \Gamma_{\mathcal{P}}) = \alpha^2 - \alpha^4,\ \ i = 1,2,3,4.$

$(d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (0,1,2,0),\ \ i = 1,2,3,4.$

$(|\Gamma_{\mathcal{P}}[1;P_i]|, |\Gamma_{\mathcal{P}}[2;P_i]|, |\Gamma_{\mathcal{P}}[3;P_i]|, |\Gamma_{\mathcal{P}}[4;P_i]|) = (0,1,3,1),\ \ i = 1,2,3,4.$

$$\Rightarrow\ \ \begin{array}{l} (\mathbf{\Gamma}^{\mathbf{1}}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_3P_4}\} \\ (\mathbf{\Gamma}^{\mathbf{2}}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_3},\ \mathbf{P_2P_4}\} \\ (\mathbf{\Gamma}^{\mathbf{3}}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_4},\ \mathbf{P_2P_3}\} \end{array}$$

$(\mathbf{14}) \qquad (\mathbf{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3},\ \mathbf{P_2P_3P_4}\}$

$\Omega^\alpha(\Gamma_{\mathcal{P}}) = 2\alpha^2 - \alpha^4,$
$\Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_1\}}) = \alpha^3,\ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = \alpha^2,\ \ i = 2,3,\ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_4\}}) = 2\alpha^2 - \alpha^3.$

$I_\alpha(P_1, \Gamma_{\mathcal{P}}) = 2\alpha^2 - \alpha^3 - \alpha^4,\ \ I_\alpha(P_i, \Gamma_{\mathcal{P}}) = \alpha^2 - \alpha^4,\ \ i = 2,3,\ \ I_\alpha(P_4, \Gamma_{\mathcal{P}}) = \alpha^3 - \alpha^4.$

$(d_{1,1}, d_{2,1}, d_{3,1}, d_{4,1}) = (0,2,3,0),\ \ (d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (0,1,2,0),\ \ i = 2,3,$
$(d_{1,4}, d_{2,4}, d_{3,4}, d_{4,4}) = (0,0,1,0).$

$(|\Gamma_{\mathcal{P}}[1;P_1]|, |\Gamma_{\mathcal{P}}[2;P_1]|, |\Gamma_{\mathcal{P}}[3;P_1]|, |\Gamma_{\mathcal{P}}[4;P_1]|) = (0,2,3,1),$
$(|\Gamma_{\mathcal{P}}[1;P_i]|, |\Gamma_{\mathcal{P}}[2;P_i]|, |\Gamma_{\mathcal{P}}[3;P_i]|, |\Gamma_{\mathcal{P}}[4;P_i]|) = (0,1,3,1),\ \ i = 2,3,$
$(|\Gamma_{\mathcal{P}}[1;P_4]|, |\Gamma_{\mathcal{P}}[2;P_4]|, |\Gamma_{\mathcal{P}}[3;P_4]|, |\Gamma_{\mathcal{P}}[4;P_4]|) = (0,0,3,1).$

$\Rightarrow\ \ (\mathbf{\Gamma}^{\mathbf{1}}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3},\ \mathbf{P_2P_3P_4}\}$

$(\mathbf{15})\qquad (\mathbf{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3P_4},\ \mathbf{P_2P_3P_4}\}$

$\Omega^\alpha(\Gamma_{\mathcal{P}}) = \alpha^2 + 2\alpha^3 - 2\alpha^4,$
$\Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = \alpha^3,\ \ i=1,2,\ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_j\}}) = \alpha^2,\ \ j=3,4.$

$I_\alpha(P_i,\Gamma_{\mathcal{P}}) = \alpha^2+\alpha^3-2\alpha^4,\ \ i=1,2,\ \ I_\alpha(P_j,\Gamma_{\mathcal{P}}) = 2\alpha^3-2\alpha^4,\ \ j=3,4.$

$(d_{1,i},d_{2,i},d_{3,i},d_{4,i}) = (0,1,3,0),\ \ i=1,2,$
$(d_{1,j},d_{2,j},d_{3,j},d_{4,j}) = (0,0,2,0),\ \ j=3,4.$

$(|\Gamma_{\mathcal{P}}[1;P_i]|,|\Gamma_{\mathcal{P}}[2;P_i]|,|\Gamma_{\mathcal{P}}[3;P_i]|,|\Gamma_{\mathcal{P}}[4;P_i]|) = (0,1,3,1),\ \ i=1,2,$
$(|\Gamma_{\mathcal{P}}[1;P_j]|,|\Gamma_{\mathcal{P}}[2;P_j]|,|\Gamma_{\mathcal{P}}[3;P_j]|,|\Gamma_{\mathcal{P}}[4;P_j]|) = (0,0,3,1),\ \ j=3,4.$

$\Rightarrow\ \ (\mathbf{\Gamma}^{\mathbf{1}}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_2P_3P_4},\ \mathbf{P_1P_3P_4}\}$

$(\mathbf{16})\qquad (\mathbf{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3P_4}\}$

$\Omega^\alpha(\Gamma_{\mathcal{P}}) = \alpha^2 + \alpha^3 - \alpha^4,$
$\Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_1\}}) = 0,\ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_2\}}) = \alpha^3,\ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = \alpha^2,\ \ i=3,4.$

$I_\alpha(P_1,\Gamma_{\mathcal{P}}) = \alpha^2+\alpha^3-\alpha^4,\ \ I_\alpha(P_2,\Gamma_{\mathcal{P}}) = \alpha^2-\alpha^4,\ \ I_\alpha(P_i,\Gamma_{\mathcal{P}}) = \alpha^3-\alpha^4,\ \ i=3,4.$

$(d_{1,1},d_{2,1},d_{3,1},d_{4,1}) = (0,1,3,1),\ \ (d_{1,2},d_{2,2},d_{3,2},d_{4,2}) = (0,1,2,0),$
$(d_{1,i},d_{2,i},d_{3,i},d_{4,i}) = (0,0,1,0),\ \ i=3,4.$

$(|\Gamma_{\mathcal{P}}[1;P_1]|,|\Gamma_{\mathcal{P}}[2;P_1]|,|\Gamma_{\mathcal{P}}[3;P_1]|,|\Gamma_{\mathcal{P}}[4;P_1]|) = (0,1,3,1),$
$(|\Gamma_{\mathcal{P}}[1;P_2]|,|\Gamma_{\mathcal{P}}[2;P_2]|,|\Gamma_{\mathcal{P}}[3;P_2]|,|\Gamma_{\mathcal{P}}[4;P_2]|) = (0,1,2,1),$
$(|\Gamma_{\mathcal{P}}[1;P_i]|,|\Gamma_{\mathcal{P}}[2;P_i]|,|\Gamma_{\mathcal{P}}[3;P_i]|,|\Gamma_{\mathcal{P}}[4;P_i]|) = (0,0,2,1),\ \ i=3,4.$

$\Rightarrow\ \ (\mathbf{\Gamma}^{\mathbf{1}}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2},\ \mathbf{P_1P_3P_4}\}$

$(\mathbf{17})\qquad (\mathbf{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2P_3},\ \mathbf{P_1P_2P_4},\ \mathbf{P_1P_3P_4},\ \mathbf{P_2P_3P_4}\}$

$\Omega^\alpha(\Gamma_{\mathcal{P}}) = 4\alpha^3 - 3\alpha^4,\ \ \Omega^\alpha(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = \alpha^3,\ \ i=1,2,3,4.$

$I_\alpha(P_i,\Gamma_{\mathcal{P}}) = 3\alpha^3-3\alpha^4,\ \ i=1,2,3,4.$

$(d_{1,i},d_{2,i},d_{3,i},d_{4,i}) = (0,0,3,0),\ \ i=1,2,3,4.$

$(|\Gamma_{\mathcal{P}}[1;P_i]|,|\Gamma_{\mathcal{P}}[2;P_i]|,|\Gamma_{\mathcal{P}}[3;P_i]|,|\Gamma_{\mathcal{P}}[4;P_i]|) = (0,0,3,1),\ \ i=1,2,3,4.$

$\Rightarrow\ \ (\mathbf{\Gamma}^{\mathbf{1}}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2P_3},\ \mathbf{P_1P_2P_4},\ \mathbf{P_1P_3P_4},\ \mathbf{P_2P_3P_4}\}$

$(\mathbf{18})$ $\quad (\boldsymbol{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2P_3},\ \mathbf{P_1P_2P_4},\ \mathbf{P_1P_3P_4}\}$

$\Omega^{\alpha}(\Gamma_{\mathcal{P}}) = 3\alpha^3 - 2\alpha^4,\ \ \Omega^{\alpha}(\Gamma_{\mathcal{P}\setminus\{P_1\}}) = 0,\ \ \Omega^{\alpha}(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = \alpha^3,\ \ i = 2, 3, 4.$

$I_{\alpha}(P_1, \Gamma_{\mathcal{P}}) = 3\alpha^3 - 2\alpha^4,\ \ I_{\alpha}(P_i, \Gamma_{\mathcal{P}}) = 2\alpha^3 - 2\alpha^4,\ \ i = 2, 3, 4.$

$(d_{1,1}, d_{2,1}, d_{3,1}, d_{4,1}) = (0, 0, 3, 1),\ \ (d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (0, 0, 2, 0),\ \ i = 2, 3, 4.$

$(|\Gamma_{\mathcal{P}}[1; P_1]|, |\Gamma_{\mathcal{P}}[2; P_1]|, |\Gamma_{\mathcal{P}}[3; P_1]|, |\Gamma_{\mathcal{P}}[4; P_1]|) = (0, 0, 3, 1),$
$(|\Gamma_{\mathcal{P}}[1; P_i]|, |\Gamma_{\mathcal{P}}[2; P_i]|, |\Gamma_{\mathcal{P}}[3; P_i]|, |\Gamma_{\mathcal{P}}[4; P_i]|) = (0, 0, 2, 1),\ \ i = 2, 3, 4.$

$\Rightarrow\ \ (\boldsymbol{\Gamma}^{\mathbf{1}}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2P_3},\ \mathbf{P_1P_2P_4},\ \mathbf{P_1P_3P_4}\}$

$(\mathbf{19})$ $\quad (\boldsymbol{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2P_3},\ \mathbf{P_1P_2P_4}\}$

$\Omega^{\alpha}(\Gamma_{\mathcal{P}}) = 2\alpha^3 - \alpha^4,\ \ \Omega^{\alpha}(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = 0,\ \ i = 1, 2,\ \ \Omega^{\alpha}(\Gamma_{\mathcal{P}\setminus\{P_j\}}) = \alpha^3,\ \ j = 3, 4.$

$I_{\alpha}(P_i, \Gamma_{\mathcal{P}}) = 2\alpha^3 - \alpha^4, i = 1, 2,\ \ I_{\alpha}(P_j, \Gamma_{\mathcal{P}}) = \alpha^3 - \alpha^4,\ \ j = 3, 4.$

$(d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (0, 0, 2, 1),\ \ i = 1, 2,\ \ (d_{1,j}, d_{2,j}, d_{3,j}, d_{4,j}) = (0, 0, j, 0),\ \ j = 3, 4.$

$(|\Gamma_{\mathcal{P}}[1; P_i]|, |\Gamma_{\mathcal{P}}[2; P_i]|, |\Gamma_{\mathcal{P}}[3; P_i]|, |\Gamma_{\mathcal{P}}[4; P_i]|) = (0, 0, 2, 1),\ \ i = 1, 2,$
$(|\Gamma_{\mathcal{P}}[1; P_j]|, |\Gamma_{\mathcal{P}}[2; P_j]|, |\Gamma_{\mathcal{P}}[3; P_j]|, |\Gamma_{\mathcal{P}}[4; P_j]|) = (0, 0, 1, 1),\ \ j = 3, 4.$

$\Rightarrow\ \ (\boldsymbol{\Gamma}^{\mathbf{1}}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2P_3},\ \mathbf{P_1P_2P_4}\}$

$(\mathbf{20})$ $\quad (\boldsymbol{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2P_3P_4}\}$

$\Omega^{\alpha}(\Gamma_{\mathcal{P}}) = \alpha^4,\ \ \Omega^{\alpha}(\Gamma_{\mathcal{P}\setminus\{P_i\}}) = 0,\ \ i = 1, 2, 3, 4.$

$I_{\alpha}(P_i, \Gamma_{\mathcal{P}}) = \alpha^4,\ \ i = 1, 2, 3, 4.$

$(d_{1,i}, d_{2,i}, d_{3,i}, d_{4,i}) = (0, 0, 0, 1),\ \ i = 1, 2, 3, 4.$

$(|\Gamma_{\mathcal{P}}[1; P_i]|, |\Gamma_{\mathcal{P}}[2; P_i]|, |\Gamma_{\mathcal{P}}[3; P_i]|, |\Gamma_{\mathcal{P}}[4; P_i]|) = (0, 0, 0, 1),\ \ i = 1, 2, 3, 4.$

$\Rightarrow\ \ (\boldsymbol{\Gamma}_{\mathcal{P}})_{\mathbf{0}} = \{\mathbf{P_1P_2P_3P_4}\}$
