

Exponentially small asymptotic estimates for the splitting of separatrices to whiskered tori with quadratic and cubic frequencies ^{*}

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Abstract

We study the splitting of invariant manifolds of whiskered tori with two or three frequencies in nearly-integrable Hamiltonian systems. We consider 2-dimensional tori with a frequency vector $\omega = (1, \Omega)$ where Ω is a quadratic irrational number, or 3-dimensional tori with a frequency vector $\omega = (1, \Omega, \Omega^2)$ where Ω is a cubic irrational number. Applying the Poincaré–Melnikov method, we find exponentially small asymptotic estimates for the maximal splitting distance between the stable and unstable manifolds associated to the invariant torus, showing that such estimates depend strongly on the arithmetic properties of the frequencies. In the quadratic case, we use the continued fractions theory to establish a certain arithmetic property, fulfilled in 24 cases, which allows us to provide asymptotic estimates in a simple way. In the cubic case, we focus our attention to the case in which Ω is the so-called cubic golden number (the real root of $x^3 + x - 1 = 0$), obtaining also asymptotic estimates. We point out the similitudes and differences between the results obtained for both the quadratic and cubic cases.

Keywords: splitting of separatrices, Melnikov integrals, quadratic and cubic frequencies.

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1 Introduction

1.1 Background and objectives

The aim of this paper is to introduce a methodology for measuring the exponentially small splitting of separatrices in a perturbed Hamiltonian system, associated to an ℓ -dimensional whiskered torus (invariant hyperbolic torus) with an algebraic frequency vector, quadratic in the case $\ell = 2$, and cubic in the case $\ell = 3$.

As the unperturbed system, we consider an integrable Hamiltonian H_0 with $\ell + 1$ degrees of freedom having ℓ -dimensional whiskered tori with coincident stable and unstable whiskers. In general, for a perturbed Hamiltonian

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$H = H_0 + \mu H_1$ where μ is small, the whiskers do not coincide anymore, giving rise to the phenomenon called *splitting of separatrices*, discovered by Poincaré [Poi90]. In order to give a measure for the splitting, one often describes it by a periodic vector function $\mathcal{M}(\theta)$, $\theta \in \mathbb{T}^\ell$, usually called *splitting function*, giving the distance between the invariant manifolds in the complementary directions, on a transverse section. The most popular tool to measure the splitting is the *Poincaré–Melnikov method*, introduced in [Poi90] and rediscovered later by Melnikov and Arnold [Mel63, Arn64]. This method provides a first order approximation

$$\mathcal{M}(\theta) = \mu M(\theta) + \mathcal{O}(\mu^2), \quad (1)$$

where $M(\theta)$ is called the *Melnikov function* and is defined by an integral. In fact, it was established [Eli94, DG00] that both the splitting and the Melnikov functions are the gradients of scalar functions: the *splitting potential* and the *Melnikov potential*, denoted $\mathcal{L}(\theta)$ and $L(\theta)$ respectively. This result implies the existence of *homoclinic orbits* (i.e. intersections between the stable and unstable whiskers) in the perturbed system.

We focus our attention on a concrete torus with an ℓ -dimensional frequency vector of *fast frequencies*:

$$\omega_\varepsilon = \frac{\omega}{\sqrt{\varepsilon}}, \quad \varepsilon > 0, \quad (2)$$

with a relation between the parameters, of the form $\mu = \varepsilon^p$ for some $p > 0$. Thus, we have a *singular* perturbation problem, and the interest for this situation lies in its relation to the normal form in the vicinity of a simple resonance [Nie00, DG01], of a nearly-integrable Hamiltonian $\mathcal{K} = \mathcal{K}_0 + \varepsilon \mathcal{K}_1$. In such a singular problem, one can give upper bounds for the splitting, showing that it is *exponentially small* with respect to ε . The first of such upper bounds was obtained by Neishtadt [Nei84] in one and a half degrees of freedom i.e. for 1 frequency, and later this was extended to the case of 2 or more frequencies (see for instance [Sim94, Gal94, BCG97, BCF97, DGJS97, DGS04]).

The problem of establishing *lower bounds* for the exponentially small splitting, or even *asymptotic estimates*, is more difficult, but some results have been obtained also by several methods. The difficulty lies in the fact that the Melnikov function is exponentially small in ε and the error of the method could overcome the main term in (1). Then, an additional study is required in order to validate the Poincaré–Melnikov method. In the case of 1 frequency, the first result providing an asymptotic estimate for the exponentially small splitting was obtained by Lazutin [Laz03] in 1984, for the Chirikov standard map, using complex parameterizations of the invariant manifolds. The same technique was used to justify the Poincaré–Melnikov method in a Hamiltonian with one and a half degrees of freedom [DS92, DS97, Gel97] or an area-preserving map [DR98]. In fact, when the Poincaré–Melnikov approach cannot be validated, other techniques can be applied to get exponentially small estimates, such as complex matching [Bal06, OSS03, MSS11a, MSS11b], or “beyond all orders” asymptotic methods [Lom00], or continuous averaging [Tre97, PT00].

For 2 or more frequencies, it turns out that *small divisors* appear in the splitting function and, as first noticed by Lochak [Loc92], the arithmetic properties of the frequency vector ω play an important rôle. This was established by Simó [Sim94], and rigorously proved in [DGJS97] for the quasi-periodically forced pendulum. A different technique was used by Lochak, Marco and Sauzin [Sau01, LMS03], and Rudnev and Wiggins [RW00], namely the parametrization of the whiskers as solutions of Hamilton–Jacobi equation, to obtain exponential small estimates of the splitting, and the existence of transverse homoclinic orbits for some intervals of the perturbation parameter ε . Besides, it was shown in [DG04] the continuation of the exponentially small estimates and the transversality of the splitting, for all sufficiently small values of ε , under a certain condition on the phases of the perturbation. Otherwise, homoclinic bifurcations can occur, studied by Simó and Valls [SV01] in the Arnold’s example. The quoted papers considered the case of 2 frequencies, and assuming in most cases that the frequency ratio is the famous *golden mean* $\Omega_1 = (\sqrt{5} - 1)/2$. A generalization to some other quadratic frequency ratios was studied in [DG03]. For a more complete background and references concerning exponentially small splitting, see for instance [DGS04].

The main objective of this paper is to develop a unified methodology in order to generalize the results on exponentially small splitting to frequency vectors ω in \mathbb{R}^2 or \mathbb{R}^3 , in order to obtain asymptotic estimates for the *maximal splitting distance* (and, consequently, to show the existence of splitting), and to emphasize the dependence of such estimates on the arithmetic properties of the frequencies. Namely, we consider two possibilities:

- *quadratic frequencies*: $\ell = 2$ and $\omega = (1, \Omega)$, where Ω is a quadratic irrational number;
- *cubic frequencies*: $\ell = 3$ and $\omega = (1, \Omega, \Omega^2)$, where Ω is a cubic irrational number whose two conjugates are not real.

Such frequency vectors satisfy a *Diophantine condition*,

$$|\langle k, \omega \rangle| \geq \frac{\gamma}{|k|^{\ell-1}}, \quad \forall k \in \mathbb{Z}^\ell \setminus \{0\} \quad (3)$$

with some $\gamma > 0$, in both the quadratic and the cubic cases. We point out that $\ell - 1$ is the minimal possible exponent for Diophantine inequalities in \mathbb{R}^ℓ (see for instance [LM88, ap. 4])

One of the goals of this paper is to show, for the above frequencies, that we can detect the integer vectors $k \in \mathbb{Z}^\ell \setminus \{0\}$ providing an approximate equality in (3), i.e. giving the “least” small divisors (relatively to the size of $|k|$). We call such vectors k the *primary resonances* of ω , and other vectors the *secondary* ones. We show that, if a certain arithmetic condition is fulfilled (see the separation condition (36)), then the harmonics associated to such primary vectors k are the dominant ones in the splitting function $\mathcal{M}(\theta)$, for each small enough value of the perturbation parameter ε .

In the quadratic case, the required arithmetic condition (36) can be formulated in terms of the *continued fraction* of Ω , which is (eventually) periodic, and in fact, we can restrict ourselves to the case of *purely periodic* continued fractions. There are 24 numbers satisfying (36), all of them having 1-periodic or 2-periodic continued fractions,

$$\Omega_a = [\overline{a}], \quad a = 1, \dots, 13, \quad \text{and} \quad \Omega_{1,a} = [\overline{1, a}], \quad a = 2, \dots, 12 \quad (4)$$

(this includes the golden number $\Omega_1 = [1, 1, 1, \dots] = (\sqrt{5} - 1)/2$).

In the cubic case, there is no standard continued fraction theory, but a particular study can be carried out for each cubic irrational Ω . We consider in this paper the *cubic golden number* (see for instance [HK00]):

$$\Omega \approx 0.6823, \quad \text{the real root of } x^3 + x - 1 = 0, \quad (5)$$

but we stress that a similar approach could be carried out for other cases.

In the main result of this paper (see Theorem 1), we establish exponentially small asymptotic estimates for the maximal splitting distance, valid in all the cases (4–5). In this way, we show that the results provided in [DG03] for some quadratic frequencies are extended to other cases, including a particular case of cubic frequencies. As far as we know, this is the first result providing asymptotic estimates (and, hence, lower bounds) for the exponentially small splitting of separatrices with 3 frequencies. To avoid technicalities, we put emphasis on the constructive part of the proofs, using the arithmetic properties of the frequencies in order to provide a unified methodology which can be applied to both the quadratic and the cubic cases, stressing the similarities and differences between them. We determine, for every ε small enough, the dominant harmonic of the Melnikov function $M(\theta)$, associated to a primary resonance, and consequently we obtain an estimate for the maximal value of this function.

In a further step, the first order approximation has to be validated showing that the dominant harmonics of the splitting function $\mathcal{M}(\theta)$ correspond to the dominant harmonics of the Melnikov function, as done in [DG04]. Besides, one can show in the cases (4–5) that the invariant manifolds intersect along transverse homoclinic orbits, with an exponentially small angle. To obtain this, one needs to consider the “next” dominant harmonics (at least 2 ones in the quadratic case and at least 3 ones in the cubic case, provided their associated vectors k are linearly independent), which can be carried out for the frequency vectors considered. Nevertheless, in some cases the secondary resonances have to be taken into account giving rise to more involved estimates. We only provide here the main ideas, and rigorous proofs will be published elsewhere.

1.2 Setup and main result

In order to formulate our main result, let us describe the Hamiltonian considered, which is analogous to the one considered in [DGS04] and other related works. In symplectic coordinates $(x, y, \varphi, I) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}^\ell \times \mathbb{R}^\ell$,

$$\begin{aligned} H(x, y, \varphi, I) &= H_0(x, y, I) + \mu H_1(x, \varphi), \\ H_0(x, y, I) &= \langle \omega_\varepsilon, I \rangle + \frac{1}{2} \langle \Lambda I, I \rangle + \frac{y^2}{2} + \cos x - 1, \quad H_1(x, \varphi) = h(x) f(\varphi), \end{aligned} \quad (6)$$

with

$$h(x) = \cos x, \quad f(\varphi) = \sum_{\substack{k \in \mathcal{Z} \\ k_l \geq 0}} e^{-\rho|k|} \cos(\langle k, \varphi \rangle - \sigma_k), \quad (7)$$

where the restriction in the sum is introduced in order to avoid repetitions. The Hamiltonian (6–7) is a generalization of the Arnold example (introduced in [Arn64] to illustrate the transition chain mechanism in Arnold diffusion). It provides a model for the behavior of a nearly-integrable Hamiltonian system $\mathcal{K} = \mathcal{K}_0 + \varepsilon\mathcal{K}_1$ in the vicinity of a *simple resonance*, after carrying out one step of *resonant normal form* (see for instance [Nie00, DG01]). In this way, our unperturbed Hamiltonian H_0 and the perturbation μH_1 play the rôle of the truncated normal form and the remainder respectively; in its turn the truncated normal form is an $\mathcal{O}(\varepsilon)$ -perturbation of the initial Hamiltonian \mathcal{K}_0 , making the hyperbolicity appear (a rescaling leads to the fast frequencies (2)). The parameters ε and μ should not be considered as independent, but linked by a relation of the type $\mu = \varepsilon^p$.

Notice that the unperturbed system H_0 consists of the pendulum given by $P(x, y) = y^2/2 + \cos x - 1$ and ℓ rotors with fast frequencies: $\dot{\varphi} = \omega_\varepsilon + \Lambda I$, $\dot{I} = 0$. The pendulum has a hyperbolic equilibrium at the origin, and the (upper) separatrix can be parameterized by $(x_0(s), y_0(s)) = (4 \arctan e^s, 2/\cosh s)$, $s \in \mathbb{R}$. The rotors system (φ, I) has the solutions $\varphi = \varphi_0 + (\omega_\varepsilon + \Lambda I_0)t$, $I = I_0$. Consequently, H_0 has an ℓ -parameter family of ℓ -dimensional whiskered invariant tori, with coincident stable and unstable whiskers. Among the family of whiskered tori, we will focus our attention on the torus located at $I = 0$, whose frequency vector is ω_ε as in (2), in our case a quadratic or cubic frequency vector (for $\ell = 2$ or $\ell = 3$ respectively). We also assume the condition of *isoenergetic nondegeneracy*

$$\det \begin{pmatrix} \Lambda & \omega \\ \omega^\top & 0 \end{pmatrix} \neq 0. \quad (8)$$

When adding the perturbation μH_1 , the *hyperbolic KAM theorem* can be applied (see for instance [Nie00]) thanks to the Diophantine condition (3) and the isoenergetic nondegeneracy (8). For μ small enough, the whiskered torus persists with some shift and deformation, as well as its local whiskers.

In general, for $\mu \neq 0$ the (global) whiskers do not coincide anymore, and one can introduce a *splitting function* $\mathcal{M}(\theta)$, $\theta \in \mathbb{T}^\ell$, giving the distance between the whiskers in the complementary directions, on a transverse section ([DG00, §5.2]; see also [Eli94]). Applying the Poincaré–Melnikov method, the first order approximation (1) is given by the (vector) *Melnikov function* $M(\theta)$. Both functions $\mathcal{M}(\theta)$ and $M(\theta)$ turn to be gradients of the (scalar) *splitting potential* $\mathcal{L}(\theta)$ and *Melnikov potential* $L(\theta)$, respectively. The latter one can be defined as follows:

$$L(\theta) = - \int_{-\infty}^{\infty} [h(x_0(t)) - h(0)] f(\theta + \omega_\varepsilon t) dt, \quad M(\theta) = \nabla L(\theta). \quad (9)$$

Notice that $L(\theta)$ is obtained by integrating H_1 along a trajectory of the unperturbed homoclinic manifold, starting at the point of the section $s = 0$ with phase θ .

In order to emphasize the rôle played by the arithmetic properties of the splitting, we have chosen for the perturbation the special form given in (7). This form was already considered in [DG04], and allows us to deal with the Melnikov function and obtain asymptotic estimates for the splitting. Notice that the constant $\rho > 0$ in the Fourier expansion of $f(\varphi)$ in (7) gives the complex width of analyticity of this function. The phases σ_k can be chosen arbitrarily for the purpose of this paper.

Now we can formulate our main result, providing asymptotic estimates for the *maximal splitting distance* in both the quadratic and cubic cases.

Theorem 1 (main result) *For the Hamiltonian system (6–7) with $\ell+1$ degrees of freedom, satisfying the isoenergetic condition (8), assume that ε is small enough and $\mu = \varepsilon^p$ with $p > 3$. For $\ell = 2$, if Ω is one of the 24 quadratic numbers (4), and for $\ell = 3$, if Ω is the cubic golden number (5), the following asymptotic estimate holds:*

$$\max_{\theta \in \mathbb{T}^\ell} |\mathcal{M}(\theta)| \sim \frac{\mu}{\varepsilon^{1/\ell}} \exp \left\{ - \frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/2\ell}} \right\}$$

where $C_0 = C_0(\Omega, \rho)$ is a positive constant, defined in (27). Concerning the function $h_1(\varepsilon)$,

- (a) in the quadratic case $\ell = 2$, it is periodic in $\ln \varepsilon$, with $\min h_1(\varepsilon) = 1$ and $\max h_1(\varepsilon) = A_1$, with a constant $A_1 = A_1(\Omega)$ defined in (33);
- (b) in the cubic case $\ell = 3$, it satisfies the bound $0 < A_0^- \leq h_1(\varepsilon) \leq A_1^+$, with constants $A_0^- = A_0^-(\Omega)$ and $A_1^+ = A_1^+(\Omega)$ defined in (34).

Remarks.

1. The periodicity in $\ln \varepsilon$ of the function $h_1(\varepsilon)$ in the quadratic case (a) was first established in [DGJS97] for the quasi-periodically forced pendulum, assuming that the frequency ratio is the golden number Ω_1 . Previously, the existence of an oscillatory behavior with lower and upper bounds had been shown in [Sim94].
2. In contrast to the quadratic case, it turns out in the cubic case (b) the function $h_1(\varepsilon)$ is not periodic in $\ln \varepsilon$ and has a more complicated form (see Figure 3, where one can conjecture that $h_1(\varepsilon)$ is a quasiperiodic function).
3. The exponent $p > 3$ in the relation $\mu = \varepsilon^p$ can be improved in some special cases. For instance, if in (7) one considers $h(x) = \cos x - 1$, then the asymptotic estimates are valid for $p > 2$. This is related to the fact that, in this case, the invariant torus remains fixed under the perturbation and only the whiskers deform [DGS04].

This paper is organized as follows. In Section 2 we study the arithmetic properties of quadratic and cubic frequencies, and in Section 3 we find, for the frequencies considered in (4–5), an asymptotic estimate of the dominant harmonic of the splitting potential, together with a bound of the remaining harmonics which allows us to provide an asymptotic estimate for the maximal splitting distance, as established in Theorem 1.

2 Arithmetic properties of quadratic and cubic frequencies

2.1 Iteration matrices and resonant sequences

We review in this section the technic developed in [DG03] for studying the resonances of quadratic frequencies ($\ell = 2$), showing that it admits a direct generalization to the case of cubic frequencies ($\ell = 3$).

In the 2-dimensional case, we consider a *quadratic frequency vector* $\omega \in \mathbb{R}^2$, i.e. its frequency ratio is a quadratic irrational number. Of course, we can assume without loss of generality that the vector has the form $\omega = (1, \Omega)$.

On the other hand, in the 3-dimensional case we consider a *cubic frequency vector* $\omega \in \mathbb{R}^3$, i.e. the frequency ratios generate a cubic field (an algebraic number field of degree 3). In order to simplify our exposition, we assume that the vector has the form $\omega = (1, \Omega, \Omega^2)$, where Ω is a cubic irrational number, hence the cubic field is $\mathbb{Q}(\Omega)$.

Any quadratic or cubic frequency vector $\omega \in \mathbb{R}^\ell$ satisfies the *Diophantine condition* (3), with the minimal exponent $\ell - 1$, see for instance [Cas57]. With this in mind, we define the “*numerators*”

$$\gamma_k := |\langle k, \omega \rangle| \cdot |k|^{\ell-1}, \quad k \in \mathbb{Z}^\ell \setminus \{0\}, \quad (10)$$

provided a norm $|\cdot|$ for integer vectors has been chosen (for quadratic vectors, it was used in [DG03] the norm $|\cdot| = |\cdot|_1$, i.e. the sum of absolute values of the components of the vector; but for cubic vectors it will be more convenient to use we use the Euclidean norm $|\cdot| = |\cdot|_2$). Our goal is to provide a classification of the integer vectors k , according to the size of γ_k , in order to find the primary resonances (i.e. the integer vectors k for which γ_k is smallest and, hence, fitting best the Diophantine condition (3)), and study their separation with respect to the secondary resonances.

The key point is to use a result by Koch [Koc99]: for a vector $\omega \in \mathbb{R}^\ell$ whose frequency ratios generate an algebraic field of degree ℓ , there exists a *unimodular* matrix T (a square matrix with integer entries and determinant ± 1) having the eigenvector ω with associated eigenvalue λ of modulus > 1 , and such the other $\ell - 1$ eigenvalues are simple and of modulus < 1 . This result is valid for any dimension ℓ , and is usually applied in the context of renormalization theory (see for instance [Koc99, Lop02]), since the iteration of the matrix T provides successive rational approximations to the direction of the vector ω . Notice that the matrix T satisfying the conditions above is not unique (for instance, any power T^j , with j positive, also satisfies them). We will assume without loss of genericity that λ is positive ($\lambda > 1$).

In this paper, we are not interested in finding approximations to ω , but rather to the *quasi-resonances* of ω , which lie close to the orthogonal hyperplane $\langle \omega \rangle^\perp$. With this aim, we consider the matrix $U = (T^{-1})^\top$, which satisfies the following *fundamental equality*:

$$\langle Uk, \omega \rangle = \langle k, U^\top \omega \rangle = \frac{1}{\lambda} \langle k, \omega \rangle. \quad (11)$$

We say that an integer vector k is *admissible* if $|\langle k, \omega \rangle| < 1/2$. We restrict ourselves to the set \mathcal{A} of admissible vectors, since for any $k \notin \mathcal{A}$ we have $|\langle k, \omega \rangle| > 1/2$ and $\gamma_k > |k|^{\ell-1}/2$. We see from (11) that if $k \in \mathcal{A}$, then also $Uk \in \mathcal{A}$. We say that k is *primitive* if $k \in \mathcal{A}$ but $U^{-1}k \notin \mathcal{A}$. We also deduce from (11) that k is primitive if and only if

$$\frac{1}{2\lambda} < |\langle k, \omega \rangle| < \frac{1}{2}. \quad (12)$$

Since the first component of ω is equal to 1, it is clear that any admissible vectors can be presented in the form

$$\begin{aligned} k^0(j) &= (-\text{rint}(j\Omega), j), & j &= \mathbb{Z} \setminus \{0\} & (\ell = 2), \\ k^0(j) &= (-\text{rint}(j_1\Omega + j_2\Omega^2), j_1, j_2), & j &= (j_1, j_2) \in \mathbb{Z}^2 \setminus \{0\} & (\ell = 3) \end{aligned}$$

(we denote $\text{rint}(x)$ the closest integer to x). If $k^0(j) \in \mathbb{Z}^\ell$ is primitive, we also say that $j \in \mathbb{Z}^{\ell-1}$ is primitive, and denote \mathcal{P} be the set of such primitives. Now we define, for each $j \in \mathcal{P}$, the following *resonant sequence* of integer vectors:

$$s(j, n) := U^n k^0(j), \quad n = 0, 1, 2, \dots \quad (13)$$

It turns out that such resonant sequences cover the whole set \mathcal{A} of admissible vectors, providing a classification of them. The properties of such a classification follow from Proposition 2 for the case of quadratic frequencies, and from Proposition 3 for the case of cubic frequencies.

2.2 Properties of quadratic frequencies

It is well-known that all *quadratic irrational numbers* $\Omega \in (0, 1)$, i.e. the real roots of quadratic polynomials with rational coefficients, have the continued fraction

$$\Omega = [a_1, a_2, a_3, \dots], \quad a_i \in \mathbb{Z}^+,$$

that is *eventually periodic*, i.e. periodic starting with some element a_i . In fact, as we see below we can restrict ourselves to the numbers with *purely periodic* continued fractions and denote them according to their periodic part; for an m -periodic continued fraction, we write $\Omega_{a_1, \dots, a_m} = [\overline{a_1, \dots, a_m}]$. For example, the famous *golden number* is $\Omega_1 = [\overline{1}] = (\sqrt{5} - 1)/2$, and the *silver number* is $\Omega_2 = [\overline{2}] = \sqrt{2} - 1$.

For a quadratic frequency $\omega = (1, \Omega)$, the matrix T provided by Koch's result [Koc99] can be constructed directly from the continued fraction of Ω . The quadratic numbers (4), considered in this paper, have 1-periodic or 2-periodic continued fractions. Let us write their matrix $T = T(\Omega)$ with ω as an eigenvector, and the associated eigenvalue $\lambda = \lambda(\Omega) > 1$:

$$\begin{aligned} \text{for } \Omega = \Omega_a, & \quad T = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}, & \quad \lambda = \frac{1}{\Omega}; \\ \text{for } \Omega = \Omega_{1,a}, & \quad T = \begin{pmatrix} a+1 & 1 \\ a & 1 \end{pmatrix}, & \quad \lambda = \frac{1}{1-\Omega}. \end{aligned}$$

Remark. In what concerns the contents of this paper, it is enough to consider quadratic numbers with purely periodic continued fractions, due to the equivalence of any quadratic number $\widehat{\Omega}$, with an eventually periodic continued fraction, to some Ω with a purely periodic one: $\widehat{\Omega} = \frac{c+d\Omega}{a+b\Omega}$, with integers a, b, c, d such that $ad - bc = \pm 1$. Then, it can be shown that the same results apply to both numbers Ω and $\widehat{\Omega}$ for ε small enough. For instance, the results for the golden number Ω_1 also apply to the *noble numbers* $\widehat{\Omega} = [b_1, \dots, b_n, \overline{1}]$. We point out that the threshold in ε of validity of the results, not considered in this paper, would depend on the non-periodic part of the continued fraction.

Now we consider the resonant sequences defined in (13). For the matrix T , let v_2 be a second eigenvector (with eigenvalue σ/λ of modulus < 1 , where $\sigma = \det T = \pm 1$); hence ω, v_2 are a basis of eigenvectors. For the matrix $U = (T^{-1})^\top$, let u_1, u_2 be a basis of eigenvectors with eigenvalues $1/\lambda$ and $\sigma\lambda$ respectively. It is well-known that $\langle u_2, \omega \rangle = \langle u_1, v_2 \rangle = 0$. For any primitive integer $j \in \mathcal{P}$, we define the quantities

$$r_j := \langle k^0(j), \omega \rangle, \quad p_j := \langle k^0(j), v_2 \rangle.$$

The properties of the quadratic frequencies can be summarized in the following proposition, whose proof is given in [DG03].

Proposition 2 *For any primitive $j \in \mathcal{P}$, there exists the limit*

$$\gamma_j^* = \lim_{n \rightarrow \infty} \gamma_{s(j,n)} = |r_j| K_j, \quad K_j = \left| \frac{p_j}{\langle u_2, v_2 \rangle} u_2 \right| = \left| k^0(j) - \frac{r_j}{\langle u_1, \omega \rangle} u_1 \right|,$$

and one has:

- (a) $\gamma_{s(j,n)} = \gamma_j^* + \mathcal{O}(\lambda^{-2n}), \quad n \geq 0;$
- (b) $|s(j,n)| = K_j \lambda^n + \mathcal{O}(\lambda^{-n}), \quad n \geq 0;$
- (c) $\gamma_j^* > \frac{(1+\Omega)|j| - a}{2\lambda}, \quad a = \frac{1}{2} \left(1 + \frac{|u_1|}{|\langle u_1, \omega \rangle|} \right).$

Since the lower bounds (c) for the “limit numerators” γ_j^* are increasing with respect to the primitive j , we can select the minimal of them, corresponding to some primitive j_0 . We denote

$$\gamma^* := \liminf_{|k| \rightarrow \infty} \gamma_k = \min_{j \in \mathcal{P}} \gamma_j^* = \gamma_{j_0}^* > 0. \quad (14)$$

The corresponding sequence $s_0(n) := s(j_0, n)$ gives us *the primary resonances*, and we call *secondary resonances* the integer vectors belonging to any of the remaining sequences $s(j, n)$, $j \neq j_0$.

We introduce *normalized numerators* $\tilde{\gamma}_k$ and their limits $\tilde{\gamma}_j^*$, $j \in \mathcal{P}$, after dividing by γ^* , and in this way $\tilde{\gamma}_{j_0}^* = 1$. We also define a parameter B_0 measuring the *separation* between primary and secondary resonances:

$$\tilde{\gamma}_k := \frac{\gamma_k}{\gamma^*}, \quad \tilde{\gamma}_j^* := \frac{\gamma_j^*}{\gamma^*}, \quad B_0 := \min_{j \in \mathcal{P} \setminus \{j_0\}} (\tilde{\gamma}_j^*)^{1/2}, \quad (15)$$

where we included the square root for convenience, see (36). We are implicitly assuming the hypothesis that the primitive j_0 is unique, and hence $B_0 > 1$. In fact, this happens for all the cases we have explored.

2.3 Properties of cubic frequencies

Now, we consider a frequency vector of the form $\omega = (1, \Omega, \Omega^2)$, where Ω is a cubic irrational number. If we consider the matrix T given by Koch’s result [Koc99], mentioned in Section 2.1, we can distinguish two possible cases for its three eigenvalues $\lambda, \lambda_2, \lambda_3$ (recall that $\lambda > 1$ is the eigenvalue with eigenvector ω):

- *the real case*: the three eigenvalues $\lambda, \lambda_2, \lambda_3$ are real;
- *the complex case*: only the eigenvalue λ is real, and the other two ones λ_2, λ_3 are a pair of complex conjugate numbers.

These two cases are often called *totally real* and *non-totally real* respectively. In this paper we only consider cubic frequency vectors in the complex or non-totally real case.

Remark. The reason to restrict ourselves to the complex case is that the remaining two (complex) eigenvalues have the same modulus. As we see below, it is natural to extend the results for quadratic frequencies to cubic frequencies

of complex type. Instead, the study of the real case would require a different approach, since the behavior of the associated small divisors turns out to be different from the complex case considered here.

Unlike the 2-dimensional quadratic frequencies, in the case of 3-dimensional cubic frequencies there is no standard theory of continued fractions providing a direct construction of the matrix T (however, there are some multidimensional continued fractions algorithms, which applied to the pair (Ω, Ω^2) could be helpful to provide T , see for instance [HK01, KLM07]). Fortunately, for a given concrete cubic frequency vector it is not hard to find the matrix T by inspection, as we do in Section 2.4 for the cubic golden number. Other examples of cubic frequencies and their associated matrices are given in [Cha02] (see also [Loc92] for an account of examples and results concerning cubic frequencies).

As in Section 2.2, we are going to establish the properties of the resonant sequences (13). Let us consider a basis of eigenvectors of T , writing the two complex ones in terms of real and imaginary parts: $\omega, v_2 + i v_3, v_2 - i v_3$, with eigenvalues λ, λ_2 and $\lambda_3 = \overline{\lambda_2}$ respectively. Notice that $|\lambda_2| = \lambda^{-1/2}$; we denote $\phi := \arg(\lambda_2)$.

In a similar way, we consider for the matrix $U = (T^{-1})^\top$ a basis $u_1, u_2 + i u_3, u_2 - i u_3$ with eigenvalues $\lambda^{-1}, \lambda_2^{-1}$ and $\lambda_3^{-1} = \overline{\lambda_2^{-1}}$ respectively. In this way, we avoid working with complex vectors. One readily sees that $\langle u_2, \omega \rangle = \langle u_3, \omega \rangle = 0$, i.e. u_2 and u_3 span the resonant plane $\langle \omega \rangle^\perp$. Other useful equalities are: $\langle u_1, v_2 \rangle = \langle u_1, v_3 \rangle = 0$, $\langle u_2, v_2 \rangle = -\langle u_3, v_3 \rangle$, $\langle u_2, v_3 \rangle = \langle u_3, v_2 \rangle$. We define Z_1, Z_2 and θ through the formulas

$$\frac{1}{2}(|u_2|^2 + |u_3|^2) = Z_1, \quad \frac{1}{2}(|u_2|^2 - |u_3|^2) = Z_2 \cos \theta, \quad \langle u_2, u_3 \rangle = Z_2 \sin \theta. \quad (16)$$

For any primitive j , we define the quantities

$$r_j := \langle k^0(j), \omega \rangle, \quad p_j := \langle k^0(j), v_2 \rangle, \quad q_j := \langle k^0(j), v_3 \rangle, \quad (17)$$

and E_j, ψ_j through the formulas

$$\frac{\langle v_2, u_2 \rangle p_j + \langle v_2, u_3 \rangle q_j}{\langle v_2, u_2 \rangle^2 + \langle v_2, u_3 \rangle^2} = E_j \cos \psi_j, \quad \frac{\langle v_2, u_3 \rangle p_j - \langle v_2, u_2 \rangle q_j}{\langle v_2, u_2 \rangle^2 + \langle v_2, u_3 \rangle^2} = E_j \sin \psi_j. \quad (18)$$

The following proposition extends the results, given in Proposition 2 for the quadratic case, to the complex cubic case.

Proposition 3 *For any primitive $j = (j_1, j_2) \in \mathcal{P}$, the sequence of numerators $\gamma_{s(j, \cdot)}$ oscillates as $n \rightarrow \infty$ between two values,*

$$\gamma_j^- = \gamma_j^* (1 - \delta), \quad \gamma_j^+ = \gamma_j^* (1 + \delta), \quad (19)$$

where we define

$$\gamma_j^* = |r_j| K_j, \quad K_j = E_j^2 Z_1, \quad \delta = \frac{Z_2}{Z_1} < 1.$$

We also have:

- (a) $\gamma_{s(j, n)} = \gamma_j^* (1 + \delta \cos[2n\phi + 2\psi_j - \theta]) + \mathcal{O}(\lambda^{-3n/2})$;
- (b) $|s(j, n)|^2 = K_j (1 + \delta \cos[2n\phi + 2\psi_j - \theta]) \cdot \lambda^n + \mathcal{O}(\lambda^{-n/2})$;
- (c) $\gamma_j^- \geq \frac{1 - \delta}{2\lambda(1 + \delta)} \left[|j| - \frac{|u_1|}{2|\langle u_1, \omega \rangle|} \right]^2$.

Proof. We present the primitive vector associated to j in the basis u_1, u_2, u_3 :

$$k^0(j) = c_1 u_1 + c_2 u_2 + c_3 u_3,$$

and taking scalar products with ω, v_2 and v_3 and solving a linear system, one can obtain the values of the coefficients:

$$c_1 = \frac{r_j}{\langle u_1, \omega \rangle}, \quad c_2 = E_j \cos \psi_j, \quad c_3 = E_j \sin \psi_j, \quad (20)$$

where the definitions (17–18) have been taken into account. Now, we apply the iteration matrix U . Using the identities

$$U^n u_2 = \lambda^{n/2} [\cos(n\phi) u_2 + \sin(n\phi) u_3], \quad U^n u_3 = \lambda^{n/2} [-\sin(n\phi) u_2 + \cos(n\phi) u_3],$$

we find

$$s(j, n) = U^n k^0(j) = \lambda^{n/2} E_j [\cos(n\phi + \psi_j) u_2 + \sin(n\phi + \psi_j) u_3] + \mathcal{O}(\lambda^{-n}),$$

and we deduce, according to the definitions (16),

$$|s(j, n)|^2 = \lambda^n E_j^2 (Z_1 + Z_2 \cos[2n\phi + 2\psi_j - \theta]) + \mathcal{O}(\lambda^{-n/2}),$$

which gives (b). Multiplying by $|\langle s(j, n), \omega \rangle| = |r_j| \lambda^{-n}$, we obtain $\gamma_{s(j, n)}$ as given in (a). This implies the asymptotic bounds introduced in (19).

Finally, one easily sees that

$$|k^0(j) - c_1 u_1|^2 = |c_2 u_2 + c_3 u_3|^2 = E_j^2 (Z_1 + Z_2 \cos[2\psi_j - \theta]) \leq E_j^2 (Z_1 + Z_2),$$

and hence using (20) and (12), and also that $|k^0(j)| \geq |j|$, we get

$$K_j = E_j^2 Z_1 \geq \frac{|k^0(j) - c_1 u_1|^2}{1 + \delta} \geq \frac{1}{1 + \delta} \left[|j| - \frac{|u_1|}{2|\langle u_1, \omega \rangle|} \right]^2,$$

which implies the lower bound given in (c). \square

As we can see in (a), the existence of limit of the sequences $\gamma_{s(j, n)}$ stated in Proposition 2 for the quadratic case, is replaced here by an *oscillatory limit behavior*, with a lower limit $\liminf_{n \rightarrow \infty} \gamma_{s(j, n)} \geq \gamma_j^-$ and an upper limit $\limsup_{n \rightarrow \infty} \gamma_{s(j, n)} \leq \gamma_j^+$. Notice that such values of the limits are exact if the phase $\phi/2\pi$, that appears in (a), is irrational.

Selecting the primitive $j_0 \in \mathcal{P}$ which gives the minimal limits, we have the *primary resonances*, and we denote them by $s_0(n) := s(j_0, n)$, and we call *secondary resonances* the integer vectors belonging to any of the remaining sequences $s(j, n)$, $j \neq j_0$. Such primary resonances can easily be detected thanks to Proposition 3(c): although γ_j^\pm are

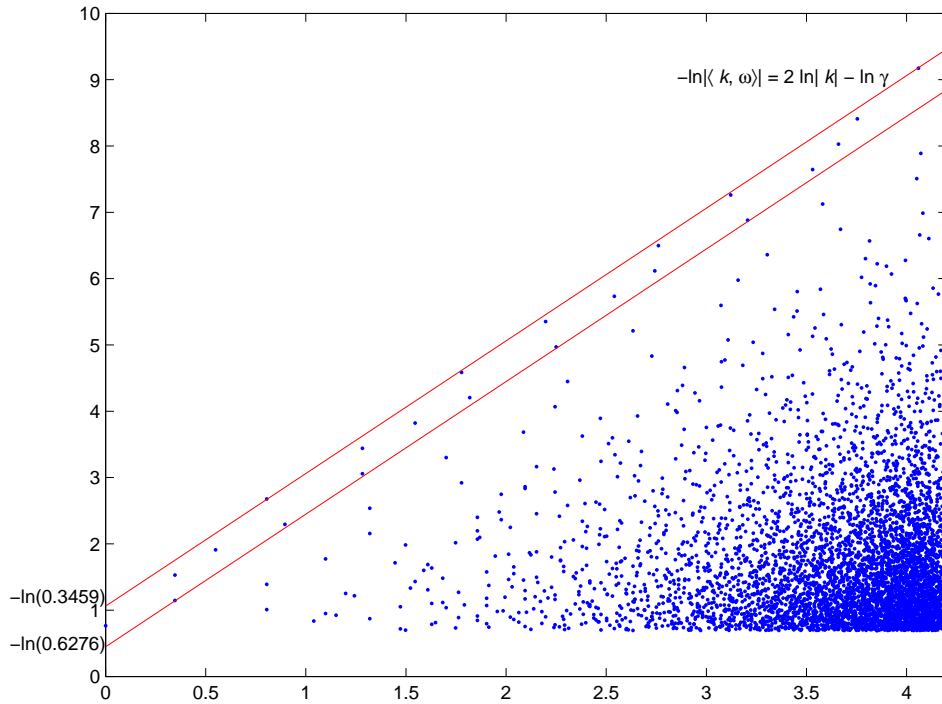


Figure 1: Points $(\ln |k|, -\ln |\langle k, \omega \rangle|)$.

not increasing in general with respect to $|j|$, we have an increasing lower bound, which implies that $\lim_{|j| \rightarrow \infty} \gamma_j^\pm = \infty$, and then one has to check only a finite number of primitive vectors j in order to find the minimal γ_j^- and γ_j^+ and, hence, the primary resonances.

As in Section 2.2, we define normalized values $\tilde{\gamma}_k, \tilde{\gamma}_j^*, \tilde{\gamma}_j^\pm$, after dividing by the minimal among the values γ_j^* ,

$$\tilde{\gamma}_k := \frac{\gamma_k}{\gamma^*}, \quad \tilde{\gamma}_j^* := \frac{\gamma_j^*}{\gamma^*}, \quad \tilde{\gamma}_j^\pm := \frac{\gamma_j^\pm}{\gamma^*}, \quad \text{where } \gamma^* := \min_{j \in \mathcal{P}} \gamma_j^* = \gamma_{j_0}^*. \quad (21)$$

We also introduce a parameter B_0^- , as a measure for the *separation* between primary and secondary resonances:

$$B_0^- := \min_{j \in \mathcal{P} \setminus \{j_0\}} \left(\frac{\tilde{\gamma}_j^-}{\tilde{\gamma}_{j_0}^+} \right)^{1/3} \quad (22)$$

(compare with (15), and see also (36)). Notice that the distinction between primary and secondary resonances makes sense if $B_0^- > 1$, i.e. the interval $[\tilde{\gamma}_{j_0}^-, \tilde{\gamma}_{j_0}^+]$ has no intersection with any other interval $[\tilde{\gamma}_j^-, \tilde{\gamma}_j^+]$, $j \neq j_0$ (as happens in the cubic golden case, see the next section).

2.4 The cubic golden frequency vector

Now, we assume that Ω is the *cubic golden number*: the real root of $x^3 + x - 1 = 0$. We have $\Omega \approx 0.6823$. In this case, the matrix T can easily be found by inspection. We have

$$T = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad U = (T^{-1})^\top = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

with the eigenvalue $\lambda = 1/\Omega \approx 1.4656$.

It is not hard to compute the data provided by Proposition 3 in this concrete case. In particular, we have

$$\phi = \arg(\lambda_2) = -\arctan \frac{4\sqrt{31}}{\Omega(6\Omega^2 + 9\Omega + 4)} + \pi \approx \frac{13\pi}{22} \quad (23)$$

and, from Proposition 3(b), we have the following approximately periodic behaviors: $\gamma_{s(j,n+22)} \approx \gamma_{s(j,n)}$, and $|s(j,n+22)| \approx \lambda^{11} |s(j,n)|$. Other relevant parameters are: $\gamma^* = \frac{2}{31}(5 + \Omega + 4\Omega^2) \approx 0.4867$ and $\delta = (3 - 2\Omega)\sqrt{2 - \Omega + \Omega^2}/(5 + \Omega + 4\Omega^2) \approx 0.2895$. In Table 1, we write down the values γ_j^* , as well as the bounds γ_j^- and γ_j^+ , for the resonant sequences induced by a few primitives $k_0(j)$, and a lower bound for all other primitives. The smallest ones correspond to the primitive vector $k_0([0, 1]) = [0, 0, 1]$ (primary resonances). The parameter introduced in (22), indicating the separation between the primary and the secondary resonances, is $B_0^- \approx 1.1824$.

Additionally, it is interesting to visualize such a separation in the following way. Taking logarithm in both hands of the Diophantine condition (3), we can write it as

$$-\ln |\langle k, \omega \rangle| \leq 2 \ln |k| - \ln \gamma.$$

If we draw all the points with coordinates $(\ln |k|, -\ln |\langle k, \omega \rangle|)$ (see Figure 1), we can see a sequence of points lying between the two straight lines $-\ln |\langle k, \omega \rangle| = 2 \ln |k| - \ln \gamma_{[0,1]}^\pm$. Such points correspond to integer vectors belonging to the sequence of primary resonances: $k = s_0(n)$, $n \geq 0$.

$k_0(j)$	γ_j^-	γ_j^*	γ_j^+
$[0, 0, 1]$	0.3459	0.4867	0.6276
$[-1, 2, 0]$	1.0376	1.4602	1.8829
$[-2, 1, 2]$	3.1127	4.3807	5.6488
$ j \geq 3$	≥ 1.2742		

Table 1: Numerical data for the cubic golden frequency vector

3 Asymptotic estimates for the maximal splitting distance

In order to provide asymptotic estimates (or lower bounds) for the splitting, we start with the first order approximation, given by the Poincaré–Melnikov method. It is convenient for us to work with the (scalar) Melnikov potential L and the splitting potential \mathcal{L} , but we state our main result in terms of the splitting function $\mathcal{M} = \nabla\mathcal{L}$, which gives a measure of the splitting distance between the invariant manifolds of the whiskered torus. Notice also that the nondegenerate critical points of \mathcal{L} correspond to simple zeros of \mathcal{M} , and give rise to transverse homoclinic orbits to the whiskered torus.

In the present paper, we restrict ourselves to present the constructive part of the proofs, which corresponds to find, for every sufficiently small ε , the dominant harmonics of the Fourier expansion of the Melnikov potential $L(\theta)$, as well as to provide bounds for the sum of the remaining terms of that expansion. The final step, to ensure that the Poincaré–Melnikov method (1) predicts correctly the size of splitting in our singular case $\mu = \varepsilon^p$, can be worked out simply by showing that the asymptotic estimates of the dominant harmonics are large enough to overcome the harmonics of the error term. This final step is analogous to the one done in [DG04] for the case of the golden number Ω_1 (using the upper bounds for the error term provided in [DGS04]), and will be published elsewhere for all cases considered in Theorem 1.

First, we are going to find in Section 3.1 an exponentially small asymptotic estimate for the dominant harmonic among the ones associated to primary resonances, given by a function $h_1(\varepsilon)$ in the exponent. We also provide an estimate for the sum of all other (primary or secondary) harmonics. This can be done jointly for both the quadratic ($\ell = 2$) and cubic ($\ell = 3$) cases. In Section 3.2, we establish a condition ensuring that the dominant harmonic among all harmonics is given by a primary resonance. This condition is fulfilled for the frequencies (4–5). To complete the proof of Theorem 1, we show that the different arithmetic properties of quadratic and cubic frequencies lead to different properties of the function $h_1(\varepsilon)$: periodic (with respect to $\ln \varepsilon$) in the quadratic case, and a more complicated bounded function in the cubic case.

3.1 Dominant harmonics of the splitting potential

We put our functions f and h , defined in (7), into the integral (9) and get the Fourier expansion of the Melnikov potential,

$$L(\theta) = \sum_{k \in \mathbb{Z} \setminus \{0\}} L_k \cos(\langle k, \theta \rangle - \sigma_k), \quad L_k = \frac{2\pi |\langle k, \omega_\varepsilon \rangle| e^{-\rho|k|}}{\sinh |\frac{\pi}{2} \langle k, \omega_\varepsilon \rangle|}.$$

Using (2) and (10), we present the coefficients in the form

$$L_k = \alpha_k e^{-\beta_k}, \quad \alpha_k \approx \frac{4\pi\gamma_k}{|k|^{\ell-1}\sqrt{\varepsilon}}, \quad \beta_k = \rho|k| + \frac{\pi\gamma_k}{2|k|^{\ell-1}\sqrt{\varepsilon}}, \quad (24)$$

where an exponentially small term has been neglected in the denominator of α_k . For any given ε , the harmonics with largest coefficients $L_k(\varepsilon)$ correspond essentially to the smallest exponents $\beta_k(\varepsilon)$. Thus, we have to study the dependence on ε of such exponents.

With this aim, we introduce for any X, Y (and a fixed $\ell = 2, 3$) the function

$$G(\varepsilon; X, Y) := \frac{Y^{1/\ell}}{\ell} \left[(\ell - 1) \left(\frac{\varepsilon}{X} \right)^{1/2\ell} + \left(\frac{X}{\varepsilon} \right)^{(\ell-1)/2\ell} \right]. \quad (25)$$

One easily checks that this function has its minimum at $\varepsilon = X$, and the corresponding minimum value is $G(X; X, Y) = Y^{1/\ell}$. Then, the exponents $\beta_k(\varepsilon)$ in (24) can be presented in the form

$$\beta_k(\varepsilon) = \frac{C_0}{\varepsilon^{1/2\ell}} g_k(\varepsilon),$$

where we define

$$g_k(\varepsilon) := G(\varepsilon; \varepsilon_k, \tilde{\gamma}_k), \quad \varepsilon_k := D_0 \frac{\tilde{\gamma}_k^2}{|k|^{2\ell}}, \quad (26)$$

$$C_0 = \ell \left(\frac{\rho}{\ell-1} \right)^{(\ell-1)/\ell} \cdot \left(\frac{\pi\gamma^*}{2} \right)^{1/\ell}, \quad D_0 = \left(\frac{(\ell-1)\pi\gamma^*}{2\rho} \right)^2, \quad (27)$$

with $\gamma^* = \gamma_{j_0}^*$ and $\tilde{\gamma}_k$ given in (14–15) and (21), for $\ell = 2$ and $\ell = 3$ respectively. Consequently, for all k we have $\beta_k \geq \frac{C_0 \tilde{\gamma}_k^{1/\ell}}{\varepsilon^{1/2\ell}}$, which provides the maximum value of the coefficient $L_k(\varepsilon)$ of the harmonic given by the integer vector k . Recall that for $k = s(j, n)$, belonging to the resonant sequence generated by a given primitive $j \in \mathcal{P}$ (see definition (13)), the (normalized) numerators $\tilde{\gamma}_k$ tend to a limit $\tilde{\gamma}_j^*$ (see Proposition 2 for the quadratic case), or oscillate between two limit values $\tilde{\gamma}_j^\pm$, with $\tilde{\gamma}_j^- < \tilde{\gamma}_j^* < \tilde{\gamma}_j^+$ (see Proposition 3 for the cubic case).

The primary integer vectors k , belonging to the sequence $s_0(n) = s(j_0, n)$, play an important rôle here, since they give the smallest limit $\tilde{\gamma}_{j_0}^* = 1$. Consequently, they give the dominant harmonics of the Melnikov potential, at least for ε close to their minimum points ε_k . Our aim is to show that this happens also for any ε (small enough) not necessarily close to ε_k , under a condition ensuring that the separation between the primary and secondary resonances is large enough (see the separation condition (36)). For the sequence of primary resonances, the asymptotic behavior of the functions $g_k(\varepsilon)$ in (26), $k = s_0(n)$, is obtained from the main terms, as $n \rightarrow \infty$, given by Propositions 2 and 3. Thus, we can write

$$g_{s_0(n)}(\varepsilon) \approx g_n^*(\varepsilon) := G(\varepsilon; \varepsilon_n^*, b_n), \quad \varepsilon_n^* := \frac{D_0}{K_{j_0}^{2\ell/(\ell-1)}} \left(\frac{1}{b_n \lambda^{\ell n}} \right)^{2/(\ell-1)}, \quad (28)$$

where we define, in order to unify the notation,

$$\begin{aligned} b_n &= 1 && \text{in the quadratic case;} \\ b_n &= 1 + \delta \cos[2n\phi + 2\psi_{j_0} - \theta] && \text{in the cubic case.} \end{aligned} \quad (29)$$

Notice that (28) relies in the approximations $\tilde{\gamma}_{s_0(n)} \approx b_n$ and $|s_0(n)|^{\ell-1} \approx K_{j_0} b_n \lambda^n$, as well as the fact that $\tilde{\gamma}_{j_0}^* = 1$.

Now we define, for any given ε , the function $h_1(\varepsilon)$ as the minimum of the values $g_n^*(\varepsilon)$, which takes place for some index $N = N(\varepsilon)$,

$$h_1(\varepsilon) := \min_{n \geq 0} g_n^*(\varepsilon) = g_N^*(\varepsilon). \quad (30)$$

As we show in the next result, the function $h_1(\varepsilon)$ indicates for any ε the size of the dominant harmonic among the primary resonances, given by the integer vector $k = s_0(N)$. Besides, we are going to establish an asymptotic estimate for the sum of all remaining coefficients in the Fourier expansion of the splitting function. This second estimate is written in terms of the function

$$h_2(\varepsilon) := \min_{k \neq s_0(N)} g_k(\varepsilon). \quad (31)$$

See Figure 2 as an illustration for the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$, corresponding to the case of the quadratic vector given by $\Omega_{1,2}$, and Figure 3 for $h_1(\varepsilon)$ in the case of the cubic golden vector.

Notice that this result is stated in terms of the Fourier coefficients of the splitting function $\mathcal{M} = \nabla \mathcal{L}$. We write, for the splitting potential,

$$\mathcal{L}(\theta) = \sum_{k \in \mathcal{Z} \setminus \{0\}} \mathcal{L}_k \cos(\langle k, \theta \rangle - \tau_k),$$

with upper bounds for $|\mathcal{L}_k - \mu L_k|$ and $|\tau_k - \sigma_k|$. Then, for the splitting function we have $|\mathcal{M}_k| = |k| \mathcal{L}_k$.

Proposition 4 *For ε small enough and $\mu = \varepsilon^p$ with $p > 3$, one has:*

$$(a) \quad |\mathcal{M}_{s_0(N)}| \sim \mu |s_0(N)| L_{s_0(N)} \sim \frac{\mu}{\varepsilon^{1/\ell}} \exp \left\{ -\frac{C_0 h_1(\varepsilon)}{\varepsilon^{1/2\ell}} \right\};$$

$$(b) \quad \sum_{k \neq s_0(N)} |\mathcal{M}_k| \sim \frac{\mu}{\varepsilon^{1/\ell}} \exp \left\{ -\frac{C_0 h_2(\varepsilon)}{\varepsilon^{1/2\ell}} \right\}.$$

Sketch of the proof. At first order in μ , for the coefficients of the splitting function we can write

$$|\mathcal{M}_k| \sim \mu |k| L_k = \mu |k| \alpha_k e^{-\beta_k}, \quad (32)$$

where we have neglected the error term in the Melnikov approximation (1), and we have used the expression (24) for the coefficients of the Melnikov potential. As mentioned throughout this section, the main behavior of the coefficients L_k is given by the exponents β_k , which have been written in (26) in terms of the functions $g_k(\varepsilon)$. In particular, the coefficient associated to the dominant harmonic, among the primary resonances, $L_{s_0(N)}$ with $N = N(\varepsilon)$, can be expressed in terms of the function $h_1(\varepsilon)$ introduced in (30).

Now, we consider the remaining factors in (32). We see from (24) that such factors can be written as $|k| \alpha_k \sim |k|^{-(\ell-2)} \sqrt{\varepsilon}$. For $k = s_0(N)$, let us show that they turn out to be polynomial with respect to ε , with a concrete exponent to be determined. First, we use that in (28) we have $\varepsilon_n^* \sim \lambda^{-2\ell n/(\ell-1)}$. Then, for a given ε the coefficient $N = N(\varepsilon)$ giving the dominant harmonic is such that ε_N^* (the minimum of the function g_N^*) is close to ε , and hence $\lambda^N \sim \varepsilon^{-(\ell-1)/2\ell}$. On the other hand, we deduce from Propositions 2(b) and 3(b) that $|s_0(N)|^{\ell-1} \sim \lambda^N$. Putting the obtained estimates together, we get $|s_0(N)| \alpha_{s_0(N)} \sim \varepsilon^{-1/\ell}$, which provides the polynomial factor in part (a). The estimate obtained is valid for the dominant coefficient of the Melnikov function. To get the analogous estimate for the splitting function, one has to bound the corresponding coefficient of the error term in (1), showing that it is also exponentially small and dominated by the main term in the approximation. This works as in [DG04], where the case of the golden number was considered, and we omit the details here.

The proof of part (b) can be carried out in similar terms. For the second dominant harmonic, we get an exponentially small estimate with the function $h_2(\varepsilon)$, defined in (31). This estimate is also valid if one considers the whole sum in (b), since the terms of this sum can be bounded by a geometric series, and hence it can be estimated by its dominant term (see [DG04] for more details). \square

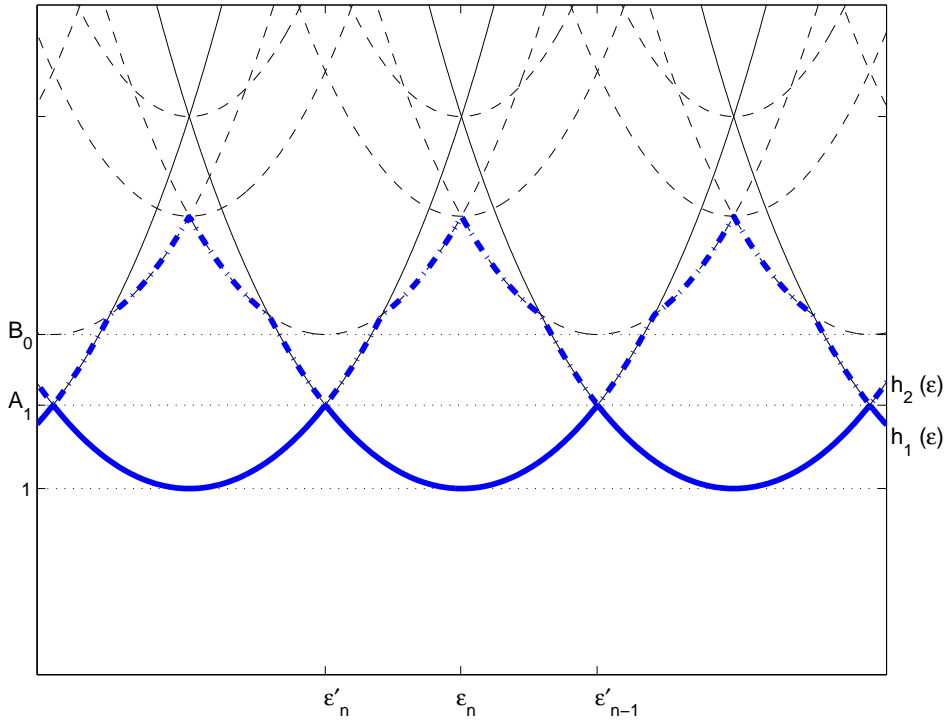


Figure 2: Graphs of the functions $g_k(\varepsilon)$, $h_1(\varepsilon)$, $h_2(\varepsilon)$ for the quadratic number $\Omega_{1,2}$, using a logarithmic scale for ε

3.2 Study of the functions $h_1(\varepsilon)$ and $h_2(\varepsilon)$

To conclude the proof of Theorem 1, we show in this section the different properties of the function $h_1(\varepsilon)$ for the quadratic and cubic cases, and establish a condition, fulfilled in all cases (4-5), ensuring that $h_2(\varepsilon) \leq h_1(\varepsilon)$ for any ε .

Lemma 5

- (a) In the quadratic case $\ell = 2$, the function $h_1(\varepsilon)$ is $4 \ln \lambda$ -periodic in $\ln \varepsilon$, with $\min h_1(\varepsilon) = A_0$ and $\max h_1(\varepsilon) = A_1$, with constants

$$A_0 = 1, \quad A_1 = \frac{1}{2} \left(\frac{1}{\sqrt{\lambda}} + \sqrt{\lambda} \right). \quad (33)$$

- (b) In the cubic case $\ell = 3$, the function $h_1(\varepsilon)$ satisfies the bound $0 < A_0^- \leq h_1(\varepsilon) \leq A_1^+$, with constants

$$A_0^- = (1 - \delta)^{1/3}, \quad A_1^+ = \frac{(1 + \delta)^{1/3}}{3} \left[2 \left(\frac{\sqrt{\lambda} + 1}{2\lambda} \right)^{1/6} + \left(\frac{2\lambda}{\sqrt{\lambda} + 1} \right)^{1/3} \right]. \quad (34)$$

Proof. We use that the functions $g_n^*(\varepsilon)$ and the values ε_n^* satisfy the following scaling properties:

$$g_{n+1}^*(\varepsilon) = \left(\frac{b_{n+1}}{b_n} \right)^{1/\ell} \cdot g_n^* \left(\frac{\varepsilon_n^*}{\varepsilon_{n+1}^*} \cdot \varepsilon \right), \quad \varepsilon_{n+1}^* = \left(\frac{b_n}{b_{n+1}\lambda^\ell} \right)^{2/(\ell-1)} \cdot \varepsilon_n^*. \quad (35)$$

In the quadratic case ($\ell = 2$), we have $b_n = b_{n+1} = 1$, and hence (35) becomes

$$g_{n+1}^*(\varepsilon) = g_n^*(\lambda^4 \varepsilon), \quad \varepsilon_{n+1}^* = \frac{\varepsilon_n^*}{\lambda^4},$$

where we have (ε_n^*) as a geometric sequence, and the functions $g_n^*(\varepsilon)$ are just translations of the initial one, if we use a logarithmic scale for ε (see Figure 2). It is easy to check that the intersection between the graphs of the functions g_n^*

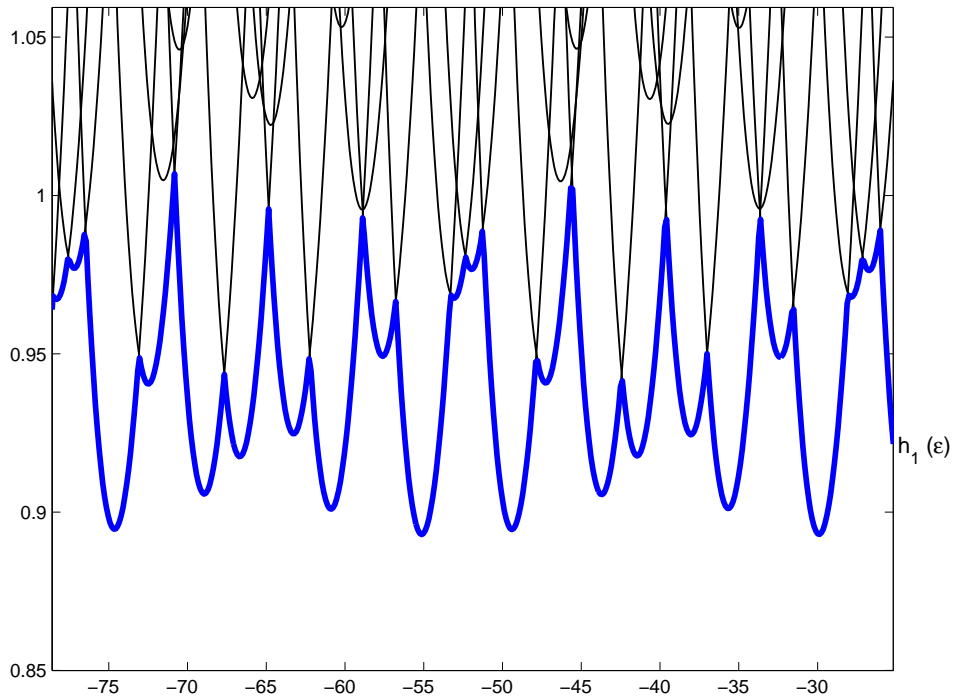


Figure 3: Graphs of the functions $g_n^*(\varepsilon)$ and $h_1(\varepsilon)$ for the cubic golden number $\Omega \approx 0.6823$, using a logarithmic scale for ε .

and g_{n+1}^* takes place at $\varepsilon'_n := \varepsilon_n^*/\lambda^2$. Thus, for $\varepsilon \in [\varepsilon'_n, \varepsilon'_{n-1}]$ we have $N(\varepsilon) = n$, and hence $h_1(\varepsilon) = g_n^*(\varepsilon)$. We can obtain $h_1(\varepsilon)$ from any given interval $[\varepsilon'_n, \varepsilon'_{n-1}]$ by extending it as a $4 \ln \lambda$ -periodic function of $\ln \varepsilon$, and it is clear that its minimum and maximum values are $h_1(\varepsilon_n^*) = 1$ and $h_1(\varepsilon'_n) = h_1(\varepsilon'_{n-1}) = A_1$ respectively.

The cubic case ($\ell = 3$) becomes more cumbersome, because the function $h_1(\varepsilon)$ is not periodic in $\ln \varepsilon$, due to the oscillating quantities b_n, b_{n+1} in (35); notice that $\frac{1-\delta}{1+\delta} < \frac{b_{n+1}}{b_n} < \frac{1+\delta}{1-\delta}$. Nevertheless, we are going to obtain a periodic upper bound for the function $h_1(\varepsilon)$. Let us introduce the functions

$$g_n^+(\varepsilon) := G(\varepsilon; \varepsilon_n^+, 1 + \delta), \quad \varepsilon_n^+ := \frac{D_0}{K_{j_0}^3} \cdot \frac{1}{(1 + \delta)\lambda^{3n}},$$

obtained from (28), by replacing the oscillatory factors b_n by the constant $1 + \delta$ (and taking $\ell = 3$). One can check that the graphs of the functions $g_n^*(\varepsilon)$ and $g_n^+(\varepsilon)$ have no intersection if $b_n < 1 + \delta$, and coincide if $b_n = 1 + \delta$, which implies that $g_n^*(\varepsilon) \leq g_n^+(\varepsilon)$ for any ε . As in (30), we can define

$$h_1^+(\varepsilon) := \min_{n \geq 0} g_n^+(\varepsilon),$$

and it is clear that we have the upper bound $h_1(\varepsilon) \leq h_1^+(\varepsilon)$ for any ε . By a similar argument to the one used in the quadratic case, we can establish the periodicity in $\ln \varepsilon$ of the function $h_1^+(\varepsilon)$, and we find its maximum value. Indeed, one can check from the expression (25), with $\ell = 3$, that the graphs of the functions g_n^+ and g_{n+1}^+ intersect at

$$\varepsilon'_n := \left(\frac{\sqrt{\lambda} + 1}{2\lambda} \right)^2 \cdot \varepsilon_n^+, \quad \text{with } g_n^+(\varepsilon'_n) = g_{n+1}^+(\varepsilon'_n) = A_1^+.$$

Thus, we have $h_1^+(\varepsilon) = g_n^+(\varepsilon)$ for $\varepsilon \in [\varepsilon'_n, \varepsilon'_{n-1}]$, and we can extend it as a $3 \ln \lambda$ -periodic function in $\ln \varepsilon$, whose maximum value is A_1^+ , which provides an upper bound for $h_1(\varepsilon)$. On the other hand, since $g_n^*(\varepsilon) \geq b_n^{1/3}$ it is clear that $h_1(\varepsilon) \geq A_0^-$. \square

Remark. In general, for the cubic case the function $h_1(\varepsilon)$ is not periodic in $\ln \varepsilon$, but we can conjecture that it is quasiperiodic due to the oscillating quantities b_n introduced in (29). In fact, using the approximation (23) for the angle ϕ , we have $b_{n+22} \simeq b_n$, which gives $66 \ln \lambda$ as an approximate period for $h_1(\varepsilon)$ (see Figure 3).

As said in Section 3.1, the function $h_1(\varepsilon)$ is related to the dominant harmonic among the primary resonances, corresponding to the integer vector $s_0(N)$, with $N = N(\varepsilon)$. In order to ensure that this harmonic provides the maximal splitting distance, we need that $h_1(\varepsilon) \leq g_k(\varepsilon)$ also for secondary harmonics k . To have this inequality, the separation between the primary and secondary resonances has to be large enough. Recalling that the separations B_0, B_0^- for quadratic and cubic frequencies were defined in (15) and (22) respectively, we impose the “*separation condition*”:

$$\begin{aligned} B_0 &\geq A_1 && \text{in the quadratic case;} \\ B_0^- &\geq A_1^+ && \text{in the cubic case.} \end{aligned} \tag{36}$$

A numerical exploration of this condition, among quadratic frequency vectors given by purely periodic continued fractions, indicates that the 24 cases considered in (4) are all the ones satisfying it. On the other hand, the condition is fulfilled in the case of the cubic golden vector considered in (5), since $B_0^- \approx 1.1824$ and $A_1^+ \approx 1.0909$, as can be checked from the numerical data given in Section 2.4.

Notice that, under the separation condition (36), we have $h_2(\varepsilon) < h_1(\varepsilon)$, unless ε is very close to some concrete values where there is a change in the dominant harmonic given by $N = N(\varepsilon)$ (in the quadratic case, this happens for ε close to the geometric sequence (ε'_n) ; see Figure 2).

Now we can complete the proof of Theorem 1. Indeed, according to Proposition 4 the separation condition implies that for any ε sufficiently small, we have the estimate

$$\max_{\theta \in \mathbb{T}^\ell} |\mathcal{M}(\theta)| \sim |\mathcal{M}_{s_0(N)}|,$$

since the coefficient of the dominant harmonic $k = s_0(N)$, $N = N(\varepsilon)$, is greater or equal than the sum of all other harmonics.

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