

KINEMATIC REDUCTION AND THE HAMILTON-JACOBI EQUATION

MARÍA BARBERO-LIÑÁN, MANUEL DE LEÓN, JUAN C. MARRERO,
DAVID MARTÍN DE DIEGO, AND MIGUEL C. MUÑOZ-LECANDA

ABSTRACT. A close relationship between the classical Hamilton-Jacobi theory and the kinematic reduction of control systems by decoupling vector fields is shown in this paper. The geometric interpretation of this relationship relies on new mathematical techniques for mechanics defined on a skew-symmetric algebroid. This geometric structure allows us to describe in a simplified way the mechanics of nonholonomic systems with both control and external forces.

Dedicated to Tudor Ratiu on the occasion of his 60th birthday

1. INTRODUCTION

The reduction of mechanical control systems to kinematic systems is very interesting and useful for solving control problems such as optimal control problems [3] and for designing suitable control laws (see [21], Chapter 8 in [4] and references therein). For instance, the planning motion for the associated kinematic system determines trajectories of the mechanical control system. Thus the methodologies to find these trajectories have been simplified because the kinematic reduction gives rise to a first-order control-linear system defined on the configuration manifold. Hence in general it is easier to solve or to analyze the kinematic system. If the mechanical control system is reducible to a kinematic one, then the controlled trajectories of this kinematic system under reparametrization define solutions of the original second-order problem on the phase space. An interesting particular case is the one

Date: October 28, 2011.

This work has been partially supported by MICINN (Spain) Grants MTM 2007-62478, MTM2008-00689, MTM2009-08166 and MTM2009-13383; SOLSUV 2008-01000238 and ProID20100210 of the Canarian government, 2009SGR1338 of the Catalan government, IRSES project GEOMECH (246981) within the 7th European Community Framework Program, and project “Ingenio Mathematica” (i-MATH) No. CSD 2006-00032 (Consolider-Ingenio 2010). MBL has been financially supported by Juan de la Cierva fellowship from MICINN.

defined by kinematic reductions of order 1. This kind of reductions define a decoupling vector field. Unfortunately, there is not a systematic procedure for finding such kinematic reductions.

The philosophy of kinematic reductions of order 1 seems, in a first approach, quite similar to the standard Hamilton-Jacobi theory. This theory, that appeared with the dawn of analytical mechanics, is a valuable tool for the exact integration of Hamilton's equations, for instance using the technique of separation of variables (see [1] and references therein). In many cases, the Hamilton-Jacobi theory allows us to simplify the integration of Hamilton's equations or, at least, to find some particular solutions. To be more precise, consider a configuration manifold Q and a hamiltonian function $H : T^*Q \rightarrow \mathbb{R}$. The Hamilton-Jacobi equation can be written as

$$H\left(q, \frac{\partial W}{\partial q}\right) = \text{constant}$$

for some function $W : Q \rightarrow \mathbb{R}$. If we find such a function W , then the integration of the associated Hamilton's equations (with initial conditions along $dW(Q)$) is reduced to knowing the integral curves of a vector field X_H^{dW} on Q . This vector field is given by $X_H^{dW} = T\tau_{T^*Q} \circ X_H \circ dW \in \mathfrak{X}(Q)$, where $\tau_{T^*Q} : T^*Q \rightarrow Q$ is the canonical projection and X_H is the hamiltonian vector field associated to H . Hence, from the integration of a vector field on the configuration space it is possible to recover some of the solutions of the original hamiltonian system. Recent developments in Hamilton-Jacobi theory are described in [5, 6, 13, 14, 19, 22]. Of course, the possible similarities with the theory of kinematic reductions are now clearer.

One of the main objectives in our paper is to carefully study the underlying geometry of the kinematic reduction theory by showing the close relation with the classical Hamilton-Jacobi theory. Moreover, advantage of recent developments in Hamilton-Jacobi theory for nonholonomic systems on skew-symmetric algebroids [2, 14] even with external forces will be really useful to obtain a full novel theory of kinematic reduction for this type of systems. It is important to highlight this is not an arbitrary generalization since the mechanics on algebroids [8, 9, 11, 17, 20] is particularly relevant for the class of Lagrangian systems invariant under the action of a Lie group of symmetries including as a particular case nonholonomic dynamics (see [7] for a survey on the subject; see also [15, 20, 23]).

The main results of this paper can be summarized in the following points:

- A description of nonholonomic mechanics in terms of the Levi-Civita connection associated to a fibered riemannian metric defined on the vector subbundle determined by the nonholonomic constraints.
- A deduction of the Hamilton-Jacobi equation for nonholonomic systems in terms of the induced Levi-Civita connection.
- An affine connection approach in presence of control forces.
- A description of Hamilton-Jacobi equation with controls.
- Relationship between Hamilton-Jacobi equation and kinematic reductions by decoupling vector fields.

It is interesting to observe that our approach allows us to extend the theory of kinematic reduction to controlled system with symmetries as for instance, nonholonomic Lagrange-Poincaré equations, etc.

In the sequel, all the manifolds are real, second countable and C^∞ . The maps are assumed to be also C^∞ . Sum over all repeated indices is understood.

2. SKEW-SYMMETRIC ALGEBROIDS

In this section we introduce the notion of a skew-symmetric algebroid on a vector bundle $\tau_D : D \rightarrow M$. It is known that this geometric structure covers many interesting cases in mechanics, as for instance, nonholonomic mechanics (see [14]). Similarly to the intrinsic definition of the Euler-Lagrange equations for a Lagrangian function $L : TM \rightarrow \mathbb{R}$ obtained by the canonical structures on it (standard Lie bracket, exterior differential...), it is possible to determine the motion equations for a Lagrangian $L : D \rightarrow \mathbb{R}$ using the differential geometric structures naturally induced by the skew-symmetric algebroid structure. We will show that this generalization is quite useful in applications and clarifies the dynamics of systems with nonholonomic constraints. Let us first introduce the notion of a skew-symmetric algebroid.

Definition 2.1. A *skew-symmetric algebroid structure* on the vector bundle $\tau_D : D \rightarrow M$ is a \mathbb{R} -bilinear bracket $[\cdot, \cdot]_D : \Gamma(\tau_D) \times \Gamma(\tau_D) \rightarrow \Gamma(\tau_D)$ on the space $\Gamma(\tau_D)$ of sections of τ_D and a vector bundle morphism $\rho_D : D \rightarrow TM$, so-called **anchor map**, such that:

(i) $[\cdot, \cdot]_D$ is skew-symmetric, that is,

$$[X, Y]_D = -[Y, X]_D, \quad \text{for } X, Y \in \Gamma(\tau_D).$$

(ii) If we also denote by $\rho_D : \Gamma(\tau_D) \rightarrow \mathfrak{X}(M)$ the morphism of $C^\infty(M)$ -modules induced by the anchor map then

$$[X, fY]_D = f[X, Y]_D + \rho_D(X)(f)Y, \quad \text{for } X, Y \in \Gamma(\tau_D) \text{ and } f \in C^\infty(M).$$

If the bracket $[\![\cdot, \cdot]\!]_D$ satisfies the Jacobi identity, we have that the pair $([\![\cdot, \cdot]\!]_D, \rho_D)$ is a **Lie algebroid structure** on the vector bundle $\tau_D : D \rightarrow M$.

If $([\![\cdot, \cdot]\!]_D, \rho_D)$ is a skew-symmetric algebroid structure on the vector bundle $\tau_D : D \rightarrow M$, then an **almost differential** d^D of sections of $\Lambda^k \tau_{D^*}$ being $\tau_{D^*} : D^* \rightarrow M$ the vector bundle projection of the dual bundle D^* is defined as follows

$$\begin{aligned} (d^D \alpha)(X_0, X_1, \dots, X_k) &= \sum_{i=0}^k (-1)^i \rho_D(X_i) (\alpha(X_0, \dots, \hat{X}_i, \dots, X_k)) \\ &+ \sum_{i < j} (-1)^{i+j} \alpha([\![X_i, X_j]\!]_D, X_0, X_1, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_k) \end{aligned}$$

for $\alpha \in \Gamma(\Lambda^k \tau_{D^*})$ and $X_0, X_1, \dots, X_k \in \Gamma(\tau_D)$.

In general $(d^D)^2 \neq 0$. Indeed, $([\![\cdot, \cdot]\!]_D, \rho_D)$ is a Lie algebroid structure on the vector bundle $\tau_D : D \rightarrow M$ if and only if $(d^D)^2 = 0$ (see [17, 18, 20, 25] for more details about the Lie algebroids).

Suppose that (x^i) are local coordinates on M and that $\{e_A\}$ is a local basis of the space of sections $\Gamma(\tau_D)$, then

$$[\![e_A, e_B]\!]_D = \mathfrak{C}_{AB}^C e_C, \quad \rho_D(e_A) = (\rho_D)_A^i \frac{\partial}{\partial x^i}.$$

The functions $\mathfrak{C}_{AB}^C, (\rho_D)_A^i \in C^\infty(M)$ are called the **local structure functions** of the skew-symmetric algebroid on $\tau_D : D \rightarrow M$.

If $\{e^A\}$ is the dual basis of $\{e_A\}$, then

$$\begin{aligned} d^D F &= (\rho_D)_A^i \frac{\partial F}{\partial x^i} e^A, \\ d^D \kappa &= \left\{ (\rho_D)_A^i \frac{\partial \kappa_B}{\partial x^i} - \frac{1}{2} \mathfrak{C}_{AB}^C \kappa_C \right\} e^A \wedge e^B, \end{aligned}$$

where $F \in C^\infty(M)$ and $\kappa = \kappa_B e^B \in \Gamma(\tau_{D^*})$.

A **ρ_D -admissible curve** is a curve $\gamma : I \subseteq \mathbb{R} \rightarrow D$ such that

$$\frac{d(\tau_D \circ \gamma)}{dt}(t) = \rho_D(\gamma(t)).$$

Given $X \in \Gamma(\tau_D)$, the **integral curves** of the section X are those curves $\sigma : I \subseteq \mathbb{R} \rightarrow M$ such that satisfy

$$\dot{\sigma} = \rho_D(X) \circ \sigma.$$

That is, they are the integral curves of the associated vector field $\rho_D(X) \in \mathfrak{X}(M)$. If σ is an integral curve of X , then $X \circ \sigma$ is a ρ_D -admissible curve. Locally, the integral curves are characterized as the

solutions of the following system of differential equations

$$\dot{x}^i = (\rho_D)_A^i X^A(x),$$

where $X = X^A e_A$.

Consider now the vector space over \mathbb{R}

$$H^0(d^D) = \{f \in C^\infty(M) \mid d^D f = 0\}.$$

If M is connected and D is a transitive skew-symmetric algebroid, that is,

$$\rho_D(D_x) = T_x M \quad \text{for all } x \in M,$$

then

$$H^0(d^D) \simeq \mathbb{R}. \quad (2.1)$$

It is important to stress that condition (2.1) holds if the skew-symmetric algebroid has a connected base space and is completely nonholonomic, that is,

$$\text{Lie}_x^{(\infty)}(\rho_D(D)) = T_x M$$

for all $x \in M$. See [14] for more details.

3. BUNDLE METRICS ON SKEW-SYMMETRIC ALGEBROIDS AND NEWTONIAN SYSTEMS

3.1. The Levi-Civita connection. Let $\mathcal{G}^D : D \times_M D \rightarrow \mathbb{R}$ be a non-degenerate bundle metric on a skew-symmetric algebroid $(D, [\cdot, \cdot]_D, \rho_D)$. Given this bundle metric we can construct a unique torsion-less connection $\nabla^{\mathcal{G}^D}$ on D which is metric with respect to \mathcal{G} (see [9] and references therein, for the standard case of Lie algebroids). The following construction mimics the classical construction of the Levi-Civita connection for a riemannian metric on a differentiable manifold.

We will denote by $b_{\mathcal{G}^D} : D \rightarrow D^*$ the vector bundle isomorphism induced by \mathcal{G}^D and by $\#_{\mathcal{G}^D} : D^* \rightarrow D$ the inverse morphism. The bundle metric can be locally written as $\mathcal{G}^D = (\mathcal{G}^D)_{AB} e^A \otimes e^B$.

The **Levi-Civita connection** $\nabla^{\mathcal{G}^D} : \Gamma(\tau_D) \times \Gamma(\tau_D) \rightarrow \Gamma(\tau_D)$ associated to the bundle metric \mathcal{G}^D is defined by the formula:

$$\begin{aligned} 2\mathcal{G}^D(\nabla_X^{\mathcal{G}^D} Y, Z) &= \rho_D(X)(\mathcal{G}^D(Y, Z)) + \rho_D(Y)(\mathcal{G}^D(X, Z)) \\ &\quad - \rho_D(Z)(\mathcal{G}^D(X, Y)) + \mathcal{G}^D(X, [Z, Y]_D) \\ &\quad + \mathcal{G}^D(Y, [Z, X]_D) - \mathcal{G}^D(Z, [Y, X]_D) \end{aligned}$$

for $X, Y, Z \in \Gamma(\tau_D)$.

Alternatively, $\nabla^{\mathcal{G}^D}$ is determined by

$$\begin{aligned} [X, Y]_D &= \nabla_X^{\mathcal{G}^D} Y - \nabla_Y^{\mathcal{G}^D} X \quad (\text{symmetry}) \\ \rho_D(X)(\mathcal{G}^D(Y, Z)) &= \mathcal{G}^D(\nabla_X^{\mathcal{G}^D} Y, Z) + \mathcal{G}^D(Y, \nabla_X^{\mathcal{G}^D} Z) \quad (\text{metricity}), \end{aligned}$$

These two properties allow to determine *Christoffel symbols* of the connection $\nabla^{\mathcal{G}^D}$ that satisfy

$$\nabla_{e_B}^{\mathcal{G}^D} e_C = \Gamma_{BC}^A e_A.$$

More details about how to compute the Christoffel symbols are given in Section 5.3 if a \mathcal{G}^D -orthogonal basis of $\Gamma(\tau_D)$ is taken.

Additionally, we have the notion of derivative along an admissible curve. If $\gamma: I \subseteq \mathbb{R} \rightarrow D$ is a ρ_D -admissible curve and

$$\Gamma(\gamma) = \{X: I \subseteq \mathbb{R} \rightarrow D \mid X \text{ is } \mathcal{C}^\infty \text{ and } X(t) \in D_{\gamma(t)} \forall t \in I\}$$

is the set of sections along γ , then the induced *covariant derivative* $\nabla_\gamma: \Gamma(\gamma) \rightarrow \Gamma(\gamma)$ can be defined as the mapping from $X \in \Gamma(\gamma)$ to $\nabla_\gamma X \in \Gamma(\gamma)$ with local expression

$$\nabla_\gamma X = \left[\frac{dX^C}{dt} + \Gamma_{AB}^C y^A X^B \right] e_C, \quad (3.1)$$

if $\gamma(t) = (x^i(t), y^A(t))$ and $X = X^A e_A$.

3.2. Geodesics. Given the bundle metric \mathcal{G}^D , a ρ_D -admissible curve $\gamma: I \subseteq \mathbb{R} \rightarrow D$ on D is said to be a *geodesic* if

$$\nabla_\gamma \gamma = 0.$$

If the local expression of γ is

$$\gamma(t) = (x^i(t), y^A(t)),$$

then γ is a geodesic if and only if

$$\frac{dx^i}{dt} = (\rho_D)^i_C y^C, \quad \frac{dy^C}{dt} = -\Gamma_{AB}^C y^A y^B.$$

The geodesics are just the integral curves of a vector field on D , called *the geodesic spray* $\xi_{\mathcal{G}^D}$, whose local expression is

$$\xi_{\mathcal{G}^D} = (\rho_D)^i_C y^C \frac{\partial}{\partial x^i} - \Gamma_{AB}^C y^A y^B \frac{\partial}{\partial y^C}.$$

Note that if $a \in D$, then there exists a unique geodesic $\gamma_a: (-\epsilon, \epsilon) \subseteq \mathbb{R} \rightarrow D$ such that $\gamma_a(0) = a$. If $\sigma_a = \tau_D \circ \gamma_a: (-\epsilon, \epsilon) \subseteq \mathbb{R} \rightarrow M$ is the base curve of γ_a then, since γ_a is ρ_D -admissible, we have that

$$\dot{\sigma}_a(t) = \rho_D(\gamma_a(t)), \quad \forall t \in I.$$

The associated *symmetric product* is defined as follows:

$$\langle X : Y \rangle_{\mathcal{G}^D} = \nabla_X^{\mathcal{G}^D} Y + \nabla_Y^{\mathcal{G}^D} X, \quad X, Y \in \Gamma(\tau_D).$$

The symmetric product on Riemannian manifolds is a fundamental tool in controllability results, kinematic reduction of mechanical systems and in the characterization of geodesic invariance of distributions (see

[4]). These results can be extended to our setting of skew-symmetric algebroids.

Lemma 3.1. *If $X, Y \in \Gamma(\tau_D)$, then $\langle X : Y \rangle_{\mathfrak{g}^D}^{\mathbf{v}} = [X^{\mathbf{v}}, [\xi_{\mathfrak{g}^D}, Y^{\mathbf{v}}]]$, where $X^{\mathbf{v}}$ is the natural vertical lift of the section X of τ_D .*

Proof. This is proved locally, similarly to the proof of this result on Riemannian manifolds in Lemma B.3 in [4].

Locally, $X = X^A e_A$ and $Y = Y^B e_B$, then

$$X^{\mathbf{v}} = X^A \frac{\partial}{\partial y^A}, \quad Y^{\mathbf{v}} = Y^B \frac{\partial}{\partial y^B}.$$

Let us compute,

$$\begin{aligned} [\xi_{\mathfrak{g}^D}, Y^{\mathbf{v}}] &= \left[(\rho_D)^i_C y^C \frac{\partial}{\partial x^i} - \Gamma_{AB}^C y^A y^B \frac{\partial}{\partial y^C}, Y^E \frac{\partial}{\partial y^E} \right] \\ &= (\rho_D)^i_C y^C \frac{\partial Y^E}{\partial x^i} \frac{\partial}{\partial y^E} - Y^B (\rho_D)^i_B \frac{\partial}{\partial x^i} \\ &+ Y^B (\Gamma_{AB}^C + \Gamma_{BA}^C) y^A \frac{\partial}{\partial y^C} = -Y^B (\rho_D)^i_B \frac{\partial}{\partial x^i} \\ &+ \left((\rho_D)^i_A y^A \frac{\partial Y^C}{\partial x^i} + Y^B (\Gamma_{AB}^C + \Gamma_{BA}^C) y^A \right) \frac{\partial}{\partial y^C}. \end{aligned}$$

Then,

$$\begin{aligned} [X^{\mathbf{v}}, [\xi_{\mathfrak{g}^D}, Y^{\mathbf{v}}]] &= \left[X^A \frac{\partial}{\partial y^A}, -Y^B (\rho_D)^i_B \frac{\partial}{\partial x^i} + \left((\rho_D)^i_A y^A \frac{\partial Y^C}{\partial x^i} \right. \right. \\ &+ \left. \left. Y^B (\Gamma_{AB}^C + \Gamma_{BA}^C) y^A \right) \frac{\partial}{\partial y^C} \right] = \left(X^A (\rho_D)^i_A \frac{\partial Y^C}{\partial x^i} \right. \\ &+ \left. Y^B (\rho_D)^i_B \frac{\partial X^C}{\partial x^i} + X^A Y^B (\Gamma_{AB}^C + \Gamma_{BA}^C) \right) \frac{\partial}{\partial y^C}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \langle X : Y \rangle_{\mathfrak{g}^D} &= \nabla_X^{\mathfrak{g}^D} Y + \nabla_Y^{\mathfrak{g}^D} X = X^A (\rho_D)^i_A \frac{\partial Y^C}{\partial x^i} e_C + X^A Y^B \Gamma_{AB}^C e_C \\ &+ Y^B (\rho_D)^i_B \frac{\partial X^C}{\partial x^i} e_C + X^A Y^B \Gamma_{BA}^C e_C = \left(X^A (\rho_D)^i_A \frac{\partial Y^C}{\partial x^i} \right. \\ &+ \left. Y^B (\rho_D)^i_B \frac{\partial X^C}{\partial x^i} + X^A Y^B (\Gamma_{AB}^C + \Gamma_{BA}^C) \right) e_C. \end{aligned}$$

This proves the equality since the vertical lift of this section is equal to the local expression of $[X^{\mathbf{v}}, [\xi_{\mathfrak{g}^D}, Y^{\mathbf{v}}]]$ we just computed above. \square

Now, we can extend the characterization of geodesically invariant distributions already known on Riemannian manifolds [4] to skew-symmetric algebroids. A subbundle \mathfrak{D} of a skew-symmetric algebroid $(D, [\cdot, \cdot]_D, \rho_D)$ is *geodesically invariant* if for any geodesic $\gamma: I \subseteq \mathbb{R} \rightarrow D$ with initial condition $\gamma(0) \in \mathfrak{D}(\sigma(0))$, then $\gamma(t) \in \mathfrak{D}(\sigma(t))$ for any $t \in I$, where $\sigma = \tau_D \circ \gamma$.

Theorem 3.2. *Let $(D, [\cdot, \cdot]_D, \rho_D)$ be a skew-symmetric algebroid and \mathcal{G}_D be a bundle metric over D . Let \mathfrak{D} be a subbundle of D and $\tau_{\mathfrak{D}} = \tau_D|_{\mathfrak{D}}$. The following statements are equivalent:*

- (i) \mathfrak{D} is geodesically invariant.
- (ii) The restriction of the geodesic spray $\xi_{\mathcal{G}_D}$ to \mathfrak{D} is tangent to \mathfrak{D} .
- (iii) If $X, Y \in \Gamma(\tau_{\mathfrak{D}})$, then $\langle X, Y \rangle_{\mathcal{G}_D} \in \Gamma(\tau_{\mathfrak{D}})$.

Proof. (i) \Leftrightarrow (ii) The integral curves of $\xi_{\mathcal{G}_D}$ are the geodesics. This proves the equivalence.

(ii) \Rightarrow (iii) Assume that $X, Y \in \Gamma(\tau_{\mathfrak{D}})$. Then the restrictions of X^{\vee}, Y^{\vee} to \mathfrak{D} are tangent to \mathfrak{D} . By hypothesis, $(\xi_{\mathcal{G}_D})|_{\mathfrak{D}}$ is tangent to \mathfrak{D} . Thus, $[\xi_{\mathcal{G}_D}, Y^{\vee}]|_{\mathfrak{D}}$ is tangent to \mathfrak{D} . As $X^{\vee}|_{\mathfrak{D}}$ is also tangent to \mathfrak{D} , we have that $[X^{\vee}, [\xi_{\mathcal{G}_D}, Y^{\vee}]]|_{\mathfrak{D}}$ is tangent to \mathfrak{D} . By Lemma 3.1, $([X^{\vee}, [\xi_{\mathcal{G}_D}, Y^{\vee}]])|_{\mathfrak{D}} = (\langle X: Y \rangle_{\mathcal{G}_D}^{\vee})|_{\mathfrak{D}}$. Thus $\langle X: Y \rangle_{\mathcal{G}_D} \in \Gamma(\tau_{\mathfrak{D}})$ because X^{\vee} restricted to \mathfrak{D} are tangent to \mathfrak{D} if and only if $X \in \Gamma(\tau_{\mathfrak{D}})$.

(iii) \Rightarrow (i) Let $\{X_1, \dots, X_r\} = \{X_a\}_{a=1, \dots, r}$ be a local basis for $\Gamma(\tau_{\mathfrak{D}})$. It can be extended into a local basis for $\Gamma(\tau_D)$, which is given by

$$\{X_1, \dots, X_r, X_{r+1}, \dots, X_n\} = \{X_a, X_{\alpha}\},$$

for $a = 1, \dots, r; \alpha = r+1, \dots, n$. Hence,

$$\nabla_{X_a}^{\mathcal{G}_D} X_b = \Gamma_{ab}^c X_c + \Gamma_{ab}^{\alpha} X_{\alpha}, \quad \text{for } a, b \in \{1, \dots, r\}.$$

By assumption, $\langle X_a: X_b \rangle_{\mathcal{G}_D} \in \Gamma(\tau_{\mathfrak{D}})$, then $\Gamma_{ab}^{\alpha} + \Gamma_{ba}^{\alpha} = 0$ for all $a, b \in \{1, \dots, r\}, \alpha \in \{r+1, \dots, n\}$ because

$$\langle X_a: X_b \rangle_{\mathcal{G}_D} = \nabla_{X_a}^{\mathcal{G}_D} X_b + \nabla_{X_b}^{\mathcal{G}_D} X_a = (\Gamma_{ab}^c + \Gamma_{ba}^c) X_c + (\Gamma_{ab}^{\alpha} + \Gamma_{ba}^{\alpha}) X_{\alpha}.$$

Now, let $\gamma: I \rightarrow D$ be a geodesic such that $\gamma(0) \in \mathfrak{D}(\sigma(0))$ where $\sigma = \tau_D \circ \gamma$. Suppose that

$$\gamma(t) = \sum_{a=1}^r u^a(t) X_a(\sigma(t)) + \sum_{\alpha=r+1}^n u^{\alpha}(t) X_{\alpha}(\sigma(t)) = \sum_{i=1}^n u^i(t) X_i(\sigma(t)),$$

then $u^{\alpha}(0) = 0$ for all $\alpha \in \{r+1, \dots, n\}$. As γ is a geodesic,

$$0 = \nabla_{\dot{\gamma}(t)}^{\mathcal{G}_D} \gamma(t) = \sum_{k=1}^n \left(\frac{du^k}{dt} + \Gamma_{ij}^k(\sigma(t)) u^i(t) u^j(t) \right) X_k(\sigma(t)).$$

Then

$$\frac{du^k}{dt} + (\Gamma_{ij}^k \circ \sigma)u^i u^j = 0, \quad \forall k \in \{1, \dots, n\}. \quad (3.2)$$

As $\Gamma_{ab}^\alpha + \Gamma_{ba}^\alpha = 0$ for all $a, b \in \{1, \dots, r\}$, $\alpha \in \{r+1, \dots, n\}$, we have that $u^\alpha(t) = 0$ for all $t \in I$ is a solution of the differential equation (3.2) for $\alpha = r+1, \dots, n$:

$$\begin{aligned} \frac{du^\alpha}{dt} + (\Gamma_{ij}^\alpha \circ \sigma)u^i u^j &= \frac{du^\alpha}{dt} + (\Gamma_{ab}^\alpha \circ \sigma)u^a u^b + (\Gamma_{a\beta}^\alpha \circ \sigma)u^a u^\beta \\ &+ (\Gamma_{\delta b}^\alpha \circ \sigma)u^\delta u^b + (\Gamma_{\delta\beta}^\alpha \circ \sigma)u^\delta u^\beta \\ &= (\Gamma_{ab}^\alpha \circ \sigma)u^a u^b = 0. \end{aligned}$$

As the initial condition for u^α is $u^\alpha(0) = 0$ for all $\alpha \in \{r+1, \dots, n\}$, it follows that $u^\alpha(t) = 0$ is the unique solution for the differential equations in (3.2). Thus,

$$\gamma(t) = \sum_{a=1}^r u^a(t)X_a(\sigma(t)) \in \mathfrak{D}(\sigma(t)), \quad \forall t \in I \subseteq \mathbb{R}.$$

Hence \mathfrak{D} is geodesically invariant. \square

3.3. Newtonian systems. Given a bundle map $\mathcal{F} : D \rightarrow D$ (that is, $\tau_D \circ \mathcal{F} = \tau_D$) we define a *newtonian system* as the triple $(D, \mathcal{G}^D, \mathcal{F})$. This newtonian system induces the system of differential equations:

$$\nabla_{\gamma(t)}^{\mathcal{G}^D} \gamma(t) = \mathcal{F}(\gamma(t)), \quad t \in I, \quad (3.3)$$

where the solutions are curves $\gamma : I \subseteq \mathbb{R} \rightarrow D$ which are ρ_D -admissible.

Given local coordinates (x^i, y^A) associated with the basis $\{e_A\}$ for sections of D , Equations (3.3) can be written as

$$\begin{aligned} \dot{x}^i &= (\rho_D)_A^i y^A, \\ \dot{y}^C &= -\Gamma_{AB}^C y^A y^B + \mathcal{F}^C(x, y), \end{aligned} \quad (3.4)$$

where $\mathcal{F}(x^j, y^B) = (x^j, \mathcal{F}^A(x^j, y^B))$.

Observe that Equations (3.4) are the equations of the integral curves of a vector field $\xi_{\mathcal{G}^D, \mathcal{F}}$ on D . Locally, this vector field is given by

$$\xi_{\mathcal{G}^D, \mathcal{F}} = (\rho_D)_A^i y^A \frac{\partial}{\partial x^i} + (-\Gamma_{AB}^C y^A y^B + \mathcal{F}^C) \frac{\partial}{\partial y^C}.$$

Remark 3.3. The map \mathcal{F} could be given by a section $F \in \Gamma(\tau_D)$ such that $\mathcal{F} = F \circ \tau_D$. An interesting particular case is when we have a potential function $V : M \rightarrow \mathbb{R}$ and F is the section $-\text{grad}_{\mathcal{G}^D} V \in \Gamma(\tau_D)$ given by

$$\mathcal{G}^D(\text{grad}_{\mathcal{G}^D} V, X) = \rho_D(X)(V) = dV(\rho_D(X)), \quad \text{for every } X \in \Gamma(\tau_D).$$

In particular, the solutions of this newtonian system $(D, \mathcal{G}^D, F = -\text{grad}_{\mathcal{G}^D} V)$ are equivalent to the solutions of the Euler-Lagrange equations on a skew-symmetric algebroid with Lagrangian $L : D \rightarrow \mathbb{R}$:

$$L(v) = \frac{1}{2} \mathcal{G}^D(v, v) - V(\tau_D(v)). \quad (3.5)$$

Therefore, these solutions are ρ_D -admissible curves $\gamma : I \rightarrow D$ such that

$$\nabla_{\gamma(t)}^{\mathcal{G}^D} \gamma(t) + \text{grad}_{\mathcal{G}^D} V(\tau_D(\gamma(t))) = 0. \quad (3.6)$$

Locally, those solutions satisfy

$$\begin{aligned} \dot{x}^i &= (\rho_D)^i_A y^A, \\ \dot{y}^C &= -\Gamma_{AB}^C y^A y^B - (\mathcal{G}^D)^{CB} (\rho_D)_B^i \frac{\partial V}{\partial x^i}, \end{aligned}$$

where $(\mathcal{G}^D)^{AB}$ are the entries of the inverse matrix of $((\mathcal{G}^D)_{AB})$. In this case, the vector field $\xi_{\mathcal{G}^D, F}$ on D is given by $\xi_{\mathcal{G}^D, V} = \xi_{\mathcal{G}^D} + (\text{grad}_{\mathcal{G}^D} V)^\vee$, where $(\text{grad}_{\mathcal{G}^D} V)^\vee$ is the vertical lift to D of the section $\text{grad}_{\mathcal{G}^D} V \in \Gamma(\tau_D)$. \diamond

Example 3.4. If $D = TM$, $[[\cdot, \cdot]]_D = [\cdot, \cdot]$ the standard Lie bracket on M , $\rho_D = \text{Id}_{TM}$ and \mathcal{G}^D is a riemannian metric on M , then Equations (3.6) are the classical **Euler-Lagrange equations** for the mechanical lagrangian $L : TM \rightarrow \mathbb{R}$.

Example 3.5. Given a regular distribution D on TM and a riemannian metric \mathcal{G}^{TM} , we consider the riemannian orthogonal decomposition $TM = D \oplus D^\perp$ and the associated orthogonal projectors $P : TM \rightarrow D$ and $Q : TM \rightarrow D^\perp$, see [2, 14]. Denote also by $\iota_D : D \hookrightarrow TM$ the canonical inclusion. We induce by restriction a bundle metric $\mathcal{G}^D : D \times_M D \rightarrow \mathbb{R}$ and an skew-symmetric algebroid structure on D as follows:

$$[[X, Y]]_D = P[\iota_D(X), \iota_D(Y)], \quad \rho_D(X) = \iota_D(X),$$

where $X, Y \in \Gamma(\tau_D)$. Note that in this example, X, Y are vector fields on M taking values on D . Moreover, the Levi-Civita connection $\nabla^{\mathcal{G}^D}$ coincides with the constrained connection $\nabla_X^D Y := P(\nabla_X^{\mathcal{G}^D} Y)$ defined, for instance, in [4] if ∇^D is restricted to $\Gamma(\tau_D)$.

For this particular skew-symmetric algebroid structure, Equations (3.6) correspond with the equations of the nonholonomic system determined by the constraints induced by the distribution D and the mechanical lagrangian (3.5). These equations are also called in the literature *Lagrange-D'Alembert's equations*.

Consider a basis of \mathcal{G}^D -orthogonal vector fields $\{X_A, X_\alpha\}$, $1 \leq A \leq m = \text{rank } D$, $m + 1 \leq \alpha \leq n = \dim M$, adapted to the decomposition $TM = D \oplus D^\perp$. In other words, $D_x = \text{span} \{X_A(x)\}$ and $D_x^\perp =$

span $\{X_\alpha(x)\}$. Observe that for the induced coordinates (x^i, y^A, y^α) on TM the nonholonomic constraints are rewritten as $y^\alpha = 0$, $m + 1 \leq \alpha \leq n$. That is, the induced coordinates on D are given by (x^i, y^A) . Therefore, the skew-symmetric algebroid structure induced on the vector subbundle $D \rightarrow M$ is locally described by:

$$\begin{aligned} \llbracket X_A, X_B \rrbracket_D &= P[X_A, X_B] = P(\mathcal{C}_{AB}^C X_C + \mathcal{C}_{AB}^\beta X_\beta) = \mathcal{C}_{AB}^C X_C, \\ \rho_D(X_A) &= X_A, \end{aligned}$$

where $X_A = (\rho_D)_A^i \frac{\partial}{\partial x^i}$, $1 \leq A \leq m$.

The **Lagrange-D'Alembert's equations** are

$$\begin{aligned} \dot{x}^i &= (\rho_D)_A^i y^A, \\ \dot{y}^C &= -\Gamma_{AB}^C y^A y^B - (\mathcal{G}^D)^{CB} (\rho_D)_B^i \frac{\partial V}{\partial x^i}. \end{aligned} \quad (3.7)$$

Example 3.6. Our theory is not only restricted to lagrangian systems defined on the tangent bundle TM or nonholonomic systems determined by a regular distribution on TM . The techniques described in this paper by means of skew-symmetric algebroids are general enough to cover the most important cases of reduction of mechanical systems subjected or not to nonholonomic constraints.

As a particular example, we include in our analysis the case of Lie algebras \mathfrak{g} of finite dimension (it is clear that \mathfrak{g} is a Lie algebroid over a single point). Now, suppose that (l, \mathfrak{d}) is a nonholonomic Lagrangian system on \mathfrak{g} , where $l : \mathfrak{g} \rightarrow \mathbb{R}$ is a Lagrangian function defined by $l(\xi) = \frac{1}{2} \langle \mathbb{I} \xi, \xi \rangle$, $\mathbb{I} : \mathfrak{g} \rightarrow \mathfrak{g}^*$ is a symmetric positive definite inertia operator and \mathfrak{d} is a vector subspace of \mathfrak{g} . We have the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{d}^\perp,$$

where $\mathfrak{d}^\perp = \{\xi' \in \mathfrak{g} \mid \langle \mathbb{I} \xi', \xi \rangle = 0 \forall \xi \in \mathfrak{d}\}$. Take now an adapted basis to this decomposition $\{e_A, e_\alpha\}$ where $\mathfrak{d} = \text{span}\{e_A\}$ and $\mathfrak{d}^\perp = \text{span}\{e_\alpha\}$. Then, the **Euler-Poincaré-Suslov equations** for (l, \mathfrak{d}) are

$$\dot{y}^C = -\Gamma_{AB}^C y^A y^B,$$

where $\{y^A, y^\alpha\}$ are the global coordinates on \mathfrak{g} induced by the basis $\{e_A, e_\alpha\}$.

Example 3.7. Similarly, more involved situations can be recovered using our techniques. For instance, nonholonomic systems on Atiyah algebroids associated with principal G -bundles. For the sake of simplicity, we will consider the particular case when the principal G -bundle is trivial. In such a case, the Atiyah algebroid is a vector bundle of the form

$$\tau_A : A = \mathfrak{g} \times TM \longrightarrow M,$$

where \mathfrak{g} is the Lie algebra of the Lie group G and M is a smooth manifold. The Lie bracket of the space $\Gamma(\tau_A)$ is characterized by the following condition

$$[[(\xi, X), (\xi', X')]_A = ([\xi, \xi']_{\mathfrak{g}}, [X, X']) ,$$

for $\xi, \xi' \in \mathfrak{g}$ and $X, X' \in \mathfrak{X}(M)$. The anchor map ρ_A is the canonical projection onto the second factor.

Suppose now that D is a vector subbundle of A over M of constant rank (the constraint bundle) such that

$$M \ni x \longrightarrow D_V(x) := D(x) \cap (\mathfrak{g} \times \{0_{T_x M}\}) \subseteq \mathfrak{g} \times T_x M$$

is a vector subbundle of A . Then we can choose a local basis $\{\xi_a\}_{1 \leq a \leq r}$ of $\Gamma(\tau_{D_V})$, with $\xi_a : U \subseteq M \longrightarrow \mathfrak{g}$ smooth maps, and a local basis $\{X_A\} = \{\xi_a, (\eta_\alpha, Y_\alpha)\}$ of $\Gamma(\tau_D)$, with $\eta_\alpha : U \subseteq M \longrightarrow \mathfrak{g}$ and $Y_\alpha \in \mathfrak{X}(U)$.

Moreover, if (x^i) are local coordinates on $U \subseteq M$ and $Y_\alpha = Y_\alpha^i \frac{\partial}{\partial x^i}$, the **Lagrange-D'Alembert-Poincaré equations** are:

$$\begin{aligned} \dot{x}^i &= Y_\alpha^i y^\alpha, \\ \dot{y}^c &= -\Gamma_{AB}^c y^A y^B - \frac{\partial V}{\partial x^i} Y_\alpha^i (\mathcal{G}^D)^{c\alpha}, \\ \dot{y}^\alpha &= -\Gamma_{AB}^\alpha y^A y^B - \frac{\partial V}{\partial x^i} Y_\beta^i (\mathcal{G}^D)^{\alpha\beta}, \end{aligned}$$

where (x^i, y^c, y^α) are the corresponding local coordinates on \mathcal{D} .

Note that in the particular case when M is a single point, we recover the **Euler-Poincaré Suslov equations**.

Geometric interpretations of **nonholonomic LR** systems or **non-holonomic systems with semidirect product symmetry** may also be given using skew-symmetric algebroids deduced from Lie algebroids (see [8] for more details).

4. HAMILTON-JACOBI EQUATION

The next result is a direct consequence of Equations (3.6). See [2] for an extension of this result (in a hamiltonian context) for many different types of mechanical systems (nonholonomic dynamics, dissipative systems...).

Proposition 4.1. *Let $(D, [\cdot, \cdot]_D, \rho_D)$ be a skew-symmetric algebroid and consider a newtonian system determined by $(D, \mathcal{G}^D, \mathcal{F})$. Take an arbitrary section $X \in \Gamma(\tau_D)$ then, the following conditions on X are equivalent:*

(i) If $\sigma : I \longrightarrow M$ is an integral curve of the vector field $\rho_D(X)$ that is,

$$\dot{\sigma}(t) = \rho_D(X)(\sigma(t)), \quad (4.1)$$

then $\gamma = X \circ \sigma : I \longrightarrow D$ is a solution of

$$\nabla_{\gamma(t)}^{\mathcal{G}^D} \gamma(t) = \mathcal{F}(\gamma(t)).$$

(ii) X satisfies

$$\nabla_X^{\mathcal{G}^D} X = \mathcal{F} \circ X.$$

Proof. Let $\sigma : I \rightarrow M$ be an integral curve of the vector field $\rho_D(X)$. Then, the result follows immediately from the fact that

$$\nabla_{X \circ \sigma(t)}^{\mathcal{G}^D} (X \circ \sigma(t)) = (\nabla_X^{\mathcal{G}^D} X) \circ \sigma(t).$$

□

This result is analogous to the Hamilton-Jacobi theory already described in a Lagrangian framework for a free mechanical system in [5] and for a nonholonomic one in [6], but adapted to skew-symmetric algebroid structures.

Remark 4.2. An analogous result can be written replacing the section X by an entire distribution spanned by sections. ◇

Remark 4.3. In standard riemannian geometry (that is, $D = TM$ equipped with the standard Lie bracket and without external forces $\mathcal{F} \equiv 0$) the vector fields X satisfying $\nabla_X^{\mathcal{G}^D} X = 0$ are called auto-parallel vector fields and are obviously connected with the solutions of the geodesic equations. ◇

From now on, we only consider mechanical problems given by (D, \mathcal{G}^D, V) as described in Remark 3.3. Specializing Proposition 4.1 to this kind of problems and to vector fields X verifying an extra condition $i_X d^D(b_{\mathcal{G}}(X)) = 0$ we obtain a new expression of this Proposition. Indeed, the following Theorem can be compared with the classical expression of the Hamilton-Jacobi equation proposed in [14] for the hamiltonian function $h : D^* \longrightarrow \mathbb{R}$:

$$h(\kappa) = \mathcal{G}^{D^*}(\kappa, \kappa) + V(\tau_{D^*}(\kappa)),$$

where $\kappa \in D^*$ and $\mathcal{G}^{D^*} : D^* \times_M D^* \longrightarrow \mathbb{R}$ is the induced bundle metric on the dual bundle.

Theorem 4.4. Let $(D, \llbracket \cdot, \cdot \rrbracket_D, \rho_D)$ be a skew-symmetric algebroid and consider a mechanical problem determined by (D, \mathcal{G}^D, V) . Take a section $X \in \Gamma(\tau_D)$ such that $i_X d^D(b_{\mathcal{G}^D}(X)) = 0$. Under this hypothesis, the following conditions are equivalent:

(i) If $\sigma : I \rightarrow M$ is an integral curve of the vector field $\rho_D(X)$, that is,

$$\dot{\sigma}(t) = \rho_D(X)(\sigma(t)),$$

then $\gamma = X \circ \sigma : I \rightarrow D$ is a solution of

$$\nabla_{\gamma(t)}^{\mathcal{G}^D} \gamma(t) + \text{grad}_{\mathcal{G}^D} V(\tau_D(\gamma(t))) = 0.$$

(ii) X satisfies **the Hamilton-Jacobi differential equation**

$$d^D \left(\frac{1}{2} \mathcal{G}^D(X, X) + V \right) = 0. \quad (4.2)$$

If, additionally, the skew-symmetric algebroid $(D, [\cdot, \cdot]_D, \rho_D)$ is completely nonholonomic and M is connected or if $H^0(d^D) \simeq \mathbb{R}$, then Equation (4.2) is equivalent to

$$\frac{1}{2} \mathcal{G}^D(X, X) + V = \text{constant}.$$

Proof. First, we study how to express the condition $i_X d^D(b_{\mathcal{G}^D}(X)) = 0$ in terms of the Levi-Civita connection associated to \mathcal{G}^D . Let $Y \in \Gamma(\tau_D)$,

$$\begin{aligned} 0 &= d^D(b_{\mathcal{G}^D}(X))(X, Y) \\ &= \rho_D(X)(\mathcal{G}^D(X, Y)) - \rho_D(Y)(\mathcal{G}^D(X, X)) - \mathcal{G}^D(X, [X, Y]_D) \\ &= \mathcal{G}^D(\nabla_X^{\mathcal{G}^D} X, Y) + \mathcal{G}^D(\nabla_X^{\mathcal{G}^D} Y, X) - \mathcal{G}^D(\nabla_Y^{\mathcal{G}^D} X, X) - \mathcal{G}^D(\nabla_Y^{\mathcal{G}^D} X, X) \\ &\quad - \mathcal{G}^D(X, \nabla_X^{\mathcal{G}^D} Y - \nabla_Y^{\mathcal{G}^D} X) \\ &= \mathcal{G}^D(\nabla_X^{\mathcal{G}^D} X, Y) - \mathcal{G}^D(\nabla_Y^{\mathcal{G}^D} X, X). \end{aligned}$$

Therefore, the condition $i_X d^D(b_{\mathcal{G}^D}(X)) = 0$ is alternatively written as

$$\mathcal{G}^D(\nabla_X^{\mathcal{G}^D} X, Y) = \mathcal{G}^D(\nabla_Y^{\mathcal{G}^D} X, X) \text{ for every } Y \in \Gamma(\tau_D). \quad (4.3)$$

We only need to check that both condition (ii) in Proposition 4.1 and Theorem 4.4 are equivalent. If we examine the Hamilton-Jacobi differential equation

$$d^D \left(\frac{1}{2} \mathcal{G}^D(X, X) + V \right) = 0,$$

then for any $Y \in \Gamma(\tau_D)$ and $X \in \Gamma(\tau_D)$ satisfying $i_X d^D(b_{\mathcal{G}^D}(X)) = 0$ we have that

$$\begin{aligned} 0 &= d^D \left(\frac{1}{2} \mathcal{G}^D(X, X) + V \right)(Y) \\ &= \frac{1}{2} \rho_D(Y)(\mathcal{G}^D(X, X)) + \rho_D(Y)(V) \\ &= \mathcal{G}^D(\nabla_Y^{\mathcal{G}^D} X, X) + \rho_D(Y)(V) \\ &= \mathcal{G}^D(\nabla_X^{\mathcal{G}^D} X, Y) + \mathcal{G}^D(\text{grad}_{\mathcal{G}^D} V, Y). \end{aligned}$$

In the last equality we have used condition (4.3). Therefore Equation (4.2) is written as

$$\nabla_X^{\mathcal{G}^D} X + \text{grad}_{\mathcal{G}^D} V = 0 .$$

(See also [2, 14]). \square

From Theorem 4.4, it is clear that we need to find sections X satisfying $i_X d^D(b_{\mathcal{G}^D}(X)) = 0$. The most simple candidate to be a solution is $X = \text{grad}_{\mathcal{G}^D} f$ with $f : M \rightarrow \mathbb{R}$. Note that if $(d^D)^2(f) = 0$, then $i_X d^D(b_{\mathcal{G}^D}(X)) = 0$. After some straightforward calculations, this condition is equivalent to:

$$[\rho_D(Y), \rho_D(Z)](f) = (\rho_D \llbracket Y, Z \rrbracket_D)(f) .$$

This condition is always true if the bracket $\llbracket \cdot, \cdot \rrbracket_D$ satisfies the Jacobi identity, that is, if the pair $(\llbracket \cdot, \cdot \rrbracket_D, \rho_D)$ is a **Lie algebroid structure** on the vector bundle $\tau_D : D \rightarrow M$ (see [14]).

5. MECHANICAL CONTROL SYSTEMS AND KINEMATIC REDUCTIONS

Assume that the newtonian system determined by $(D, \mathcal{G}^D, \mathcal{F})$ also contains some input forces. We model this set of input forces by a vector subbundle $\mathcal{D}^{(c)}$ of D^* . Locally, $\mathcal{D}^{(c)} = \text{span} \{\theta^1, \dots, \theta^k\}$, where $\theta^l \in \Gamma(\tau_{D^*})$, $1 \leq l \leq k$. Denote by $\mathcal{D}_{(c)}$ the vector subbundle of D defined by $\mathcal{D}_{(c)} = \sharp_{\mathcal{G}^D}(\mathcal{D}^{(c)})$. Therefore, locally $\mathcal{D}_{(c)} = \text{span} \{Y_1, \dots, Y_k\}$ where $Y_l = \sharp_{\mathcal{G}^D} \theta^l$, $Y_l \in \Gamma(\tau_D)$, $1 \leq l \leq k$. The vector fields Y_1, \dots, Y_k are called the **control sections** or **input sections**.

The equations of motion for a newtonian system with input sections are as follows

$$\nabla_{\gamma(t)}^{\mathcal{G}^D} \gamma(t) - \mathcal{F}(\gamma(t)) \in \mathcal{D}_{(c)}(\gamma(t)), \quad \forall t \in I \subseteq \mathbb{R}, \quad (5.1)$$

where $\gamma : I \rightarrow D$ is a ρ_D -admissible curve.

In terms of the control sections, Equation (5.1) can be rewritten as follows:

$$\nabla_{\gamma(t)}^{\mathcal{G}^D} \gamma(t) - \mathcal{F}(\gamma(t)) = \sum_{l=1}^k u^l(t) Y_l(\tau_D(\gamma(t))) \quad (5.2)$$

for some $u : I \subseteq \mathbb{R} \rightarrow \mathbb{R}^k$, playing the role of controls. The corresponding local equations are

$$\begin{aligned} \dot{x}^i &= (\rho_D)_A^i y^A, \\ \dot{y}^C &= -\Gamma_{AB}^C y^A y^B + \mathcal{F}^C(x, y) + \sum_{l=1}^k u^l Y_l(x). \end{aligned}$$

Note that if $Y \in \Gamma(\tau_{\mathcal{D}_{(c)}})$ then the integral curves of the vector field $\xi_{\mathcal{G}^D, \mathcal{F}} + Y^\vee$ on D are solutions of the previous equations, where Y^\vee is the vertical lift of the section Y .

Remark 5.1. It is also possible to study the more realistic case when U is a proper subset of \mathbb{R}^k . In this case, the controls u take value in a proper set of \mathbb{R}^k (i.e., not all the linear combinations of controls are allowed). Our procedure can be adapted to that particular control set. But for geometrical clarity in this paper we only consider the control distribution $\mathcal{D}_{(c)}$. \diamond

Definition 5.2. *The 4-tuple $(D, \mathcal{G}^D, \mathcal{F}, \mathcal{D}_{(c)})$ is called a **mechanical control system on a skew-symmetric algebroid**.*

Consider the orthogonal decomposition $D = \mathcal{D}_{(c)} \oplus \mathcal{D}_{(c)}^\perp$ induced by the bundle metric \mathcal{G}^D , with associated orthogonal projectors $P_{(c)} : D \rightarrow \mathcal{D}_{(c)}$ and $Q_{(c)} : D \rightarrow \mathcal{D}_{(c)}^\perp$.

The following proposition is a direct consequence of the definition of mechanical control systems.

Proposition 5.3. *An admissible curve $\gamma : I \rightarrow D$ is solution of Equation (5.1) if and only if $\gamma : I \rightarrow D$ satisfies*

$$Q_{(c)} \left(\nabla_{\gamma(t)}^{\mathcal{G}^D} \gamma(t) - \mathcal{F}(\gamma(t)) \right) = 0 \quad \forall t \in I \subseteq \mathbb{R}.$$

5.1. Kinematic reduction of mechanical control systems on a skew-symmetric algebroid. Now we introduce the notion of a kinematic reduction (see [4]).

Given an **skew-symmetric algebroid structure** on the vector bundle $\tau_D : D \rightarrow M$, we define a **driftless system** as the set $(M, \mathfrak{D}, \bar{U})$, where \mathfrak{D} is a vector subbundle of D locally spanned by $\{X_1, \dots, X_{k'}\}$, with $X_\alpha \in \Gamma(\tau_D)$, $1 \leq \alpha \leq k'$, and $\bar{U} \subset \mathbb{R}^{k'}$ is the set of admissible controls. For a section $X = \sum_{\alpha=1}^{k'} \bar{u}^\alpha X_\alpha \in \Gamma(\tau_D)$, remember that an **integral curve of X** is a curve $\sigma : I \subset \mathbb{R} \rightarrow M$ such that

$$\dot{\sigma}(t) = \sum_{\alpha=1}^{k'} \bar{u}^\alpha(t) \rho_D(X_\alpha)(\sigma(t)), \quad (5.3)$$

where $(\bar{u}^1(t), \dots, \bar{u}^{k'}(t)) = (\bar{u}^1(\sigma(t)), \dots, \bar{u}^{k'}(\sigma(t))) \in \bar{U}$ for all t . It can also be said that the pair (σ, \bar{u}) is a solution to the driftless system.

Observe that for each pair $(\sigma(t), \bar{u}(t))$ we have the curve $\gamma : I \rightarrow \mathfrak{D} \subseteq D$ defined by

$$\gamma(t) = \sum_{\alpha=1}^{k'} \bar{u}^\alpha(t) X_\alpha(\sigma(t)).$$

In the sequel, we will denote by $\tau_{\mathfrak{D}}$ the restriction $(\tau_D)|_{\mathfrak{D}}$.

Definition 5.4 (Kinematic reduction). *Let $(D, \mathcal{G}^D, \mathcal{F}, \mathcal{D}_{(c)})$ be a mechanical control system. A driftless system $(M, \mathfrak{D}, \bar{U})$ is called a **kinematic reduction** of $(D, \mathcal{G}^D, \mathcal{F}, \mathcal{D}_{(c)})$ if for every solution $(\sigma(t), \bar{u}(t))$ of (5.3) there exists a pair $(\gamma(t), u(t))$ solution of (5.2), where $\gamma(t) = \sum_{\alpha=1}^{k'} \bar{u}^\alpha(t) X_\alpha(\sigma(t))$.*

The rank of a kinematic reduction is the rank of the distribution \mathfrak{D} . Rank-one kinematic reductions are particularly interesting. A section X of $\Gamma(\tau_D)$ is called a **decoupling section** if the rank-one kinematic system induced by $\mathfrak{D} = \text{span} \{X\}$ is a kinematic reduction.

Definition 5.5 (Kinematic controllability). *A mechanical control system $(D, \mathcal{G}^D, \mathcal{F}, \mathcal{D}_{(c)})$ is **kinematically controllable** if it possesses decoupling sections $\{X_1, \dots, X_{k'}\}$ whose involutive closure has maximum rank.*

When a system is kinematically controllable, motion planning is possible by using concatenations of integral curves of the decoupling vector fields. Those curves must be reparametrized in such a way that each segment begins and ends with zero velocity, see [4] for more details.

We have the following adaptation of the results in Section 4.

Proposition 5.6. *Let $(D, [\cdot, \cdot]_D, \rho_D)$ be a skew-symmetric algebroid and consider a mechanical control problem determined by $(D, \mathcal{G}^D, \mathcal{F}, \mathcal{D}_{(c)})$. Consider a driftless system $(M, \mathfrak{D}, \bar{U})$. For all $X \in \Gamma(\tau_{\mathfrak{D}})$, the following conditions are equivalent:*

(i) *if $\sigma : I \rightarrow M$ is an integral curve of $\rho_D(X)$, that is,*

$$\dot{\sigma}(t) = \rho_D(X)(\sigma(t)),$$

then $\gamma = X \circ \sigma : I \rightarrow D$ is an admissible curve solution of

$$Q_{(c)}(\nabla_{\gamma(t)}^{\mathcal{G}^D} \gamma(t) - \mathcal{F}(\gamma(t))) = 0.$$

(ii) $Q_{(c)} \left(\nabla_X^{\mathcal{G}^D} X - \mathcal{F} \circ X \right) = 0$.

Under extra assumptions, we have an alternative way to write condition (ii) in Proposition 5.6. This provides us with a new characterization of kinematic reductions in terms of the affine connection of the given mechanical control system. (See also [4, 21]).

Proposition 5.7. *If $Q_{(c)}(\mathcal{F}) = 0$ and $\mathfrak{D} = \text{span} \{X_1, \dots, X_{k'}\}$, then the following conditions are equivalent:*

(i) *For all $X \in \Gamma(\tau_{\mathfrak{D}})$, $Q_{(c)} \left(\nabla_X^{\mathcal{G}^D} X - \mathcal{F} \circ X \right) = Q_{(c)} \left(\nabla_X^{\mathcal{G}^D} X \right) = 0$.*

- (ii) For all $\alpha, \beta, \gamma \in \{1, \dots, k'\}$, $Q_{(c)}(X_\alpha) = 0$, $Q_{(c)}(\langle X_\beta : X_\gamma \rangle_{\mathfrak{g}_D}) = 0$. In other words, for all $\alpha, \beta, \gamma \in \{1, \dots, k'\}$, $X_\alpha \in \Gamma(\tau_{\mathcal{D}(c)})$ and $\langle X_\beta : X_\gamma \rangle_{\mathfrak{g}_D} \in \Gamma(\tau_{\mathcal{D}(c)})$.
- (iii) $(M, \mathfrak{D}, \bar{U})$ is a kinematic reduction of $(D, \mathfrak{g}^D, \mathcal{F}, \mathcal{D}(c))$.

Proof. (i) \Rightarrow (ii) Observe that for each $\alpha = 1, \dots, k'$ we have that $fX_\alpha \in \Gamma(\tau_{\mathfrak{D}})$ for all $f \in C^\infty(M)$. If (i) holds, we have

$$0 = Q_{(c)}\left(\nabla_{fX_\alpha}^{\mathfrak{g}^D} fX_\alpha\right) = f^2 Q_{(c)}\left(\nabla_{X_\alpha}^{\mathfrak{g}^D} X_\alpha\right) + f\rho_D(X_\alpha)(f)Q_{(c)}(X_\alpha)$$

for every $f \in C^\infty(M)$. Then, taking suitable functions f , we obtain $Q_{(c)}(X_\alpha) = 0$ and $Q_{(c)}(\nabla_{X_\alpha}^{\mathfrak{g}^D} X_\alpha) = 0$. Now, using the polarization identity we have that

$$\langle X_\alpha : X_\beta \rangle_{\mathfrak{g}_D} = \frac{1}{2} \left(\nabla_{X_\alpha + X_\beta}^{\mathfrak{g}^D} (X_\alpha + X_\beta) - \nabla_{X_\alpha}^{\mathfrak{g}^D} X_\alpha - \nabla_{X_\beta}^{\mathfrak{g}^D} X_\beta \right).$$

Therefore, $Q_{(c)}(\langle X_\alpha : X_\beta \rangle_{\mathfrak{g}_D}) = 0$.

(ii) \Rightarrow (iii) By definition $\mathfrak{D} \subseteq D$. As (ii) is true, then statement (ii) in Proposition 5.6 is satisfied for every $X \in \Gamma(\tau_D)$. Hence statement (i) in Proposition 5.6 is true and we can conclude that $(M, \mathfrak{D}, \bar{U})$ is a kinematic reduction of $(D, \mathfrak{g}^D, \mathcal{F}, \mathcal{D}(c))$ according to Definition 5.4.

(iii) \Rightarrow (i) As $(M, \mathfrak{D}, \bar{U})$ is a kinematic reduction of $(D, \mathfrak{g}^D, \mathcal{F})$, statement (i) in Proposition 5.6 is satisfied. Hence the result follows. \square

A straightforward corollary of Proposition 5.7 is the following one:

Corollary 5.8. *If $Q_{(c)}(\mathcal{F}) = 0$ and $\mathfrak{D} = \text{span}\{X_1\}$, then the following conditions are equivalent:*

- (i) $Q_{(c)}\left(\nabla_{X_1}^{\mathfrak{g}^D} X_1 - \mathcal{F} \circ X_1\right) = Q_{(c)}\left(\nabla_{X_1}^{\mathfrak{g}^D} X_1\right) = 0$.
- (ii) $(M, \mathfrak{D}, \bar{U})$ is a rank-one kinematic reduction of $(D, \mathfrak{g}^D, \mathcal{F}, \mathcal{D}(c))$.

In the sequel assume that \mathcal{F} comes from a potential function $V : M \rightarrow \mathbb{R}$, that is, $\mathcal{F} = -\text{grad}_{\mathfrak{g}^D} V \circ \tau_D$. Then Theorem 4.4 can be adapted to control systems as follows.

Theorem 5.9. *Let $(D, [\cdot, \cdot]_D, \rho_D)$ be a skew-symmetric algebroid and consider a control problem determined by $(D, \mathfrak{g}^D, V, \mathcal{D}(c))$.*

Take a section $X \in \Gamma(\tau_D)$ such that $i_X d^D(\flat_{\mathfrak{g}^D}(X))(Y) = 0$ for all $Y \in \Gamma(\tau_{\mathcal{D}(c)}^\perp)$. Under this hypothesis, the following conditions are equivalent:

- (i) *If $\sigma : I \rightarrow M$ is an integral curve of $\rho_D(X)$, that is,*

$$\dot{\sigma}(t) = \rho_D(X)(\sigma(t)),$$

then $\gamma = X \circ \sigma : I \rightarrow D$ is an admissible curve solution of

$$Q_{(c)}(\nabla_{\gamma(t)}^{\mathcal{G}^D} \gamma(t) + \text{grad}_{\mathcal{G}^D} V(\tau_D(\gamma(t)))) = 0.$$

(ii) X satisfies **the Hamilton-Jacobi differential equation**

$$d^D \left(\frac{1}{2} \mathcal{G}^D(X, X) + V \right) (Y) = 0 \text{ for all } Y \in \Gamma(\tau_{\mathcal{D}_{(c)}^\perp}). \quad (5.4)$$

Proof. Following similar arguments that in the proof of Theorem 4.4 it is easy to prove that the condition $i_X d^D(b_{\mathcal{G}^D}(X))(Y) = 0$ for all $Y \in \Gamma(\tau_{\mathcal{D}_{(c)}^\perp})$ is equivalent to

$$\mathcal{G}^D(\nabla_X^{\mathcal{G}^D} X, Y) = \mathcal{G}^D(\nabla_Y^{\mathcal{G}^D} X, X).$$

Finally, observe now that

$$\begin{aligned} 0 &= d^D \left(\frac{1}{2} \mathcal{G}^D(X, X) + V \right) (Y) \\ &= \mathcal{G}^D(\nabla_Y^{\mathcal{G}^D} X, X) + \rho_D(Y)(V) \\ &= \mathcal{G}^D(\nabla_X^{\mathcal{G}^D} X, Y) + \mathcal{G}^D(\text{grad}_{\mathcal{G}^D} V, Y), \end{aligned}$$

The last expression is equivalent to the equation

$$Q_{(c)}(\nabla_X^{\mathcal{G}^D} X + \text{grad}_{\mathcal{G}^D} V) = 0.$$

□

Hence, we have just extended the notion of decoupling sections for mechanical control systems with nonzero potential since Theorem 5.9 gives sufficient and necessary conditions to have a kinematic reduction of such a mechanical control system.

Remark 5.10. Note that the condition $i_X d^D(b_{\mathcal{G}^D}(X))(Y) = 0$ in the hypothesis of Theorem 5.9 is $\mathcal{C}^\infty(M)$ -linear in $Y \in \mathcal{D}_{(c)}^\perp$. Hence only for a basis of vector fields in $\mathcal{D}_{(c)}^\perp$ the condition must be checked. However that same condition is not $\mathcal{C}^\infty(M)$ -linear in $X \in \Gamma(\tau_D)$. \diamond

This theorem plays a key role to define in the future the geometric notion of motion planning for mechanical control systems with non-zero potentials, only known so far for zero potentials [4]. In that sense it will be useful to have some notion of reparametrization of integral curves of sections $X \in \Gamma(\tau_D)$.

Proposition 5.11. *Let X be a nonzero section in $\Gamma(\tau_{\mathcal{D}_{(c)}})$ such that for all $Y \in \Gamma(\tau_{\mathcal{D}_{(c)}^\perp})$, $i_X d^D(b_{\mathcal{G}^D}(X))(Y) = 0$. For a nonzero function f in $\mathcal{C}^\infty(M)$, the section $fX \in \Gamma(\tau_{\mathcal{D}_{(c)}})$ satisfies $i_{fX} d^D(b_{\mathcal{G}^D}(fX))(Y) = 0$ for all $Y \in \Gamma(\tau_{\mathcal{D}_{(c)}^\perp})$ if and only if $\rho_D(Y)(f) = 0$ for all $Y \in \Gamma(\tau_{\mathcal{D}_{(c)}^\perp})$.*

Proof. Let us rewrite the condition $i_{fX}d^D(\flat_{\mathcal{G}^D}(fX))(Y) = 0$ as follows

$$\begin{aligned} 0 &= \mathcal{G}^D(\nabla_{fX}^{\mathcal{G}^D} fX, Y) - \mathcal{G}^D(\nabla_Y^{\mathcal{G}^D} fX, fX) \\ &= f^2 \left(\mathcal{G}^D(\nabla_X^{\mathcal{G}^D} X, Y) - \mathcal{G}^D(\nabla_Y^{\mathcal{G}^D} X, X) \right) + f\rho_D(X)(f)\mathcal{G}^D(X, Y) \\ &\quad - f\rho_D(Y)(f)\mathcal{G}^D(X, X) = -f\rho_D(Y)(f)\mathcal{G}^D(X, X) \end{aligned}$$

because $X \in \Gamma(\tau_{\mathcal{D}(c)})$. From here, the equivalence is straightforward. \square

As a consequence of Proposition 5.11, Theorem 5.9 can also be written for such a fX .

From Theorem 5.9, we establish a connection between decoupling sections in the sense defined in [4] and the solutions to Hamilton-Jacobi differential equation.

Corollary 5.12. *Let $(D, \llbracket \cdot, \cdot \rrbracket_D, \rho_D)$ be a skew-symmetric algebroid and consider a control problem determined by $(D, \mathcal{G}^D, V, \mathcal{D}(c))$ where V satisfies that $\text{grad}_{\mathcal{G}^D} V \in \Gamma(\tau_{\mathcal{D}(c)})$. If a section $X \in \Gamma(\tau_{\mathcal{D}(c)})$ is a decoupling section, then the following conditions are equivalent:*

- (i) $i_X d^D(\flat_{\mathcal{G}^D}(X))(Y) = 0$ for all $Y \in \Gamma(\tau_{\mathcal{D}(c)}^\perp)$,
- (ii) $0 = \mathcal{G}^D(\nabla_Y^{\mathcal{G}^D} X, X)$ for all $Y \in \Gamma(\tau_{\mathcal{D}(c)}^\perp)$,
- (iii) X satisfies **the Hamilton-Jacobi differential equation**

$$d^D \left(\frac{1}{2} \mathcal{G}^D(X, X) + V \right) (Y) = 0 \text{ for all } Y \in \Gamma(\tau_{\mathcal{D}(c)}^\perp).$$

Proof. The condition $i_X d^D(\flat_{\mathcal{G}^D}(X))(Y) = 0$ for all $Y \in \Gamma(\tau_{\mathcal{D}(c)}^\perp)$ can be rewritten as

$$\mathcal{G}^D(\nabla_X^{\mathcal{G}^D} X, Y) = \mathcal{G}^D(\nabla_Y^{\mathcal{G}^D} X, X)$$

for all $Y \in \Gamma(\tau_{\mathcal{D}(c)}^\perp)$.

As X is a decoupling section, both X and $\nabla_X^{\mathcal{G}^D} X \in \Gamma(\tau_{\mathcal{D}(c)})$ by Corollary 5.8. Hence the left-hand side of the above equality is zero and (i) is equivalent to (ii).

Condition (iii) can be rewritten as follows:

$$\begin{aligned} 0 &= d^D \left(\frac{1}{2} \mathcal{G}^D(X, X) + V \right) (Y) = \frac{1}{2} \rho_D(Y) \mathcal{G}^D(X, X) + \rho_D(Y)(V) \\ &= \mathcal{G}^D(\nabla_Y^{\mathcal{G}^D} X, X) + \mathcal{G}^D(\text{grad}_{\mathcal{G}^D} V, Y), \end{aligned}$$

for all $Y \in \Gamma(\tau_{\mathcal{D}(c)}^\perp)$. Hence the equivalence between (ii) and (iii) is clear because of the hypothesis for V . \square

Note that the property of being a decoupling section is preserved by $\mathcal{C}^\infty(M)$ -multiplication. However, the condition $i_X d^D(b_{\mathcal{G}^D}(X))(Y) = 0$ for all $Y \in \Gamma(\tau_{\mathcal{D}_{(c)}}^\perp)$ is not $\mathcal{C}^\infty(M)$ -linear on X . Proposition 5.11 has already characterized those functions f such that $i_{fX} d^D(b_{\mathcal{G}^D}(fX))(Y) = 0$ for all $Y \in \Gamma(\tau_{\mathcal{D}_{(c)}}^\perp)$.

It might also be the case that a decoupling section $X \in \Gamma(\tau_{\mathcal{D}_{(c)}})$ does not satisfy (ii) in Corollary 5.12, but there might exist functions $f \in \mathcal{C}^\infty(M)$ such that fX satisfies (ii). Let $Y \in \Gamma(\tau_{\mathcal{D}_{(c)}}^\perp)$,

$$\begin{aligned} \mathcal{G}^D(\nabla_Y^{\mathcal{G}^D} fX, fX) &= f\mathcal{G}^D(\rho_D(Y)(f)X + f\nabla_Y^{\mathcal{G}^D} X, X) \\ &= f\rho_D(Y)(f)\mathcal{G}^D(X, X) + f^2\mathcal{G}^D(\nabla_Y^{\mathcal{G}^D} X, X). \end{aligned} \quad (5.5)$$

Condition (ii) in Corollary 5.12 would be satisfied if the following partial differential equation for f has solutions:

$$f\rho_D(Y)(f)\mathcal{G}^D(X, X) + f^2\mathcal{G}^D(\nabla_Y^{\mathcal{G}^D} X, X) = 0$$

for all $Y \in \Gamma(\tau_{\mathcal{D}_{(c)}}^\perp)$. The chances to find a solution depend on the particular examples under study, see Section 5.3.

5.2. Maximally reducible systems. If $(M, \mathfrak{D}, \bar{U})$ is a kinematic reduction of $(D, \mathcal{G}^D, \mathcal{F}, \mathcal{D}_{(c)})$, then any solution of (5.3) can be followed by a solution of (5.1). In this section we consider when the converse is also possible in such a way that we can talk about ‘‘equivalence’’ of controlled trajectories as mentioned in the introduction.

Definition 5.13 (Maximal reducibility). *A mechanical control system $(D, \mathcal{G}^D, \mathcal{F}, \mathcal{D}_{(c)})$ being $\mathcal{D}_{(c)}$ a subbundle of D is **maximally reducible** to a driftless system $(M, \mathfrak{D}, \bar{U})$ if*

- (i) $(M, \mathfrak{D}, \bar{U})$ is a kinematic reduction of $(D, \mathcal{G}^D, \mathcal{F}, \mathcal{D}_{(c)})$, and
- (ii) for every solution $(\gamma(t), u(t))$ of Equations (5.2) satisfying $\gamma(0) \in \mathcal{D}_{(\tau_D \circ \gamma)(0)}$, there exists a control $\bar{u} \in \bar{U}$ such that $(\tau_D \circ \gamma(t), \bar{u}(t))$ is a solution of Equation (5.3).

The characterization of maximally reducible mechanical control systems defined on skew-symmetric algebroids is given by the following result. This is a generalization of the notion of being maximally reducible systems proved in [4] on Riemannian manifolds.

Theorem 5.14. *Let $(D, \mathcal{G}^D, \mathcal{F}, \mathcal{D}_{(c)})$ be a mechanical control system such that $\mathcal{D}_{(c)}$ has locally constant rank and $\mathcal{F} = 0$. This mechanical control system is maximally reducible to a driftless system $(M, \mathfrak{D}, \bar{U})$ if and only if the following two conditions hold:*

- (i) $\mathfrak{D} = \mathcal{D}_{(c)}$,
(ii) $\mathcal{D}_{(c)}$ is geodesically invariant, that is, $\text{Sym}^{(\infty)}(\mathcal{D}_{(c)}) = \mathcal{D}_{(c)}$,

where $\text{Sym}^{(\infty)}(\mathcal{D}_{(c)})$ is the smallest distribution containing $\mathcal{D}_{(c)}$ and closed under the symmetric product $\langle \cdot : \cdot \rangle_{\mathcal{G}^D}$.

Proof. It follows the same lines as the proof of Theorem 8.27 in [4].

Suppose that the rank of $\mathfrak{D} \subseteq D$ is k' . Note that $X \in \Gamma(\tau_D)$ is a section of \mathfrak{D} if and only if the vertical lift of X restricted to \mathfrak{D} is tangent to \mathfrak{D} .

Assume that $(D, \mathcal{G}^D, 0, \mathcal{D}_{(c)})$ is maximally reducible to a driftless system $(M, \mathfrak{D}, \bar{U})$.

If $\gamma : I \subseteq \mathbb{R} \rightarrow D$ is a geodesic for the bundle metric \mathcal{G}^D with initial condition $\gamma(0) \in \mathfrak{D}_{\sigma(0)}$, then γ is a solution of Equations (5.2) with zero controls. Thus, by hypothesis, there exist controls $\bar{u}^\alpha : I \rightarrow \bar{U}$ such that
$$\gamma(t) = \sum_{\alpha=1}^{k'} \bar{u}^\alpha(t) X_\alpha(\sigma(t)),$$
 where $\sigma = \tau_D \circ \gamma : I \subseteq \mathbb{R} \rightarrow M$. Therefore, $\gamma(I) \subseteq \mathfrak{D}$ and we can conclude that \mathfrak{D} is geodesically invariant.

It remains to prove that $\mathfrak{D} = \mathcal{D}_{(c)}$.

Remember that $\mathcal{D}_{(c)} = \text{span}\{Y_1, \dots, Y_k\}$. Then it is sufficient to prove that Y_s is a section of \mathfrak{D} for $s = 1, \dots, k$ in order to obtain $\mathcal{D}_{(c)} \subset \mathfrak{D}$. Having in mind that $\xi_{\mathcal{G}^D}$ is the geodesic spray on D and Y_s^\vee is the vertical lift of $Y_s \in \Gamma(\tau_{\mathcal{D}_{(c)}})$ to $\mathcal{D}_{(c)}$, it is clear that the integral curves of $\xi_{\mathcal{G}^D} + Y_s^\vee$ are solutions of Equations (5.2). By assumption, $\xi_{\mathcal{G}^D} + Y_s^\vee$ restricted to \mathfrak{D} is tangent to \mathfrak{D} . On the other hand, \mathfrak{D} is geodesically invariant, what implies that $\xi_{\mathcal{G}^D}$ restricted to \mathfrak{D} is tangent to \mathfrak{D} . Hence, $(Y_s^\vee)|_{\mathfrak{D}}$ is tangent to \mathfrak{D} and Y_s is a section of \mathfrak{D} .

Next, we will see that $\mathfrak{D} \subseteq \mathcal{D}_{(c)}$. Let a be a vector in \mathfrak{D} . We consider the geodesic $\tilde{\gamma} : \tilde{I} \subseteq \mathbb{R} \rightarrow D$ with initial condition $\tilde{\gamma}(0) = a$. Hence, $\tilde{\gamma}(\tilde{I}) \subseteq \mathfrak{D}$ because \mathfrak{D} is geodesically invariant. Now, take the curve $\gamma : I \subseteq \mathbb{R} \rightarrow \mathfrak{D}$ given by

$$\gamma(t) = t\tilde{\gamma}\left(\frac{t^2}{2}\right), \quad \forall t \in I.$$

Note that $\tau_D \circ \gamma = \tau_D \circ \tilde{\gamma} \circ \tau$, where $\tau : \mathbb{R} \rightarrow \mathbb{R}$ is the map defined by $\tau(t) = t^2/2$, and $\gamma(0) = 0$. From (3.1) it is straightforward to prove that

$$\left(\nabla_{\gamma(0)}^{\mathcal{G}^D} \gamma\right)(0) = a.$$

On the other hand, define a curve $\tilde{\sigma}$ on M given by $\tilde{\sigma} = \tau_D \circ \tilde{\gamma}$ such that

$$\gamma(t) = \sum_{\alpha=1}^{k'} \bar{u}^\alpha(t) X_\alpha((\tau_D \circ \gamma)(t)) = \sum_{\alpha=1}^{k'} \bar{u}^\alpha(t) X_\alpha((\tilde{\sigma} \circ \tau)(t)).$$

Then, it is clear that

$$\sum_{\alpha=1}^{k'} \bar{u}^\alpha(t) \rho_D(X_\alpha)((\tilde{\sigma} \circ \tau)(t)) = \rho_D(\gamma(t)) = \frac{d}{dt}(\tilde{\sigma} \circ \tau) = \frac{d}{dt}(\tau_D \circ \gamma),$$

what implies that $(\tilde{\sigma} \circ \tau, \bar{u})$ is a solution of Equations (5.3). Therefore, there exist controls u such that (γ, u) is a solution of Equations (5.2). In particular,

$$a = \nabla_{\gamma(0)}^{\mathfrak{G}^D} \gamma(0) \in \mathcal{D}_{(c)}.$$

The other implication is proved as follows: First, we prove that $(M, \mathfrak{D}, \bar{U})$ is a kinematic reduction of $(D, \mathfrak{G}^D, \mathcal{F}, \mathcal{D}_{(c)})$. Let $\sigma: I \subseteq \mathbb{R} \rightarrow M$ be a curve such that it satisfies $\dot{\sigma}(t) = \sum_{\alpha=1}^{k'} \bar{u}^\alpha(t) \rho_D(X_\alpha)(\sigma(t))$, i.e. Equation (5.3). Take $\gamma(t) = \sum_{\alpha=1}^{k'} \bar{u}^\alpha(t) X_\alpha(\sigma(t))$ and note that $\gamma(I) \subseteq \mathfrak{D}$ and $\tau_D \circ \gamma = \sigma$. Moreover,

$$\frac{d}{dt}(\tau_D \circ \gamma) = \dot{\sigma} = \sum_{\alpha=1}^{k'} \bar{u}^\alpha(t) \rho_D(X_\alpha)(\sigma(t)) = \rho_D(\gamma(t)).$$

Thus γ is ρ_D -admissible.

Now we have to prove that $\nabla_{\gamma(t)}^{\mathfrak{G}^D} \gamma(t) \in \mathcal{D}_{(c)}(\sigma(t))$. By hypothesis, $\mathfrak{D} = \mathcal{D}_{(c)}$ and $\mathcal{D}_{(c)}$ is geodesically invariant. Then by Theorem 3.2 we conclude that $\nabla_{\gamma(t)}^{\mathfrak{G}^D} \gamma(t) \in \mathcal{D}_{(c)}(\sigma(t))$ for all $t \in I$. Hence, there exist controls such that γ is a solution of Equation (5.2).

It remains to prove condition (ii) in Definition 5.13. Let $(\gamma, u): I \rightarrow D \times U$ be a solution to Equation (5.3), then $Q_{(c)}(\nabla_{\gamma(t)}^{\mathfrak{G}^D} \gamma(t)) = 0$, that is,

$$\nabla_{\gamma(t)}^{\mathfrak{G}^D} \gamma(t) \in \mathcal{D}_{(c)}(\sigma(t)) = \mathfrak{D}(\sigma(t))$$

where $\sigma = \tau_D \circ \gamma$.

By assumption $\gamma(0) \in \mathfrak{D}(\sigma(0))$ and γ is an integral curve of $\xi_{\mathfrak{G}^D} + X^\vee$, where $\xi_{\mathfrak{G}^D}$ is the geodesic spray associated with \mathfrak{G}^D and $X \in \Gamma(\tau_{\mathfrak{D}})$.

Note that $X^\vee|_{\mathfrak{D}}$ is tangent to \mathfrak{D} and $(\xi_{\mathfrak{G}^D})|_{\mathfrak{D}}$ is also tangent to \mathfrak{D} because \mathfrak{D} is geodesically invariant. Hence, if $\gamma(0) \in \mathfrak{D}(\sigma(0))$, then the integral curve of $\xi_{\mathfrak{G}^D} + X^\vee$ with initial condition $\gamma(0)$ is entirely contained in \mathfrak{D} , that is, $\gamma(t) \in \mathfrak{D}(\sigma(t))$ for all $t \in I$.

Then there exist $\bar{u}: I \rightarrow \bar{U}$ such that $\gamma(t) = \sum_{\alpha=1}^{k'} \bar{u}^\alpha(t) X_\alpha(\sigma(t))$ and

$$\rho_D(\gamma(t)) = \sum_{\alpha=1}^{k'} \bar{u}^\alpha(t) \rho_D(X_\alpha)(\sigma(t)).$$

Using the fact that γ is ρ_D -admissible it immediately follows that $\dot{\sigma}(t) = \sum_{\alpha=1}^{k'} \bar{u}^\alpha(t) \rho_D(X_\alpha)(\sigma(t))$ for all $t \in I$. \square

Let us provide some results to find decoupling sections for the mechanical control systems under consideration.

Corollary 5.15. *If a mechanical control system $(D, \mathcal{G}^D, \mathcal{F}, \mathcal{D}_{(c)})$ with $\mathcal{F} = 0$ is maximally reducible, then all control sections are decoupling.*

Proof. This follows immediately from Theorem 5.14 and the definition of decoupling sections. \square

The converse is not necessarily true. But it is true when $\mathcal{D}_{(c)}$ has locally constant rank equal to one.

Corollary 5.16. *Let $(D, \mathcal{G}^D, \mathcal{F}, \mathcal{D}_{(c)})$ be a mechanical control system where $\mathcal{D}_{(c)}$ has rank one and $\mathcal{F} = 0$. The following statements are equivalent:*

- (i) *There exist decoupling sections for $(D, \mathcal{G}^D, \mathcal{F}, \mathcal{D}_{(c)})$.*
- (ii) *The mechanical control system $(D, \mathcal{G}^D, \mathcal{F}, \mathcal{D}_{(c)})$ is maximally reducible to a driftless system defined by decoupling sections.*

Proof. It is straightforward from Theorem 5.14. \square

5.3. Examples. The first two examples are specific cases of Example 3.4. As is shown, the example in Section 5.3.1 is not maximally reducible but admits rank-one kinematic reductions. Particular solutions to Hamilton-Jacobi differential equation are found using Theorem 5.9 and Corollary 5.12. In Section 5.3.2 we compute particular solutions to Hamilton-Jacobi differential equation for a maximally reducible system. The snakeboard described in Section 5.3.3 has nonholonomic constraints. Hence the use of skew-symmetric algebroids to find solutions to Hamilton-Jacobi differential equations is very natural.

5.3.1. Planar rigid body with a variable-direction thruster. We refer to [4, Section 7.4.2] for a detailed description of the system. The configuration space is $M = \mathbb{R}^2 \times \mathbb{S}^1$. Consider local coordinates (x, y, θ) . Here the distribution D is the entire tangent space TM and $V = 0$. We are in the case explained in Example 3.4.

The riemannian metric is

$$\mathcal{G}^{TM} = Jd\theta \otimes d\theta + m(dx \otimes dx + dy \otimes dy).$$

The control vector fields in $\mathcal{D}_{(c)}$ are

$$Y_1 = \frac{\cos \theta}{m} \frac{\partial}{\partial x} + \frac{\sin \theta}{m} \frac{\partial}{\partial y}, \quad Y_2 = -\frac{\sin \theta}{m} \frac{\partial}{\partial x} + \frac{\cos \theta}{m} \frac{\partial}{\partial y} - \frac{h}{J} \frac{\partial}{\partial \theta}.$$

In this example

$$\mathcal{D}_{(c)}^\perp = \text{span}_{\mathcal{C}^\infty} \left\{ Y_3 : = -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y} + \frac{1}{h} \frac{\partial}{\partial \theta} \right\}.$$

Note that $\{Y_1, Y_2, Y_3\}$ is a \mathcal{G}^{TM} -orthogonal basis. The skew-symmetric algebroid structure is defined as follows $D = TM$, $[\cdot, \cdot]_{TM}$ is the usual Lie bracket and $\rho_{TM} = \text{Id}_{TM}$. In our adapted basis we have

$$\begin{aligned} \llbracket Y_1, Y_2 \rrbracket_{TM} &= [Y_1, Y_2] = \frac{h}{J + mh^2} Y_2 + \frac{h^3}{(J + mh^2)J} Y_3, \\ \llbracket Y_1, Y_3 \rrbracket_{TM} &= [Y_1, Y_3] = -\frac{J}{h(J + mh^2)} Y_2 - \frac{h}{J + mh^2} Y_3, \\ \llbracket Y_2, Y_3 \rrbracket_{TM} &= [Y_2, Y_3] = \frac{mh^2 + J}{hJ} Y_1. \end{aligned}$$

The non-zero Christoffel symbols for the associated Levi-Civita $\nabla^{\mathcal{G}^{TM}}$ connection are

$$\begin{aligned} \Gamma_{22}^1 &= \frac{h}{J}, & \Gamma_{23}^1 &= \frac{mh}{J}, & \Gamma_{32}^1 &= -\frac{1}{h}, \\ \Gamma_{33}^1 &= -\frac{m}{h}, & \Gamma_{21}^2 &= -\frac{h}{J + mh^2}, & \Gamma_{31}^2 &= \frac{J}{h(J + mh^2)}, \\ \Gamma_{21}^3 &= -\frac{h^3}{J(J + mh^2)}, & \Gamma_{31}^3 &= \frac{h}{J + mh^2}. \end{aligned}$$

Let us compute the symmetric products of the control vector fields

$$\begin{aligned} \langle Y_1 : Y_1 \rangle_{g^D} &= 0, & \langle Y_1 : Y_2 \rangle_{g^D} &= \Gamma_{21}^2 Y_2 + \Gamma_{21}^3 Y_3, \\ \langle Y_2 : Y_2 \rangle_{g^D} &= 2\Gamma_{22}^1 Y_1. \end{aligned}$$

By Corollary 5.8, Y_1 and Y_2 are decoupling vector fields. However, $\mathcal{D}_{(c)} = \text{span}\{Y_1, Y_2\}$ is not geodesically invariant because $\text{Sym}^{(1)}\mathcal{D}_{(c)} \not\subseteq \mathcal{D}_{(c)}$. Then according to Theorem 5.14 the mechanical control system is not maximally reducible to a driftless control system. As the involutive closure of the decoupling vector fields Y_1 and Y_2 has maximum rank, the system is kinematic controllable, see Definition 5.5. Then the motion planning is feasible.

Let us see if Y_1 and Y_2 satisfy condition (ii) in Corollary 5.12, that is, $0 = \mathcal{G}^{TM}(\nabla_Y^{\mathcal{G}^{TM}} X, X)$ for all $Y \in \mathcal{D}_{(c)}^\perp = \text{span}\{Y_3\}$.

$$\begin{aligned}\mathcal{G}^{TM}(\nabla_{Y_3}^{\mathcal{G}^{TM}} Y_1, Y_1) &= \Gamma_{31}^2 \mathcal{G}^{TM}(Y_2, Y_1) + \Gamma_{31}^3 \mathcal{G}^{TM}(Y_3, Y_1) = 0, \\ \mathcal{G}^{TM}(\nabla_{Y_3}^{\mathcal{G}^{TM}} Y_2, Y_2) &= \Gamma_{32}^1 \mathcal{G}^{TM}(Y_1, Y_2) = 0\end{aligned}$$

since $\{Y_1, Y_2, Y_3\}$ is a \mathcal{G}^{TM} -orthogonal basis. Thus Y_1 and Y_2 are both solutions to Hamilton-Jacobi differential equation because of Corollary 5.12.

Let us try to find more vector fields solution to Hamilton-Jacobi differential equation. In order to do this we have to find functions $f \in \mathcal{C}^\infty(M)$ such that fY_1 and fY_2 also satisfy condition (ii) in Corollary 5.12. According to Proposition 5.11 f must satisfy

$$Y_3(f) = 0 \Leftrightarrow -\sin\theta \frac{\partial f}{\partial x} + \cos\theta \frac{\partial f}{\partial y} + \frac{1}{h} \frac{\partial f}{\partial \theta} = 0.$$

For instance, $f(x, y, \theta) = g(x - h \cos\theta, y - h \sin\theta)$ where $g: \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies the above partial differential equation. Hence, all vector fields fY_1 and fY_2 are solutions to Hamilton-Jacobi differential equation, but not necessarily their linear combinations.

So far we have found some particular solutions of Hamilton-Jacobi differential equation. Let us consider now the most general section in $\Gamma(\tau_D)$, $X = \alpha_1 Y_1 + \alpha_2 Y_2 + \alpha_3 Y_3$. Let us find functions $\alpha_1, \alpha_2, \alpha_3$ such that condition $i_X(d^D(b_{\mathcal{G}^D}(X)))(Y) = 0$ for all $Y \in \mathcal{D}_{(c)}^\perp$ in Theorem 5.9 is satisfied, that is,

$$\begin{aligned}0 &= -\mathcal{G}^D(Y_1, Y_1)(\alpha_1 Y_3(\alpha_1) + \alpha_2 \alpha_1 \Gamma_{32}^1 + \alpha_3 \alpha_1 \Gamma_{33}^1) \\ &\quad - \mathcal{G}^D(Y_2, Y_2)(\alpha_2 Y_3(\alpha_2) + \alpha_1 \alpha_2 \Gamma_{31}^2) \\ &\quad + \mathcal{G}^D(Y_3, Y_3)(\alpha_1 Y_1(\alpha_3) + \alpha_2 Y_2(\alpha_3) + \alpha_1 \alpha_2 \Gamma_{21}^3).\end{aligned}$$

It can be proved that this condition and Hamilton-Jacobi differential equation (5.4) are satisfied, for instance, by

$$\begin{aligned}g(x - h \cos\theta, y - h \sin\theta)Y_3, &\quad g(x - h \cos\theta, y - h \sin\theta)Y_2 + \alpha_3 Y_3, \\ g(x - h \cos\theta, y - h \sin\theta)Y_1, &\quad f((x - h \cos\theta)/h)Y_2 + \alpha_3 Y_3.\end{aligned}$$

5.3.2. *Robotic leg.* We refer to [4, Section 7.4.1] for a detailed description of this system. The configuration manifold is $M = \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{S}^1$ with local coordinates (r, θ, ψ) . The riemannian metric for the system is

$$\mathcal{G}^{TM} = m(dr \otimes dr + r^2 d\theta \otimes d\theta) + Jd\psi \otimes d\psi,$$

where m is the mass of the particle on the end of the extensible leg and J is the moment of inertia of the base rigid body about the pivot point.

The control vector fields that span $\mathcal{D}_{(c)}$ are

$$Y_1 = \frac{1}{mr^2} \frac{\partial}{\partial \theta} - \frac{1}{J} \frac{\partial}{\partial \psi}, \quad Y_2 = \frac{1}{m} \frac{\partial}{\partial r}.$$

There are no constraints on the system, then as in the previous example $D = TM$, $\llbracket \cdot, \cdot \rrbracket_{TM}$ is the usual Lie bracket and $\rho_{TM} = \text{Id}_{TM}$.

The \mathcal{G}^D -orthogonal distribution to $\mathcal{D}_{(c)}$ is spanned by

$$Y_3 = \frac{\partial}{\partial \psi} + \frac{\partial}{\partial \theta}.$$

Note that $\{Y_1, Y_2, Y_3\}$ is a \mathcal{G}^D -orthogonal basis. From now on we consider coordinates adapted to this basis. Then the non-zero Christoffel symbols are:

$$\begin{aligned} \Gamma_{11}^2 &= -\frac{1}{mr^3}, & \Gamma_{12}^1 &= \frac{J}{mr(J+mr^2)}, & \Gamma_{12}^3 &= \frac{1}{mr(J+mr^2)}, \\ \Gamma_{13}^2 &= -\frac{1}{r}, & \Gamma_{21}^1 &= -\frac{J}{mr(J+mr^2)}, & \Gamma_{21}^3 &= -\frac{1}{mr(J+mr^2)}, \\ \Gamma_{23}^1 &= \frac{Jr}{J+mr^2}, & \Gamma_{23}^3 &= \frac{r}{J+mr^2}, & \Gamma_{31}^2 &= -\frac{1}{r}, \\ \Gamma_{32}^1 &= \frac{Jr}{J+mr^2}, & \Gamma_{32}^3 &= \frac{r}{J+mr^2}, & \Gamma_{33}^2 &= -rm. \end{aligned}$$

From here, it is easy to compute the following symmetric products:

$$\langle Y_1 : Y_1 \rangle_{\mathcal{G}^D} = -\frac{2}{mr^3} Y_2, \quad \langle Y_1 : Y_2 \rangle_{\mathcal{G}^D} = \langle Y_2 : Y_2 \rangle_{\mathcal{G}^D} = 0.$$

According to Theorem 5.14 the system is maximally reducible. Thus all the $\mathcal{C}^\infty(M)$ -linear combination of Y_1 and Y_2 are decoupling sections and we can apply Corollary 5.12 to identify those decoupling sections that are solutions to Hamilton-Jacobi differential equation:

$$\begin{aligned} 0 &= \mathcal{G}^D(\nabla_{Y_3}^{\mathcal{G}^D}(\alpha_1 Y_1 + \alpha_2 Y_2), \alpha_1 Y_1 + \alpha_2 Y_2) \\ &= \mathcal{G}^D(Y_1, Y_1) (\alpha_1 Y_3(\alpha_1) + \alpha_1 \alpha_2 \Gamma_{32}^1) + \mathcal{G}^D(Y_2, Y_2) (\alpha_2 Y_3(\alpha_2) + \alpha_1 \alpha_2 \Gamma_{31}^2). \end{aligned}$$

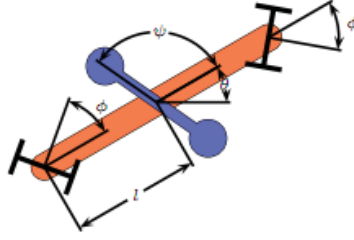
It can be proved that if $\alpha_i(r, \theta, \psi) = f_i(r, \theta - \psi)$ either for $i = 1$ or $i = 2$ and smooth functions $f_i: \mathbb{R}^+ \times \mathbb{S}^1 \rightarrow \mathbb{R}$, then the section $\alpha_1 Y_1 + \alpha_2 Y_2$ is a solution to Hamilton-Jacobi differential equation.

By Theorem 5.9, we can also check that

$$f(r, \psi - \theta) Y_1 + \alpha_3 Y_3, \quad f(r, \psi - \theta) Y_3$$

are solutions to Hamilton-Jacobi differential equation.

5.3.3. *Snakeboard*. We refer to [4, Section 13.4] and [21] for a detailed description of this system. The configuration manifold is $M = SE(2) \times \mathbb{S}^1 \times \mathbb{S}^1$ with local coordinates $(x, y, \theta, \psi, \phi)$. Consider the following physical parameters: the mass m_c of coupler, mass m_r of rotor, mass m_w of each wheel assembly, inertia J_c of coupler about center of mass, inertia J_r of rotor about center of mass, inertia J_w of wheel assembly about center of mass, distance l from coupler center of mass to wheel assembly.



The Riemannian metric in the given coordinates is

$$\mathcal{G} = (m_c + m_r + 2m_w)(dx \otimes dx + dy \otimes dy) + (J_c + J_r + 2(J_w + m_w l^2))d\theta \otimes d\theta + J_r d\psi \otimes d\psi + 2J_w d\phi \otimes d\phi + J_r(d\theta \otimes d\psi + d\psi \otimes d\theta).$$

The constraints require that the wheels of the snakeboard roll without slipping. These constraints define a nonholonomic distribution D . As computed in [4, Lemma 13.11] a \mathcal{G} -orthogonal basis for D on $M_D = \{(x, y, \theta, \psi, \phi) \mid \phi \neq \pm \frac{\pi}{2}\}$ is

$$\begin{aligned} X_1 &= l \cos \phi \left(\cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} \right) - \sin \phi \frac{\partial}{\partial \theta} = l \cos \phi V_1 - \sin \phi \frac{\partial}{\partial \theta}, \\ X_2 &= a(\phi) V_1 - b(\phi) \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \psi}, \\ X_3 &= \frac{\partial}{\partial \phi}, \end{aligned}$$

where

$$a(\phi) = \frac{J_r l \cos \phi \sin \phi}{c_1(\phi)}, \quad b(\phi) = \frac{J_r \sin^2 \phi}{c_1(\phi)}$$

and

$$c_1(\phi) = (m_c + m_r + 2m_w)l^2 \cos^2 \phi + (J_c + J_r + 2(J_w + m_w l^2)) \sin^2 \phi.$$

Let us define a skew-symmetric algebroid structure on the vector bundle $\tau_D: D \rightarrow M_D$. If $\{e_1, e_2, e_3\}$ is a local basis of $\Gamma(\tau_D)$, then

$$\rho_D(e_1) = X_1, \quad \rho_D(e_2) = X_2, \quad \rho_D(e_3) = X_3,$$

and

$$\begin{aligned} \llbracket e_1, e_1 \rrbracket_D &= \llbracket e_2, e_2 \rrbracket_D = \llbracket e_3, e_3 \rrbracket_D = 0, \\ \llbracket e_1, e_2 \rrbracket_D &= 0, \\ \llbracket e_1, e_3 \rrbracket_D &= \frac{\mathcal{G}([X_1, X_3], X_1)}{\mathcal{G}(X_1, X_1)} e_1 + \frac{\mathcal{G}([X_1, X_3], X_2)}{\mathcal{G}(X_2, X_2)} e_2, \\ \llbracket e_2, e_3 \rrbracket_D &= \frac{\mathcal{G}([X_2, X_3], X_1)}{\mathcal{G}(X_1, X_1)} e_1 + \frac{\mathcal{G}([X_2, X_3], X_2)}{\mathcal{G}(X_2, X_2)} e_2. \end{aligned}$$

Hence the non-vanishing local structure functions for the bracket $\llbracket \cdot, \cdot \rrbracket_D$ are

$$\mathcal{C}_{13}^1, \mathcal{C}_{13}^2, \mathcal{C}_{23}^1, \mathcal{C}_{23}^2.$$

The bundle metric on the skew-symmetric algebroid is given by $\mathcal{G}^D = \mathcal{G}|_{D \times_{M_D} D}$ as explained in Example 3.5. We can construct the Levi-Civita connection $\nabla^{\mathcal{G}^D}$ associated to the bundle metric \mathcal{G}^D having in mind its properties:

$$\begin{aligned} \Gamma_{BC}^A &= \frac{1}{2} g^{AA} (g_{CC} \mathcal{C}_{AB}^C + g_{AA} \mathcal{C}_{BC}^A + g_{BB} \mathcal{C}_{AC}^B) \\ &+ \frac{1}{2} g^{AA} \left(-(\rho_D)_A^i \frac{\partial}{\partial x^i} (g_{BC} \delta^{BC}) + (\rho_D)_B^i \frac{\partial}{\partial x^i} (g_{AC} \delta^{AC}) \right. \\ &\left. + (\rho_D)_C^i \frac{\partial}{\partial x^i} (g_{AB} \delta^{AB}) \right), \end{aligned}$$

for a \mathcal{G}^D -orthogonal basis, where $g^{AA} = 1/g_{AA}$.

The non-vanishing Christoffel symbols are

$$\begin{aligned} \Gamma_{13}^1 &= \mathcal{C}_{13}^1 + \frac{1}{2} g^{11} c'_1(\phi), & \Gamma_{23}^1 &= \frac{1}{2} \mathcal{C}_{23}^1 + \frac{1}{2} g^{11} g_{22} \mathcal{C}_{13}^2, \\ \Gamma_{31}^1 &= \frac{1}{2} g^{11} c'_1(\phi), & \Gamma_{32}^1 &= \frac{1}{2} g^{11} g_{22} \mathcal{C}_{13}^2 - \frac{1}{2} \mathcal{C}_{23}^1, \\ \Gamma_{13}^2 &= \frac{1}{2} g^{22} g_{11} \mathcal{C}_{23}^1 + \frac{1}{2} \mathcal{C}_{13}^2, & \Gamma_{23}^2 &= \frac{1}{2} g^{22} g'_{22}(\phi) + \mathcal{C}_{23}^2, \\ \Gamma_{31}^2 &= \frac{1}{2} g^{22} g_{11} \mathcal{C}_{23}^1 - \frac{1}{2} \mathcal{C}_{13}^2, & \Gamma_{32}^2 &= \frac{1}{2} g^{22} g'_{22}(\phi), \\ \Gamma_{11}^3 &= g^{33} g_{11} \mathcal{C}_{31}^1, & \Gamma_{12}^3 &= -\frac{1}{2} g^{33} (g_{22} \mathcal{C}_{13}^2 + g_{11} \mathcal{C}_{23}^1), \\ \Gamma_{21}^3 &= -\frac{1}{2} g^{33} (g_{11} \mathcal{C}_{23}^1 + g_{22} \mathcal{C}_{13}^2), & \Gamma_{22}^3 &= \frac{1}{2} g^{33} g_{22} \mathcal{C}_{32}^2. \end{aligned}$$

The mechanical control system on the skew-symmetric algebroid is given by $(D, \mathcal{G}^D, 0, \mathcal{D}_{(c)})$, where

$$\mathcal{D}_{(c)} = \left\{ \frac{c_1(\phi)}{J_r c_2(\phi)} X_2, \frac{1}{2J_w} X_3 \right\},$$

where $c_2(\phi) = (m_c + m_r + 2m_\omega) l^2 \cos^2 \phi + (J_c + 2(J_w + m_\omega l^2)) \sin^2 \phi$.

The control forces are torques, one actuating the rotor and the other one actuating the wheels. It is easy to prove that

$$\mathcal{D}_{(c)}^\perp = \{X_1\}.$$

In order to find solutions to Hamilton-Jacobi differential equation, let us first check if there exist any decoupling section of $\Gamma(\mathcal{D}_{(c)})$ so that we can use Corollary 5.12.

$$\begin{aligned}\langle X_2 : X_2 \rangle_{\mathcal{G}^D} &= 2\Gamma_{22}^3 X_3 \in \mathcal{D}_{(c)}, \\ \langle X_3 : X_3 \rangle_{\mathcal{G}^D} &= 0 \in \mathcal{D}_{(c)}.\end{aligned}$$

Hence both control sections are decoupling. As in the previous example the mechanical control system is not maximally reducible to a driftless system because

$$\langle X_2 : X_3 \rangle_{\mathcal{G}^D} = (\Gamma_{23}^1 + \Gamma_{32}^1)X_1 + (\Gamma_{23}^2 + \Gamma_{32}^2)X_2 \notin \mathcal{D}_{(c)}.$$

Let us check if the decoupling sections satisfy condition (ii) in Corollary 5.12.

$$\begin{aligned}\mathcal{G}^D(\nabla_{X_1}^{\mathcal{G}^D} X_2, X_2) &= \Gamma_{12}^3 \mathcal{G}^D(X_3, X_2) = 0, \\ \mathcal{G}^D(\nabla_{X_1}^{\mathcal{G}^D} X_3, X_3) &= \Gamma_{13}^1 \mathcal{G}^D(X_1, X_3) + \Gamma_{13}^2 \mathcal{G}^D(X_2, X_3) = 0\end{aligned}$$

because $\{X_1, X_2, X_3\}$ is \mathcal{G}^D -orthogonal basis. Thus X_2 and X_3 are both solutions to Hamilton-Jacobi differential equation because of Corollary 5.12. Let us see if these solutions can generate more decoupling sections being solution to Hamilton-Jacobi differential equation. As proved in Proposition 5.11, fX_2 and fX_3 with $f \in \mathcal{C}^\infty(M)$ are solutions to Hamilton-Jacobi differential equation if and only if

$$X_1(f) = 0 \Leftrightarrow l \cos \phi \left(\cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y} \right) - \sin \phi \frac{\partial f}{\partial \theta} = 0.$$

If we take any function

$$f(x, y, \theta, \psi, \phi) = g \left(\frac{2x \sin \phi}{l} + 2 \cos \phi \sin \theta, y - l \frac{\cos \phi}{\sin \phi} \cos \theta, \psi, \phi \right),$$

where $g: \mathbb{R}^4 \rightarrow \mathbb{R}$ is a solution to the partial differential equation $X_1(f) = 0$, then fX_2 and fX_3 are also solutions to Hamilton-Jacobi differential equation.

Consider now a general section $\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3$ in $\Gamma(\tau_D)$. Condition $i_X d^D(b_{\mathcal{G}^D}(X))(Y) = 0$ for all $Y \in \mathcal{D}_{(c)}^\perp$ in Theorem 5.9 becomes

$$\begin{aligned}\mathcal{G}^D(X_1, X_1)(\alpha_2 X_2(\alpha_1) + \alpha_3 X_3(\alpha_1) + \alpha_3 \alpha_1 \Gamma_{31}^1 + \alpha_2 \alpha_3 \Gamma_{23}^1 + \alpha_3 \alpha_2 \Gamma_{32}^1) \\ - \mathcal{G}^D(X_2, X_2)(\alpha_2 X_1(\alpha_2) + \alpha_3 \alpha_2 \Gamma_{13}^2) \\ - \mathcal{G}^D(X_3, X_3)(\alpha_3 X_1(\alpha_3) + \alpha_1 \alpha_3 \Gamma_{11}^3 + \alpha_3 \alpha_2 \Gamma_{12}^3) = 0.\end{aligned}$$

For instance, for any $\alpha_1 \in \mathcal{C}^\infty(M)$ $\alpha_1 X_1$ satisfies this condition. It will be a solution to Hamilton-Jacobi differential equation if it fulfills (5.4),

that is,

$$\begin{aligned} 0 &= \mathcal{G}^D(\nabla_{\alpha_1 X_1}^{\mathcal{G}^D} \alpha_1 X_1, X_1) = \alpha_1 X_1(\alpha_1) \mathcal{G}^D(X_1, X_1) + \alpha_1^2 \Gamma_{11}^3 \mathcal{G}^D(X_3, X_1) \\ &= \alpha_1 X_1(\alpha_1) \mathcal{G}^D(X_1, X_1). \end{aligned}$$

Equivalently, $X_1(\alpha_1) = 0$. Thus $\alpha_1 X_1$ is also a solution to Hamilton-Jacobi differential equation for any

$$\alpha_1(x, y, \theta, \psi, \phi) = g \left(\frac{2x \sin \phi}{l} + 2 \cos \phi \sin \theta, y - l \frac{\cos \phi}{\sin \phi} \cos \theta, \psi, \phi \right)$$

where $g: \mathbb{R}^4 \rightarrow \mathbb{R}$.

6. FUTURE WORK

The results in this paper extend the notion of decoupling sections for mechanical systems with nonzero potential. A decoupling section for those mechanical systems is a section that satisfies the assumption and condition (ii) in Theorem 5.9. The future research line consists of taking advantage of this geometric description of decoupling sections to do motion planning for mechanical systems with nonzero potential. One of the key points to succeed in motion planning is that not any reparametrization of decoupling sections in the sense of Theorem 5.9 is again decoupling in the sense of Theorem 5.9. In order to define the suitable notion of reparametrization Proposition 5.11 seems to be useful.

REFERENCES

- [1] R. Abraham, J.E. Marsden: *Foundations of Mechanics*, Second Edition, Benjamin, New York, 1978.
- [2] P. Balseiro, J.C. Marrero, D. Martín de Diego, E. Padrón: Unified framework for mechanics. Hamilton-Jacobi equation and applications, *Nonlinearity* **23** (2010), 1887–1918
- [3] M. Barbero-Liñán, M.C. Muñoz Lecanda. Strict abnormal extremals in nonholonomic and kinematic control systems, Special issue “Nonholonomic constraints in Mechanics and Optimal Control Theory” in *Discrete Contin. Dyn. Syst. Ser. S* 3(1), 2010.
- [4] F. Bullo, A.D. Lewis: *Geometric Control of Mechanical Systems: Modeling, Analysis, and Design for Simple Mechanical Control Systems*. *Texts in Applied Mathematics*, Springer Verlag, New York 2005.
- [5] J. F. Cariñena, X. Gracia, G. Marmo, E. Martínez, M. C. Muñoz-Lecanda, N. Roman-Roy: Geometric Hamilton-Jacobi theory. *Int. J. Geom. Methods Mod. Phys.* **3** (7) (2006), 1417–1458.
- [6] J. F. Cariñena, X. Gracia, G. Marmo, E. Martínez, M. C. Muñoz-Lecanda, N. Roman-Roy: Geometric Hamilton-Jacobi Theory for Nonholonomic Dynamical Systems, *Int. J. Geom. Methods Mod. Phys.* **7** (3) (2010), 431–454.

- [7] J. Cortés, M. de León, J.C. Marrero, D. Martín de Diego, E. Martínez: A survey of Lagrangian mechanics and control on Lie algebroids and groupoids, *Int. J. Geom. Methods Mod. Phys.* **3** (2006), no. 3, 509–558.
- [8] J. Cortés, M. de León, J.C. Marrero, E. Martínez: Nonholonomic Lagrangian systems on Lie algebroids, *Discrete and Continuous Dynamical Systems: Series A*, **24** (2) (2009), 213–271.
- [9] J. Cortés, E. Martínez: Mechanical control systems on Lie algebroids, *IMA J. Math. Control. Inform.* **21** (2004), 457–492.
- [10] K. Grabowska, J. Grabowski: Variational calculus with constraints on general algebroids, *J. Phys. A: Math Theoret.*, **41** (2008), 175–204.
- [11] K. Grabowska, J. Grabowski, P. Urbański: Geometrical mechanics on algebroids, *Int. J. Geom. Methods Mod. Phys.* **3** (3) (2006), 559–575.
- [12] J. Grabowski, M. de León, J.C. Marrero, D. Martín de Diego: Nonholonomic constraints: a new viewpoint, *J. Math. Phys.* **50** (2009), no. 1, 013520, (17 pp)
- [13] D. Iglesias, M. de León, D. Martín de Diego: Towards a Hamilton-Jacobi Theory for Nonholonomic Mechanical Systems, *J. Phys. A: Math. Theor.* **41** (2008) 015205 (14pp).
- [14] M. de León, J.C. Marrero, D. Martín de Diego: Linear almost Poisson structures and Hamilton-Jacobi theory. Applications to nonholonomic Mechanics, *Journal of Geometric Mechanics* **2** (2) (2010), 159–198
- [15] M. de León, J.C. Marrero, E. Martínez: Lagrangian submanifolds and dynamics on Lie algebroids, *J. Phys. A: Math. Gen.* **38** (2005), R241–R308.
- [16] M. de León, P.R. Rodrigues: *Methods of Differential Geometry in Analytical Mechanics*, North Holland Math. Series **152** (Amsterdam, 1996).
- [17] P. Libermann: Lie algebroids and Mechanics, *Arch. Math. (Brno)*, **32** (1996), 147–162.
- [18] K. Mackenzie: *General Theory of Lie groupoids and Lie algebroids in differential geometry*, London Mathematical Society Lecture Note Series, No. 124, 2005.
- [19] J.C. Marrero, D. Sosa: The Hamilton-Jacobi equation on Lie affgebroids, *Int. J. Geom. Meth. Mod. Phys.* **3** (3) (2006) 605–622.
- [20] E. Martínez: Lagrangian mechanics on Lie algebroids, *Acta Appl. Math.* **67** (2001), no. 3, 295–320.
- [21] M.C. Muñoz-Lecanda, F.J. Yañiz-Fernández: Mechanical control systems and kinematic systems. *IEEE Trans. Automat. Control* **53** (5) (2008), 1297–1302.
- [22] T. Ohsawa, A.M. Bloch: Nonholonomic Hamilton-Jacobi equation and Integrability, *Journal of Geometric Mechanics*, **1**(4)(2009), 461–481.
- [23] M. Popescu, P. Popescu: Geometric objects defined by almost Lie structures, *Proc. Workshop on Lie algebroids and related topics in Differential Geometry (Warsaw)* **54** (Warsaw: Banach Center Publications) (2001), 217–233.
- [24] H.J. Sussmann: Orbits of families of vector fields and integrability of distributions, *Transactions of the American Mathematical Society*, **180**, (1973), 171–188.
- [25] A. Weinstein: Lagrangian Mechanics and Groupoids, in “*Mechanics day*” (Waterloo, ON, 1992), *Fields Institute Communications* **7**, American Mathematical Society, (1996), 207–231.

M. BARBERO-LIÑÁN: INSTITUTO DE CIENCIAS MATEMÁTICAS, CSIC-UAM-UC3M-UCM, C/NICOLÁS CABRERA 13-15, 28049 MADRID, SPAIN

E-mail address: mbarbero@icmat.es

M. DE LEÓN: INSTITUTO DE CIENCIAS MATEMÁTICAS, CSIC-UAM-UC3M-UCM, C/NICOLÁS CABRERA 13-15, 28049 MADRID, SPAIN

E-mail address: mdeleon@icmat.es

JUAN C. MARRERO: UNIDAD ASOCIADA ULL-CSIC, GEOMETRÍA DIFERENCIAL Y MECÁNICA GEOMÉTRICA, DEPARTAMENTO DE MATEMÁTICA FUNDAMENTAL, FACULTAD DE MATEMÁTICAS, UNIVERSIDAD DE LA LAGUNA, LA LAGUNA, TENERIFE, CANARY ISLANDS, SPAIN

E-mail address: jcmarrer@ull.es

D. MARTÍN DE DIEGO: INSTITUTO DE CIENCIAS MATEMÁTICAS, CSIC-UAM-UC3M-UCM, C/NICOLÁS CABRERA 13-15, 28049 MADRID, SPAIN

E-mail address: david.martin@icmat.es

MIGUEL C. MUÑOZ-LECANDA: DEPARTAMENTO DE MATEMÁTICA APLICADA IV, EDIFICIO C-3, CAMPUS NORTE UPC, C/ JORDI GIRONA 1. 08034 BARCELONA, SPAIN

E-mail address: matmcm1@ma4.upc.edu