GEOMETRIC APPROACH TO PONTRYAGIN'S MAXIMUM PRINCIPLE

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Abstract

Since the second half of the 20th century, Pontryagin's Maximum Principle has been widely discussed and used as a method to solve optimal control problems in medicine, robotics, finance, engineering, astronomy. Here, we focus on the proof and on the understanding of this Principle, using as much geometric ideas and geometric tools as possible. This approach provides a better and clearer understanding of the Principle and, in particular, of the role of the abnormal extremals. These extremals are interesting because they do not depend on the cost function, but only on the control system. Moreover, they were discarded as solutions until the nineties, when examples of strict abnormal optimal curves were found. In order to give a detailed exposition of the proof, the paper is mostly self–contained, which forces us to consider different areas in mathematics such as algebra, analysis, geometry.

Key words: *Pontryagin's Maximum Principle, perturbation vectors, tangent perturbation cones, optimal control problems.*

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1 Introduction

The importance of Pontryagin's Maximum Principle as a method to find solutions to optimal control problems is the main justification for this work. The use and the comprehension of this Principle does not always gather together. The understanding of this Maximum Principle never finishes as shows the continuous wide number of references in this topic [2, 8, 16, 18, 19, 24, 44, 51, 52, 53, 64, 70, 71, 72] and references therein. We try to contribute to this process through a differential geometric approach.

When we can interfere in the evolution of a dynamical system, we deal with a control system; that is, a differential equation depending on parameters, which are called controls. The way we interfere in the control system consists of changing the controls arbitrarily. In optimal control problems, the controls are chosen such that the integral of a given cost function is minimized. That functional to be minimized can correspond with the time, the energy, the length of a path or other magnitude related to the system.

In general, to find a solution to an optimal control problem is not straightforward. A valuable tactic to deal with these problems is to restrict the candidates to be solution through necessary conditions for optimality, such as those given by Pontryagin's Maximum Principle. This technique is used in a wide range of disciplines, as for instance engineering [28, 30, 38, 55], aerospace [69], robotics [35, 37, 62], medicine [47], economics [45, 57], traffic flow [36]. Nevertheless, it is worth remarking that the Maximum Principle does not give sufficient conditions to compute an optimal trajectory; it only provides necessary conditions. Thus only candidates to be optimal trajectories are found, called extremals. To determine if they are optimal or not, other results related to the existence of solutions for these problems are needed. See [2, 8, 34, 53] for more details.

In 1958 the International Congress of Mathematicians was held in Edinburgh, Scotland, where for the first time L. S. Pontryagin talked publicly about the Maximum Principle. This Principle was developed by a research group on automatic control created by Pontryagin in the fifties. He was engaged in applied mathematics by his friend A. Andronov and because scientists in the Steklov Mathematical Institute were asked to carry out applied research, especially in the field of aircraft dynamics.

At the same time, in the regular seminars on automatic control in the Institute of Automatics and Telemechanics, A. Feldbaum introduced Pontryagin and his collegues to the time–optimization problem. This allowed them to study how to find the best way of piloting an aircraft in order to defeat a zenith fire point in the shortest time as a time–optimization problem.

Since the equations for modelling the aircraft's problem are nonlinear and the control of the rear end of the aircraft runs over a bounded subset, it was necessary to reformulate the calculus of variations known at that time. Taking into account ideas suggested by E. J. McShane in [58], Pontryagin and his collaborators managed to state and prove the Maximum Principle, which was published in Russian in 1961 and translated into English [64] the following year. See [17] for more historical remarks.

Initially the approach to optimal control problems was from the point of view of the differential equations [8, 53, 64, 76], but later the approach was from the differential geometry [2, 21, 44, 70]. Furthermore, the Maximum Principle is being modified to study stochastic control systems [12, 42] and discrete control systems [29, 39, 43]. Lately, the Skinner–Rusk formulation [67] has been applied to study optimal control problem for non–autonomous control systems, obtaining again the necessary conditions of Pontryagin's Maximum Principle, as long as the differentiability with respect to controls is assumed [9]. This formulation is suitable to deal with implicit optimal control problems that come up in engineering problems described by the descriptor systems [60, 61]. This Principle also admits a presymplectic formalism that gives weaker necessary conditions for optimality [11, 32, 33].

Therefore, it is concluded that Pontryagin's Maximum Principle has had and still has a great impact in optimal control theory. The references mentioned show that the research is still active as for the understanding and also for the applications of the Maximum Principle.

A symplectic Hamiltonian formalism to optimal control problems is provided by the necessary conditions stated in Pontryagin's Maximum Principle. The solutions to the problem are in the phase space manifold of the system, but the Maximum Principle relates solutions to a lift to the cotangent bundle of that manifold. Thus, in order to find candidates to be optimal solutions, not only the controls but also the momenta must be chosen appropriately so that the necessary conditions in the Maximum Principle are fulfilled. These conditions are, in fact, first–order necessary conditions and they are not always enough to determine the evolution of all the degrees of freedom in the problem. That is why sometimes it is necessary to use the high order Maximum Principle [14, 46, 48, 49]. But, even when we succeed in finding the controls and the momenta in such a way that Hamilton's equations can be integrated to obtain a trajectory on the manifold, the controls and the momenta are not necessarily unique. In other words, different controls and different momenta can give the same trajectory on the manifold, although the necessary conditions in the Maximum Principle will be satisfied in different ways. The momenta and the controls determine different kinds of trajectories, which can be abnormal, normal, strict abnormal, strict normal and singular. We point out that these different kinds of extremals do not provide a partition of the set of trajectories in the manifold, because it may happen that a trajectory admits more than one lift to the momenta space so that the trajectory is in two different categories.

For years, abnormal extremals were discarded because it was thought that they could not be optimal [41, 68]. The idea was that abnormal extremals were isolated curves and thus it was impossible to consider any variation of these curves. However, in [59] it is proved that there exist abnormal minimizers by giving an example in subRiemannian geometry. Furthermore, in [56] the strict abnormal minimizers are characterized in a general way, studying the length–minimizing problem in subRiemannian geometry when there are only two controls. To be more precise, a large enough set with abnormal extremals is given and it contains strict abnormal curves that are locally optimal for the considered control–linear system. Here began a new interest in the abnormal extremals [3, 5, 6, 20, 51, 52]. What makes these extremals more special is that the abnormality does not depend on the cost function. Hence, the abnormal extremals can be determined exclusively using the geometry of the control system. Thus abnormality and controllability must be closely related. In fact, in order to have abnormal minimizers, the system cannot be controllable. In control theory, controllability is still one of the properties under active research [1, 7] and the same happens with abnormality in optimal control theory. Moreover, the controllability is related with the reachable set. Thus, as first pointed out in [24, 52], the geometry of the reachable set also helps to characterize the abnormal extremals.

On the other hand, the cost function is essential to prove that abnormal extremals are abnormal minimizers, as pointed out in §3.4. That is why the existence or non–existence of abnormal minimizers is only known for specific control problems, mainly control–linear and control–affine systems with control–quadratic cost functions or for time–optimal control problems [4, 6, 19, 26, 27, 77].

How the necessary conditions of Pontryagin's Maximum Principle are satisfied determines the kind of extremals obtained, in particular, the abnormal ones. That is why the thorough proof of the Maximum Principle given here gives insights into the geometric understanding of the abnormality. Any chance we have along the report to make a comment about abnormality will be made because that might help to characterize strict abnormality in the future.

In this paper, we go through the entire proof of Pontryagin's Maximum Principle translating it into a geometric framework, but preserving the outline of the original proof. All details have been carefully proved, making us to go into the details of concepts such as time–dependent variational equations and their properties, separation conditions given by hyperplanes and convexity. All this is included as appendices in order not to disturb the continuous evolution of the concepts here given. Nevertheless, we assume some knowledge in differential geometry, such as the core chapters of [54], differential equations [25, 31, 40], and convexity [13, 65].

The control systems in this report are given by a vector field along a projection, that is defined in §2 together with its properties. In the heart of the report there are two big parts corresponding with two different statements of Pontryagin's Maximum Principle. In §3 and §4, it is studied the optimal control problem with both the time interval and the endpoints given. If the final time is not given and the endpoints are not fixed but they must be in specific submanifolds, then the problem is studied in §5 and §6. These four sections have been written in an analogous way. First of all, two different but equivalent statements of the optimal control problems are given. The so–called extended system is the useful one in §4 and §6 because the functional to be minimized is included as a new coordinate of the system. The last subsection in §3 and §5 explains the associated Hamiltonian problem that leads to the statements of Maximum Principle. In this way, the proof is just in §4 and §6.

One part of the proof of Pontryagin's Maximum Principle consists of perturbing the given optimal curve, therefore we introduce in §3.3 and §5.2 how this curve can be perturbed depending on the known data. Above all, it is important the complete proof of Proposition 3.12, although known, to our knowledge, there is not a self–contained proof of it in the literature.

The appendices contain essential results for the core of the report and also some explanation to make clear some well–known ideas related to time–dependent vector fields in Appendix B, the reachable set and the tangent perturbation cone in Appendix C.

The study of the time–dependent variational equations treated in Appendix B gives a clear picture of the flows of the complete lift and of the cotangent lift of a time–dependent vector field via Propositions B.1, B.2, B.4, B.5. These results although known, to our knowledge, have not appeared in the literature.

Appendix C devotes to the careful study of the connection between the reachable set and the tangent perturbation cone, because the proof of Pontryagin's Maximum Principle suggests that all the perturbation vectors generate a linear approximation of the reachable set in some sense. That sense will become clear in Proposition C.1, which proves a result assumed as true in the literature.

To summarize the main contributions of the paper are:

- The proof of Proposition 3.12, that is useful to prove Pontryagin's Maximum Principle. All the proofs of this Proposition in the literature, to our knowledge, are not written carefully enough. This Proposition is adapted for Pontryagin's Maximum Principle without fixing the final time in Proposition 5.9.

- The complete proof of Pontryagin's Maximum Principle in a symplectic framework as in [70], but here we include all the necessary results and the analytical reasoning, which has been sketched in great detail.

- The highlight of the properties concerning the abnormal extremals that can be deduced from the classical result in [64].
- An analytic result necessary in Pontryagin's Maximum Principle, which is proved in Proposition A.7. This result is used in [64], but without proving it.
- The intrinsic study of the flows of the complete lift and the cotangent lift of a time–dependent vector field in Appendix B, including the proofs of Propositions B.1, B.2, B.4 and B.5.
- The geometric understanding of the interpretation of the tangent perturbation cone as linear approximation of the reachable set in Appendix C, including the proof of Proposition C.1.

As for the future and actual research line, we point out that all the effort to elaborate this work is being used to enlighten the research, from a geometric point of view, on abnormal and strict abnormal extremals in optimal control problems in general [10], and for mechanical systems [11]. The study of strict abnormal minimizers impose us to consider different cost functions, because only the property of being an abnormal extremal depends exclusively on the geometry of the control system. That makes the problem of searching for strict abnormal minimizers much harder and the possible forthcoming results will be valid only for determined optimal control problems.

To conclude this introduction, we remark that Pontryagin's Maximum Principle provides first–order necessary conditions for optimality. These conditions are not always enough to determine the controls for abnormal and singular extremals, then high order Maximum Principle is necessary [49]. The Maximum Principle works with linear approximation of the trajectories, whereas in the high order Maximum Principle high order perturbations must be considered [14, 15, 46, 48, 49]. The way to construct the proof is the same as in Pontryagin's Maximum Principle, but now the tangent perturbation cones are bigger since not only linear approximation of the trajectories are considered. In the same way we have provided a geometric meaning to most of the elements in Pontryagin's Maximum Principle, we expect to give a geometric version of high order Maximum Principle suggested by [49], focusing on abnormality.

The origin of this report was a series of seminars and talks with Professor Andrew D. Lewis during his stay in our Department on sabbatical during the first term of 2005. We tried to understand the details of the proof as a way to work on some aspects of controllability and accesibility of control systems with a cost function, [24, 44], and where abnormal solutions are in the accesibility sets.

In the sequel, unless otherwise stated, all the manifolds are real, second countable and $\mathcal{C}^\infty$ and the maps are assumed to be $\mathcal{C}^\infty$. Sum over repeated indices is understood.

2 General setting

From the differential geometric viewpoint a control system is understood as a vector field depending on parameters. Properties about how the integral curves of differential equations depending on parameters evolve are explained in [25, 31, 40, 50] and used in §3.3 and §5.2.

Let M be a differentiable manifold of dimension m and U be a set in $\mathbb{R}^k$. Consider the trivial Euclidean bundle $\pi\colon M \times U \to M$.

Definition 2.1. *A* ***vector field*** X ***on*** M ***along the projection*** π *is a mapping* $X\colon M \times U \to TM$ *such that* X *is continuous on* $M \times U$*, continuously differentiable on* M *for every* $u \in U$ *and* $\tau_M \circ X = \pi$*, where* $\tau_M\colon TM \to M$ *is the canonical tangent projection.*

The set of vector fields along the projection π is denoted by $\mathfrak{X}(\pi)$. If (V, x^i) is a local chart at x in M, then locally a vector field X along the projection is given by $f^i \partial/\partial x^i$, where f^i are functions defined on $V \times U$.

Let $I = [a, b] \subset \mathbb{R}$ be a closed interval, $(\gamma, u)\colon I \to M \times U$ is an integral curve of X if $\dot{\gamma}(t) = X(\gamma(t), u(t))$. All these elements come together in Diagram (2.1).

$$\begin{array}{ccc} & & TM \\ & \overset{X}{\nearrow} & \downarrow{\scriptstyle \tau_M} \\ M \times U & \xrightarrow{\pi} & M \\ {\scriptstyle (\gamma,u)}\uparrow & \overset{\gamma}{\nearrow} & \\ I & & \end{array} \tag{2.1}$$

In other words, X is a vector field depending on parameters in U. In this work, the parameters are called *controls* and are assumed to be measurable mappings $u\colon I \to U$ such that $\operatorname{Im} u$ is bounded. Given the parameter u, we have a time–dependent vector field on M,

$$\begin{aligned} X^{\{u\}}\colon I \times M &\longrightarrow TM \\ (t, x) &\longmapsto X^{\{u\}}(t, x) = X(x, u(t)). \end{aligned} \tag{2.2}$$

For an integral curve (γ, u) of X, it is said that γ is an integral curve of $X^{\{u\}}$, as shown in the following commutative diagram:

$$\begin{array}{ccc} I \times M & \xrightarrow{X^{\{u\}}} & TM \\ {\scriptstyle (\gamma,\,\mathrm{Id})}\uparrow & \overset{\dot{\gamma}}{\nearrow} & \uparrow{\scriptstyle X} \\ I & \xrightarrow[(\gamma,u)]{} & M \times U \end{array} \tag{2.3}$$

That is, $X^{\{u\}} \circ (\gamma, \mathrm{Id}) = \dot{\gamma} = X \circ (\gamma, u)$.

A differentiable time–dependent vector field X has associated the *time dependent flow* or *evolution operator of* X defined as

$$\begin{aligned} \Phi^X\colon \quad I \times I \times M &\longrightarrow M \\ (t, s, x) &\longmapsto \Phi^X(t, s, x) = \Phi^X_{(s,x)}(t) \end{aligned}$$

where $\Phi^X_{(s,x)}$ is the integral curve of X with initial condition x at time s. See Appendix B.1 for more details. Moreover, the evolution operator defines a diffeomorphism on M that is used in the following section $\Phi^X_{(t,s)}\colon M \to M$, $x \mapsto \Phi^X_{(t,s)}(x) = \Phi^X_{(s,x)}(t)$.

As the controls $u\colon I \to U$ are measurable and bounded, the vector fields $X^{\{u\}}$ are measurable on t, and for a fixed t, they are differentiable on M. Hence, the notion of Carathéodory vector fields must be considered [25, 31] from now on. Then, we only consider absolutely continuous curves $\gamma\colon I \to M$ to be *generalized integral curves* of the vector field $X^{\{u\}}$; that is, they only satisfy $\dot{\gamma} = X \circ (\gamma, u)$ at points where γ is derivable, which happens almost everywhere. The existence and uniqueness of these integral curves are guaranteed once the parameter is fixed because of the theorems of existence and uniqueness of differential equations depending on parameters. For more details about absolute continuity, see Appendix A and [25, 31, 74].

3 Pontryagin's Maximum Principle for fixed time and fixed endpoints

We particularize the general setting described in §2 for optimal control theory. To make clear we are in a specific case the manifold is denoted by Q, instead of M.

3.1 Statement of optimal control problem and notation

Let Q be a differentiable manifold of dimension m and $U \subset \mathbb{R}^k$ a subset. Let us consider the trivial Euclidean bundle $\pi\colon Q \times U \to Q$.

Let X be a vector field along the projection $\pi\colon Q \times U \to Q$ as in Definition 2.1. If (V, x^i) is a local chart at a point in Q, the local expression of the vector field is $X = f^i \partial/\partial x^i$ where f^i are functions defined on $V \times U$.

Let $I \subset \mathbb{R}$ be an interval and $(\gamma, u)\colon I \to Q \times U$ be a curve. Given $F\colon Q \times U \to \mathbb{R}$, let us consider the functional

$$\mathcal{S}[\gamma, u] = \int_I F(\gamma, u)\, dt$$

defined on curves (γ, u) with a compact interval as domain. The function $F\colon Q \times U \to \mathbb{R}$ is continuous on $Q \times U$ and continuously differentiable with respect to Q on $Q \times U$.

Statement 3.1. ***(Optimal Control Problem,*** *OCP) Given the elements Q, U, X, F, $I = [a, b]$ and the endpoint conditions x_a, $x_b \in Q$, consider the following problem.*

Find (γ^*, u^*) such that

(1) endpoint conditions: $\gamma^*(a) = x_a$, $\gamma^*(b) = x_b$,

(2) γ^* is an integral curve of $X^{\{u^*\}}$: $\dot{\gamma}^*(t) = X(\gamma^*(t), u^*(t))$, $t \in I$, and

(3) minimal condition: $\mathcal{S}[\gamma^*, u^*]$ is minimum over all curves (γ, u) satisfying (1) *and* (2).

The tuple $(Q, U, X, F, I, x_a, x_b)$ denotes the *optimal control problem.* The function F is called the *cost function* of the problem. The mappings $u\colon I \to U$ are called *controls.*

Comments:

1. The curves considered in the previous statement satisfy the same properties as the generalized integral curves of vector fields along a projection described in §2. That is, γ is absolutely continuous and the controls u are measurable and bounded.

2. Locally, condition (2) is equivalent to the fact that the curve (γ^*, u^*) satisfies the differential equation $\dot{x}^i = f^i$.

3.2 The extended problem

Taking into account the elements defining the optimal control problem and their properties, we state an equivalent problem.

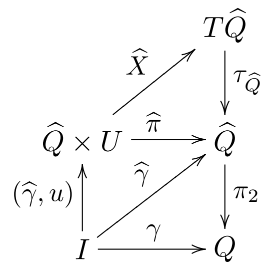
Given the *OCP* $(Q, U, X, F, I, x_a, x_b)$, let us consider $\widehat{Q} = \mathbb{R} \times Q$ and the trivial Euclidean bundle $\widehat{\pi}\colon \widehat{Q} \times U \to \widehat{Q}$.

Let $\widehat{X}$ be the following vector field along the projection $\widehat{\pi}\colon \widehat{Q} \times U \to \widehat{Q}$:

$$\widehat{X}(x^0, x, u) = F(x, u)\partial/\partial x^0|_{(x^0, x, u)} + X(x, u),$$

where x^0 is the natural coordinate on $\mathbb{R}$. According to Equation (2.2), this vector field can be rewritten as $\widehat{X}^{\{u\}}$.

Given a curve $(\widehat{\gamma}, u) = ((x^0 \circ \widehat{\gamma}, \gamma), u) \colon I \to \widehat{Q} \times U$ such that $\widehat{\gamma}$ is absolutely continuous and u is measurable and bounded, the previous elements come together in the following diagram:



where π_2 is the projection of $\widehat{Q}$ onto Q.

Statement 3.2. ***(Extended Optimal Control Problem, $\widehat{\textbf{OCP}}$)*** *Given the above–mentioned OCP $(Q, U, X, F, I, x_a, x_b)$, $\widehat{Q}$ and $\widehat{X}$, consider the following problem.*

Find $(\widehat{\gamma}^*, u^*)$ such that

(1) endpoint conditions: $\widehat{\gamma}^*(a) = (0, x_a)$, $\gamma^*(b) = x_b$,

(2) $\widehat{\gamma}^*$ is an integral curve of $\widehat{X}^{\{u^*\}}$: $\dot{\widehat{\gamma}}^*(t) = \widehat{X}(\widehat{\gamma}^*(t), u^*(t))$, $t \in I$, and

(3) minimal condition: ${\gamma^*}^0(b)$ is minimum over all curves $(\widehat{\gamma}, u)$ satisfying (1) *and* (2)*.*

The tuple $(\widehat{Q}, U, \widehat{X}, I, x_a, x_b)$ denotes the *extended optimal control problem.*

1. The functional ${\gamma^*}^0(b)$ to be minimized in the $\widehat{OCP}$ is equal to the functional defined in the OCP. That is to say, we have

$$\widehat{\mathcal{S}}[\widehat{\gamma}, u] = \gamma^0(b) = \int_a^b F(\gamma, u) dt = \mathcal{S}[\gamma, u]$$

   for curves $(\widehat{\gamma}, u)$.

2. Locally, the condition (2) is equivalent to the fact that the curve $(\widehat{\gamma}^*, u^*)$ satisfies the differential equations $\dot{x}^0 = F$, $\dot{x}^i = f^i$.

The elements in the problem $(\widehat{M}, U, \widehat{X}, I, x_a, x_b)$ satisfy properties analogous to the ones fulfilled by the elements in the problem $(M, U, X, F, I, x_a, x_b)$, but for different spaces; see §2, §3.1 for more details about the properties.

3.3 Perturbation and associated cones

The following constructions can be defined for any vector field depending on parameters—see §2—in particular, for those vector fields defining a control system. In order not to make the notation harder, we will construct everything on M, but the same can be done on $\widehat{M}$ or on any other convenient manifold, as for instance the tangent bundle TQ for the mechanical case.

3.3.1 Elementary perturbation vectors: class I

Now we study how integral curves of the time–dependent vector field $X^{\{u\}}\colon M \times I \to TM$, introduced in §2, change when the control u is perturbed in a small interval.

In the sequel, a measurable and bounded control $u\colon I = [a,b] \to U$ and an absolutely continuous integral curve $\gamma\colon I \to M$ of $X^{\{u\}}$ are given. Let $\pi_1 = \{t_1, l_1, u_1\}$, where t_1 is a Lebesgue time in (a,b) always for the $X \circ (\gamma, u)$—i.e. it satisfies Equation (A.17)—$l_1 \in \mathbb{R}^+$, $u_1 \in U$. From now on, to simplify, t_1 is called just a Lebesgue time. For every $s \in \mathbb{R}^+$ small enough such that $a < t_1 - l_1 s$, consider $u[\pi_1^s]\colon I \to U$ defined by

$$u[\pi_1^s](t) = \begin{cases} u_1, & t \in [t_1 - l_1 s, t_1], \\ u(t), & \text{elsewhere.} \end{cases}$$

Definition 3.3. *The function $u[\pi_1^s]$ is called an **elementary perturbation of** u **specified by the data** $\pi_1 = \{t_1, l_1, u_1\}$. It is also called a **needle–like variation**.*

Associated to $u[\pi_1^s]$, consider the mapping $\gamma[\pi_1^s]\colon I \to M$, the generalized integral curve of $X^{\{u[\pi_1^s]\}}$ with initial condition $(a, \gamma(a))$.

Given $\epsilon > 0$, define the map

$$\begin{array}{rcl} \varphi_{\pi_1}\colon I \times [0,\epsilon] & \longrightarrow & M \\ (\ t\ ,\ s\ ) & \longmapsto & \varphi_{\pi_1}(t,s) = \gamma[\pi_1^s](t) \end{array}$$

For every $t \in I$, $\varphi^t_{\pi_1}\colon [0,\epsilon] \to M$ is given by $\varphi^t_{\pi_1}(s) = \varphi_{\pi_1}(t,s)$.

As the controls are assumed to be measurable and bounded, it makes sense to define the distance between two controls $u, \overline{u}\colon I \to U$ as follows

$$d(u, \overline{u}) = \int_I \|u(t) - \overline{u}(t)\| \ dt$$

where $\|\cdot\|$ is the usual norm in $\mathbb{R}^k$. Here, a bounded control $u\colon I \to U$ means that there exists a compact set in U that contains Im u. The control $u[\pi_1^s]$ depends continuously on the parameters s and $\pi_1 = \{t_1, l_1, u_1\}$; that is, given $\epsilon > 0$ there exists $\delta > 0$ such that if $|t_1 - t_2| < \delta$, $|l_1 - l_2| < \delta$, $\|u_1 - u_2\| < \delta$, $|s_1 - s_2| < \delta$, then $d(u[\pi_1^{s_1}], u[\pi_2^{s_2}]) < \epsilon$.

Hence the curve $\varphi^t_{\pi_1}$ depends continuously on s and $\pi_1 = \{t_1, l_1, u_1\}$, then it converges uniformly to γ as s tends to 0. See [25, 31] for more details of the differential equations depending continuously on parameters.

Let us prove that the curve $\varphi^{t_1}_{\pi_1}$ has a tangent vector at $s = 0$. Let $u[\pi_1^s]$ be an elementary perturbation of u specified by $\pi_1 = \{t_1, l_1, u_1\}$ and consider the curve $\varphi^{t_1}_{\pi_1}\colon [0,\epsilon] \to M$, $\varphi^{t_1}_{\pi_1}(s) = \gamma[\pi_1^s](t_1)$.

Proposition 3.4. *If t_1 is a Lebesgue time, then the curve $\varphi^{t_1}_{\pi_1}\colon [0,\epsilon] \to M$ is differentiable at $s = 0$. Its tangent vector is $[X(\gamma(t_1), u_1) - X(\gamma(t_1), u(t_1))]\ l_1$.*

Proof. It is enough to prove that for every differentiable function $g\colon M \to \mathbb{R}$, there exists

$$A = \lim_{s\to 0} \frac{g(\varphi^{t_1}_{\pi_1}(s)) - g(\varphi^{t_1}_{\pi_1}(0))}{s}.$$

As this is a derivation on the functions defined on a neighbourhood of $\gamma(t_1)$, it is enough to prove the proposition for the coordinate functions x^i of a local chart at $\gamma(t_1)$. Thus take $g = x^i$,

$$\begin{aligned} A &= \lim_{s\to 0} \frac{(x^i \circ \varphi^{t_1}_{\pi_1})(s) - (x^i \circ \varphi^{t_1}_{\pi_1})(0)}{s} = \lim_{s\to 0} \frac{(x^i \circ \gamma[\pi_1^s])(t_1) - (x^i \circ \gamma)(t_1)}{s} \\ &= \lim_{s\to 0} \frac{\gamma^i[\pi_1^s](t_1) - \gamma^i(t_1)}{s}. \end{aligned}$$

As γ is an absolutely continuous integral curve of $X^{\{u\}}$, $\dot{\gamma}(t) = X(\gamma(t), u(t))$ at every Lebesgue time. Then integrating

$$\gamma^i(t_1) - \gamma^i(a) = \int_a^{t_1} f^i(\gamma(t), u(t))dt$$

and similarly for $\gamma[\pi_1^s]$ and $u[\pi_1^s]$. Observe that $\gamma[\pi_1^s](t) = \gamma(t)$ and $u[\pi_1^s](t) = u(t)$ for $t \in [a, t_1 - l_1 s)$. Then,

$$\begin{aligned} A &= \lim_{s\to 0} \frac{\int_a^{t_1} f^i(\gamma[\pi_1^s](t), u[\pi_1^s](t))dt - \int_a^{t_1} f^i(\gamma(t), u(t))dt}{s} \\ &= \lim_{s\to 0} \frac{\int_{t_1-l_1 s}^{t_1} f^i(\gamma[\pi_1^s](t), u_1)dt - \int_{t_1-l_1 s}^{t_1} f^i(\gamma(t), u(t))dt}{s}. \end{aligned}$$

As t_1 is a Lebesgue time, we use Equation (A.17):

$$\int_{t-h}^{t} X(\gamma(s), u(s))ds = hX(\gamma(t), u(t)) + o(h)$$

in such a way that

$$\begin{aligned} A &= \lim_{s\to 0} \frac{f^i(\gamma[\pi_1^s](t_1), u_1)l_1 s - f^i(\gamma(t_1), u(t_1))l_1 s + o(s)}{s} \\ &= \lim_{s\to 0} [f^i(\gamma[\pi_1^s](t_1), u_1) - f^i(\gamma(t_1), u(t_1))]\, l_1. \end{aligned}$$

As f^i is continuous on M, we have

$$\begin{aligned} A &= \lim_{s\to 0} [f^i(\gamma[\pi_1^s](t_1), u_1) - f^i(\gamma(t_1), u(t_1))]\, l_1 = [f^i(\lim_{s\to 0} \gamma[\pi_1^s](t_1), u_1) - f^i(\gamma(t_1), u(t_1))]\, l_1 \\ &= [f^i(\gamma(t_1), u_1) - f^i(\gamma(t_1), u(t_1))]\, l_1 = [(X(\gamma(t_1), u_1) - X(\gamma(t_1), u(t_1)))\ \ l_1]\, (x^i). \end{aligned}$$

□

Definition 3.5. *The tangent vector* $v[\pi_1] = (X(\gamma(t_1), u_1) - X(\gamma(t_1), u(t_1)))\ l_1 \in T_{\gamma(t_1)}M$ *is the* ***elementary perturbation vector associated to the perturbation data*** $\pi_1 = \{t_1, l_1, u_1\}$. *It is also called a* ***perturbation vector of class I****.*

Comments:

(a) The previous proof shows the importance of defining perturbations only at Lebesgue times, otherwise the elementary perturbation vectors may not exist.

(b) Observe that if we change $\pi_1 = \{t_1, l_1, u_1\}$ for $\pi_2 = \{t_1, l_2, u_1\}$, then $v[\pi_1] = (l_1/l_2)\ v[\pi_2]$. If $v[\pi_1]$ is a perturbation vector of class I and $\lambda \in \mathbb{R}^+$, then $\lambda\ v[\pi_1]$ is also a perturbation vector of class I with perturbation data $\{t_1, \lambda\ l_1, u_1\}$.

(c) We write $\mathrm{L}(w)g$ for the derivative of the function g in the direction given by the vector $w \in T_xM$. Due to Proposition 3.4, for every differentiable function $g\colon M \to \mathbb{R}$ we have

$$\frac{g(\varphi_{\pi_1}^{t_1}(s)) - g(\gamma(t_1)) - s\ \mathrm{L}(v[\pi_1])g}{s} \underset{s\to 0}{\longrightarrow} 0.$$

Hence

$$g\left(\varphi_{\pi_1}^{t_1}(s)\right) = g\left(\gamma(t_1)\right) + s\, \mathrm{L}(v[\pi_1])g + o(s).$$

If (x^i) are local coordinates of a chart at $\gamma(t_1)$,

$$x^i\left(\varphi_{\pi_1}^{t_1}(s)\right) = x^i\left(\gamma(t_1)\right) + s\, v[\pi_1]^i + o(s).$$

That is,

$$\left(\varphi_{\pi_1}^{t_1}\right)^i(s) = \gamma^i(t_1) + s\,v[\pi_1]^i + o(s).$$

Now, if we identify the open set of the local chart and the tangent space to M at $\gamma(t_1)$ with the same space $\mathbb{R}^m$, we write the following linear approximation

$$\varphi_{\pi_1}^{t_1}(s) = \gamma(t_1) + s\,v[\pi_1] + o(s). \tag{3.4}$$

The initial condition for the velocity given by the elementary perturbation vector evolves along the reference trajectory γ through the integral curves of the complete lift $\left(X^T\right)^{\{u\}}$ of $X^{\{u\}}$, as explained in Appendix B.2. Note that $\varphi_{\pi_1}^{t}(s) = \Phi_{(t,t_1)}^{X^{\{u\}}}\left(\varphi_{\pi_1}^{t_1}(s)\right)$ for $t \geq t_1$ because of the definition of φ_{π_1} and $u[\pi_1^s]$.

Proposition 3.6. *Let $V[\pi_1]\colon [t_1, b] \to TM$ be the integral curve of the complete lift $\left(X^T\right)^{\{u\}}$ of $X^{\{u\}}$ with initial condition $(t_1, (\gamma(t_1), v[\pi_1]))$. For every Lebesgue time $t \in (t_1, b]$, $V[\pi_1](t)$ is the tangent vector to the curve $\varphi_{\pi_1}^t\colon [0, \epsilon] \to M$ at $s = 0$.*

Proof. The proof follows from Proposition B.1 and the definition of the curves considered. □

3.3.2 Perturbation vectors of class II

The control can be perturbed twice instead of only once, in fact it may be modified a finite number of times. If t_2 is a Lebesgue time greater than t_1 and we perturb the control with $\pi_1 = \{t_1, l_1, u_1\}$ and $\pi_2 = \{t_2, l_2, u_2\}$, then we obtain the perturbation data $\pi_{12} = \{(t_1, t_2), (l_1, l_2), (u_1, u_2)\}$, which is given by

$$u[\pi_{12}^s](t) = \begin{cases} u_1, & t \in [t_1 - l_1 s, t_1], \\ u_2, & t \in [t_2 - l_2 s, t_2], \\ u(t), & \text{elsewhere} \end{cases}$$

for every $s \in \mathbb{R}^+$ small enough such that $[t_1 - l_1 s, t_1] \cap [t_2 - l_2 s, t_2] = \emptyset$. Then $\gamma[\pi_{12}^s]\colon I \longrightarrow M$ is the generalized integral curve of $X^{\{u[\pi_{12}^s]\}}$ with initial condition $(a, \gamma(a))$. Observe that $\gamma[\pi_{12}^0](t) = \gamma(t)$. Consider the curve $\varphi_{\pi_{12}}^{t_2}\colon [0, \epsilon] \to M$ given by $\varphi_{\pi_{12}}^{t_2}(s) = \gamma[\pi_{12}^s](t_2)$.

Proposition 3.7. *Let t_1, t_2 be Lebesgue times such that $t_1 < t_2$. The vector tangent to $\varphi_{\pi_{12}}^{t_2}\colon [0, \epsilon] \to M$ at $s = 0$ is $v[\pi_2] + V[\pi_1](t_2)$, where $V[\pi_1]\colon [t_1, b] \to TM$ is the generalized integral curve of $\left(X^T\right)^{\{u\}}$ with initial condition $(t_1, (\gamma(t_1), v[\pi_1]))$.*

Proof. Here we perturb the control first with π_1 along γ and we obtain $u[\pi_1^s]$. Then we perturb this last control with the other perturbation data, π_2, along $\gamma[\pi_1^s]$. Then the superindeces of the tangent vectors denote the curve along which the perturbation is made. As in the proof of Proposition 3.4,

$$\begin{aligned} A &= \lim_{s \to 0} \frac{(x^i \circ \varphi_{\pi_{12}}^{t_2})(s) - (x^i \circ \varphi_{\pi_{12}}^{t_2})(0)}{s} = \lim_{s \to 0} \frac{(x^i \circ \gamma[\pi_{12}^s])(t_2) - (x^i \circ \gamma)(t_2)}{s} \\ &= \lim_{s \to 0} \frac{\gamma^i[\pi_{12}^s](t_2) - \gamma^i(t_2)}{s} = \lim_{s \to 0} \left(\frac{\gamma^i[\pi_{12}^s](t_2) - \gamma^i[\pi_1^s](t_2)}{s} + \frac{\gamma^i[\pi_1^s](t_2) - \gamma^i(t_2)}{s} \right). \end{aligned}$$

We understand $\gamma[\pi_{12}^s]$ as the result of perturbing $\gamma[\pi_1^s]$ with π_2, and use the linear approximation in Equation (3.4) for $\gamma[\pi_{12}^s](t_2)$ and $\gamma[\pi_1^s](t_2)$ according to Proposition 3.4.

$$\varphi_{\pi_{12}}^{t_2}(s) = \gamma[\pi_{12}^s](t_2) = \gamma[\pi_1^s](t_2) + s\,v[\pi_2]^{\gamma[\pi_1^s]} + o(s),$$

$$\gamma[\pi_1^s](t_2) = \gamma(t_2) + s\,V[\pi_1]^{\gamma}(t_2) + o(s).$$

Then

$$A = \lim_{s\to 0}\left(\frac{s(v[\pi_2]^{\gamma[\pi_1^s]})^i}{s} + \frac{s(V[\pi_1]^{\gamma})^i(t_2)}{s}\right) = \lim_{s\to 0}\left((v[\pi_2]^{\gamma[\pi_1^s]})^i + (V[\pi_1]^{\gamma})^i(t_2)\right).$$

As $\gamma[\pi_1^s]$ depends on s and s tends to 0, $A = \mathrm{L}\,(v[\pi_2]^{\gamma} + V[\pi_1]^{\gamma}(t_2))\,x^i$. □

Considering identifications similar to the ones used to write Equation (3.4), we have

$$\varphi_{\pi_{12}}^{t_2}(s) = \gamma(t_2) + sv[\pi_2] + sV[\pi_1](t_2) + o(s).$$

Now we define how the control changes when it is perturbed twice at the same time. If t_1 is a Lebesgue time, $\pi_1' = \{t_1, l_1', u_1'\}$ and $\pi_1'' = \{t_1, l_1'', u_1''\}$ are perturbation data, then $\pi_{11} = \{(t_1,t_1),(l_1',l_1''),(u_1',u_1'')\}$ is a perturbation data given by

$$u[\pi_{11}^s](t) = \begin{cases} u_1', & t\in[t_1-(l_1'+l_1'')s, t_1-l_1''s],\\ u_1'', & t\in[t_1-l_1''s, t_1],\\ u(t), & \text{elsewhere.}\end{cases}$$

for every $s\in\mathbb{R}^+$ small enough such that $a < t_1-(l_1'+l_1'')s$. Then $\gamma[\pi_{11}^s]\colon I \longrightarrow M$ is the generalized integral curve of $X^{\{u[\pi_{11}^s]\}}$ with initial condition $(a,\gamma(a))$. Observe that $\gamma[\pi_{11}^0](t) = \gamma(t)$. Consider the curve $\varphi_{\pi_{11}}^{t_1}\colon[0,\epsilon]\to M$, defined by $\varphi_{\pi_{11}}^{t_1}(s) = \gamma[\pi_{11}^s](t_1)$.

Proposition 3.8. *Let t_1 be a Lebesgue time. The vector tangent to $\varphi_{\pi_{11}}^{t_1}\colon[0,\epsilon]\to M$ at $s=0$ is $v[\pi_1'] + v[\pi_1'']$, where $v[\pi_1']$ and $v[\pi_1'']$ are the perturbation vectors of class I associated to π_1' and π_1'', respectively.*

Proof. As in the proof of Proposition 3.4

$$A = \lim_{s\to 0}\frac{(x^i\circ\varphi_{\pi_{11}}^{t_1})(s) - (x^i\circ\varphi_{\pi_{11}}^{t_1})(0)}{s} = \lim_{s\to 0}\frac{\gamma^i[\pi_{11}^s](t_1)-\gamma^i(t_1)}{s}.$$

As γ is an absolutely continuous integral curve of $X^{\{u\}}$, $\dot\gamma(t) = X(\gamma(t),u(t))$ at every Lebesgue time. Then, we integrate

$$\gamma^i(t_1) - \gamma^i(a) = \int_a^{t_1} f^i(\gamma(t),u(t))dt$$

and similarly for $\gamma[\pi_{11}^s]$ and $u[\pi_{11}^s]$. Observe that $\gamma[\pi_{11}^s](t) = \gamma(t)$ and $u[\pi_{11}^s](t) = u(t)$ for $t\in[a, t_1-(l_1'+l_1'')s)$. Then,

$$\begin{aligned}
A &= \lim_{s\to 0}\frac{\int_a^{t_1} f^i(\gamma[\pi_{11}^s](t), u[\pi_{11}^s](t))dt - \int_a^{t_1} f^i(\gamma(t),u(t))dt}{s}\\
&= \lim_{s\to 0}\frac{\int_{t_1-(l_1'+l_1'')s}^{t_1} f^i(\gamma[\pi_{11}^s](t), u[\pi_{11}^s](t))dt - \int_{t_1-(l_1'+l_1'')s}^{t_1} f^i(\gamma(t),u(t))dt}{s}\\
&= \lim_{s\to 0}\left(\frac{\int_{t_1-(l_1'+l_1'')s}^{t_1-l_1''s}\left(f^i(\gamma[\pi_1'^s](t),u_1') - f^i(\gamma(t),u(t))\right)dt}{s}\right.\\
&+ \left.\frac{\int_{t_1-l_1''s}^{t_1}\left[f^i(\gamma[\pi_{11}^s](t),u_1'') - f^i(\gamma(t),u(t))\right]dt}{s}\right).
\end{aligned}$$

As t_1 and $t_1 - l_1''s$ are Lebesgue times and $a < t_1 - (l_1' + l_1'')s$ for a small enough s, Equation (A.17) is used. Now we have

$$\begin{aligned} A &= \lim_{s\to 0} \left\{ \frac{f^i(\gamma[\pi_1'^s](t_1 - l_1''s), u_1')l_1's - f^i(\gamma(t_1 - l_1''s), u(t_1 - l_1''s))l_1's}{s} \right. \\ &+ \left. \frac{f^i(\gamma[\pi_{11}^s](t_1), u_1'')l_1''s - f^i(\gamma(t_1), u(t_1))l_1''s}{s} \right\} \\ &= \lim_{s\to 0} \Big(\left[f^i(\gamma[\pi_1'^s](t_1 - l_1''s), u_1') - f^i(\gamma(t_1 - l_1''s), u(t_1 - l_1''s)) \right] l_1' \\ &+ \left(f^i(\gamma[\pi_{11}^s](t_1), u_1'') - f^i(\gamma(t_1), u(t_1)) \right) l_1'' \Big) . \end{aligned}$$

As f^i is continuous on $M \times U$, we have

$$\begin{aligned} A &= \left(f^i \left(\lim_{s\to 0} \gamma[\pi_1'^s](t_1 - l_1''s), u_1' \right) - f^i \left(\lim_{s\to 0} \gamma(t_1 - l_1''s), \lim_{s\to 0} u(t_1 - l_1''s) \right) \right) l_1' \\ &+ \left(f^i \left(\lim_{s\to 0} \gamma[\pi_{11}^s](t_1), u_1'' \right) - f^i \left(\gamma(t_1), u(t_1) \right) \right) l_1'' = \left[f^i(\gamma(t_1), u_1') - f^i(\gamma(t_1), u(t_1)) \right] l_1' \\ &+ \left(f^i(\gamma(t_1), u_1'') - f^i(\gamma(t_1), u(t_1)) \right) l_1'' = \mathrm{L} \left(v[\pi_1'] + v[\pi_1''] \right) (x^i). \end{aligned}$$

□

Analogous to the linear approximation (3.4), we have

$$\varphi_{\pi_{11}}^{t_1}(s) = \gamma(t_1) + sv[\pi_1'] + sv[\pi_1''] + o(s).$$

If we perturb the control r times, $\pi = \{\pi_1, \ldots, \pi_r\}$, with $a < t_1 \leq \ldots \leq t_r < b$, then $\gamma[\pi^s](t)$ is the generalized integral curve of $X^{\{u[\pi^s]\}}$ with initial condition $(a, \gamma(a))$. Consider the curve $\varphi_\pi^t \colon [0, \epsilon] \to M$ for $t \in [t_r, b]$ given by $\varphi_\pi^t(s) = \gamma[\pi^s](t)$.

Corollary 3.9. *For $t \in [t_r, b]$, the vector tangent to the curve $\varphi_\pi^t \colon [0, \epsilon] \to M$ at $s = 0$ is $V[\pi_1](t) + \ldots + V[\pi_r](t)$, where $V[\pi_i] \colon [t_i, b] \to TM$ is the generalized integral curve of $\left(X^T\right)^{\{u\}}$ with initial condition $(t_i, (\gamma(t_i), v[\pi_i]))$ for $i = 1, \ldots, r$.*

This corollary may be easily proved by induction using Propositions 3.4, 3.7, 3.8, where all the possibilities of combination of perturbation data have been studied. If w is the vector tangent to φ_π^t at $s = 0$, the perturbation data will be denoted by π_w. Bearing in mind the different combination of vectors in Definition D.2, we have the following definition.

Definition 3.10. *The conic non–negative combinations of perturbation vectors of class I and displacements by the flow of $X^{\{u\}}$ of perturbation vectors of class I are called* ***perturbation vectors of class II****.*

3.3.3 Perturbation cones

Considering all the elementary perturbation vectors, we define a closed convex cone at every time containing at least all displacements of these vectors. To transport all the elementary perturbation vectors, the pushforward of the flow of the vector field $X^{\{u\}}$ is used. See Appendix B. Observe that the second comment after Definition 3.5 guarantees that the set of elementary perturbation vectors is a cone.

Definition 3.11. *For $t \in (a, b]$, the* ***tangent perturbation cone*** K_t *is the smallest closed convex cone in $T_{\gamma(t)}M$ that contains all the displacements by the flow of $X^{\{u\}}$ of all the elementary perturbations vectors from all Lebesgue times τ smaller than t:*

$$K_t = \overline{\operatorname{conv}\left(\bigcup_{\substack{a<\tau\leq t \\ \tau \text{ is a Lebesgue time}}} (\Phi^{X^{\{u\}}}_{(t,\tau)})_*(\mathcal{V}_\tau) \right)},$$

where $\mathcal{V}_\tau$ denotes the set of elementary perturbation vectors at τ and $\operatorname{conv}(A)$ means the convex hull of the set A.

To prove the following statement, we use results in Appendices D and E; precisely Proposition D.4, D.5 and Corollary E.2.

Proposition 3.12. *Let $t \in (a, b]$. If v is a nonzero vector in the interior of K_t, then there exists $\epsilon > 0$ such that for every $s \in (0, \epsilon)$ there are $s' > 0$ and a perturbation of the control $u[\pi^s]$ such that $\gamma[\pi^s](t) = \gamma(t) + s'v$.*

Proof. As v is interior to K_t, by Proposition D.5, item (d), v is in the interior of the cone

$$\mathcal{C} = \operatorname{conv}\left(\bigcup_{\substack{a<\tau\leq t \\ \tau \text{ is a Lebesgue time}}} \left(\Phi^{X^{\{u\}}}_{(t,\tau)}\right)_* \mathcal{V}_\tau \right),$$

where $\mathcal{V}_\tau$ is the cone of perturbation vectors of class I at time τ. Hence, v can be expressed as a convex finite combination of perturbation vectors of class I by Proposition D.4.

Let (W, x^i) be a local chart of M at $\gamma(t)$. We suppose that the image of the local chart and W are identified locally with an open set of $\mathbb{R}^m$. Through the local chart we also identify $T_{\gamma(t)}M$ with $\mathbb{R}^m$. We consider the affine hyperplane Π orthogonal to v at the endpoint of the vector v and identify Π with $\mathbb{R}^{m-1}$.

A "closed" cone denotes a closed cone without the vertex. Observe that such a cone is not closed, that is why we use the inverted commas. We can choose a "closed" convex cone $\widetilde{\mathcal{C}}$ contained in the interior of $\mathcal{C}$ such that v lies in the interior of $\widetilde{\mathcal{C}}$ and $\langle w, v\rangle > 0$ for every $w \in \widetilde{\mathcal{C}}$. For example, we can consider a circular cone with axis v satisfying the two previous conditions, as assumed from now on. Hence

$$\Pi \cap \widetilde{\mathcal{C}} = v + \overline{B(0, R)},$$

where $\overline{B(0, R)}$ is the closure of an open ball in the subspace orthogonal to v, denoted by $v^\perp$. For $r \in v^\perp$, we will write r instead of $0v + r$ as a vector in $\mathbb{R}^m$.

Let us construct a diffeomorphism from the cone $\widetilde{\mathcal{C}}$ to a cylinder of $\mathbb{R}^m$. If $w \in \widetilde{\mathcal{C}}$, the orthogonal decomposition of w induced by v and $v^\perp$ is

$$w = \frac{\langle w, v\rangle}{\|v\|}\frac{v}{\|v\|} + \left(w - \frac{\langle w, v\rangle}{\langle v, v\rangle}v\right) = \frac{\langle w, v\rangle}{\langle v, v\rangle}\left[v + \left(\frac{\langle v, v\rangle}{\langle w, v\rangle}w - v\right)\right].$$

Observe that $\frac{\langle v,v\rangle}{\langle w,v\rangle}w - v$ is a vector in $\overline{B(0, R)} \subset v^\perp$. Considering the "closed" cone $\widetilde{\mathcal{C}}$ without the vertex, we have the map

$$\begin{aligned} g\colon \widetilde{\mathcal{C}} &\longrightarrow \mathbb{R}^+ \times \overline{B(0, R)} \\ w &\longmapsto \left(\frac{\langle w, v\rangle}{\langle v, v\rangle}, \frac{\langle v, v\rangle}{\langle w, v\rangle}w - v\right) = (s, r), \end{aligned}$$

that is a $\mathcal{C}^\infty$ diffeomorphism with inverse given by

$$\begin{array}{rccc} g^{-1}\colon & \mathbb{R}^+ \times \overline{B(0,R)} & \longrightarrow & \widetilde{\mathcal{C}} \\ & (s,r) & \longmapsto & s(v+r) = w. \end{array}$$

Note that g and g^{-1} can be extended to an open cone, without the vertex, containing $\widetilde{\mathcal{C}}$, so the condition that g is diffeomorphism is clear.

If we truncate $\widetilde{\mathcal{C}}$ by the affine hyperplane Π, we obtain a bounded convex set $\widetilde{\mathcal{C}}_v$. The restriction of g to $\widetilde{\mathcal{C}}_v$ is $g_v\colon \widetilde{\mathcal{C}}_v \to (0,1] \times \overline{B(0,R)}$, that is also a $\mathcal{C}^\infty$ diffeomorphism with inverse $g_v^{-1}\colon (0,1] \times \overline{B(0,R)} \to \widetilde{\mathcal{C}}_v$.

If $r \in \overline{B(0,R)}$, then $w_0 = v+r$ is interior to $\mathcal{C}$. Hence, associated to w_0 we have a perturbation π_{w_0} of the control u. Let $\gamma[\pi^s_{w_0}]\colon I \to M$ be the generalized integral curve of $X^{\{u[\pi^s_{w_0}]\}}$ with initial condition $(a, \gamma(a))$ and consider the map

$$\begin{array}{rccc} \Gamma\colon & [0,1] \times \overline{B(0,R)} & \longrightarrow & M \\ & (s,r) & \longmapsto & \Gamma(s,r) = \gamma[\pi^s_{w_0}](t) \\ & (0,r) & \longmapsto & \Gamma(0,r) = \gamma(t), \end{array}$$

which is continuous because $\gamma[\pi^s_{w_0}](t)$ depends continuously on s and $\pi^s_{w_0}$ and

$$\lim_{(s,r)\to(0,r_0)} \Gamma(s,r) = \gamma(t) = \Gamma(0,r_0).$$

Hence, for every $\epsilon > 0$, there exist $\delta_1, \delta_2 > 0$ such that if $|s| < \delta_1$ and $\|r\| < \delta_2$, then $\|\Gamma(s,r) - \Gamma(0,0)\| = \|\gamma[\pi^s_{w_0}](t) - \gamma(t)\| < \epsilon$.

Taking ϵ such that $B(\gamma(t), \epsilon)$ is contained in W, there exist $\delta_1, \delta_2 > 0$ such that if $|s| < \delta_1$ and $\|r\| < \delta_2$, then $\gamma[\pi^s_{w_0}](t) \in W$.

We consider now the map

$$\begin{array}{rccc} \Delta\colon & [0,\delta_1] \times \overline{B(0,\delta_2)} & \longrightarrow & T_{\gamma(t)}M \simeq \mathbb{R}^m \\ & (s,r) & \longmapsto & \Delta(s,r) = \gamma[\pi^s_{w_0}](t) - \gamma(t) \\ & (0,r) & \longmapsto & \Delta(0,r) = 0 \end{array}$$

that is continuous because $\lim_{(s,r)\to(0,r_0)} \Delta(s,r) = 0 = \Delta(0,r_0)$. Remember that we have identified W with $\mathbb{R}^m$ via the local chart. With this in mind and using Equation (3.4), we can write

$$\gamma[\pi^s_{w_0}](t) - \gamma(t) = s(v+r) + o_r(s),$$

where $o_r(s) \in \mathbb{R}^m$.

We are going to show that, taking (s,r) in an adequate subset, $\Delta(s,r)$ lies in the interior of the cone $\widetilde{\mathcal{C}}$.

Take a section of the cone through a plane containing v and w, and compute the distance from the endpoint of w to the boundary of the cone $\widetilde{\mathcal{C}}$. This is given by

$$\frac{s\ (R - \|r\|)}{\sqrt{1 + \left(\frac{R}{\|v\|}\right)^2}}.$$

This is the maximum value for the radius of an open ball centered at the endpoint of $s(v+r)$ to be contained in $\widetilde{\mathcal{C}}$.

Define the function

$$\begin{array}{rccc} \Theta\colon & [0,\delta_1] \times \overline{B(0,\delta_2)} & \longrightarrow & \mathbb{R}^m \\ & (s,r) & \longmapsto & \left(\gamma[\pi^s_{w_0}](t) - \gamma(t) - s(v+r)\right)/s \ = \ o_r(s)/s \\ & (0,r) & \longmapsto & 0. \end{array}$$

which is continuous because $\lim_{(s,r)\to(0,r_0)} \Theta(s,r) = 0 = \Theta(0,r_0)$. Take

$$\epsilon = \frac{R-\delta_2}{\sqrt{1+\left(\frac{R}{\|v\|}\right)^2}},$$

then there exist $\bar{\delta}_1, \bar{\delta}_2 > 0$ such that, if $|s| < \bar{\delta}_1$ and $\|r\| < \bar{\delta}_2$, then $\|\Theta(s,r)\| = \|o_r(s)/s\| < \epsilon$.

If $(s,r) \in (0,\bar{\delta}_1) \times \overline{B(0,\bar{\delta}_2)}$, then

$$\|\Delta(s,r) - s(v+r)\| = \|s(v+r) + o_r(s) - s(v+r)\| = \|o_r(s)\| \leq s\epsilon < s\frac{R-\|r\|}{\sqrt{1+\left(\frac{R}{\|v\|}\right)^2}}$$

since $\|r\| \leq \bar{\delta}_2 < \delta_2 < R$. Thus we conclude that $\Delta(s,r) = s(v+r) + o_r(s)$ is in the interior of the cone $\widetilde{\mathcal{C}}$ for every $(s,r) \in (0,\bar{\delta}_1) \times \overline{B(0,\bar{\delta}_2)}$.

Now, for $s \in (0,\bar{\delta}_1)$, we define the continuous mapping

$$\begin{array}{rcl} G_s\colon \overline{B(0,\bar{\delta}_2)} & \longrightarrow & \overline{B(0,R)} \subset \mathbb{R}^{m-1} \\ r & \longmapsto & G_s(r) = (\pi_2 \circ g \circ \Delta)(s,r), \end{array} \tag{3.5}$$

where $\pi_2\colon \mathbb{R}^+ \times \overline{B(0,R)} \to \overline{B(0,R)}$, $\pi_2(s,r) = r$. Observe that for $r_0 \in \overline{B(0,\bar{\delta}_2)}$ we have

$$\lim_{(s,r)\to(0,r_0)} G_s(r) = \lim_{(s,r)\to(0,r_0)} \left[\frac{\langle v,v\rangle}{s\langle v,v\rangle + \langle o(s),v\rangle}(s(v+r)+o(s)) - v\right] = r$$

and

$$(g\circ\Delta)(s,r) = g(\gamma[\pi^s_{w_0}](t) - \gamma(t)) = g(s(v+r)+o_r(s)) = (s',r'). \tag{3.6}$$

Suppose that there exists $r \in \overline{B(0,R)}$ such that $G_s(r) = 0$. Then applying g^{-1} to (3.6), we have

$$\Delta(s,r) = \gamma[\pi^s_{w_0}](t) - \gamma(t) = g^{-1}(s',0) = s'v. \tag{3.7}$$

Hence, to conclude the proof we need to show that there exists r with $G_s(r) = 0$ for s small enough. To apply Corollary E.2, there must exist $r' \in B(0,\bar{\delta}_2)$ such that $\|G_s(r) - r\| < \|r - r'\|$ for every $r \in \partial\left(\overline{B(0,\bar{\delta}_2)}\right)$. We will show that the condition is fulfilled for $r' = 0$.

Consider the mapping

$$\begin{array}{rccl} \mathcal{G}\colon & [0,\bar{\delta}_1] \times \overline{B(0,\bar{\delta}_2)} & \longrightarrow & \overline{B(0,R)} \subset \mathbb{R}^{m-1} \\ & (s,r) & \longmapsto & \mathcal{G}(s,r) = G_s(r) - r \\ & (0,r) & \longmapsto & \mathcal{G}(0,r) = 0. \end{array}$$

For $r_0 \in \overline{B(0,\bar{\delta}_2)}$, we have $\lim_{(s,r)\to(0,r_0)} \mathcal{G}(s,r) = \lim_{(s,r)\to(0,r_0)} G_s(r) - r = 0$. Thus $\mathcal{G}$ is continuous.

Given $r_0 \in \partial\left(\overline{B(0,\bar{\delta}_2)}\right)$, take $\epsilon = \bar{\delta}_2/2$, then there exist $\delta_0(0,r_0)$, $\delta_1(0,r_0) > 0$ such that if $|s| < \delta_0(0,r_0)$ and $\|r - r_0\| < \delta_1(0,r_0)$, then $\|\mathcal{G}(s,r) - \mathcal{G}(0,r_0)\| < \bar{\delta}_2/2$. Hence $\left\{B(r_0,\delta_1(0,r_0)) \,|\, r_0 \in \partial\left(\overline{B(0,\bar{\delta}_2)}\right)\right\}$ is an open covering of the boundary of $\overline{B(0,\bar{\delta}_2)}$; $\partial\overline{B(0,\bar{\delta}_2)}$. As this is a compact set, there exists a finite subcovering,

$$\{B(r_1,\delta_1(0,r_1)),\ldots,B(r_l,\delta(0,r_l))\}.$$

Take δ as the minimum of $\{\delta_0(0, r_1), \ldots, \delta_0(0, r_l)\}$. Let us see that, for every $(s, r) \in [0, \delta] \times \overline{\partial B(0, \overline{\delta}_2)}$, $\|G_s(r) - r\| < \|r\|$. As r is in an open set of the finite subcovering,

$$\|\mathcal{G}(s, r)\| = \|G_s(r) - r\| < \frac{\overline{\delta}_2}{2} < \overline{\delta}_2 = \|r\|.$$

Hence, using Corollary E.2, for every $s \in (0, \delta)$, the set $G_s(\overline{B(0, \overline{\delta}_2)})$ covers the origin; that is, there exists $r \in \overline{B(0, \overline{\delta}_2)}$ such that

$$G_s(r) = (\pi_2 \circ g \circ \Delta)(s, r) = 0.$$

Then, because of the definition of the mapping G_s in Equation (3.5) and Equations (3.6) and (3.7), there exists $s' \in \mathbb{R}^+$ such that

$$\gamma[\pi^s_{w_0}](t) = \gamma(t) + s'v.$$

To finish the proof we only need to take $\pi^s = \pi^s_{w_0}$. In other words, we have a trajectory coming from a perturbation of the control that meets the ray generated by v, as wanted. □

3.4 Pontryagin's Maximum Principle in the symplectic formalism for the optimal control problem

In this section, the OCP is transformed into a Hamiltonian problem that will allow us to state Pontryagin's Maximum Principle.

Given the OCP $(Q, U, X, F, I, x_a, x_b)$ and the $\widehat{OCP}$ $(\widehat{Q}, U, \widehat{X}, I, x_a, x_b)$, let us consider the cotangent bundle $T^*\widehat{Q}$ with its natural symplectic structure that will be denoted by ω. If $(\widehat{x}, \widehat{p}) = (x^0, x, p_0, p) = (x^0, x^1, \ldots, x^m, p_0, p_1, \ldots, p_m)$ are local natural coordinates on $T^*\widehat{Q}$, the form ω has as its local expression $\omega = dx^0 \wedge dp_0 + dx^i \wedge dp_i$.

For each $u \in U$, $H^u \colon T^*\widehat{Q} \to \mathbb{R}$ is the Hamiltonian function defined by

$$H^u(\widehat{p}) = H(\widehat{p}, u) = \langle \widehat{p}, \widehat{X}(\widehat{x}, u)\rangle = p_0 F(x, u) + \sum_{i=1}^{m} p_i f^i(x, u),$$

where $\widehat{p} \in T^*_{\widehat{x}}\widehat{Q}$. The tuple $(T^*\widehat{Q}, \omega, H^u)$ is a Hamiltonian system. Using the notation in (2.2), the associated Hamiltonian vector field $Y^{\{u\}}$ satisfies the equation

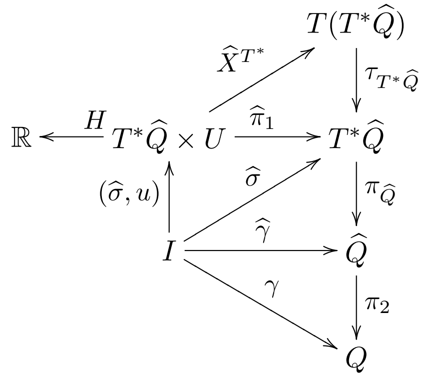
$$i(Y^{\{u\}})\omega = dH^u.$$

Thus we get a family of Hamiltonian systems parameterized by u, $H \colon T^*\widehat{Q} \times U \to \mathbb{R}$, and the associated Hamiltonian vector field $Y \colon T^*\widehat{Q} \times U \to T(T^*\widehat{Q})$ which is a vector field along the projection $\widehat{\pi}_1 \colon T^*\widehat{Q} \times U \to T^*\widehat{Q}$. Its local expression is

$$Y(\widehat{p}, u) = \left(F(x, u)\frac{\partial}{\partial x^0} + f^i(x, u)\frac{\partial}{\partial x^i} + 0\frac{\partial}{\partial p_0} + \left(-p_0 \frac{\partial F}{\partial x^i}(x, u) - p_j \frac{\partial f^j}{\partial x^i}(x, u) \right) \frac{\partial}{\partial p_i} \right)_{(\widehat{x}, \widehat{p}, u)}.$$

It should be noted that $Y = \widehat{X}^{T^*}$ is the cotangent lift of $\widehat{X}$. See Appendix B.3 for definition and properties of the cotangent lift.

Given a curve $(\widehat{\lambda}, u) \colon I \to T^*\widehat{Q} \times U$ such that it is absolutely continuous on $T^*\widehat{Q}$, it is measurable and bounded on U, and $\widehat{\gamma} = \pi_{\widehat{Q}} \circ \widehat{\lambda}$; if $\pi_{\widehat{Q}} \colon T^*\widehat{Q} \to \widehat{Q}$ is the natural projection, the previous elements come together in the following diagram:



Statement 3.13. ***(Hamiltonian Problem,*** *HP**)*** *Given the OCP $(Q, U, X, F, I, x_a, x_b)$, and the equivalent $\widehat{OCP}$ $(\widehat{Q}, U, \widehat{X}, I, x_a, x_b)$, consider the following problem.*

Find $(\widehat{\sigma}^*, u^*)$ such that

(1) $\widehat{\gamma}^*(a) = (0, x_a)$ *and* $\gamma^*(b) = x_b$, *if* $\widehat{\gamma}^* = \pi_{\widehat{Q}} \circ \widehat{\sigma}^*$, $\gamma^* = \pi_2 \circ \widehat{\gamma}^*$.

(2) $\dot{\widehat{\sigma}}^*(t) = \widehat{X}^{T^*}(\widehat{\sigma}^*(t), u^*(t))$, $t \in I$.

The tuple $(T^*\widehat{Q}, U, \widehat{X}^{T^*}, I, x_a, x_b)$ denotes the *Hamiltonian problem* as it has just been defined and the elements satisfy the same properties as in §2.

Comments: The Hamiltonian problem satisfies analogous conditions to those satisfied by the OCP and the $\widehat{OCP}$ defined in §3.1 and §3.2 respectively.

1. Given $(\widehat{\sigma}, u)$, the function $u\colon I \to U$ allows us to construct a time–dependent vector field on $T^*\widehat{Q}$, $(\widehat{X}^{T^*})^{\{u\}}\colon T^*\widehat{Q} \times I \to T(T^*\widehat{Q})$, defined by

$$(\widehat{X}^{T^*})^{\{u\}}(\widehat{x}, \widehat{p}, t) = \widehat{X}^{T^*}(\widehat{x}, \widehat{p}, u(t)).$$

   Condition (2) shows that $\widehat{\sigma}^*$ is an integral curve of $(\widehat{X}^{T^*})^{\{u^*\}}$.

2. The vector field $(\widehat{X}^{T^*})^{\{u\}}$ is $\pi_{\widehat{Q}}$–projectable and projects onto $\widehat{X}^{\{u\}}$. Thus if $\widehat{\sigma}$ is an integral curve of $(\widehat{X}^{T^*})^{\{u\}}$, $\widehat{\gamma} = \pi_{\widehat{Q}} \circ \widehat{\sigma}$ is an integral curve of $\widehat{X}^{\{u\}}$.

3. Locally, conditions (1) and (2) are equivalent to the fact that the curve $(\widehat{\sigma}, u)$ satisfies the Hamilton equations of the system $(T^*\widehat{M}, \omega, H^u)$,

$$\begin{aligned}
\dot{x}^0 &= \frac{\partial H^u}{\partial p_0} = F \\
\dot{x}^i &= \frac{\partial H^u}{\partial p_i} = f^i \\
\dot{p}_0 &= -\frac{\partial H^u}{\partial x^0} = 0 \Rightarrow p_0 = \mathrm{ct} && (3.8) \\
\dot{p}_i &= -\frac{\partial H^u}{\partial x^i} = -p_0 \frac{\partial F}{\partial x^i} - p_j \frac{\partial f^j}{\partial x^i}, && (3.9)
\end{aligned}$$

   and satisfies the conditions $\widehat{\gamma}(a) = (0, x_a)$, $\gamma(b) = x_b$.

   In the literature of optimal control, the system of differential equations given by Equations (3.8), (3.9) is called the *adjoint system*. In differential geometry, the adjoint system is the differential equations satisfied by the fiber coordinates of an integral curve of the cotangent lift of a vector field on Q. See Appendix B.3 for more details.

Note that there is no initial condition for $\widehat{p} = (p_0, p_1, \ldots, p_m)$, hence HP is not a Cauchy problem.

Comment: So far we have considered a fixed control $u \in U$. Therefore we have been working with a family of Hamiltonian systems on the manifold $(T^*\widehat{Q}, \omega)$ given by the Hamiltonians $\{H^u | \, u \in U\}$.

Given $u\colon I \to U$, then we consider the Hamiltonian $H^{u(t)}$. The equation of the Hamiltonian vector field for the Hamiltonian system $(T^*\widehat{Q}, \omega, H^{u(t)})$ is

$$i(Y^{\{u(t)\}})\omega = d_{\widehat{Q}} H^{u(t)},$$

where $d_{\widehat{Q}}$ is the exterior differential on the manifold $T^*\widehat{Q}$. Observe that we have studied the system defined by $(T^*\widehat{Q}, \omega, H^{u(t)})$ as an autonomous system by fixing the time t. The Hamiltonian vector field obtained $Y^{\{u(t)\}}$ is a time–dependent vector field whose integral curves satisfy the equation

$$\dot{\widehat{\sigma}}(t) = Y^{\{u(t)\}}(\widehat{\sigma}(t)), \quad t \in I. \tag{3.10}$$

Observe that $Y^{\{u(t)\}} = (\widehat{X}^{T^*})^{\{u(t)\}}$.

Now we are ready to state Pontryagin's Maximum Principle that provides the necessary conditions, which are in general not sufficient, to find solutions of the optimal control problem.

Theorem 3.14. ***(Pontryagin's Maximum Principle, PMP)***
If $(\widehat{\gamma}^*, u^*)\colon I \to \widehat{Q} \times U$ is a solution of the extended optimal control problem, Statement 3.2, such that $\widehat{\gamma}^*$ is absolutely continuous and u^* is measurable and bounded, then there exists $(\widehat{\sigma}^*, u^*)\colon I \to T^*\widehat{Q} \times U$ such that:

1. *it is a solution of the Hamiltonian problem, that is, it satisfies Equation (3.10) and the initial conditions $\widehat{\gamma}^*(a) = (0, x_a)$ and $\gamma^*(b) = x_b$, if $\gamma^* = \pi_2 \circ \widehat{\gamma}^*$;*

2. *$\widehat{\gamma}^* = \pi_{\widehat{Q}} \circ \widehat{\sigma}^*$;*

3. *(a) $H(\widehat{\sigma}^*(t), u^*(t)) = \sup_{u \in U} H(\widehat{\sigma}^*(t), u)$ almost everywhere;*
   (b) $\sup_{u \in U} H(\widehat{\sigma}^*(t), u)$ is constant everywhere;
   (c) $\widehat{\sigma}^*(t) \neq 0 \in T^*_{\widehat{\gamma}^*(t)}\widehat{Q}$ for each $t \in [a, b]$;
   (d) $\sigma_0^*(t)$ is constant and $\sigma_0^*(t) \leq 0$.

Comments:

1. There exists an abuse of notation between $\widehat{\sigma}(t) \in T^*\widehat{M}$ and $\widehat{\sigma}(t) \in T^*_{\widehat{\gamma}^*(t)}\widehat{M}$. We assume that the meaning of $\widehat{\sigma}$ in each situation will be clear from the context.

2. Condition (2) is immediately satisfied because $\widehat{\sigma}^*$ is a covector along $\widehat{\gamma}^*$.

3. Conditions $(3a)$ and $(3b)$ imply that the Hamiltonian function is constant almost everywhere for $t \in [a, b]$.

4. In item $(3a)$, if U is a closed set, then the maximum of the Hamiltonian over the controls is considered instead of the the supremum over the controls. But in condition $(3b)$ we can always consider the maximum, instead of the supremum, because item $(3a)$ guarantees that the supremum of the Hamiltonian is reached in the optimal curve. Thus, from now on and according to the classical literature, we refer to the assertion $(3a)$ as the condition of maximization of the Hamiltonian over the controls.

5. Condition ($3c$) implies that $\sigma_0^*(t) \neq 0$ or $\sigma^*(t) \neq 0 \in T^*_{\gamma^*(t)}M$ for each $t \in [a,b]$. Locally the condition ($3c$) states that for each $t \in [a,b]$ there exists a coordinate of $\widehat{\sigma}^*(t)$ nonzero, $(p_i \circ \widehat{\sigma}^*)(t) = \sigma_i^*(t) \neq 0$.

6. From the Hamilton's equations of the system $(T^*\widehat{M}, \omega, H^{u(t)})$, it is concluded that σ_0 is constant along the integral curves of $(\widehat{X}^{T^*})^{\{u(t)\}}$, since $\dot{p}_0 = 0$. Hence the first result in ($3d$) is immediate for every integral curve of $(\widehat{X}^{T^*})^{\{u(t)\}}$. As σ_0^* is constant, $\widehat{\sigma}^*$ may be normalized without loss of generality. Thus it is assumed that either $\sigma_0^* = 0$ or $\sigma_0^* = -1$ because of the second result in ($3d$).

7. Pontryagin's Maximum Principle only guarantees that given a solution of $\widehat{OCP}$ there exists a solution of HP. Hence, in principle, both problems are not equivalent.

Observe that Maximum Principle guarantees the existence of a covector along the optimal curve, but it does not say anything about the uniqueness of the covector. Indeed, this covector may not be unique. Depending on the covector we associate with the optimal curves, different kind of curves can be defined.

Definition 3.15. *A curve $(\widehat{\gamma}, u)\colon [a,b] \to \widehat{Q} \times U$ for $\widehat{OCP}$ is*

1. *an **extremal** if there exist $\widehat{\sigma}\colon [a,b] \to T^*\widehat{Q}$ such that $\widehat{\gamma} = \pi_{T\widehat{Q}} \circ \widehat{\sigma}$ and $(\widehat{\sigma}, u)$ satisfies the necessary conditions of PMP;*

2. *a **normal extremal** if it is an extremal with $\sigma_0 = -1$ and $\widehat{\sigma}$ is called a **normal lift or momenta**;*

3. *an **abnormal extremal** if it is an extremal with $\sigma_0 = 0$ and $\widehat{\sigma}$ is called an **abnormal lift or momenta**;*

4. *a **strictly abnormal extremal** if it is not a normal extremal, but it is abnormal;*

5. *a **strictly normal extremal** if it is not a abnormal extremal, but it is normal.*

In [2, 73] there are some examples of optimal control problems whose solutions are searched using Pontryagin's Maximum Principle.

Observe that if $\widehat{\gamma}\colon I \to \widehat{Q}$ is an integral curve of a vector field, there always exists a lift of $\widehat{\gamma}$ to a curve $\widehat{\sigma}\colon I \to T^*\widehat{Q}$, given an initial condition for the cofibers, which is an integral curve of the cotangent lift of the given vector field on $\widehat{Q}$. Analogously, if the system is given by a vector field along the projection $\widehat{\pi}\colon \widehat{Q} \times U \to \widehat{Q}$.

Therefore, the items 1 and 2 in Theorem 3.14 do not provide any information related with the optimality. They only ask for the fulfilment of a final condition in the integral curve. The accomplishment of this depends on the accessibility of the problem, see [23, 63].

The real contribution of PMP is the third item related with the optimality through the maximization of the Hamiltonian, that will be only satisfied if the initial conditions for the fibers are chosen suitably. This is the key element of the proof of Pontryagin's Maximum Principle. In other words, we can always find a cotangent lift of an integral curve such that conditions 1 and 2 are satisfied under the assumption of accessibility, but it is not guaranteed the fulfilment of conditions in assertion 3 in Theorem 3.14.

If we write the Hamiltonian function for the abnormal and the normal case, the difference is that the cost function does not play any role in the Hamiltonian for abnormal extremals. That is why it is said the abnormal extremals only depend on the geometry of the control system. But to determine the optimality of the abnormal extremals, the cost function is essential. In fact, for the same control system different optimal control problems can be stated depending on the cost function, in such a way that the abnormal extremals are minimizers only for some of the problems.

To conclude, the strict abnormality characterizes the abnormal extremals that are not normal. An extremal is not normal when there does not exist any covector that satisfies Hamilton's equations for normality. Thus it is necessary to know the cost function in order to prove that there exists only one kind of lift.

4 Proof of Pontryagin's Maximum Principle for fixed time and fixed endpoints

To prove Pontryagin's Maximum Principle it is necessary to use analytic results about absolute continuity and lower semicontinuity for real functions, and properties of convex cones. For the details see Appendix A and Appendix D and references therein. The reader is referred to §3.3 for results on perturbations of a trajectory in a control system.

In the literature of optimal control, the proof of the Maximum Principle has been discussed taking into account varying hypotheses, [2, 8, 18, 44, 70, 71, 72]. Most authors believe and justify that the origin of this Principle is the calculus of variations, see [76] for instance.

Proof. (Theorem 3.14: Pontryagin's Maximum Principle, PMP)
1. As $(\widehat{\gamma}^*, u^*)$ is a solution of $\widehat{OCP}$, if τ is in $[a,b]$, for every initial condition $\widehat{\sigma}_\tau$ in $T^*_{\widehat{\gamma}^*(\tau)}\widehat{Q}$, we have a solution of HP, $(\widehat{\gamma}^*, \widehat{\sigma})\colon [a,b] \to T^*\widehat{Q}$, satisfying that initial condition. The covector $\widehat{\sigma}_\tau$ must be chosen conveniently so that the remaining conditions of the PMP are satisfied.

According to §3.3, we construct the tangent perturbation cone $\widehat{K}_b$ in $T_{\widehat{\gamma}^*(b)}\widehat{Q}$ that contains all tangent vectors associated with perturbations of the trajectory $\widehat{\gamma}^*$ corresponding to variations of u^*; see Definition 3.11.

Let us consider the vector $(-1, \mathbf{0})_{\widehat{\gamma}^*(b)} \in T_{\widehat{\gamma}^*(b)}\widehat{Q}$, where the zero in bold emphasizes that 0 is a vector in $T_{\gamma^*(b)}M$. The vector $(-1, \mathbf{0})$ has the following properties:

1. the variation of $x^0(t) = \int_a^t F(\gamma^*(s), u^*(s))ds$ along $(-1, \mathbf{0})$ is negative;

2. it is not interior to $\widehat{K}_b$.

   Let us prove the second assertion. Take a local chart at $\widehat{\gamma}^*(b)$ and work on the image of the local chart, in $\mathbb{R}^{m+1}$, without changing the notation.

   If $(-1, \mathbf{0})_{\widehat{\gamma}^*(b)}$ was interior to $\widehat{K}_b$, by Proposition 3.12 there would exist a positive number ϵ, such that for every $s \in (0, \epsilon)$, there would exist a positive number s', close to s, and a perturbation of the control $u[\pi^s]$ such that

   $$\widehat{\gamma}[\pi^s](b) = (\gamma^0[\pi^s](b), \gamma[\pi^s](b)) = \widehat{\gamma}^*(b) + s'(-1, \mathbf{0}).$$

   For this perturbed trajectory we have

   $$\gamma^0[\pi^s](b) < {\gamma^*}^0(b) \quad \text{and} \quad \gamma[\pi^s](b) = \gamma^*(b).$$

   Hence there would be a trajectory, $\widehat{\gamma}[\pi^s]$, from $\gamma^*(a)$ to $\gamma^*(b)$ with less cost than $\widehat{\gamma}^*$. Hence $\widehat{\gamma}^*$ would not be optimal. In other words, $(-1, \mathbf{0})_{\widehat{\gamma}^*(b)}$ is the direction of decreasing of the functional to be minimized in the extended optimal control problem.

The second property implies that $\widehat{K}_b$ cannot be equal to $T_{\widehat{\gamma}^*(b)}\widehat{Q}$. As $(-1,\mathbf{0})_{\widehat{\gamma}^*(b)}$ is not interior to $\widehat{K}_b$, there exist a separating hyperplane of $\widehat{K}_b$ and $(-1,\mathbf{0})_{\widehat{\gamma}^*(b)}$ by Proposition D.15; that is, there exists a nonzero covector determining a separating hyperplane. Let $\widehat{\sigma}_b \in T^*_{\widehat{\gamma}^*(b)}\widehat{Q}$ be nonzero such that $\ker \widehat{\sigma}_b$ is a separating hyperplane satisfying

$$\langle \widehat{\sigma}_b, (-1,\mathbf{0})\rangle \geq 0,$$
$$\langle \widehat{\sigma}_b, \widehat{v}_b\rangle \leq 0 \quad \forall \widehat{v}_b \in \widehat{K}_b.$$

Observe that if $\widehat{\sigma}_b = 0 \in T^*_{\widehat{\gamma}^*(b)}\widehat{Q}$, $\ker \widehat{\sigma}_b$ does not determine a hyperplane, but the whole space $T^*_{\widehat{\gamma}^*(b)}\widehat{Q}$.

Given the initial condition $\widehat{\sigma}_b \in T^*_{\widehat{\gamma}^*(b)}\widehat{Q}$, there exists only one integral curve $\widehat{\sigma}^*$ of $(\widehat{X}^{T^*})^{\{u^*\}}$ such that $\widehat{\sigma}^*(b) = \widehat{\sigma}_b$. Hence $(\widehat{\sigma}^*, u^*)$ is a solution of HP.

2. Obviously, by construction, $\widehat{\gamma}^* = \pi_{\widehat{Q}} \circ \widehat{\sigma}^*$.

Now we prove that $\widehat{\sigma}^*$, the solution of HP, satisfies the remaining conditions of the PMP.

(3a) $H(\widehat{\sigma}^*(t), u^*(t)) = \sup_{u\in U} H(\widehat{\sigma}^*(t), u)$ almost everywhere.

We are going to prove the statement for every Lebesgue time, hence it will be true almost everywhere, see Appendix A for more details. Suppose that there exists a control $\widetilde{u}\colon I \to \overline{U}$ and a Lebesgue time t_1 such that u^* does not give the supremum of the Hamiltonian at t_1; that is,

$$H(\widehat{\sigma}^*(t_1), \widetilde{u}(t_1)) > H(\widehat{\sigma}^*(t_1), u^*(t_1)).$$

As $H(\widehat{p}, u) = \langle \widehat{p}, \widehat{X}(\widehat{x}, u)\rangle$,

$$\langle \widehat{\sigma}^*(t_1), \widehat{X}(\widehat{\gamma}^*(t_1), \widetilde{u}(t_1)) - \widehat{X}(\widehat{\gamma}^*(t_1), u^*(t_1))\rangle > 0;$$

that is, $\langle \widehat{\sigma}^*(t_1), \widehat{v}[\pi_1]\rangle > 0$ where $\widehat{v}[\pi_1] = \widehat{X}(\widehat{\gamma}^*(t_1), \widetilde{u}(t_1)) - \widehat{X}(\widehat{\gamma}^*(t_1), u^*(t_1))$ in $\widehat{K}_{t_1} \subset T_{\widehat{\gamma}^*(t_1)}\widehat{Q}$ is the elementary perturbation vector associated with the perturbation data $\pi_1 = \{t_1, 1, \widetilde{u}(t_1)\}$ by Proposition 3.4.

Let $\widehat{V}[\pi_1]\colon [t_1, b] \to T\widehat{Q}$ be the integral curve of $(\widehat{X}^T)^{\{u^*\}}$ with $(t_1, \widehat{\gamma}^*(t_1), \widehat{v}[\pi_1])$ as initial condition. For $\widehat{\sigma}^*$, solution of HP, the continuous function $\langle \widehat{\sigma}^*, \widehat{V}[\pi_1]\rangle\colon [t_1, b] \to \mathbb{R}$ is constant everywhere by Proposition B.5. Hence $\langle \widehat{\sigma}^*(t_1), \widehat{v}[\pi_1]\rangle > 0$ implies that $\langle \widehat{\sigma}_b, \widehat{V}[\pi_1](b)\rangle > 0$, which is a contradiction with $\langle \widehat{\sigma}_b, \widehat{v}_b\rangle \leq 0$ for every $\widehat{v}_b \in \widehat{K}_b$, since $\widehat{V}[\pi_1](b) \in \widehat{K}_b$.

Therefore,

$$H(\widehat{\sigma}^*(t), u^*(t)) = \sup_{u\in U} H(\widehat{\sigma}^*(t), u)$$

at every Lebesgue time on $[a, b]$, so almost everywhere.

(3b) $\sup_{u\in U} H(\widehat{\sigma}^*(t), u)$ is constant everywhere.

In fact, because of $(3a)$ we know that the supremum is achieved along the optimal curve, so at the end we are going to prove that $\max_{u\in U} H(\widehat{\lambda}^*(t), u)$ is constant everywhere. To simplify the notation we define the function

$$\begin{array}{rcl} \mathcal{M} \circ \widehat{\sigma}^*\colon I & \longrightarrow & \mathbb{R} \\ t & \longmapsto & \mathcal{M}(\widehat{\sigma}^*(t)) = \sup_{u\in U} H(\widehat{\sigma}^*(t), u). \end{array}$$

In order to prove $(3b)$, it is enough to see that $\mathcal{M}(\widehat{\sigma}^*(t))$ is constant everywhere.

First let us see that $\mathcal{M}\circ\widehat{\sigma}^*$ is lower semicontinuous on I. See Appendix A for details of this property. As $\mathcal{M}(\widehat{\sigma}^*(t))$ is the supremum of the Hamiltonian function with respect to control, for every $\epsilon > 0$, there exists a control $u_{\mathcal{M}}\colon I \to U$ such that

$$H(\widehat{\sigma}^*(t), u_{\mathcal{M}}(t)) \geq \mathcal{M}(\widehat{\sigma}^*(t)) - \frac{\epsilon}{2} \tag{4.11}$$

everywhere.

For each constant control $\widetilde{u}\in U$, $H^{\widetilde{u}}\circ\widehat{\sigma}^* = H(\widehat{\sigma}^*,\widetilde{u})\colon I\to\mathbb{R}$ is continuous on I. Hence for every $t_0\in I$ and $\epsilon>0$, there exists $\delta>0$ such that $\mid t-t_0 \mid<\delta$, we have

$$\mid H^{\widetilde{u}}(\widehat{\sigma}^*(t)) - H^{\widetilde{u}}(\widehat{\sigma}^*(t_0)) \mid < \frac{\epsilon}{2}.$$

If $\widetilde{u} = u_{\mathcal{M}}(t_0)$, then using the continuity of $H^{\widetilde{u}}\circ\widehat{\sigma}^*$

$$\mathcal{M}(\widehat{\sigma}^*(t)) = \sup_{u\in U} H(\widehat{\sigma}^*(t),u) \geq H(\widehat{\sigma}^*(t), u_{\mathcal{M}}(t_0)) \geq$$

$$\geq H(\widehat{\sigma}^*(t_0), u_{\mathcal{M}}(t_0)) - \frac{\epsilon}{2} \geq \mathcal{M}(\widehat{\sigma}^*(t_0)) - \epsilon.$$

The last inequality is true by evaluating Equation (4.11) at t_0. Hence $\mathcal{M}\circ\widehat{\sigma}^*$ is lower semicontinuous at every $t_0\in I$; that is, $\mathcal{M}\circ\widehat{\sigma}^*$ is lower semicontinuous on I.

The control u^* is bounded, that means $\operatorname{Im} u^*$ is contained in a compact set $D\subset U$. Let us define a new function

$$\begin{aligned} \mathcal{M}_D\colon T^*\widehat{Q} &\longrightarrow \mathbb{R} \\ \beta &\longmapsto \mathcal{M}_D(\beta) = \sup\nolimits_{\widetilde{u}\in D} H(\beta,\widetilde{u}). \end{aligned}$$

As $H(\beta,\cdot)\colon D\to\mathbb{R}$, $\widetilde{u}\mapsto H(\beta,\widetilde{u})$ is continuous by hypothesis and D is compact, for every $\beta\in T^*\widehat{Q}$ there exists a control $\widetilde{w}_\beta$ that gives us the maximum of $H(\beta,\widetilde{u})$

$$\mathcal{M}_D(\beta) = \sup_{\widetilde{u}\in D} H(\beta,\widetilde{u}) = H(\beta,\widetilde{w}_\beta). \tag{4.12}$$

Hence $\mathcal{M}_D$ is well–defined on $T^*\widehat{Q}$. The following sketch explains in a compact way the necessary steps to prove that $\mathcal{M}\circ\sigma^*$ is constant everywhere. In this sketch, the figures in bold refer to statements which are going to be proved in the next paragraphs and a.c. stands for absolutely continuous and a.e. for almost everywhere.

$$\left.\begin{array}{l}
\mathcal{M}\circ\widehat{\sigma}^* \text{ is lower semicontinuous on } I \\[4pt]
H^{\widetilde{u}}\in\mathcal{C}^1(T^*\widehat{Q}) \\
\quad\Downarrow_{\mathbf{1}} \\
H^{\widetilde{u}} \text{ is locally Lipschitz } \forall\,\widetilde{u}\in D \\
\quad\Downarrow_{\mathbf{2}} \\
\left.\begin{array}{l} \mathcal{M}_D \text{ is locally Lipschitz on } \operatorname{Im}\widehat{\sigma}^* \\ \widehat{\sigma}^* \text{ is a.c.} \end{array}\right\} \Rightarrow^{\mathbf{3}} \mathcal{M}_D\circ\widehat{\sigma}^* \text{ is a.c.} \Rightarrow \mathcal{M}_D\circ\widehat{\sigma}^* \text{ is continuous} \\
{}^{\mathbf{4}}\, \mathcal{M}_D(\widehat{\sigma}^*(t)) \leq \mathcal{M}(\widehat{\sigma}^*(t)),\ \forall\, t\in[a,b] \\
{}^{\mathbf{5}}\, \mathcal{M}_D(\widehat{\sigma}^*(t)) = \mathcal{M}(\widehat{\sigma}^*(t)) \text{ a.e.}
\end{array}\right\}\Rightarrow^{\mathbf{6}}$$

$$\left.\begin{array}{l}
\mathcal{M}_D\circ\widehat{\sigma}^* \text{ is a.c.} \\
{}^{\mathbf{7}}\, \mathcal{M}_D\circ\widehat{\sigma}^* \text{ has zero derivative}
\end{array}\right\}\Rightarrow^{\mathbf{8}}$$

$$\left.\begin{array}{l}
\Rightarrow^{\mathbf{6}}\ \text{(A. 15)}\ \mathcal{M}_D(\widehat{\sigma}^*(t)) = \mathcal{M}(\widehat{\sigma}^*(t))\ \forall\, t\in[a,b] \\
\Rightarrow^{\mathbf{8}}\ \text{(A. 16)}\ \mathcal{M}_D(\widehat{\sigma}^*(t)) \text{ is constant } \forall\, t\in[a,b]
\end{array}\right\}\Rightarrow^{\mathbf{9}} \mathcal{M}(\widehat{\sigma}^*(t)) \text{ is constant } \forall\, t\in[a,b]$$

1. $H^{\widetilde{u}} \in \mathcal{C}^1(T^*\widehat{Q}) \Rightarrow H^{\widetilde{u}}$ is locally Lipschitz $\forall\, \widetilde{u} \in D$.

The Lipschitzian property applies to functions defined on a metric space. As the property we want to prove is local, we define the distance on a local chart as is explained in Appendix A. For every $\beta \in T^*\widehat{Q}$, let (V_β, ϕ) be a local chart centered at β such that $\phi(\beta) = 0$ and $\phi(V_\beta) = B$, where B is an open ball centered at $0 \in \mathbb{R}^{2m+2}$. If β_1 and β_2 are in V_β, define $d_\phi(\beta_1, \beta_2) = d(\phi(\beta_1), \phi(\beta_2))$ where d is the Euclidean distance in $\mathbb{R}^{2m+2}$.

For every β in $T^*\widehat{Q}$, we get an open neighbourhood V_β using the local chart (V_β, ϕ). As $H^{\widetilde{u}}$ is $\mathcal{C}^1(T^*\widehat{Q})$ and $\widetilde{u}$ lies in the compact set D, by the Mean Value Theorem for every β in $T^*\widehat{Q}$ there exists an open neighbourhood V_β such that $|H^{\widetilde{u}}(\beta_1) - H^{\widetilde{u}}(\beta_2)| < K_\beta d_\phi(\beta_1, \beta_2)$ where K_β does not depend on the control $\widetilde{u}$. Thus $H^{\widetilde{u}}$ is locally Lipschitz on $T^*\widehat{Q}$. Moreover, the Lipschitz constant and the open neighbourhood V_β do not depend on the control since $\widetilde{u}$ is in a compact set.

2. $H^{\widetilde{u}}$ is locally Lipschitz $\forall\, \widetilde{u} \in D \Rightarrow \mathcal{M}_D$ is locally Lipschitz on $\operatorname{Im} \widehat{\sigma}^*$.

Let β be in $\operatorname{Im} \widehat{\sigma}^*$, there exists an open convex neighbourhood V_β such that

$$|H^{\widetilde{u}}(\beta_1) - H^{\widetilde{u}}(\beta_2)| < K_\beta d(\beta_1, \beta_2)$$

for every $\widetilde{u}$ in D and β_1, β_2 in V_β. If $\widetilde{w}_1$, $\widetilde{w}_2$ are the controls in D maximizing $H(\beta_1, \widetilde{u})$ and $H(\beta_2, \widetilde{u})$, respectively, then

$$\begin{aligned} H(\beta_1, \widetilde{w}_2) &\leq H(\beta_1, \widetilde{w}_1), \\ H(\beta_2, \widetilde{w}_1) &\leq H(\beta_2, \widetilde{w}_2). \end{aligned}$$

Moreover, $H^{\widetilde{w}_1}$ and $H^{\widetilde{w}_2}$ are Lipschitz on V_β since the Lipschitz constant and the neighbourhood is independent of the control. Then using the last inequalities

$$\begin{aligned} -K_\beta d(\beta_1, \beta_2) &\leq H^{\widetilde{w}_2}(\beta_1) - H^{\widetilde{w}_2}(\beta_2) \leq H^{\widetilde{w}_1}(\beta_1) - H^{\widetilde{w}_2}(\beta_2) \\ &\leq H^{\widetilde{w}_1}(\beta_1) - H^{\widetilde{w}_1}(\beta_2) \leq K_\beta d(\beta_1, \beta_2). \end{aligned}$$

Observe that by Equation (4.12), $H^{\widetilde{w}_1}(\beta_1) - H^{\widetilde{w}_2}(\beta_2) = \mathcal{M}_D(\beta_1) - \mathcal{M}_D(\beta_2)$. Hence

$$|\mathcal{M}_D(\beta_1) - \mathcal{M}_D(\beta_2)| \leq K_\beta d(\beta_1, \beta_2), \quad \forall\, \beta_1, \beta_2 \in V_\beta; \tag{4.13}$$

that is, $\mathcal{M}_D$ is locally Lipschitz on $\operatorname{Im} \widehat{\sigma}^*$. As $\widehat{\sigma}^*$ is absolutely continuous, $\operatorname{Im} \widehat{\sigma}^*$ is compact. Thus we may choose a Lipschitz constant independent of the point β. Hence

$$|\mathcal{M}_D(\beta_1) - \mathcal{M}_D(\beta_2)| \leq K d(\beta_1, \beta_2), \quad \forall\, \beta_1, \beta_2 \in V_\beta.$$

3. $\mathcal{M}_D$ is locally Lipschitz on $\operatorname{Im} \widehat{\sigma}^*$ and $\widehat{\sigma}^*$ is absolutely continuous $\Rightarrow \mathcal{M}_D \circ \widehat{\sigma}^* \colon I \to \mathbb{R}$ is absolutely continuous $\Rightarrow \mathcal{M}_D \circ \widehat{\sigma}^* \colon I \to \mathbb{R}$ is continuous.

For every $t \in I$, let us consider the neighbourhood $V_{\widehat{\sigma}^*(t)}$ where Equation (4.13) is satisfied. As $\operatorname{Im} \widehat{\sigma}^*$ is a compact set,

- there exists a finite open subcovering $V_{\widehat{\sigma}^*(t_1)}, \ldots, V_{\widehat{\sigma}^*(t_r)}$ of $\{V_{\widehat{\sigma}^*(t)} \;;\; t \in I\}$, and
- there exists a Lebesgue number l of the subcovering; that is, for every two points in an open ball of diameter l there exists an open set of the finite subcovering containing both points.

For the Lebesgue number l, by the uniform continuity of $\widehat{\sigma}^*$, there exists a $\delta_l > 0$ such that for each t_1, t_2 in I with $|t_2 - t_1| < \delta_l$, then $d(\widehat{\sigma}^*(t_2), \widehat{\sigma}^*(t_1)) < l$. Thus there exists an open set of the finite subcovering containing $\widehat{\sigma}^*(t_1)$ and $\widehat{\sigma}^*(t_2)$.

On the other hand, taken $\epsilon > 0$ the absolutely continuity of $\widehat{\sigma}^*$ determines a $\delta_\epsilon > 0$.

To prove the absolute continuity of $\mathcal{M}_D \circ \widehat{\sigma}^*$, take $\delta = \min\{\delta_l, \delta_\epsilon\}$. Then, for every finite number of nonoverlapping subintervals (t_{i_1}, t_{i_2}) of I such that $\sum_{i=1}^n |t_{i_2} - t_{i_1}| < \delta$,

$$\sum_{i=1}^n |\mathcal{M}_D(\widehat{\sigma}^*(t_{i_2})) - \mathcal{M}_D(\widehat{\sigma}^*(t_{i_1}))| \leq \sum_{i=1}^n K d(\widehat{\sigma}^*(t_{i_2}), \widehat{\sigma}^*(t_{i_1})) \leq K\epsilon.$$

In the first step we use that $\delta < \delta_l$ to guarantee that $\widehat{\sigma}^*(t_{i_2})$ and $\widehat{\sigma}^*(t_{i_1})$ are contained in the same open set of the finite subcovering of $\operatorname{Im}\widehat{\sigma}^*$. That allows us to use the property of being locally Lipschitzian. Secondly, we use that $\delta < \delta_\epsilon$ to apply the absolute continuity of $\widehat{\sigma}^*$.

As $\mathcal{M}_D \circ \widehat{\sigma}^*$ is absolutely continuous on I, $\mathcal{M}_D \circ \widehat{\sigma}^*$ is continuous on I.

4. $\mathcal{M}_D(\widehat{\sigma}^*(t)) \leq \mathcal{M}(\widehat{\sigma}^*(t))$ everywhere.

Observe that

$$\mathcal{M}_D(\widehat{\sigma}^*(t)) = \sup_{u \in D} H(\widehat{\sigma}^*(t), u) \leq \sup_{u \in U} H(\widehat{\sigma}^*(t), u) = \mathcal{M}(\widehat{\sigma}^*(t)),$$

for each $t \in I$.

5. $\mathcal{M}(\widehat{\sigma}^*(t)) = \mathcal{M}_D(\widehat{\sigma}^*(t))$ almost everywhere.

For each $t \in I$ there exists a control $w(t)$ maximizing $H(\widehat{\sigma}^*(t), u)$ over the controls in D because of condition (3a),

$$\mathcal{M}_D(\widehat{\sigma}^*(t)) = \sup_{u \in D} H(\widehat{\sigma}^*(t), u) = H(\widehat{\sigma}^*(t), w(t)).$$

As $u^*(t) \in D$ for each $t \in I$,

$$\sup_{u \in D} H(\widehat{\sigma}^*(t), u) = \sup_{u \in U} H(\widehat{\sigma}^*(t), u) = \mathcal{M}(\widehat{\sigma}^*(t)) = H(\widehat{\sigma}^*(t), u^*(t))$$

almost everywhere by (3a). Thus $\mathcal{M}(\widehat{\sigma}^*(t)) = \mathcal{M}_D(\widehat{\sigma}^*(t))$ a.e..

6. Applying Proposition A.7, we have $\mathcal{M}_D(\widehat{\sigma}^*(t)) = \mathcal{M}(\widehat{\sigma}^*(t))$ everywhere on I, because $\mathcal{M}_D \circ \widehat{\sigma}^*$ is continuous on I, $\mathcal{M} \circ \widehat{\sigma}^*$ is lower semicontinuous, $\mathcal{M}_D(\widehat{\sigma}^*(t)) \leq \mathcal{M}(\widehat{\sigma}^*(t))$ everywhere and $\mathcal{M}_D(\widehat{\sigma}^*(t)) = \mathcal{M}(\widehat{\sigma}^*(t))$ almost everywhere.

7. $\mathcal{M}_D \circ \widehat{\sigma}^*$ has zero derivative.

As $\mathcal{M}_D \circ \widehat{\sigma}^*$ is absolutely continuous on I, by Corollary A.4 it has a derivative almost everywhere. As the intersection of two sets of full measure is not empty, see Appendix A, there exists a $t_0 \in I$ such that $\mathcal{M}_D \circ \widehat{\sigma}^*$ is derivable at t_0 and $\mathcal{M}_D(\widehat{\sigma}^*(t_0)) = H(\widehat{\sigma}^*(t_0), u^*(t_0))$. For each $t \neq t_0$, by the definition of $\mathcal{M}_D$, we have

$$\mathcal{M}_D(\widehat{\sigma}^*(t)) = \sup_{u \in D} H(\widehat{\sigma}^*(t), u) \geq H(\widehat{\sigma}^*(t), u^*(t_0))$$

because $u^*(t_0) \in D$. Thus $\mathcal{M}_D(\widehat{\sigma}^*(t)) - \mathcal{M}_D(\widehat{\sigma}^*(t_0)) \geq H(\widehat{\sigma}^*(t), u^*(t_0)) - H(\widehat{\sigma}^*(t_0), u^*(t_0))$.

If $t - t_0 > 0$,

$$\frac{\mathcal{M}_D(\widehat{\sigma}^*(t)) - \mathcal{M}_D(\widehat{\sigma}^*(t_0))}{t - t_0} \geq \frac{H(\widehat{\sigma}^*(t), u^*(t_0)) - H(\widehat{\sigma}^*(t_0), u^*(t_0))}{t - t_0}.$$

Let us compute the right derivative of $\mathcal{M}_D \circ \widehat{\sigma}^*$ at t_0

$$\begin{aligned} \left.\frac{d(\mathcal{M}_D \circ \widehat{\sigma}^*)}{dt}\right|_{t=t_0^+} &= \lim_{t \to t_0^+} \frac{\mathcal{M}_D(\widehat{\sigma}^*(t)) - \mathcal{M}_D(\widehat{\sigma}^*(t_0))}{t - t_0} \geq \lim_{t \to t_0^+} \frac{H^{u^*(t_0)}(\widehat{\sigma}^*(t)) - H^{u^*(t_0)}(\widehat{\sigma}^*(t_0))}{t - t_0} \\ &= \mathrm{L}_{\widehat{X}^{T^*\{u^*(t_0)\}}_{\widehat{\sigma}^*(t_0)}} H^{u^*(t_0)} = 0 \end{aligned}$$

since $i\left(\widehat{X}^{T^*\{u^*(t_0)\}}_{\widehat{\sigma}^*(t_0)}\right)\omega = \left(dH^{u^*(t_0)}\right)_{\widehat{\sigma}^*(t_0)}$.

Similarly, if $t - t_0 < 0$,

$$\left.\frac{d(\mathcal{M}_D \circ \widehat{\sigma}^*)}{dt}\right|_{t=t_0^-} \leq 0.$$

Hence the derivative of $\mathcal{M}_D \circ \widehat{\sigma}^*$ is zero almost everywhere.

8. Applying Theorem A.5, $\mathcal{M}_D \circ \widehat{\sigma}^*$ is constant everywhere, because $\mathcal{M}_D \circ \widehat{\sigma}^*$ is absolutely continuous.

9. As $\mathcal{M}_D(\widehat{\sigma}^*(t))$ and $\mathcal{M}(\widehat{\sigma}^*(t))$ coincide everywhere, $\mathcal{M} \circ \widehat{\sigma}^*$ is constant everywhere on I.

(3c) $\widehat{\sigma}^*(t) \neq 0 \in T^*_{\widehat{\gamma}^*(t)}\widehat{Q}$ for each $t \in [a, b]$.

Let us suppose that there exists $\tau \in [a, b]$ such that $\widehat{\sigma}^*(\tau) = 0 \in T^*_{\widehat{\gamma}^*(\tau)}\widehat{Q}$. As $\widehat{\sigma}^*$ is a generalized integral curve of $(\widehat{X}^{T^*})^{\{u^*\}}$, a linear vector field over $\widehat{X}$, we have $\widehat{\sigma}^*(t) = 0$ for each $t \in [a, b]$. As there exists at least a time such that $\widehat{\sigma}^*(\tau) \neq 0$, we arrive at a contradiction. Hence $\widehat{\sigma}^*(t) \neq 0$ for each $t \in [a, b]$.

(3d) $\sigma_0^*(t)$ is constant, $\sigma_0^*(t) \leq 0$.

From the equations satisfied by the generalized integral curves of $(\widehat{X}^{T^*})^{\{u^*\}}$, we have p_0 is constant. It was seen that $\langle \widehat{\sigma}_b, (-1, \mathbf{0})\rangle \geq 0$ is equivalent to $(p_0 \circ \widehat{\sigma}^*)(b) = \sigma_0(b) \leq 0$. Hence $\sigma_0 \leq 0$ for each $t \in [a, b]$. □

Comment: As $\widehat{\sigma}_b$ is determined up to multiply by a positive real number, we may assume that $\sigma_0 \in \{-1, 0\}$.

The way in which perturbations have been used in this proof gives some clue concerning the fact that the tangent perturbation cone is understood as an approximation of the reachable set defined in Appendix C. A precise meaning of this approximation is explained in Appendix C.

The covector in the proof has been chosen such that

$$\begin{aligned} &\langle \widehat{\sigma}_b, (-1, \mathbf{0})\rangle \geq 0, \\ &\langle \widehat{\sigma}_b, \widehat{v}_b\rangle \leq 0 \quad \forall\, \widehat{v}_b \in \widehat{K}_b. \end{aligned}$$

In the abnormal case $\sigma_0 = 0$ and the first inequality is satisfied with equality. Thus the covector is contained in the separating hyperplane. It would be interesting to determine geometrically what else must happen in order to have abnormal minimizers.

5 Pontryagin's Maximum Principle for nonfixed time and nonfixed endpoints

Now, that Pontryagin's Maximum Principle has been proved for time and endpoints fixed, let us state the different problems related to Pontryagin's Maximum Principle with nonfixed time and nonfixed endpoints.

5.1 Statement of the optimal control problem with time and endpoints nonfixed

We consider the elements Q, U, X, F, $\mathcal{S}$ and π_2 with the same properties as in §2, §3.1. Let S_a and S_f be submanifolds of Q.

Statement 5.1. ***(Free Optimal Control Problem,*** *FOCP) Given the elements Q, U, X, F, and the disjoint submanifolds of Q, S_a and S_f, consider the following problem.*

Find $b \in \mathbb{R}$ and $(\gamma^*, u^*)\colon [a, b] \to Q \times U$ such that

(1) endpoint conditions: $\gamma^*(a) \in S_a$, $\gamma^*(b) \in S_f$,

(2) γ^* is an integral curve of $X^{\{u^*\}}$: $\dot{\gamma}^* = X^{\{u^*\}} \circ (\gamma^*, \mathrm{id})$, and

(3) minimal condition: $\mathcal{S}[\gamma^*, u^*] = \int_a^b F(\gamma^*(t), u^*(t))dt$ is minimum over all curves (γ, u) satisfying (1) *and* (2).

The tuple (Q, U, X, F, S_a, S_f) denotes the *free optimal control problem.*

Statement 5.2. ***(Extended Free Optimal Control Problem,*** $\widehat{\textbf{FOCP}}$*) Given the FOCP, (Q, U, X, F, S_a, S_f), and the elements $\widehat{Q}$ and $\widehat{X}$ defined in §3.2, consider the following problem.*

Find $b \in \mathbb{R}$ and $(\widehat{\gamma}^*, u^*)\colon [a, b] \to \widehat{Q} \times U$, with $\gamma^* = \pi_2 \circ \widehat{\gamma}^*$, such that

(1) endpoint conditions: $\widehat{\gamma}^*(a) \in \{0\} \times S_a$, $\gamma^*(b) \in S_f$,

(2) $\widehat{\gamma}^*$ is an integral curve of $\widehat{X}^{\{u^*\}}$: $\dot{\widehat{\gamma}}^* = \widehat{X}^{\{u^*\}} \circ (\widehat{\gamma}^*, \mathrm{id})$, and

(3) minimal condition: ${\gamma^*}^0(b)$ is minimum over all curves $(\widehat{\gamma}, u)$ satisfying (1) *and* (2).

The tuple $(\widehat{Q}, U, \widehat{X}, S_a, S_f)$ denotes the *extended free optimal control problem.*

5.2 Perturbation of the time and the endpoints

In this case of nonfixed time and nonfixed endpoint optimal control problems, we not only modify the control as explained in §3.3, but also modify the final time and the endpoint conditions. As was mentioned in §3.3, the following constructions obtained from perturbing the final time and the endpoint conditions are also general for any vector field depending on parameters.

5.2.1 Time perturbation vectors and associated cones

We study how to perturb the interval of definition of the control taking advantage of the fact that the final time is another unknown for the free optimal control problems.

Let X be a vector field on M along the projection $\pi\colon M \times U \to M$, $I \subset \mathbb{R}$ be a closed interval and $(\gamma, u)\colon I = [a, b] \to M \times U$ a curve such that γ is an integral curve of $X^{\{u\}}$.

Let $\pi_\pm = \{\tau, l_\tau, \delta\tau, u_\tau\}$, where τ is a Lebesgue time in (a, b) for $X \circ (\gamma, u)$, $l_\tau \in \mathbb{R}^+ \cup \{0\}$, $\delta\tau \in \mathbb{R}$, $u_\tau \in U$. For every $s \in \mathbb{R}^+$ small enough such that $a < \tau - (l_\tau - \delta\tau)s$, consider $u[\pi^s_\pm]\colon [a, b + \delta\tau s] \to U$ defined by

$$u[\pi^s_\pm](t) = \begin{cases} u(t), & t \in [a, \tau - (l_\tau - \delta\tau)s], \\ u_\tau, & t \in (\,\tau - (l_\tau - \delta\tau)s, \tau + \delta\tau s\,]\,, \\ u(t), & t \in (\,\tau + \delta\tau s, b + \delta\tau s\,]\,, \end{cases}$$

if $\delta\tau < 0$, and by

$$u[\pi^s_\pm](t) = \begin{cases} u(t), & t \in [a, \tau - (l_\tau - \delta\tau)s], \\ u_\tau, & t \in (\,\tau - (l_\tau - \delta\tau)s, \tau + \delta\tau s\,]\,, \\ u(t - \delta\tau s), & t \in (\,\tau + \delta\tau s, b + \delta\tau s\,]\,, \end{cases}$$

if $\delta\tau \geq 0$.

Definition 5.3. *The function* $u[\pi_\pm^s]$ *is called a* ***perturbation of*** u ***specified by the data*** $\pi_\pm = \{\tau, l_\tau, \delta\tau, u_\tau\}$.

Associated to $u[\pi_\pm^s]$ we consider the mapping $\gamma[\pi_\pm^s]\colon [a, b+\delta\tau s] \to M$, the generalized integral curve of $X^{\{u[\pi_\pm^s]\}}$ with initial condition $(a, \gamma(a))$.

Given $\epsilon > 0$, define

$$\begin{array}{rcl} \varphi_{\pi_\pm}\colon [\tau, b]\times[0,\epsilon] & \longrightarrow & M \\ (t,s) & \longmapsto & \varphi_{\pi_\pm}(t,s) = \gamma[\pi_\pm^s](t+\delta\tau s) \end{array}$$

For every $t \in [\tau, b]$, $\varphi_{\pi_\pm}^t\colon [0,\epsilon] \to M$ is given by $\varphi_{\pi_\pm}^t(s) = \varphi_{\pi_\pm}(t,s)$.

As explained in §3.3, the control $u[\pi_\pm^s]$ depends continuously on the parameters s and $\pi_\pm = \{\tau, l_\tau, \delta\tau, u_\tau\}$. Hence the curve $\varphi_{\pi_\pm}^t$ depends continuously on s and $\pi_\pm = \{\tau, l_\tau, \delta\tau, u_\tau\}$, then it converges uniformly to γ as s tends to 0. See [25, 31] for more details of the differential equations depending continuously on parameters.

Let us prove that the curve $\varphi_{\pi_\pm}^\tau$ has a tangent vector at $s = 0$; cf. Proposition 3.4.

Proposition 5.4. *Let* τ *be a Lebesgue time. If* $u[\pi_\pm^s]$ *is the perturbation of the control* u *specified by the data* $\pi_\pm = \{\tau, l_\tau, \delta\tau, u_\tau\}$ *such that* $\tau + s\delta\tau$ *is a Lebesgue time, then the curve* $\varphi_{\pi_\pm}^\tau\colon [0,\epsilon] \to M$ *is differentiable at* $s = 0$. *Its tangent vector is*

$$X(\gamma(\tau), u(\tau))\, \delta\tau + [X(\gamma(\tau), u_\tau) - X(\gamma(\tau), u(\tau))]\; l_\tau.$$

Proof. As in the proof of Proposition 3.4, we compute the limit

$$A = \lim_{s\to 0} \frac{(x^i \circ \varphi_{\pi_\pm}^\tau)(s) - (x^i \circ \varphi_{\pi_\pm}^\tau)(0)}{s} = \lim_{s\to 0} \frac{\gamma^i[\pi_\pm^s](\tau + \delta\tau s) - \gamma^i(\tau)}{s}$$

As γ is an absolutely continuous integral curve of $X^{\{u\}}$, $\dot\gamma(t) = X(\gamma(t), u(t))$ at every Lebesgue time. Then by integration

$$\gamma^i(\tau) - \gamma^i(a) = \int_a^\tau f^i(\gamma(t), u(t))dt$$

and similarly for $\gamma[\pi_\pm^s]$ and $u[\pi_\pm^s]$. Observe that $\gamma[\pi_\pm^s](t) = \gamma(t)$ and $u[\pi_\pm^s](t) = u(t)$ for $t \in [a, \tau - (l_\tau - \delta\tau)s]$.

Here, we should consider three different possibilities

- if $0 \le \delta\tau \le l_\tau$, then $\tau - (l_\tau - \delta\tau)s < \tau < \tau + \delta\tau s$;
- if $\delta\tau < 0$, then $\tau - (l_\tau - \delta\tau)s < \tau + \delta\tau s < \tau$;
- if $0 < l_\tau < \delta\tau$, then $\tau < \tau - (l_\tau - \delta\tau)s < \tau + \delta\tau s$.

We prove the proposition for the first case and the other cases follow analogously.

$$\begin{aligned} A &= \lim_{s\to 0} \frac{\int_a^{\tau+\delta\tau s} f^i(\gamma[\pi_\pm^s](t), u[\pi_\pm^s](t))dt - \int_a^\tau f^i(\gamma(t), u(t))dt}{s} \\ &= \lim_{s\to 0} \frac{\int_{\tau-(l_\tau-\delta\tau)s}^{\tau+\delta\tau s} f^i(\gamma[\pi_\pm^s](t), u_\tau)dt - \int_{\tau-(l_\tau-\delta\tau)s}^{\tau} f^i(\gamma(t), u(t))dt}{s}. \end{aligned}$$

We need $\tau + \delta\tau s$ to be a Lebesgue time in order to use Equation (A.17).

$$\begin{aligned} A &= \lim_{s\to 0} \frac{f^i(\gamma[\pi^s_\pm](\tau+\delta\tau s), u_\tau) l_\tau s - f^i(\gamma(\tau), u(\tau))(l_\tau - \delta\tau)s + o(s)}{s} \\ &= \lim_{s\to 0} f^i(\gamma[\pi^s_\pm](\tau+\delta\tau s), u_\tau) l_\tau - f^i(\gamma(\tau), u(\tau))(l_\tau - \delta\tau). \end{aligned}$$

As f^i is continuous on M, we have

$$\begin{aligned} A &= [f^i(\gamma(\tau), u_\tau) - f^i(\gamma(\tau), u(\tau))]\, l_\tau + f^i(\gamma(\tau), u(\tau))\ \delta\tau \\ &= \mathrm{L}([X(\gamma(\tau), u(\tau))\ \delta\tau + (X(\gamma(\tau), u_\tau) - X(\gamma(\tau), u(\tau)))\ \ l_\tau])(x^i). \end{aligned}$$

□

Definition 5.5. *The tangent vector*

$$v[\pi_\pm] = X(\gamma(\tau), u(\tau))\ \delta\tau + [X(\gamma(\tau), u_\tau) - X(\gamma(\tau), u(\tau))]\ \ l_\tau$$

is the ***perturbation vector associated to the perturbation data*** $\pi_\pm = \{\tau, l_\tau, \delta\tau, u_\tau\}$.

If we disturb the control r times at r different Lebesgue times as in §3.3.1 and also the domain of the curve (γ, u) as just described, that is, $\pi = \{\pi_1, \ldots, \pi_r, \pi_\pm\}$, with $a < t_1 \leq \ldots \leq t_r \leq \tau < b$, then $\gamma[\pi^s]$ is the generalized integral curve of $X^{\{u[\pi^s]\}}$ with initial condition $(a, \gamma(a))$. Consider the curve $\varphi^t_\pi \colon [0,\epsilon] \to M$ for $t \in [\tau, b]$ given by $\varphi^t_\pi(s) = \gamma[\pi^s](t + \delta\tau s)$.

Corollary 5.6. *The vector tangent to the curve* $\varphi^t_{\pi_\pm} \colon [0,\epsilon] \to M$ *at* $s = 0$ *is* $X(\gamma(t), u(t))\ \delta\tau + V[\pi_1](t) + \ldots + V[\pi_n](t)$*, where* $V[\pi_i] \colon [t_i, b] \to TM$ *is the generalized integral curve of* $\left(X^T\right)^{\{u\}}$ *with initial condition* $(t_i, (\gamma(t_i), v[\pi_i]))$.

This corollary may be proved taking into account Propositions 3.6 and 5.4, Corollary 3.9 and Appendix B.

Now, at a Lebesgue time $t \in (a, b)$, we construct a new cone that contains the perturbation vectors in Definition 3.11 and $\pm X(\gamma(t), u(t))$.

Definition 5.7. *The* ***time perturbation cone*** $K^\pm_t$ *at every Lebesgue time t is the smallest closed cone in the tangent space at* $\gamma(t)$ *containing* K_t *and* $\pm X(\gamma(t), u(t))$*,*

$$K^\pm_t = \overline{\mathrm{conv}\left(\{\pm\alpha X(\gamma(t), u(t)) \mid \alpha \in \mathbb{R}\} \bigcup \left(\bigcup_{\substack{a<\tau\leq t \\ \tau \text{ is a Lebesgue time}}} \left(\Phi^{X\{u\}}_{(t,\tau)}\right)_* \mathcal{V}_\tau \right)\right)},$$

where $\mathcal{V}_\tau$ *denotes the set of elementary perturbation vectors at* τ*, see Definition 3.11.*

Enlarging the cone K_τ to $K^\pm_\tau$ allows us to introduce time variations.

Proposition 5.8. *If* t_2 *is a Lebesgue time greater than* t_1*, then*

$$\left(\Phi^{X\{u\}}_{(t_2,t_1)}\right)_* K^\pm_{t_1} \subset K^\pm_{t_2}.$$

Proof. We have

$$K^\pm_{t_1} = \overline{\mathrm{conv}\left(\{\pm\alpha X(\gamma(t_1), u(t_1)) \mid \alpha \in \mathbb{R}\} \bigcup \left(\bigcup_{\substack{a<\tau\leq t_1 \\ \tau \text{ is a Lebesgue time}}} \left(\Phi^{X\{u\}}_{(t_1,\tau)}\right)_* \mathcal{V}_\tau \right)\right)}.$$

Just for simplicity we use $\mathcal{C}^{\pm}_{t_1}$ to denote

$$\operatorname{conv}\left(\{\pm\alpha X(\gamma(t_1),u(t_1))\,|\,\alpha\in\mathbb{R}\}\bigcup\left(\bigcup_{\substack{a<\tau\le t_1\\ \tau \text{ is a Lebesgue time}}}\left(\Phi^{X^{\{u\}}}_{(t_1,\tau)}\right)_*\mathcal{V}_\tau\right)\right).$$

1. The set $\mathcal{C}^{\pm}_{t_1}$ being convex, if v is interior to $K^{\pm}_{t_1}$, then v is interior to $\mathcal{C}^{\pm}_{t_1}$ by Proposition D.5, item (d). Hence, by Proposition D.4

$$v=\delta t_1 X(\gamma(t_1),u(t_1))+\sum_{i=1}^{r} l_i V[\pi_i](t_1),$$

where every $V[\pi_i](t_1)$ is the transported perturbation vector $v[\pi_i]$ of class I from t_i to t_1 by the flow of $X^{\{u\}}$. By definition of the cone and the linearity of the flow, $\left(\Phi^{X^{\{u\}}}_{(t_2,t_1)}\right)_* v$ is in $K^{\pm}_{t_2}$, since $\left(\Phi^{X^{\{u\}}}_{(t_2,t_1)}\right)_*(X(\gamma(t_1),u(t_1)))=X(\gamma(t_2),u(t_2))$, because both sides of the equality are the unique solutions of the variational equation along γ associated with $X^{\{u\}}$ with initial condition $(t_1,X(\gamma(t_1),u(t_1))$. See Appendix B.2 for more details.

2. If v is in the boundary of $K^{\pm}_{t_1}$, then there exists a sequence of vectors $(v_j)_{j\in\mathbb{N}}$ in the interior of $K^{\pm}_{t_1}$ such that

$$\lim_{j\to\infty} v_j=v.$$

Due to the continuity of the flow

$$\lim_{j\to\infty}\left(\Phi^{X^{\{u\}}}_{(t_2,t_1)}\right)_* v_j=\left(\Phi^{X^{\{u\}}}_{(t_2,t_1)}\right)_* v.$$

All the elements of the convergent sequence are in the closed cone $K^{\pm}_{t_2}$, hence the limit $\left(\Phi^{X^{\{u\}}}_{(t_2,t_1)}\right)_* v$ is also in $K^{\pm}_{t_2}$.

If the interior of $K^{\pm}_{t_1}$ is empty, we consider the relative topology and the reasoning follows as before. See Appendix D for details.

□

For the time perturbation cone $K^{\pm}_{\tau}$ and the corresponding perturbation vectors, it can be proved properties analogous to the ones stated in Propositions 3.4, 3.6, 3.7 and 3.8.

Proposition 5.9. *Let $t\in(a,b)$ be a Lebesgue time. If v is a nonzero vector interior to $K^{\pm}_t$, then there exists $\epsilon>0$ such that for every $s\in(0,\epsilon)$ there are $s'>0$ and a perturbation of the control $u[\pi^s_{\pm}]$ such that $\gamma[\pi^s_{\pm}](t+s\delta t)=\gamma(t)+s'v$.*

Proof. The proof follows the same line as the proof of Proposition 3.12, but now the tangent space to M at $\gamma(t+s\delta t)$ is also identified with $\mathbb{R}^m$ through the local chart of M at $\gamma(t)$.

We use the same functions as in the proof of Proposition 3.12, but keeping in mind that $\Gamma(s,r)=\gamma[\pi^s_{w_0}](t)$ is replaced by $\Gamma(s,r)=\gamma[\pi^s_{w_0}](t+s\delta t)$. □

5.2.2 Perturbing the endpoint conditions

Now we consider that the endpoint conditions for the integral curves of $X^{\{u\}}$ varies on submanifolds of M. Let S_a be a submanifold of M and $\gamma(a)$ in S_a; consider the integral curve $\gamma\colon I \to M$ of $X^{\{u\}}$ with initial condition $(a, \gamma(a))$.

We consider the curve $\gamma[\pi^s_\pm]$ obtained from a time perturbation of the control u associated with a vector in the time perturbation cone. The initial condition is disturbed along a curve $\delta\colon [0,\epsilon] \to S_a$ with initial tangent vector v_a in $T_{\gamma(a)}S_a$ and $\delta(0) = \gamma(a)$. Taking into account Appendix B.2.1, §3.3.1 and considering that $T_{\gamma(a)}S_a$ and an open set at $\delta(a)$ are identified with $\mathbb{R}^m$, the integral curve $\gamma_{\delta(s)}[\pi^s_\pm]\colon I \to M$ of $X^{\{u[\pi^s_\pm]\}}$ with initial condition $(a, \delta(s))$ can be written as

$$\gamma_{\delta(s)}[\pi^s_\pm](t) = \gamma(t) + s\left(\Phi^{X^{\{u\}}}_{(t,a)}\right)_* v_a + sv[\pi_\pm](t) + o(s).$$

We define a cone that includes the time perturbation vectors, the elementary perturbation vectors and the vectors coming from changing the initial condition on S_a along different curves $\delta\colon [0,\epsilon] \to S_a$ through $\gamma(a)$ and contained in S_a.

Definition 5.10. *Let t be a Lebesgue time. The cone $\mathcal{K}_t$ is the smallest closed and convex cone containing the time perturbation cone at time t and the transported of the tangent space to S_a from a to t through the flow of $X^{\{u\}}$.*

$$\mathcal{K}_t = \overline{\operatorname{conv}(K_t^\pm \bigcup (\Phi^{X^{\{u\}}}_{(t,a)})_*(T_{\gamma(a)}S_a))}$$

Proposition 5.11. *Let t be a Lebesgue time in (a,b) and $S \subset M$ be a submanifold with boundary. Suppose that $\gamma(t)$ is on the boundary of S. Let T be the half–plane tangent to S at $\gamma(t)$. If $\mathcal{K}_t$ and T are not separated, then there exists a perturbation of the control $u[\pi^s_\pm]$ and $x_a \in S_a$ such that the integral curve $\gamma_{x_a}[\pi^s_\pm]$ of $X^{\{u[\pi^s_\pm]\}}$ with initial condition (a, x_a) meets S at a point in the relative interior of S.*

Proof. As $\mathcal{K}_t$ and T are not separated, by Proposition D.15 there no exists any hyperplane containing both and there is a vector v in the relative interior of both $\mathcal{K}_t$ and T. By Corollary D.16, if $\mathcal{K}_t$ and T are not separated,

$$T_{\gamma(t)}M = \mathcal{K}_t - T.$$

See Appendix D for the notation and properties. If V is an open set of a local chart at $\gamma(t)$, we identify V with $\mathbb{R}^m$ and also the tangent space at $\gamma(t)$, $T_{\gamma(t)}M$, in the same sense defined for Equation (3.4). Let us consider an orthonormal basis in $T_{\gamma(t)}M$, $\{e_1, \dots, e_m\}$. If we take $e_0 = -(e_1 + \ldots + e_m)$, the vector $0 \in T_{\gamma(t)}M$ is expressed as an affine combination of $e_0, e_1, \dots, e_m$:

$$0 = \frac{1}{m+1} e_0 + \ldots + \frac{1}{m+1} e_m.$$

Each w in $T_{\gamma(t)}M$ is written uniquely as

$$w = a^1 e_1 + \ldots + a^m e_m$$

and as an affine combination of $e_0, e_1, \dots, e_m$:

$$w = \sum_{i=0}^m b^i(w) e_i = re_0 + \sum_{i=0}^m (r + a^i) e_i \quad \text{with} \quad r = \frac{1 - \sum_{i=1}^m a^i}{m+1}.$$

Hence, we define the continuous mapping

$$\begin{array}{rcl} \mathcal{G}\colon T_{\gamma(t)}M & \longrightarrow & \mathbb{R}^{m+1} \\ w & \longmapsto & (b^0(w), b^1(w), \dots, b^m(w)). \end{array}$$

As $b^i(0) > 0$ for every $i = 0, \dots, m$, there exists an open ball $B(0, r)$ centered at 0 with radius r such that for every $w \in B(0, r)$, $b^i(w) > 0$ for $i = 0, \dots, m$. Now we consider the restriction of $\mathcal{G}$ to the closed ball $\overline{B(0,r)}$, $\mathcal{G}_{\overline{B(0,r)}}\colon \overline{B(0,r)} \to [0,1]^{m+1}$. Choose vectors $e_i^{\mathcal{K}} \in \mathcal{K}_t$ and $e_i^T \in T$ such that

$$e_i = e_i^{\mathcal{K}} - e_i^T .$$

As v lies in the relative interior of both convex sets, $e_i = (e_i^{\mathcal{K}} + v) - (e_i^T + v) = e_i^{\mathcal{K}} - e_i^T$. Then $e_i^{\mathcal{K}}$ and e_i^T can be chosen in the relative interior of $\mathcal{K}$ and T, respectively, because v is in the relative interior of both. For any $w \in \overline{B(0,r)}$,

$$w = \sum_{i=0}^m b^i(w) e_i = \sum_{i=0}^m b^i(w) \left(e_i^{\mathcal{K}} - e_i^T\right).$$

Then we can define

$$\begin{array}{rcl} F_1\colon \overline{B(0,r)} & \longrightarrow & \mathcal{K}_t \\ w & \longmapsto & F_1(w) = \sum_{i=0}^m b^i(w) e_i^{\mathcal{K}}, \end{array}$$

$$\begin{array}{rcl} F_2\colon \overline{B(0,r)} & \longrightarrow & T \\ w & \longmapsto & F_2(w) = \sum_{i=0}^m b^i(w) e_i^{T}, \end{array}$$

and let us consider the mapping

$$\begin{array}{rcl} G\colon \mathbb{R} \times \overline{B(0,r)} & \longrightarrow & \mathbb{R}^m \\ (s,w) & \longmapsto & (\gamma[\pi^s_{F_1(w)}](t) - \gamma[\pi^s_{F_2(w)}](t))/s, \end{array}$$

where $\gamma[\pi^s_{F_1(w)}]$ is the perturbation curve associated to $\pi^s_{F_1(w)}$ and $\gamma[\pi^s_{F_2(w)}](t) = \gamma(t) + sF_2(w)$ is the straight line through $\gamma(t)$ with tangent vector $F_2(w)$. As the perturbation vectors are in the relative interior of the convex cones, we use the linear approximation in (3.4) in such a way that $G(s, w) = F_1(w) - F_2(w) + o(1)$. Hence

$$\lim_{s \to 0} G(s, w) = F_1(w) - F_2(w) = w,$$

Hence, for any positive number ϵ, there exists $s_0 > 0$ such that if $s < s_0$ then $\|G(s, w) - w\| < \epsilon$. Take $\epsilon < r$, then

$$\|G(s, w) - w\| < \epsilon < r = \|w\|$$

for every w in the boundary of $\overline{B(0,r)}$. Thus the map $G_s\colon \overline{B(0,r)} \to \mathbb{R}^m$, $G_s(w) = G(s, w)$, satisfies the hypotheses of Corollary E.2 for the point 0 in $B(0, r)$. Hence, the point 0 is in the image of $\overline{B(0,r)}$ through G_s and there exists w such that $G_s(w) = 0$; that is,

$$\gamma[\pi^s_{F_1(w)}](t) = \gamma[\pi^s_{F_2(w)}](t).$$

Therefore, there exists a perturbation of the control such that the associated trajectory meets S in an interior point since $F_2(w)$ lies in the relative interior of T. □

5.3 Pontryagin's Maximum Principle with time and endpoints nonfixed

Bearing in mind the symplectic formalism introduced in §3.4, we define the corresponding Hamiltonian Problem when the time and the endpoints are nonfixed.

Statement 5.12. ***(Free Hamiltonian Problem,*** *FHP) Given the FOCP,* (Q, U, X, F, S_a, S_f), *and the equivalent* $\widehat{FOCP}$, $(\widehat{Q}, U, \widehat{X}, S_a, S_f)$, *consider the following problem.*

Find $b \in \mathbb{R}$ *and* $(\widehat{\sigma}, u)\colon [a, b] \to T^*\widehat{Q} \times U$, *with* $\widehat{\gamma} = \pi_{\widehat{Q}} \circ \widehat{\sigma}$ *and* $\gamma = \pi_2 \circ \widehat{\gamma}$, *such that*

(1) $\widehat{\gamma}(a) \in \{0\} \times S_a$, $\gamma(b) \in S_f$, *and*

(2) $\dot{\widehat{\sigma}} = (\widehat{X}^{T^*})^{\{u\}} \circ (\widehat{\sigma}, \mathrm{id})$.

The tuple $(T^*\widehat{Q}, U, \widehat{X}^{T^*}, S_a, S_f)$ denotes the *free Hamiltonian problem.*

Comments:

1. The minimum of the interval of definition of the curves is a, but the maximum is not fixed.

2. The curves γ, $\widehat{\gamma}$ and $\widehat{\sigma}$ are assumed to be absolutely continuous. So they are generalized integral curves of $X^{\{u\}}$, $\widehat{X}^{\{u\}}$ and $(\widehat{X}^{T^*})^{\{u\}}$, respectively, in the sense defined in §2.

Now, we are ready to state the Free Pontryagin's Maximum Principle that provides the necessary conditions, but in general not sufficient, for finding solutions of the free optimal control problem.

Theorem 5.13. ***(Free Pontryagin's Maximum Principle, FPMP)***
If $(\widehat{\gamma}^*, u^*)\colon [a, b] \to \widehat{Q} \times U$ *is a solution of the extended free optimal control problem, Statement 5.2, then there exists* $(\widehat{\sigma}^*, u^*)\colon [a, b] \to T^*\widehat{Q} \times U$ *such that:*

1. *it is a solution of the associated free Hamiltonian problem;*

2. $\widehat{\gamma}^* = \pi_{\widehat{Q}} \circ \widehat{\sigma}^*$*;*

3. *(a)* $H(\widehat{\sigma}^*(t), u^*(t)) = \sup_{u\in U} H(\widehat{\sigma}^*(t), u)$ *almost everywhere;*
   (b) $\sup_{u\in U} H(\widehat{\sigma}^*(t), u) = 0$ *everywhere;*
   (c) $\widehat{\sigma}^*(t) \neq 0 \in T^*_{\widehat{\gamma}^*(t)}\widehat{Q}$ *for each* $t \in [a, b]$*;*
   (d) $\sigma_0^*(t)$ *is constant,* $\sigma_0^*(t) \leq 0$*;*
   (e) transversality conditions:$\sigma^*(a) \in \mathrm{ann}\, T_{\gamma^*(a)} S_a$ *and* $\sigma^*(b) \in \mathrm{ann}\, T_{\gamma^*(b)} S_f$.

Observe that once we have the optimal solution of the $\widehat{FOCP}$, the final time and the endpoint conditions are known and fixed. We would like to apply just Theorem 3.14 in order to prove Theorem 5.13. However, this is not possible because the freedom to chose the final time and the endpoint conditions, only restricted to submanifolds, in Statement 5.2 is used in the proof to consider variations of the optimal curve that are slightly different from the variations used in the case of fixed time, see §3.3 and §5.2 to compare them.

Apart from the transversality conditions, the main difference between FPMP and PMP is the fact that the domain of the curves in the optimal control problems is unknown. That introduces a new necessary condition: the supremum of the Hamiltonian must be zero, not just constant. Then, from ($3a$) and ($3b$) it may be concluded that the Hamiltonian is zero almost everywhere. For instance, in the time optimal problems the Hamiltonian along extremals must be zero.

There are different statements of Pontryagin's Maximum Principle. In §3.4 we have considered the statement of PMP for a fixed–time problem without transversality conditions to simplify the proof, although it may be stated the PMP for the fixed–time problem with variable endpoints where the transversality conditions appear. There also exists the PMP for the free time problem with the degenerate case that the submanifolds are only a point, then the Theorem is the following one.

Theorem 5.14. ***(Free Pontryagin's Maximum Principle without variable endpoints)***
If $(\widehat{\gamma}^*, u^*)\colon [a, b] \to \widehat{Q} \times U$ *is a solution of the extended free optimal control problem, Statement 5.2 with* $S_a = \{x_a\}$ *and* $S_f = \{x_f\}$*, then there exists* $(\widehat{\sigma}^*, u^*)\colon [a, b] \to T^*\widehat{Q} \times U$ *such that:*

1. *it is a solution of the associated free Hamiltonian problem;*

2. $\widehat{\gamma}^* = \pi_{\widehat{Q}} \circ \widehat{\sigma}^*$*;*

3. (a) $H(\widehat{\sigma}^*(t), u^*(t)) = \sup_{u\in U} H(\widehat{\sigma}^*(t), u)$ *almost everywhere;*
   (b) $\sup_{u\in U} H(\widehat{\sigma}^*(t), u) = 0$ *everywhere;*
   (c) $\widehat{\sigma}^*(t) \neq 0 \in T^*_{\widehat{\gamma}^*(t)}\widehat{Q}$ *for each* $t \in [a, b]$*;*
   (d) $\sigma_0^*(t)$ *is constant,* $\sigma_0^*(t) \leq 0$*;*

The only difference with Theorem 5.13 is that the transversality conditions do not appear.

6 Proof of Pontryagin's Maximum Principle for nonfixed time and nonfixed endpoints

In the proof of Theorem 5.13 we use notions about perturbations of the trajectories of a system introduced in §5.2, but they are slightly different from the perturbations in §3.3 used to prove Theorem 3.14.

Proof. (Theorem 5.13: Free Pontryagin's Maximum Principle, FPMP)

Given a solution of the $\widehat{FOCP}$, we only need an appropriate initial condition in the fibers of $\pi_{\widehat{Q}}\colon T^*\widehat{Q} \to \widehat{Q}$ to find a solution of the FHP, because this initial condition is not given in the hypotheses of the Free Pontryagin's Maximum Principle. It is not possible to use Theorem 3.14 directly because the perturbation cones are not the same. Indeed, we need to consider changes in the interval of definition of the curves. These changes imply the inclusion of $\pm\widehat{X}(\widehat{\gamma}^*(t_1), u^*(t_1))$ in the perturbation cone at time t_1. All the times considered in this proof are Lebesgue times for the vector field giving the optimal curve.

By Proposition 5.8, for $t_2 > t_1$,

$$\left(\Phi^{\widehat{X}\{u^*\}}_{(t_2,t_1)}\right)_* \widehat{K}^{\pm}_{t_1} \subset \widehat{K}^{\pm}_{t_2}.$$

Let us consider the limit cone as follows

$$\widehat{K}^{\pm}_b = \overline{\bigcup_{\substack{a<\tau\leq b\\ \tau \text{ is a Lebesgue time}}} \left(\Phi^{\widehat{X}\{u\}}_{(b,\tau)}\right)_* \widehat{K}^{\pm}_{\tau}}.$$

Observe that it is a closed cone and it is convex because it is the union of an increasing family of convex cones. Let us show that $(-1, \mathbf{0})_{\widehat{\gamma}^*(b)}$ is not interior to $\widehat{K}^{\pm}_b$. Indeed, suppose that $(-1, \mathbf{0})_{\widehat{\gamma}^*(b)}$ is interior to the limit cone, then it will be interior to

$$\bigcup_{\substack{a<\tau<b\\ \tau \text{ is a Lebesgue time}}} (\Phi^{\widehat{X}\{u\}}_{(b,\tau)})_* \widehat{K}^{\pm}_{\tau}$$

by Proposition D.5, item (d). As we have an increasing family of cones, there exists a time τ such that $(-1, \mathbf{0})_{\widehat{\gamma}^*(b)}$ is interior to $(\Phi^{\widehat{X}\{u\}}_{(b,\tau)})_* \widehat{K}^{\pm}_{\tau}$. Let us see that this is not possible.

If $(-1, \mathbf{0})_{\widehat{\gamma}^*(b)}$ is interior to $(\Phi^{\widehat{X}\{u\}}_{(b,\tau)})_* \widehat{K}^{\pm}_{\tau}$, then, by Proposition 5.9, there exists $\epsilon > 0$ such that, for every $s \in (0, \epsilon)$, there exist $s' > 0$ and a perturbation of the control $u[\pi^s_{w_0}]$ such that

$$\widehat{\gamma}[\pi^s_{w_0}](b + s\delta\tau) = \widehat{\gamma}^*(b) + s'(-1, \mathbf{0}).$$

Hence

$$\gamma^0[\pi^s_{w0}](b+s\delta\tau) < {\gamma^*}^0(b) \quad \text{and} \quad \gamma[\pi^s_{w0}](b+s\delta\tau) = \gamma^*(b).$$

That is, the trajectory $\gamma[\pi^s_{w0}]$ arrives at the same endpoint as γ^* but with less cost. Then $\widehat{\gamma}^*$ cannot be optimal as assumed. Thus $(-1,\mathbf{0})_{\widehat{\gamma}^*(b)}$ is not interior to $\widehat{K}^{\pm}_b$.

As $(-1,\mathbf{0})_{\widehat{\gamma}^*(b)}$ is not in the interior of $\widehat{K}^{\pm}_b$, by Proposition D.15 there exists a covector $\widehat{\sigma}_b \in T^*_{\widehat{\gamma}^*(b)}\widehat{Q}$ such that

$$\begin{aligned} \langle \widehat{\sigma}_b, (-1,\mathbf{0})\rangle &\geq 0, \\ \langle \widehat{\sigma}_b, \widehat{v}_b\rangle &\leq 0 \quad \forall\, \widehat{v}_b \in \widehat{K}^{\pm}_b. \end{aligned}$$

The initial condition for the covector must not only satisfy the previous inequalities, but also the transversality conditions. In order to prove this, it is necessary to have the separability of two new cones.

(3e) Hence, the initial condition in the fibers of $T^*\widehat{Q}$ may be chosen satisfying the **transversality conditions**. We consider the manifold with boundary given by

$$M_f = \{(x^0, x) \mid x \in S_f,\ x^0 \leq {\gamma^*}^0(b)\}.$$

The set of tangent vectors to M_f at $\widehat{\gamma}^*(b)$ is the convex set whose generators are $(-1,\mathbf{0})_{\widehat{\gamma}^*(b)}$ and $T_f = \{0\} \times T_{\gamma^*(b)}S_f$.

Given $\tau \in [a,b]$, consider the following closed convex sets

$$\mathcal{K}_\tau = \overline{\operatorname{conv}(\widehat{K}^{\pm}_\tau \bigcup (\Phi^{\widehat{X}\{u^*\}}_{(\tau,a)})_*(T_a))}, \qquad \text{where } \; T_a = \{0\} \times T_{\gamma^*(a)}S_a,$$

$$\mathcal{J}_\tau = \overline{\operatorname{conv}((-1,\mathbf{0})_{\widehat{\gamma}^*(\tau)} \bigcup (\Phi^{\widehat{X}\{u^*\}}_{(b,\tau)})^{-1}_*(T_f))}, \qquad \text{where } \; T_f = \{0\} \times T_{\gamma^*(b)}S_f,$$

and the manifold M_τ obtained transporting M_f from b to τ using the flow of $\widehat{X}^{\{u^*\}}$. Observe that $\mathcal{J}_\tau$ is the closure of the set of tangent vectors to M_τ at the point $\widehat{\gamma}^*(\tau)$. We are going to show that the cones $\mathcal{K}_b$ and $\mathcal{J}_b$ are separated, using Proposition 5.11.

Observe that $\mathcal{J}_b$ is a half–plane tangent to M_f and $\widehat{\gamma}^*(b)$ is on the boundary of M_f by construction. Hence, if $\mathcal{K}_b$ and $\mathcal{J}_b$ were not separated, by Proposition 5.11 there would exist a perturbation of the control $u[\pi^s_{w0}]$ and $x_a \in S_a$ such that the integral curve $\gamma_{x_a}[\pi^s_{w0}]$ with initial condition (a, x_a) meets M_f at a point in the relative interior of M_f. Hence we have found a trajectory with less cost than the optimal one because of the definition of M_f and this is not possible because of the optimality of $\widehat{\gamma}^*$. Thus $\mathcal{K}_b$ and $\mathcal{J}_b$ are separated. So, by Proposition D.15, there exists a covector $\widehat{\sigma}_b \in T^*_{\widehat{\gamma}^*(b)}\widehat{Q}$ such that

$$\langle \widehat{\sigma}_b, \widehat{v}_b\rangle \leq 0 \quad \forall\, \widehat{v}_b \in \mathcal{K}_b, \tag{6.14}$$

$$\langle \widehat{\sigma}_b, \widehat{w}_b\rangle \geq 0 \quad \forall\, \widehat{w}_b \in \mathcal{J}_b. \tag{6.15}$$

This covector separates the vector $(-1,\mathbf{0})_{\widehat{\gamma}^*(b)} \in \mathcal{J}_b$ and the cone $\widehat{K}^{\pm}_b \subset \mathcal{K}^{\pm}_b$. Let $\widehat{\sigma}^*$ be the integral curve of $(\widehat{X}^{T^*})^{\{u^*\}}$ with initial condition $\widehat{\sigma}_b \in T_{\widehat{\gamma}^*(b)}\widehat{Q}$ at b.

As T_f is contained in $\mathcal{J}_b$, we have $\langle \widehat{\sigma}_b, \widehat{v}\rangle \geq 0$ for every $\widehat{v} \in T_f$. As T_f is a vector space, if $\widehat{v} \in T_f$, then $-\widehat{v} \in T_f$. Hence, we have

$$\langle \widehat{\sigma}_b, \widehat{v}\rangle = 0 \quad \text{for every } \; \widehat{v} \in T_f.$$

That is,

$$\langle \widehat{\sigma}_b, (0, v)\rangle = 0 \quad \text{for every } \; v \in T_{\gamma^*(b)}S_f.$$

This is equivalent to $\langle \sigma_b, v\rangle = 0$ for every $v \in T_{\gamma^*(b)}S_f$; that is, $\sigma_b = \sigma^*(b)$ is in the annihilator of $T_{\gamma^*(b)}S_f$ as wanted.

For every $\widehat{w}_b \in \mathcal{J}_b$, if $\widehat{W}: I \to T\widehat{Q}$ is the integral curve of $(\widehat{X}^T)^{\{u^*\}}$ with initial condition $\widehat{w}_b$ at time b, then by Proposition B.5 the pairing continuous natural function $\langle \widehat{\sigma}^*, \widehat{W}\rangle: I \to \mathbb{R}$ is constant everywhere and $\langle \widehat{\sigma}^*(a), \widehat{W}(a)\rangle \geq 0$ by Equation (6.15). As $(\Phi^{\widehat{X}^{\{u^*\}}}_{(b,a)})^{-1}_*(\mathcal{J}_b) = \mathcal{J}_a$ by the continuity and the linearity of the flow, the transversality condition at a is proved analogously as the transversality condition at b proved above.

Since $(\widehat{\gamma}^*, u^*)$ is a solution of the $\widehat{FOCP}$, it is also a solution of $\widehat{OCP}$ with time and endpoints fixed and given by the curve. Hence, we can apply Pontryagin's Maximum Principle for time and endpoints fixed, Theorem 3.14. If the curve $(\widehat{\gamma}^*, u^*)$ is a solution of $\widehat{OCP}$ with $I = [a, b]$ and endpoints $\widehat{\gamma}^*(a)$ and $\widehat{\gamma}^*(b)$, $(\widehat{\sigma}^*, u^*): [a, b] \to T^*\widehat{Q} \times U$ is a solution of the HP, such that $\widehat{\gamma}^* = \pi_{\widehat{Q}} \circ \widehat{\sigma}^*$, and moreover $\widehat{\sigma}^*$ satisfies that

(3a) $H(\widehat{\sigma}^*(t), u^*(t)) = \sup_{u\in U} H(\widehat{\sigma}^*(t), u)$ almost everywhere.

(3b) $\sup_{u\in U} H(\widehat{\sigma}^*(t), u)$ is constant everywhere.

(3c) $\widehat{\sigma}^*(t) \neq 0 \in T^*_{\widehat{\gamma}^*(t)}\widehat{Q}$ for every $t \in [a, b]$.

(3d) $\sigma_0^*(t)$ is constant, $\sigma_0^*(t) \leq 0$.

Observe that it only remains to prove **(3b)** of the Free Pontryagin's Maximum Principle, since **(3a)**, **(3c)** and **(3d)** are the same in both Theorems 3.14, 5.13.

(3b) Due to (3a) we already know that the supremum of the Hamiltonian is constant everywhere along $(\widehat{\sigma}^*, u^*)$. Now, let us prove that the supremum can be taken to be zero everywhere.

Take $\widehat{v}_b = \pm\widehat{X}(\widehat{\gamma}^*(b), u^*(b)) \in \widehat{K}_b^{\pm}$, let $\widehat{V}: I \to T\widehat{Q}$ be the integral curve of $(\widehat{X}^T)^{\{u^*\}}$ with initial condition $(b, \widehat{\gamma}(b), \widehat{v}_b)$, then the continuous function $\langle \widehat{\sigma}^*, \widehat{V}\rangle: I \to \mathbb{R}$ is constant everywhere by Proposition B.5. Thus,

$$\langle \widehat{\sigma}^*(t), \widehat{V}(t)\rangle = \langle \widehat{\sigma}^*(t), \pm\widehat{X}(\widehat{\gamma}^*(t), u^*(t))\rangle \leq 0 \quad \text{for every} \quad t \in I$$

by Equation (6.14), and this implies that

$$\langle \widehat{\sigma}^*(t), \widehat{X}(\widehat{\gamma}^*(t), u^*(t))\rangle = 0.$$

As $\langle \widehat{\sigma}^*(t), \widehat{X}(\widehat{\gamma}^*(t), u^*(t))\rangle = H(\widehat{\sigma}^*(t), u^*(t))$, the Hamiltonian function is zero everywhere and the supremum of the Hamiltonian function is zero everywhere by Theorem 3.14. □

Observe that the initial condition for the covector in this proof has been chosen such that the tangent spaces to the initial and final submanifolds are contained in the separating hyperplane defined by the covector. In this statement of the Maximum Principle the initial condition for the covector must satisfy more conditions than in Theorem 3.14 (namely the transversality conditions).

Appendices

This last part of the report is mainly devoted to state and prove some of the results used in the proof of the Maximum Principle and to give more understanding to some key points.

A Results on real functions

In this Appendix we focus on some necessary technicalities for the proof of Pontryagin's Maximum Principle. These are related with results from analysis and the notion of a Lebesgue point for a real function. In this paper the notion of a Lebesgue point is applied to vector fields. For more details about all this, see [66, 74, 75].

Definition A.1. *Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f\colon X \to Y$ is* ***Lipschitz*** *if there exists $K \in \mathbb{R}$ such that $d_Y(f(x_1), f(x_2)) \leq K\, d_X(x_1, x_2)$ for all $x_1, x_2 \in X$.*

A function $f\colon X \to Y$ is ***locally Lipschitz*** *if, for every $x \in X$ there exists an open neighbourhood V of x and $K \in \mathbb{R}^+$ such that $d_Y(f(x_1), f(x_2)) \leq K\, d_X(x_1, x_2)$ for all x_1 and x_2 in V.*

If M is a differentiable manifold, g is a Riemannian metric on M and $d_g\colon M \times M \to \mathbb{R}$ is the induced distance; then (M, d_g) is a metric space where the notion of Lipschitz on M can be defined. A real–valued function $F\colon M \to \mathbb{R}$ is locally Lipschitz if, for every $p \in M$ we take the local chart (V, ϕ) such that $\phi(p) = 0$, $\phi(V) = B(0, r)$ is the open ball centered at the origin with radius $r > 0$ in the standard Euclidean space, and $F \circ \phi^{-1}\colon B(0, r) \to \mathbb{R}$ is Lipschitz. That is, there exists $K \in \mathbb{R}^+$ with

$$|F(p_1) - F(p_2)| = |(F \circ \phi^{-1})(\phi(p_1)) - (F \circ \phi^{-1})(\phi(p_2))| \leq K d(\phi(p_1), \phi(p_2)), \quad \forall p_1, p_2 \in V.$$

Hence, given the local chart (V, ϕ), we define a distance $d_\phi\colon V \times V \to \mathbb{R}$ on V, $d_\phi(p_1, p_2) = d(\phi(p_1), \phi(p_2))$. Consequently, (V, ϕ) is a metric space with the topology induced by the open set V in M. This distance is equivalent to the distance induced by the Riemannian metric on V.Observe that the notion of locally Lipschitz for functions on manifolds depends on the local chart, but $\mathcal{C}^1$ functions are always locally Lipschitz.

Definition A.2. *A function $f\colon [a, b] \to \mathbb{R}$ is* ***uniformly continuous on*** *$[a, b]$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that for any $t, s \in [a, b]$ with $|t - s| < \delta$, we have $|f(t) - f(s)| < \epsilon$.*

Definition A.3. *A function $f\colon [a, b] \to \mathbb{R}$ is* ***absolutely continuous on*** *$[a, b]$ if, for every $\epsilon > 0$, there exists $\delta > 0$ such that for every finite number of nonoverlapping subintervals (a_i, b_i) of $[a, b]$ with $\sum_{i=1}^n |b_i - a_i| < \delta$, we have $\sum_{i=1}^n |f(b_i) - f(a_i)| < \epsilon$.*

We consider an interval $I = [a, b]$ in $\mathbb{R}$ with the usual Lebesgue measure. A statement is said to be satisfied *almost everywhere* if it is fulfilled in I except on a zero measure set. A measurable subset $A \subset I$ is said of full measure if $I - A$ has measure zero. Recall that if $A, B \subset I$ and $I - A$, $I - B$ have measure zero, then $A \cap B$ is not empty.

Results in [66], pp. 96, 100, 105 allow one to prove the following result.

Proposition A.4. *If f is absolutely continuous, then f has a derivative almost everywhere.*

Theorem A.5. *[[66], pp.105 and [74], pp.836] If f is absolutely continuous and $f'(t) = 0$ almost everywhere on $[a, b]$, then f is a constant function.*

Definition A.6. *A real–valued function f on a metric space (X, d) is called* ***lower semicontinuous at*** *$x_0 \in X$ if, for every $\epsilon > 0$, there exists $\delta(\epsilon, x_0) > 0$ such that $f(x) \geq f(x_0) - \epsilon$ whenever $d(x, x_0) \leq \delta(\epsilon, t_0)$.*

If f is lower semicontinuous at every point of (X, d), it is said to be ***lower semicontinuous on*** *(X, d).*

The following result is stated by Pontryagin et al. in [64], page 102, but it is neither proved nor stated as a proposition. We believe it is appropriate to write it with more detail because it is used in §4.

Proposition A.7. *Let f and g be real functions, $f, g\colon [a,b] \to \mathbb{R}$. If f is continuous, g is lower semicontinuous, $f \leq g$ and $f = g$ almost everywhere then $f = g$ everywhere.*

Proof. Let $t_0 \in [a,b]$. As g is lower semicontinuous on $[a,b]$, for every $\epsilon > 0$ there exists $\delta(\epsilon, t_0) = \delta > 0$ such that

$$g(t) \geq g(t_0) - \epsilon$$

whenever $|t - t_0| < \delta(\epsilon, t_0)$.

Since f and g coincide almost everywhere on $[a,b]$, there exists $t_1 \in (t_0 - \delta, t_0 + \delta)$ such that $f(t_1) = g(t_1)$. Moreover, $f \leq g$, so

$$f(t_0) \leq g(t_0) \leq g(t_1) + \epsilon = f(t_1) + \epsilon. \tag{A.16}$$

The continuity of f guarantees that for every $\epsilon' > 0$, there exists $\delta' > 0$ such that if $|t_1 - t_0| < \delta'$, then $f(t_1) - \epsilon' < f(t_0) < f(t_1) + \epsilon'$. Hence Equation (A.16) is rewritten as follows:

$$f(t_0) \leq g(t_0) \leq f(t_0) + \epsilon' + \epsilon.$$

As this inequality is valid for every $\epsilon, \epsilon' > 0$, $g(t_0) = f(t_0)$ for every $t_0 \in [a,b]$. Thus $f = g$ everywhere. □

A.1 Lebesgue points for a real function

After introducing the concept of measurable function and some properties of such functions, we state Lebesgue's differentiation theorem, which enables us to distinguish certain points for a measurable function. In the entire paper we consider the Lebesgue measure in $\mathbb{R}$. See the book by Zaanen [75] for more details.

Definition A.8. *A function $f\colon [a,b] \subset \mathbb{R} \to \mathbb{R}$ is **measurable** if the set $\{t \in [a,b] :\ f(t) > \alpha\}$ is measurable for every $\alpha \in \mathbb{R}$.*

Definition A.9. *A function $f\colon [a,b] \to \mathbb{R}$ is **Lebesgue integrable** over each Lebesgue measurable set of finite measure if $\nu(x) = \int_a^x f d\mu$ is well defined for every $x \in [a,b]$.*

Theorem A.10. ***(Lebesgue's Differentiation Theorem [75])*** *Let μ be the Lebesgue measure. If $f\colon [a,b] \to \mathbb{R}$ is a Lebesgue integrable function over every Lebesgue measurable set of finite measure, then for $\nu(x) = \int_a^x f d\mu$,*

$$D\nu(x_+) = D\nu(x_-) = f(x)$$

holds for μ–almost every $x \in [a,b]$, where $D\nu(x_+)$, $D\nu(x_-)$ are the right and left derivatives of ν respectively.

The equality $D\nu(x_-) = f(x)$ almost everywhere may be rewritten as follows for $h > 0$

$$\lim_{h\to 0} \frac{\nu(x-h) - \nu(x)}{-h} = f(x) \quad \text{a.e.} \Leftrightarrow \lim_{h\to 0} \frac{\int_a^{x-h} f(t)dt - \int_a^x f(t)dt}{-h} = f(x) \quad \text{a.e.} \Leftrightarrow$$

$$\Leftrightarrow \lim_{h\to 0} \frac{\int_{x-h}^x f(t)dt}{h} = f(x) \quad \text{a.e.} \Leftrightarrow \int_{x-h}^x f(t)dt = hf(x) + o(h) \quad \text{a.e.}$$

Definition A.11. *If $f\colon [a,b] \to \mathbb{R}$ is a measurable function, $x \in (a,b)$ is a **Lebesgue point for* f** *if,*

$$\lim_{h\to 0} \int_{x-h}^x \frac{f(t) - f(x)}{h} dt = 0.$$

Remark A.12. As Theorem A.10 is true almost everywhere, the set of Lebesgue points for a measurable function has full measure.

Remark A.13. Observe that if $u\colon I \to U$ is measurable and bounded, then it is integrable and the set of Lebesgue points for u has full measure. If $f\colon U \to \mathbb{R}$ is continuous, then $f \circ u\colon I \to \mathbb{R}$ is integrable, and the intersection of Lebesgue points for u and $f \circ u$ has full measure.

Note: Assume we have a manifold Q, an open set $U \subset \mathbb{R}^k$ and a continuous vector field X along the projection $\pi\colon Q \times U \to Q$. If $(\gamma, u)\colon I = [a,b] \to Q \times U$, where γ is absolutely continuous and u is measurable and bounded, then $X \circ (\gamma, u)\colon I \to TQ$ is a measurable vector field along (γ, u), in the sense that in any coordinate system its coordinate functions are measurable. A point $t \in (a,b)$ is a *Lebesgue point for* u if

$$\int_{t-h}^{t} X(\gamma(s), u(s))ds = hX(\gamma(t), u(t)) + o(h). \tag{A.17}$$

The Lebesgue points for a vector field are useful in §3.3, §5.2 and in the following appendix to guarantee the differentiability of some curves, that is, the existence of its tangent vector. See [25, 31] for more details about differential equations and measurability.

B Time–dependent variational equations

The variational equations give us an approach to how the integral curves of vector fields vary when the initial condition varies along a curve. These equations have a formulation on the tangent and the cotangent bundle. Here we are interested in studying the variational equations associated to time–dependent vector fields, and in proving some relationship between the solutions of variational equations on the tangent bundle and the ones on the cotangent bundle. See [50] for more details about these concepts.

B.1 Time–dependent vector fields

As seen in §2, control systems are associated to a time–dependent vector field through a vector field along a projection. For $I \subset \mathbb{R}$, a *differentiable time–dependent vector field* X is a mapping $X\colon I \times M \to TM$ such that each $(t,x) \in I \times M$ is assigned to a tangent vector $X(t,x)$ in T_xM. For every $(s,x) \in I \times M$, *the integral curve of* X *with initial condition* (s,x) is denoted by $\Phi^X_{(s,x)}\colon J_{(s,x)} \subset I \to M$ and satisfies

1. $\Phi^X_{(s,x)}(s) = x$.
2. $\left.\dfrac{d}{dt}\right|_t \Phi^X_{(s,x)} = X(t, \Phi^X_{(s,x)}(t))$, $t \in J$.

The domain of $\Phi^X_{(s,x)}$ is denoted by $J_{(s,x)} \subset I$ because it depends on the initial condition for the integral curves.

The *time dependent flow* or *evolution operator of* X is the mapping

$$\begin{aligned} \Phi^X\colon \quad I \times I \times M &\longrightarrow M \\ (t,s,x) &\longmapsto \Phi^X(t,s,x) = \Phi^X_{(s,x)}(t) \end{aligned} \tag{B.18}$$

defined in a maximal open neighborhood $V \times M$ of $\Delta_I \times M$, where Δ_I is the diagonal of $I \times I$, and Φ^X satisfies

1. $\Phi^X(s,s,x) = x$.

2. $\left.\dfrac{d}{dt}\right|_t \left(\Phi^X(t,s,x)\right) = X(t,\Phi^X(t,s,x))$.

To obtain the original vector field through the evolution operator, the expression in the second assertion must be evaluated at $s=t$,

$$\left.\left[\left.\frac{d}{dt}\right|_t \left(\Phi^X(t,s,x)\right)\right]\right|_{s=t} = X(t,x).$$

There is a time–independent vector field on the manifold $I \times M$ associated to X and given by $\widetilde{X}(t,x) = \partial/\partial t_{|(t,x)} + X(t,x)$. For $(t,s,x) \in V \times M$, the flow of $\widetilde{X}$ is $\Phi^{\widetilde{X}} \colon I \times I \times M \to I \times M$ such that $\Phi^{\widetilde{X}}_{(s,x)}$ is the integral curve of $\widetilde{X}$ with initial condition (s,x) at time 0 and $\Phi^{\widetilde{X}}(t,(s,x)) = (s+t, \Phi^X(s+t,(s,x)))$. The theorems in differential equations about the existence and uniqueness of solutions guarantee the existence and uniqueness of the evolution operator Φ^X defined maximally.

For $(t,s) \in V \subset I \times I$,

$$\begin{aligned} \Phi^X_{(t,s)} \colon M &\longrightarrow M \\ x &\longmapsto \Phi^X_{(t,s)}(x) = \Phi^X_{(s,x)}(t) \end{aligned}$$

is a diffeomorphism on M satisfying $\Phi^X_{(t,s)} = \Phi^X_{(t,r)} \circ \Phi^X_{(r,s)}$ for $r \in I$, such that (r,s), $(t,r) \in V$.

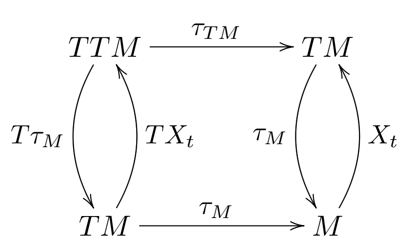
B.2 Complete lift

Fromt the evolution operator of time–dependent vector fields in Equation $(B.18)$, it is determined the evolution operator of a particular vector field on TM.

Let $X_t \colon M \to TM$ be a vector field on M such that $X_t(x) = X(t,x)$ for every $t \in I$. The *complete or tangent lift of X_t to TM* is the time–dependent vector field X_t^T on TM satisfying

$$X_t^T = \kappa_M \circ TX_t,$$

where κ_M is the canonical involution of TTM; that is, a mapping $\kappa_M \colon TTM \to TTM$ such that $\kappa_M^2 = \mathrm{Id}$ and $\tau_{TM} \circ \kappa_M = T\tau_M$. See [50] for more details in the definition. Moreover, observe that X_t is a vector field that makes the following diagram commutative:



If $(x,v) \in TM$, then $TX_t(x,v) = (x, X_t(x), T_xX_t(v)) \in T_{(x,X_t(x))}(TM)$.

Let (W, x^i) be a local chart at x in M such that $X_t = X_t^i\, \partial/\partial x^i$ where $X_t^i(x) = X^i(t,x)$ and $X^i \in \mathcal{C}^\infty(I \times W)$. If (x^i, v^i) are the induced local coordinates in TM, then locally

$$X^T(t,x,v) = X^i(t,x) \left.\frac{\partial}{\partial x^i}\right|_{(x,v)} + \frac{\partial X^i}{\partial x^j}(t,x)\, v^j \left.\frac{\partial}{\partial v^i}\right|_{(x,v)}.$$

The equations satisfied by the integral curves of X^T are called *variational equations*.

Proposition B.1. *If X is a time–dependent vector field on M and Φ^X is the evolution operator of X, then the map $\Psi\colon I \times I \times TM \to TM$ defined by*

$$\Psi(t, s, (x, v)) = \left(\Phi^X(t, s, x), T_x\Phi^X_{(t,s)}(v)\right)$$

is the evolution operator of X^T.

Proof. We have to prove that

$$\begin{cases} \Psi(s, s, (x, v)) = (x, v); \\ \left.\dfrac{d}{dt}\right|_t (\Psi(t, s, (x, v))) = X^T(t, \Psi(t, s, (x, v))). \end{cases}$$

The first item is proved easily,

$$\Psi(s, s, (x, v)) = \left(\Phi^X(s, s, x), T_x\left(\Phi^X_{(s,s)}\right)(v)\right) = (x, v).$$

As for the second assertion, we use that $\Phi^X_{(t,s)}\colon M \to M$ is a $\mathcal{C}^\infty$ diffeomorphism satisfying

$$\begin{aligned} T_t\left(T_x\Phi^X_{(t,s)}(v)\right)1 &= \left.\frac{d}{dt}\right|_t \left(T_x\Phi^X_{(t,s)}(v)\right)1 = \left(T_x\left(\left.\frac{d}{dt}\right|_t \left(\Phi^X_{(t,s)}\right)1\right)\right)(v) \\ &= T_x\left(T_t\Phi^X_{(s,x)}1\right)(v), \end{aligned}$$

and we obtain

$$\left.\frac{d}{dt}\right|_t (\Psi(t, s, x, v)) = \left(\left.\frac{d}{dt}\right|_t \left(\Phi^X(t, s, x)\right), \left.\frac{d}{dt}\right|_t \left(T_x\Phi^X_{(t,s)}(v)\right)\right)$$

$$= \left(X(t, \Phi^X(t, s, x)), \left(T_x\left(\left.\frac{d}{dt}\right|_t \left(\Phi^X_{(t,s)}\right)\right)\right)(v)\right)$$

$$= \left(X(t, \Phi^X(t, s, x)), \left(T_x\left(X_t(\Phi^X(t, s, x))\right)\right)(v)\right)$$

$$= \left(X(t, \Phi^X(t, s, x)), \left(T_{\Phi^X(t,s,x)}(X_t) \circ T_x(\Phi^X(t, s, x))\right)(v)\right)$$

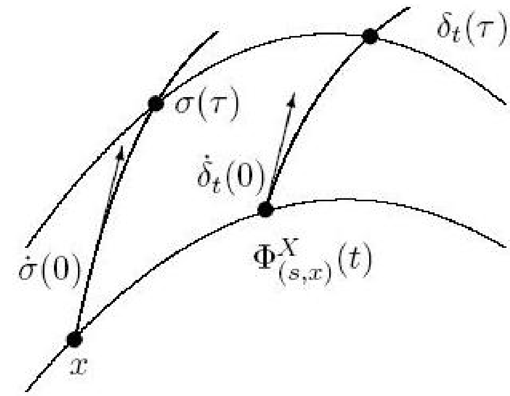
$$= \left(X(t, \Phi^X(t, s, x)), T_{\Phi^X(t,s,x)}(X_t)\left(T_x(\Phi^X_{(t,s)})(v)\right)\right) = X^T(t, \Psi(t, s, x, v)).$$

Hence, the evolution operator of X^T is the complete lift of the evolution operator of X. The integral curves of X^T are vector fields along the integral curves of X. □

B.2.1 About the geometric meaning of the complete lift

The integral curves of X^T must be understood as the linear approximation of the integral curves of X when the initial condition varies along a curve in M. This idea appears in §3.3 and §4.

Let us explain the next figure. Given an integral curve of X with initial condition (s, x), we consider a curve σ starting at the point x of the integral curve. Every point of σ can be considered as the initial condition at time s for an integral curve of X. Thus the flow of X transports the curve σ at a different curve δ_t point by point. The resultant curve is related with the complete lift of X as the following results prove.



Proposition B.2. *Let $X\colon I\times M \to TM$ be a time–dependent vector field with evolution operator Φ^X and $(s,x)\in I\times M$. For $\epsilon>0$, let $\sigma\colon(-\epsilon,\epsilon)\subset\mathbb{R}\to M$ be a $\mathcal{C}^\infty$ curve such that $\sigma(0)=x\in M$. For every $t\in I$, we define a curve $\delta_t\colon(-\epsilon,\epsilon)\to M$ such that*

1. $\delta_t(\tau)=\Phi^X_{(s,\sigma(\tau))}(t)$,
2. $\delta_s(\tau)=\sigma(\tau)$, *and*
3. $\delta_t(0)=\Phi^X_{(s,x)}(t)$.

Then $\dot{\delta}_t(0)=T_x\Phi^X_{(t,s)}(\dot{\sigma}(0))$.

Proof.

$$\begin{aligned}\dot{\delta}_t(0) &= \left(T_0\delta_t(\tau)\right)\left.\frac{d}{d\tau}\right|_0 = \left(T_0\left(\Phi^X_{(s,\sigma(\tau))}(t)\right)\right)\left.\frac{d}{d\tau}\right|_0 = \left(T_0\left(\Phi^X_{(t,s)}(\sigma(\tau))\right)\right)\left.\frac{d}{d\tau}\right|_0 \\ &= T_{\sigma(0)}\Phi^X_{(t,s)}\left(T_0(\sigma(\tau))\left.\frac{d}{d\tau}\right|_0\right) = T_{\sigma(0)}\Phi^X_{(t,s)}\left(\dot{\sigma}(0)\right) = T_x\Phi^X_{(t,s)}\left(\dot{\sigma}(0)\right).\end{aligned}$$

□

Corollary B.3. *Let X be a time–dependent vector field on M. For $x\in M$, $x\in T_xM$ and for a small enough $\epsilon>0$, let $\sigma\colon(-\epsilon,\epsilon)\subset\mathbb{R}\to M$ be a $\mathcal{C}^\infty$ curve such that $\sigma(0)=x\in M$ and $\dot{\sigma}(0)=v$. If δ_t is the curve defined in Proposition B.2, then $\dot{\delta}_{(\cdot)}(\tau)\colon I\to TM$, $t\mapsto\dot{\delta}_t(\tau)$ is the integral curve of X^T with initial condition $(s,\dot{\sigma}(\tau))$.*

Proof. The proof just comes from Propositions B.1 and B.2 and the definition of the curve δ_t. □

B.3 Cotangent lift

Given $(t,s)\in I\times I$, the evolution operator $\Phi^{X^T}_{(t,s)}$ is a diffeomorphism on TM and a linear isomorphism on the fibers on TM, so it makes sense to consider its transpose and inverse, $({}^\tau\Phi^{X^T}_{(t,s)})^{-1}=\Lambda_{(t,s)}$. It is a linear isomorphism on the fibers on T^*M and satisfies $\Lambda_{(t,s)}=\Lambda_{(t,r)}\circ\Lambda_{(r,s)}$ for $r\in I$. Hence $\Lambda\colon I\times I\times T^*M\to T^*M$ is the evolution operator of a time–dependent vector field on T^*M, called the *cotangent lift X^{T^*} of X to T^*M.*

Another vector field on T^*M may be associated to X using the concepts in Hamiltonian formalism. For every $t\in I$, the Hamiltonian system $\left(M,\omega,\widehat{X_t}\right)$ given by a symplectic manifold

(M,ω) and the Hamiltonian function $\widehat{X_t}\colon T^*M \to \mathbb{R}$, $\widehat{X_t}(p_x) = i_{X(t,x)}p_x$ where $p_x \in T^*_x M$; has associated a Hamiltonian vector field Z_t that satisfies Hamilton's equations $i_{Z_t}\omega = d\widehat{X_t}$.

In local coordinates (x,p) for T^*M, $Z\colon I \times T^*M \to TT^*M$ is given by

$$Z(t,x,p) = X^i(t,x)\left.\frac{\partial}{\partial x^i}\right|_{(x,p)} - \frac{\partial X^j}{\partial x^i}(t,x)\,p_j\left.\frac{\partial}{\partial p_i}\right|_{(x,p)}.$$

The equations satisfied by the integral curves of Z in the fibers are the *adjoint variational equations on the cotangent bundle.* In the literature, they are sometimes called *adjoint equations.*

Let us prove that both vector fields Z and X^{T^*} associated to X are the same.

Proposition B.4. *If X is a time–dependent vector field on M and Φ^X is the evolution operator of X, then $\Lambda\colon I \times I \times T^*M \to T^*M$ such that*

$$\Lambda(t,s,(x,p)) = (\Phi^X(t,s,x), \left({}^\tau T_x \Phi^X_{(t,s)}\right)^{-1}(p))$$

is the evolution operator of Z. Thus $Z = X^{T^*}$.

Proof. We have to prove that

$$\begin{cases} \Lambda(s,s,(x,p)) = (x,p), \\[2ex] \left.\dfrac{d}{dt}\right|_t (\Lambda(t,s,(x,p))) = Z(t,\Lambda(t,s,(x,p))). \end{cases}$$

The first item is proved easily,

$$\Lambda(s,s,(x,p)) = \left(\Phi^X(s,s,x), \left({}^\tau T_x \Phi^X_{(s,s)}\right)^{-1}(p)\right) = (x, \mathrm{Id}\,(p)) = (x,p).$$

As $\Phi^X_s\colon I \times M \to M$ is $\mathcal{C}^\infty$, in local coordinates we have

$$\left.\frac{d}{dt}\right|_t \left({}^\tau T_x \Phi^X_{(t,s)}\right) = {}^\tau T_x\left(\left.\frac{d}{dt}\right|_t \Phi^X_{(t,s)}\right),$$

where both mappings go from $T^*_x M$ to $T^*_{\Phi^X(t,(s,x))}M$. Now let us prove the second assertion:

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_t (\Lambda(t,s,x,p)) = \left(\left.\frac{d}{dt}\right|_t \left(\Phi^X(t,s,x)\right), \left.\frac{d}{dt}\right|_t \left(\left({}^\tau T_x \Phi^X_{(t,s)}\right)^{-1}(p)\right)\right)\\
&= \left(X(t,\Phi^X(t,s,x)), \left(-\left({}^\tau T_x \Phi^X_{(t,s)}\right)^{-1} \circ \left(\tfrac{d}{dt}\big|_t \left({}^\tau T_x \Phi^X_{(t,s)}\right)\right) \circ \left({}^\tau T_x \Phi^X_{(t,s)}\right)^{-1}\right)(p)\right)\\
&= \left(X(t,\Phi^X(t,s,x)), \left(-\left({}^\tau T_x \Phi^X_{(t,s)}\right)^{-1} \circ \left({}^\tau T_x\left(\tfrac{d}{dt}\big|_t \Phi^X_{(t,s)}\right)\right)\right)\left(\left({}^\tau T_x \Phi^X_{(t,s)}\right)^{-1}(p)\right)\right)\\
&= \left(X(t,\Phi^X(t,s,x)), -\left({}^\tau T_{\Phi^X_{(t,s)}(x)}\left(X_t \circ \Phi^X_{(t,s)} \circ \left(\Phi^X_{(t,s)}\right)^{-1}\right)\right)\left(\left({}^\tau T_x \Phi^X_{(t,s)}\right)^{-1}(p)\right)\right)\\
&= \left(X(t,\Phi^X(t,s,x)), -\left({}^\tau T_{\Phi^X_{(t,s)}(x)}(X_t)\right)\left(\left({}^\tau T_x \Phi^X_{(t,s)}\right)^{-1}(p)\right)\right) = Z(t,\Lambda(t,s,x,p)).
\end{aligned}$$

Hence, the evolution operator of Z is the cotangent lift of the evolution operator of X. Thus $Z = X^{T^*}$. □

B.4 A property for the complete and cotangent lift

The previous propositions allow us to determine an invariant function along integral curves of X.

Proposition B.5. *Let $X\colon I\times M\to TM$ be a time–dependent vector field and let $X^T\colon I\times TM\to TTM$ and $X^{T^*}\colon I\times T^*M\to TT^*M$ be the complete lift and cotangent lift of X, respectively. If $\gamma\colon I\to M$ is an integral curve of X with initial condition (s,x), $V\colon I\to TM$ is the integral curve of X^T with initial condition $(s,(x,v))$ where $v\in T_{\gamma(s)}M$, and $\Lambda\colon I\to T^*M$ is the integral curve of X^{T^*} with initial condition $(s,(x,p))$ where $p\in T^*_{\gamma(s)}M$, then*

$$\begin{array}{rcl}\langle\Lambda,V\rangle\colon I & \to & \mathbb{R}\\ t & \mapsto & \langle\Lambda(t),V(t)\rangle\end{array}$$

is constant along γ.

Proof. If Φ^X is the evolution operator of X, the evolution operators of X^T and X^{T^*} are

$$\Phi^{X^T}(t,s,(x,v))=\left(\Phi^X(t,s,x),T_x\Phi^X_{(t,s)}(v)\right),$$

$$\Phi^{X^{T^*}}(t,s,(x,p))=\left(\Phi^X(t,s,x),\left({}^{\tau}T_x\Phi^X_{(t,s)}\right)^{-1}(p)\right),$$

respectively, because of Propositions B.1 and B.4. Hence

$$\begin{aligned}\langle\Lambda(t),V(t)\rangle &= \left\langle\left({}^{\tau}T_x\Phi^X_{(t,s)}\right)^{-1}(p),T_x\Phi^X_{(t,s)}(v)\right\rangle\\ &= \left\langle{}^{\tau}\left(\left(T_x\Phi^X_{(t,s)}\right)^{-1}\right)(p),T_x\Phi^X_{(t,s)}(v)\right\rangle\\ &= \left\langle p,\left(\left(T_x\Phi^X_{(t,s)}\right)^{-1}\circ\left(T_x\Phi^X_{(t,s)}\right)\right)(v)\right\rangle=\langle p,v\rangle=\text{constant}.\end{aligned}$$

□

C The tangent perturbation cone as an approximation of the reachable set

In control systems, the reachable sets are useful to determine the accessibility and the controllability of the systems. In optimal control, the reachable set has a great importance for distinguishing the abnormal optimal curves from the normal ones [24, 52]. A key point in the proof of the Maximum Principle depends on the understanding of that linear approximation of the reachable set in a neighborhood of a point in the optimal curve. This interpretation of the tangent perturbation cone has been studied in [2], but we will study it in a great and clear detail in this appendix.

In the sequel, we explain why this interpretation of the tangent perturbation cone is feasible. Remember from §B.1 that a time–dependent vector field on M has associated the evolution operator $\Phi^X\colon I\times I\times M\to M$, $(t,s,x)\mapsto\Phi^X(t,s,x)$ as defined in Equation (B.18).

Proposition C.1. *Let X,Y be time–dependent vector fields on M, then there exists a time–dependent vector field Z such that*

$$\Phi^{X+Y}_{(t,s)}(x)=(\Phi^X_{(t,s)}\circ\Phi^Z_{(t,s)})(x)$$

and $Z=(\Phi^X_{(t,s)*})^{-1}Y=(\Phi^X_{(t,s)})^*Y$.

Proof. For any initial time s, we define the diffeomorphism $\tilde{\Phi}_s^X : I \times M \to I \times M$, $(t,x) \to (t, \Phi_s^X(t,x))$ such that $\tilde{\Phi}_s^X(s,x) = (s,x)$. We look for a time–dependent vector field Z on M such that

$$\tilde{\Phi}_s^{X+Y}(t,x) = (\tilde{\Phi}_s^X \circ \tilde{\Phi}_s^Z)(t,x) \; . \tag{C.19}$$

This expression has been assumed true in [2, 22] for $s = 0$, but it has not been carefully proved. On the left–hand side of Equation (C.19) we have

$$\tilde{\Phi}_s^{X+Y}(t,x) = (t, \Phi_s^{X+Y}(t,x))$$

and the right–hand side is

$$(\tilde{\Phi}_s^X \circ \tilde{\Phi}_s^Z)(t,x) = \tilde{\Phi}_s^X(t, \Phi_s^Z(t,x)) = (t, \Phi_s^X(t, \Phi_s^Z(t,x))) \; .$$

Thus Equation (C.19) is satisfied if and only if

$$\Phi_s^{X+Y}(t,x) = \Phi_s^X(t, \Phi_s^Z(t,x)) \quad = \quad (\Phi_s^X \circ \tilde{\Phi}_s^Z)(t,x) \; , \tag{C.20}$$

or equivalently,

$$\Phi_{(t,s)}^{X+Y} \quad = \quad \Phi_{(t,s)}^X \circ \Phi_{(t,s)}^Z \; . \tag{C.21}$$

Let us differentiate with respect to t the left–hand side of Equation (C.20),

$$\frac{d}{dt}\Phi_{(s,x)}^{X+Y}(t) = (X+Y)(t, \Phi_{(s,x)}^{X+Y}(t)) = (X+Y)(t, \Phi_s^X(t, \Phi_s^Z(t,x))) \; . \tag{C.22}$$

The differentiation with respect to time of the right–hand side of Equation (C.20), for f in $\mathcal{C}^\infty(M)$, is

$$\frac{d}{dt}(\Phi_s^X(t, \Phi_s^Z(t,x)))f = \lim_{h\to 0} \frac{f((\Phi_s^X(t+h, \Phi_s^Z(t+h,x)))) - f((\Phi_s^X(t, \Phi_s^Z(t,x))))}{h}$$

$$= \lim_{h\to 0} \left\{ \frac{(f \circ \Phi_{(t+h,s)}^X)(\Phi_{(t+h,s)}^Z(x)) - (f \circ \Phi_{(t+h,s)}^X)(\Phi_{(t,s)}^Z(x))}{h} \right.$$

$$\left. + \; \frac{(f \circ \Phi_s^X)(t+h, \Phi_s^Z(t,x)) - (f \circ \Phi_s^X)(t, \Phi_s^Z(t,x))}{h} \right\}$$

$$= Z(t, \Phi_s^Z(t,x))(f \circ \Phi_{(t,s)}^X) + X(t, \Phi_s^X(t, \Phi_s^Z(t,x)))f$$

$$= \mathrm{T}_{\Phi_s^Z(t,x)}\Phi_{(t,s)}^X Z(t, \Phi_s^Z(t,x))f + X(t, \Phi_s^X(t, \Phi_s^Z(t,x)))f \; .$$

Hence

$$\frac{d}{dt}(\Phi_s^X(t, \Phi_s^Z(t,x))) = \mathrm{T}_{\Phi_s^Z(t,x)}\Phi_{(t,s)}^X Z(t, \Phi_s^Z(t,x)) + X(t, \Phi_s^X(t, \Phi_s^Z(t,x))) \; .$$

From Equation (C.22) we have

$$\begin{aligned} X(t, \Phi_s^X(t, \Phi_s^Z(t,x))) + Y(t, \Phi_s^X(t, \Phi_s^Z(t,x))) \quad &= \quad \mathrm{T}_{\Phi_s^Z(t,x)}\Phi_{(t,s)}^X Z(t, \Phi_s^Z(t,x)) \\ &+ \quad X(t, \Phi_s^X(t, \Phi_s^Z(t,x))) \; , \end{aligned}$$

that is,

$$Y((\tilde{\Phi}_s^X \circ \tilde{\Phi}_s^Z)(t,x)) = \mathrm{T}_{\Phi_s^Z(t,x)}\Phi_{(t,s)}^X Z(t, \Phi_s^Z(t,x)) \; .$$

Remember that the pushforward of a time–dependent vector field Z is another time–dependent vector field given by

$$(\Phi^X_{(t,s)*}Z)(t,x) = \mathrm{T}_{(\Phi^X_{(t,s)})^{-1}(x)}\Phi^X_{(t,s)}(Z(t,(\Phi^X_{(t,s)})^{-1}(x)))\ .$$

Then

$$(Y\circ\tilde{\Phi}^X_s\circ\tilde{\Phi}^Z_s)(t,x) = (\Phi^X_{(t,s)*}Z)(t,\Phi^X_{(t,s)}(\Phi^Z_s(t,x)))$$

$$= (\Phi^X_{(t,s)*}Z)(\tilde{\Phi}^X_s(t,\Phi^Z_s(t,x))) = (\Phi^X_{(t,s)*}Z)(\tilde{\Phi}^X_s\circ\tilde{\Phi}^Z_s)(t,x)\ ,$$

or equivalently,

$$Y\circ\tilde{\Phi}^X_s\circ\tilde{\Phi}^Z_s = (\Phi^X_{(t,s)*}Z)\circ\tilde{\Phi}^X_s\circ\tilde{\Phi}^Z_s\ ,$$

that is, $Y = \Phi^X_{(t,s)*}Z$.

Hence $Z = (\Phi^X_{(t,s)*})^{-1}Y = (\Phi^X_{(t,s)})^*Y$. Now, going back to Equation (C.21) we have

$$\Phi^{X+Y}_{(t,s)}(x) = (\Phi^X_{(t,s)}\circ\Phi^{(\Phi^X_{(t,s)})^*Y}_{(t,s)})(x)\ . \tag{C.23}$$

□

Definition C.2. *Let M be a manifold, U be a set in $\mathbb{R}^k$ and X be a vector field along the projection $\pi\colon M\times U\to M$. The* ***reachable set from*** *$x_0\in M$* ***at time*** *$T\in I$ is the set of points described by*

$$\mathcal{R}(x_0,T) = \{x\in M \quad | \quad \text{there exists } (\gamma,u)\colon [a,b]\to M\times U \text{ such that } \dot{\gamma}(t) = X(\gamma(t),u(t)),\ \gamma(a)=x_0,\ \gamma(T)=x\}\ .$$

Once we know how to express the flow of a sum of vector fields as a composition of flows of different vector fields, we are going to show that all the integral curves used to construct the reachable set in Definition C.2 can be written as composition of flows associated with vector fields given by vectors in the tangent perturbation cone in Definition 3.11.

Each control system $X\in\mathfrak{X}(\pi)$ with the projection $\pi\colon M\times U\to M$ is a time–dependent vector field $X^{\{u\}}$ when the control is given. Consider the reference trajectory (γ,u) to be an integral curve of $X^{\{u\}}$ with initial condition x_0 at a. Take $\gamma(t_1)$ to be a reachable point from x_0 at time t_1. Let us consider another control $\tilde{u}\colon I\to U$ and the integral curve of $X^{\{\tilde{u}\}}$ with initial condition x_0 at a denoted by $\tilde{\gamma}$. Then $\tilde{\gamma}(t_1)$ is another reachable point from x_0 at time t_1.

Let us see how to reach the point $\tilde{\gamma}(t_1)$ using Equation (C.23),

$$\begin{aligned}
\tilde{\gamma}(t_1) &= \Phi^{X^{\{\tilde{u}\}}}_{(t_1,a)}(x_0) = \Phi^{X^{\{u\}}+\left(X^{\{\tilde{u}\}}-X^{\{u\}}\right)}_{(t_1,a)}(x_0)\\
&= \left(\Phi^{X^{\{u\}}}_{(t_1,a)}\circ\Phi^{\left(\Phi^{X^{\{u\}}}_{(t_1,a)}\right)^*\left(X^{\{\tilde{u}\}}-X^{\{u\}}\right)}_{(t_1,a)}\right)(x_0)\\
&= \left(\Phi^{X^{\{u\}}}_{(t_1,a)}\circ\Phi^{\left(\Phi^{X^{\{u\}}}_{(t_1,a)}\right)^*\left(X^{\{\tilde{u}\}}-X^{\{u\}}\right)}_{(t_1,a)}\circ\left(\Phi^{X^{\{u\}}}_{(t_1,a)}\right)^{-1}\circ\Phi^{X^{\{u\}}}_{(t_1,a)}\right)(x_0)\\
&= \left(\Phi^{X^{\{u\}}}_{(t_1,a)}\circ\Phi^{\left(\Phi^{X^{\{u\}}}_{(t_1,a)}\right)^*\left(X^{\{\tilde{u}\}}-X^{\{u\}}\right)}_{(t_1,a)}\circ\left(\Phi^{X^{\{u\}}}_{(t_1,a)}\right)^{-1}\right)(\gamma(t_1))\ .
\end{aligned}\tag{C.24}$$

Hence, from $\gamma(t_1)$ we can get every reachable point from x_0 at time t_1 through Equation (C.24) composing integral curves of the vector fields $X^{\{u\}}$ and $(\Phi^{X^{\{u\}}}_{(t_1,a)})^*(X^{\{\tilde{u}\}}-X^{\{u\}})\colon I\times M\to TM$, this latter with initial condition x_0 at a.

In fact this is true for any time τ in $[a, t_1]$, that is,

$$\tilde{\gamma}(\tau) = \left(\Phi^{X^{\{u\}}}_{(\tau,a)} \circ \Phi^{\left(\Phi^{X^{\{u\}}}_{(\tau,a)}\right)^* (X^{\{\tilde{u}\}} - X^{\{u\}})}_{(\tau,a)} \circ \left(\Phi^{X^{\{u\}}}_{(\tau,a)}\right)^{-1} \right) (\gamma(\tau)) \, .$$

If we compose with the flow of $X^{\{u\}}$, we get a reachable point from x_0 at time t_1 because it is a concatenation of integral curves of the dynamical system,

$$\begin{aligned} \Phi^{X^{\{u\}}}_{(t_1,\tau)}(\tilde{\gamma}(\tau)) &= \left(\Phi^{X^{\{u\}}}_{(t_1,\tau)} \circ \Phi^{X^{\{u\}}}_{(\tau,a)} \circ \Phi^{\left(\Phi^{X^{\{u\}}}_{(\tau,a)}\right)^* (X^{\{\tilde{u}\}} - X^{\{u\}})}_{(\tau,a)} \circ \left(\Phi^{X^{\{u\}}}_{(\tau,a)}\right)^{-1} \right) (\gamma(\tau)) \\ &= \left(\Phi^{X^{\{u\}}}_{(t_1,a)} \circ \Phi^{\left(\Phi^{X^{\{u\}}}_{(\tau,a)}\right)^* (X^{\{\tilde{u}\}} - X^{\{u\}})}_{(\tau,a)} \circ \left(\Phi^{X^{\{u\}}}_{(t_1,a)}\right)^{-1} \right) (\gamma(t_1)) \, . \end{aligned} \tag{C.25}$$

Hence, from $\gamma(t_1)$ we can also get reachable points from x_0 at time t_1 through composition of integral curves of the vector fields $X^{\{u\}}$ and $\left(\Phi^{X^{\{u\}}}_{(\tau,a)}\right)^* \left(X^{\{\tilde{u}\}} - X^{\{u\}}\right)$, the latter with initial condition $\gamma(a)$ at time a.

On the other hand, the tangent perturbation cone at $\gamma(t_1)$ is given by the closure of the convex hull of all the tangent vectors $\left(\Phi^{X^{\{u\}}}_{(t_1,\tau)}\right)_* \left(X^{\{\tilde{u}\}}(\tau, \gamma(\tau)) - X^{\{u\}}(\tau, \gamma(\tau))\right)$ for every Lebesgue time τ in $[a, t_1]$. These vectors are related with the vector fields $X^{\{u\}}$ through Equations (C.24) and (C.25).

In this sense, we say that the tangent perturbation cone at $\gamma(t_1)$ is an approximation of the reachable set in a neighborhood of $\gamma(t_1)$.

D Convex sets, cones and hyperplanes

We study some properties satisfied by convex sets and cones; see [13, 65] for details. Unless otherwise stated, we suppose that all the sets are in a n–dimensional vector space E. We need to define the different kinds of cones and linear combinations used in this report.

Definition D.1. *A* ***cone*** C ***with vertex at*** $0 \in E$ *satisfies that if* $v \in C$*, then* $\lambda v \in C$ *for every* $\lambda \geq 0$*.*

Definition D.2. *Given a family of vectors* $V \subset E$*.*

1. *A* ***conic non–negative combination*** *of elements in* V *is a vector of the form* $\lambda_1 v_1 + \cdots + \lambda_r v_r$*, with* $\lambda_i \geq 0$ *and* $v_i \in V$ *for all* $i \in \{1, \ldots, r\}$*.*

2. *The* ***convex cone*** *generated by* V *is the set of all conic non–negative combinations of vectors in* V*.*

3. *An* ***affine combination*** *of elements in* V *is a vector of the form* $\lambda_1 v_1 + \cdots + \lambda_r v_r$*, with* $v_i \in V$*,* $\lambda_i \in \mathbb{R}$ *for all* $i \in \{1, \ldots, r\}$ *and* $\sum_{i=1}^r \lambda_i = 1$*.*

4. *A* ***convex combination*** *of elements in* V *is a vector of the form* $\lambda_1 v_1 + \cdots + \lambda_r v_r$*, with* $v_i \in V$*,* $0 \leq \lambda_i \leq 1$ *for all* $i \in \{1, \ldots, r\}$ *and* $\sum_{i=1}^r \lambda_i = 1$*.*

Remember that a set $A \subset E$ is *convex* if, given two different elements in A, then any convex combination of them is contained in A. Thus, all the convex combination of elements in A are in A.

Definition D.3. *The* ***convex hull*** *of a set $A \subset E$,* $\operatorname{conv}(A)$*, is the smallest convex subset containing A.*

Let us prove a characterization of the convex hull that will be useful.

Proposition D.4. *The convex hull of a set A is the set of the convex combinations of elements in A.*

Proof. Let us denote by C the set of all convex combinations of elements in A. First, we prove that C is a convex set. If x, y are in C, then they are convex combinations of elements in A; that is, $x = \sum_{i=1}^{l} \lambda_i v_i$, $y = \sum_{i=1}^{r} \mu_i w_i$, with $\sum_{i=1}^{l} \lambda_i = 1$, $\sum_{i=1}^{r} \mu_i = 1$. For $s \in (0,1)$, consider

$$sx + (1-s)y = s\left(\sum_{i=1}^{l} \lambda_i v_i\right) + (1-s)\left(\sum_{i=1}^{r} \mu_i w_i\right),$$

that will be in C if the sum of the coefficients is equal to 1 and each of the coefficients lies in $[0,1]$. Observe that $s\sum_{i=1}^{l} \lambda_i + (1-s)\sum_{i=1}^{r} \mu_i = s + (1-s) = 1$ and the other condition is satisfied trivially. As C is convex and contains A, the convex hull of A is a subset of C.

Second, we prove that $C \subset \operatorname{conv}(A)$ by induction on the number of vectors in the convex combinations of elements in A. Trivially, when the convex combination is given by an element in A, it lies in the convex hull of A.

Now, suppose that a convex combination of $l-1$ elements of A is in $\operatorname{conv}(A)$, and we prove that a convex combination of l elements of A is in $\operatorname{conv}(A)$. Let

$$x = \sum_{i=1}^{l} \mu_i v_i = \sum_{i=1}^{l-1} \mu_i v_i + \mu_l v_l.$$

If $\sum_{i=1}^{l-1} \mu_i = 0$, then $\mu_l = 1$. By the first step of the induction, x is in $\operatorname{conv}(A)$. If $\sum_{i=1}^{l-1} \mu_i \in (0,1]$, then $\mu_l \in [0,1)$ and we can rewrite x as

$$x = (1-\mu_l)\left(\sum_{i=1}^{l-1} \mu_i (1-\mu_l)^{-1} v_i\right) + \mu_l v_l.$$

Observe that $\sum_{i=1}^{l-1} \mu_i (1-\mu_l)^{-1} = (1-\mu_l)(1-\mu_l)^{-1} = 1$, and so $\sum_{i=1}^{l-1} \mu_i (1-\mu_l)^{-1} v_i$ is in $\operatorname{conv}(A)$. By the first step of induction, v_l is in $\operatorname{conv}(A)$. As $(1-\mu_l) + \mu_l = 1$, x is in $\operatorname{conv}(A)$ because of the convexity of $\operatorname{conv}(A)$. Thus $C \subset \operatorname{conv}(A)$ and so $C = \operatorname{conv}(A)$. □

Proposition D.5. *Let C be a convex set. If $\overline{C}$ and $\operatorname{int} C$ are the topological closure and the interior of C, respectively, we have:*

(a) for every $x \in \operatorname{int} C$, if $y \in \overline{C}$, then $(1-\lambda)x + \lambda y \in \operatorname{int} C$ for all $\lambda \in [0,1)$;

(b) $\overline{C} = \overline{\operatorname{int} C}$;

(c) the interior of C is empty if and only if the interior of $\overline{C}$ is empty;

(d) $\operatorname{int} C = \operatorname{int} \overline{C}$.

Proof. **(a)** If $x \in \operatorname{int} C$, then there exists $\epsilon_x > 0$ such that $B(x, \epsilon_x) \subset C$, where $B(x, \epsilon_x)$ denotes the open ball centered at x of radius ϵ_x.

Observe that if $y \in \overline{C}$, for any $\epsilon > 0$, $y \in C + \epsilon B(0,1) = \{x + \epsilon z \mid x \in C,\, z \in B(0,1)\}$.

For every $\lambda \in [0,1)$, we consider $x_\lambda = (1-\lambda)x + \lambda y$. Let us compute the value of ϵ_λ such that $x_\lambda + \epsilon_\lambda B(0,1) \subset C$.

$$x_\lambda + \epsilon_\lambda B(0,1) = (1-\lambda)x + \lambda y + \epsilon_\lambda B(0,1)$$

$$\subset (1-\lambda)x + \lambda C + \lambda\epsilon B(0,1) + \epsilon_\lambda B(0,1) = (1-\lambda)x + (\lambda\epsilon + \epsilon_\lambda)B(0,1) + \lambda C.$$

If $\epsilon_\lambda = (1-\lambda)\epsilon_x - \lambda\epsilon$, then $(1-\lambda)x + (\lambda\epsilon + \epsilon_\lambda)B(0,1) \subset (1-\lambda)C$ and $x_\lambda + \epsilon_\lambda B(0,1) \subset C$. For $\epsilon > 0$ small enough, ϵ_λ is positive. Here we use the sum operation of convex sets, which is well–defined if the coefficients are positive (if C_1 and C_2 are convex sets, $\mu_1 C_1 + \mu_2 C_2$ is a convex set for all $\mu_1, \mu_2 \geq 0$).

(b) As $\text{int}\ C \subset C$, $\overline{\text{int}\ C} \subset \overline{C}$.

On the other hand, each point in the closure of C can be approached along a line segment by points in the interior of C by (a). Thus $\overline{C} \subset \overline{\text{int}\ C}$.

(c) As $\text{int}\ C \subset \text{int}\ \overline{C}$, if $\text{int}\ \overline{C}$ is empty, then $\text{int}\ C$ is empty.

Conversely, if $\text{int}\ C$ is empty, then by (b) $\overline{C}$ is empty. So C is empty and $\text{int}\ C$ is also empty.

(d) Trivially $\text{int}\ C \subset \text{int}\ \overline{C}$.

As the equality of the sets is true when they are empty because of (c), let us suppose that $\text{int}\ C$ is not empty. If $z \in \text{int}\ \overline{C}$ and take $x \in \text{int}\ C$, then there exists a small enough positive number δ such that $y = z + \delta(z - x) \in \text{int}\ \overline{C} \subset \overline{C}$.

Hence,

$$z = \frac{1}{1+\delta}y + \frac{\delta}{1+\delta}x.$$

Note that

$$0 < \frac{1}{1+\delta} < 1, \quad 0 < \frac{\delta}{1+\delta} < 1, \quad \frac{1}{1+\delta} + \frac{\delta}{1+\delta} = 1.$$

As $y \in \overline{C}$, $x \in \text{int}\ C$ and $1/(1+\delta)$ lies in $(0,1)$. By (a), $z \in \text{int}\ C$. □

Remark D.6. Consequently, if C is convex and dense, then C is the whole space.

The following paragraphs introduce elements playing an important role in the proof of Pontryagin's Maximum Principle in §4 and §6.

Definition D.7. *Let C be a cone with vertex at $0 \in E$. A **supporting hyperplane to C at** 0 is a hyperplane such that C is contained in one of the half–spaces defined by the hyperplane.*

Remark D.8. In a geometric framework, we will define a hyperplane in E as the kernel of a nonzero 1–form α in the dual space E^*. Then the hyperplane P_α associated to α is $\ker\alpha$. Hence the supporting hyperplane to C at 0 is a hyperplane P_α such that $\alpha(v) \leq 0$ for all $v \in C$. A supporting hyperplane to C at 0 is not necessarily unique.

From now on, we consider that all the cones have vertex at 0.

Definition D.9. *Let C be a cone, the **polar of** C is*

$$C^* = \{\alpha \in E^* \mid \alpha(v) \leq 0 \ , \ \forall\, v \in C\}.$$

Note that the polar of a cone is a closed and convex cone in E^*.

Definition D.10. *Let C be a cone, the set*

$$C^{**} = \{w \in E \mid \alpha(w) \leq 0 \ , \ \forall \, \alpha \in C^*\}$$

is called the ***polar of the polar of*** *C.*

Observe that $C \subset C^{**}$. The following lemma is used in the proof of the existence of a supporting hyperplane to a cone with vertex at 0.

Lemma D.11. *The cone C is closed and convex if and only if $C^{**} = C$.*

Proof. Observe that

$$C^{**} = \{w \in E \mid \alpha(w) \leq 0 \ , \ \forall \, \alpha \in C^*\} = \bigcap_{\alpha \in C^*} \{w \in E \mid \alpha(w) \leq 0\}.$$

Then $C^{**} = \overline{\text{conv}(C)}$, because of Theorem 6.20 in Rockafellar [65]: the closure of the convex hull of a set is the intersection of all the closed half–spaces containing the set. Now, the result is immediate. □

The following proposition guarantees the existence of a supporting hyperplane to a cone with vertex at 0. This result is used throughout the proof of Pontryagin's Maximum Principle in §4 and §6.

Proposition D.12. *If C is a convex and closed cone that is not the whole space, then there exists a supporting hyperplane to C at 0.*

Proof. If there is no supporting hyperplane containing the cone in one of the two half–spaces, then for all $\alpha \in E^*$ there exist v_1, $v_2 \in C$ with $\alpha(v_1) \leq 0$ and $\alpha(v_2) \geq 0$. Thus $C^* = \{0\}$ and $C^{**} = E$. Then, by Lemma D.11, $C = C^{**} = E$ in contradiction with the hypothesis on C. □

Corollary D.13. *If C is a convex cone that is not the whole space, then there exists a supporting hyperplane to C at 0.*

Proof. If $C \neq E$, then $\overline{C} \neq E$ by Proposition D.5 (d). Hence, by Proposition D.12, there exists a supporting hyperplane to $\overline{C}$ which is also a supporting hyperplane to C. □

Definition D.14. *Let C_1 and C_2 be cones with common vertex 0. They are* ***separated*** *if there exists a hyperplane P such that each cone lies in a different closed half–space defined by P. This P is called a* ***separating hyperplane of $\mathbf{C_1}$ and $\mathbf{C_2}$****.*

A point x is a *relative interior point of a set C*, if $x \in C$ and there exists a neighbourhood V of x such that $V \cap \text{aff}(V) \subseteq V$. Then, a useful characterization of separated convex cones is the following:

Proposition D.15. *The convex cones C_1 and C_2, with common vertex 0, are separated if and only if one of the two following conditions are satisfied:*

(1) there exists a hyperplane containing both C_1 and C_2,

(2) there is no point that is a relative interior point of both C_1 and C_2.

Proof. ⇒ If C_1 and C_2 are separated, then there exists a separating hyperplane P_α such that

$$\alpha(v_1) \leq 0 \quad \forall\, v_1 \in C_1,\ \alpha(v_2) \geq 0 \quad \forall\, v_2 \in C_2.$$

If $\alpha(v_i) = 0$ for all $v_i \in C_i$ and $i = 1, 2$, then we are in the first case.

If some $v_i \in C_i$ satisfies the strict inequality, then both sets do not lie in the hyperplane P_α. They lie in a different closed half–space. If the convex cones intersect, the intersection lies in the boundary of the cones and in the hyperplane. Hence, there is no point that is a relative interior point of both C_1 and C_2.

⇐ First, we are going to prove that if (1) is true, then C_1 and C_2 are separated. As there exists a hyperplane determined by α such that $\alpha(v_i) = 0$ for all $v_i \in C_i$, α determines a separating hyperplane of C_1 and C_2.

Now, we are going to prove that if (2) is true, then C_1 and C_2 are separated. As C_1 and C_2 are convex cones,

$$C_1 - C_2 = \{u \in E \mid u = v_1 - v_2,\ v_1 \in C_1,\ v_2 \in C_2\}$$

is a convex cone. Since there is no relative interior point of both C_1 and C_2, 0 does not lie in $C_1 - C_2$. By Corollary D.13 there exists a supporting hyperplane P_α to $C_1 - C_2$ such that $\alpha(v_1 - v_2) \leq 0$, that is, $\alpha(v_1) \leq \alpha(v_2)$, for all $v_1 \in C_1$, $v_2 \in C_2$.

Observe that a supporting hyperplane to $C_1 - C_2$ is a supporting hyperplane to C_1, because, taking $v_2 = 0$, $\alpha(v_1) \leq \alpha(v_2) = 0$ for all $v_1 \in C_1$.

As $\partial(C_1 - C_2) \cap C_1 \subset \partial C_1$, we consider a supporting hyperplane P_α to $C_1 - C_2$ such that $\alpha(v_1) = 0$ for some $v_1 \in \partial C_1$. Hence $\alpha(v_2) \geq \alpha(v_1) = 0$ for all $v_2 \in C_2$. As $\alpha(v_1) \leq 0$ for all $v_1 \in C_1$, α determines a separating hyperplane of C_1 and C_2. □

This proposition gives us necessary and sufficient conditions for the existence of a separating hyperplane of two convex cones with common vertex. Observe that a separating hyperplane of two cones with common vertex is also a supporting hyperplane to each cone at the vertex.

Corollary D.16. *If the convex cones C_1 and C_2 with common vertex 0 are not separated, then $E = C_1 - C_2$.*

Proof. If the cones are not separated, by Proposition D.15 there exists no any hyperplane containing both and the intersection of their relative interior is not empty.

Let us suppose that the convex cone $C_1 - C_2 \neq E$. Then, by Corollary D.13, there exists a supporting hyperplane determined by λ at the vertex such that $\lambda(v) \geq 0$ for every v in $C_1 - C_2$.

Because of the definition of cones, if $v_1 \in C_1$, then $v_1 \in C_1 - C_2$ and $\lambda(v_1) \geq 0$. Analogously, if $v_2 \in C_2$, then $-v_2 \in C_1 - C_2$ and $\lambda(-v_2) \geq 0$, that is, $\lambda(v_2) \leq 0$. □

E One corollary of Brouwer Fixed–Point Theorem

From the statement of Brouwer Fixed–point Theorem, it is possible to prove a corollary in [53] useful for the proof of Proposition 3.12.

Theorem E.1. ***(Brouwer Fixed–point Theorem)*** *Let B_1^n be the closed unit ball in $\mathbb{R}^n$. Any continuous function $G\colon B_1^n \to B_1^n$ has a fixed point.*

Corollary E.2. *Let $g\colon B_1^n \to \mathbb{R}^n$ be a continuous map. Let P be an interior point of B_1^n. If $\|g(x) - x\| < \|x - P\|$ for every x in the boundary ∂B_1^n, then the image $g(B_1^n)$ covers P.*

Proof. Without loss of generality, we assume that P is the origin of $\mathbb{R}^n$. Consider the mapping g as a continuous vector field on the unit ball B_1^n.

As $\|g(x)-x\| < \|x\|$, we are going to show that $g(x)$ makes an acute angle with the outward ray from the origin through x for every $x \in \partial B^n$. Let us consider the equality

$$\|y-z\|^2 + \|z-x\|^2 = \|y-x\|^2 + 2\langle y-z, x-z\rangle,$$

and take $y = g(x)$ and $z = 0$. Then

$$2\langle g(x), x\rangle = \|g(x)\|^2 + \|x\|^2 - \|g(x)-x\|^2 > \|g(x)\|^2 + \|x\|^2 - \|x\|^2 = \|g(x)\|^2 \geq 0.$$

Thus $g(x)$ makes an acute angle with x. So $g(x)$ has an outward radial component at every point $x \in \partial B_1^n$. The vector $-g(x)$ has a negative radial component. For a sufficiently small positive number α the function $x \to x - \alpha\, g(x)$ goes from B_1^n to B_1^n. By Theorem E.1 there exists a fixed point x_0 such that $x_0 = x_0 - \alpha\, g(x_0)$, then $\alpha\, g(x_0) = 0$ and $g(x_0) = 0$ since $\alpha \in \mathbb{R}^+$. As g is continuous and $g(x_0) = 0$, the image of a neighbourhood of x_0 covers the origin. □

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