A Comparison of Analytical Models for Resonant Inductive Coupling Wireless Power Transfer

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Abstract— Recent research in wireless power transfer (WPT) using resonant inductive coupling has demonstrated very high efficiencies (above 40%) at large distances compared to the antenna dimensions, which has exponentially increased the number of potential applications of WPT. Since resonant inductive coupling is a very multidisciplinary field, different approaches have been proposed to predict the behaviour of these systems from physical theory of resonators, reflected load theory and the circuit point of view. However, the relation between these methods is still obscure. In this article, we compare the results of these models to find the efficiency of a Resonant Inductive Coupling WPT system under Steady-State sonditions and to analyze the relation between the optimal load values obtained from this perspectives and the ones obtained using impedance matching techniques.

1. INTRODUCTION

In this paper, three different Resonant Inductive Coupling Theories are revisited in terms of Power Transfer Efficiency, namely: Coupled Mode Theory, Reflected Load Theory and Lumped Circuit Theory (Sections 2, 3 and 4 respectively). The results are then described and compared in Section 5. In Section 6, the optimal load values have been derived from the efficiency formulas previously obtained and compared to optimal load values resulting from the application of Impedance Matching Techniques.

2. COUPLED MODE THEORY

Resonant Inductive Coupling (hereafter referred to as RIC) was presented using Coupled Mode Theory Form [1, 2]. This model, which is based on the physical theory behind resonators, provides a framework to analyze a wireless power transfer system in strong coupling regime as a first order differential equation. Although it is an approximate method, it does predict very accurately the steady-state response of a Resonant Inductive Coupling link.

In this case, the two coils forming a WPT system with low losses are approximated by two resonators where their time-domain field amplitudes can be described as [3]:

$$\begin{aligned}\dot{a}_1 &= -(j\omega_1 + \Gamma_1)a_1(t) + jK_{12}a_2(t) + F_S(t)\\ \dot{a}_2 &= -(j\omega_2 + \Gamma_2 + \Gamma_L)a_2(t) + jK_{12}a_1(t)\end{aligned} \tag{1}$$

where $a_\pm$ is the mode amplitude:

$$a(t)_\pm = \sqrt{\frac{C}{2}}v(t) \pm j\sqrt{\frac{L}{2}}i(t) \tag{2}$$

$\omega_{1,2}$ are the eigenfrequencies (frequencies at which the coils resonate), $\Gamma_{1,2}$ are the rates of intrinsic decay due to the coils losses (absorption and radiative), $F_S(t)$ is the excitation applied to the first coil and K_{12} is the coupling rate between both resonant objects:

$$\Gamma_{1,2} = \frac{R_{1,2}}{2L_{1,2}}; \quad K_{12} = \frac{\sqrt{\omega_1\omega_2}k_{12}}{2} = \frac{j\sqrt{\omega_1\omega_2}M_{12}}{2\sqrt{L_1L_2}} \tag{3}$$

where ω is the resonant frequency of the system ($\omega_1 = \omega_2 = \omega$), and k_{12} is the mutual coupling between the coils.

In steady state, being $F_S(t)$ a sinusoidal function described as $F_S(t) = A_s e^{-j\omega t}$, the field amplitudes in first and secondary coils are $a_1(t) = A_1 e^{-j\omega t}$ and $a_2(t) = A_2 e^{-j\omega t}$. It can be shown that the amplitudes A_1 and A_2 verify:

$$\frac{A_2}{A_1} = \frac{jK_{12}}{\Gamma_2 + \Gamma_L} \tag{4}$$

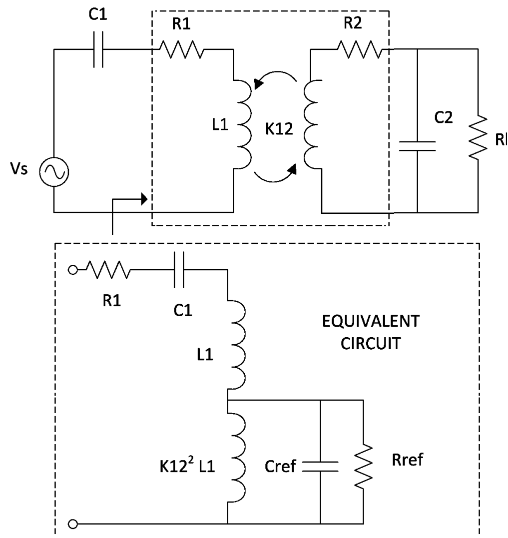

Figure 1: Reflected load theory schematic [3].

and therefore the power at the first coil, second coil and load is, respectively:

$$P_1 = 2\Gamma_1|A_1|^2; \quad P_2 = 2\Gamma_2|A_2|^2; \quad P_L = 2\Gamma_L|A_2|^2 \tag{5}$$

Finally, the efficiency can be described as the ratio between the power delivered to the load P_L and the total power delivered to the system:

$$\eta_{CMT} = \frac{P_L}{P_T} = \frac{\Gamma_L|A_2|^2}{|A_1|^2\Gamma_1 + |A_2|^2(\Gamma_2+\Gamma_L)} = \frac{\Gamma_L K_{12}^2}{\Gamma_1(\Gamma_2+\Gamma_L)^2 + (\Gamma_2+\Gamma_L)K_{12}^2} \tag{6}$$

3. REFLECTED LOAD THEORY

Reflected Load Theory has been widely used by electrical engineers to analyze transformers and it is now also used to predict RIC behaviour in the near-field [3, 4]. Reflected Load Theory states that the amount of current that flows through the primary coil is affected by the load present in the secondary coil. This load does not appear to the primary coil with the same actual value of the load, but instead as a function of the load value and the mutual impedance between primary and secondary coils.

In reflected load theory, the inductive link is described using the mutual coupling between coils $k_{12} = \frac{M_{12}}{L_1 L_2}$ and their quality factors (Q_1, Q_2):

$$Q_{s,x} = \frac{\omega L_x}{R_x}; \qquad Q_{p,x} = \frac{R_x}{\omega L_x} \tag{7}$$

where $Q_{s,x}$ and $Q_{p,x}$ represent the quality factors of an element placed in series and in parallel respectively.

At resonance frequency, the secondary coil is reflected onto the primary and the value that this coil sees is represented by R_{ref} [3]:

$$R_{ref} = k_{12}^2 \frac{L_1}{L_2} = k_{12}^2 \omega L_1 Q_{2L} \tag{8}$$

where k_{12} is the coupling between the coils and Q_{2L} is the loaded quality factor of the load $Q_{2L} = Q_2 Q_L/(Q_2 + Q_L)$.

Because at resonance the impedance of the two coils is purely resistive, the power provided by the source V_s is divided between R_1 and R_{ref} (which also divides between R_2 and R_L). This leads to the definition of the WPT RIC efficiency in Reflected Load Theory:

$$\eta_{RLT} = \frac{R_{ref}}{R_2 + R_{ref}} \frac{Q_2^2 R_2}{Q_2^2 R_2 + R_L} = \frac{k_{12}^2 Q_1 Q_{2L}}{1 + k_{12}^2 Q_1 Q_{2L}} \frac{Q_{2L}}{Q_L} \tag{9}$$

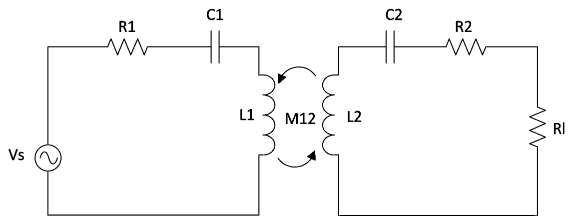

Figure 2: Lumped circuit theory schematic.

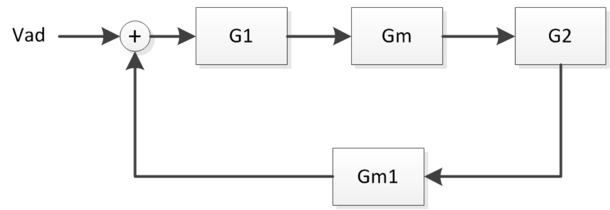

Figure 3: Lumped circuit theory — block diagram.

4. LUMPED CIRCUIT THEORY

The resonant inductive coupling between two electromagnetic resonators can also be described by Lumped circuit theory using a coupled RLC representation system. In this circuit, the capacitances and inductances model the resonant nature of the loops while the resistors model the radiative and ohmic losses.

In Resonant Inductive Coupling, the effect of the first coil to the second (mutual inductance) can be represented by a compensation source Z_{2M} on the coil. Similarly, the effect of the second coil to the first one (back EMF) can be represented also by a compensation source Z_{1M}.

$$Z_{2M} = \frac{\omega M_{12} V_{ad}}{Z_1} = \frac{\omega M_{12} V_{ad}}{R_1 + j\omega L_1 + \frac{1}{j\omega C_1}} \tag{10}$$

The power at the first coil, second coil and load is:

$$\begin{aligned} P_1 &= \frac{V_{ad}^2}{Z_1} \\ P_2 &= I_2^2 Z_2 = \left(\frac{Z_{2M}}{Z_2 + Z_L}\right)^2 Z_2 = \left(\frac{\omega M_{12} V_{ad}}{Z_1(Z_2 + Z_L)}\right)^2 Z_2 \\ P_L &= I_2^2 Z_L = \left(\frac{Z_{2M}}{Z_2 + Z_L}\right)^2 Z_L = \left(\frac{\omega M_{12} V_{ad}}{Z_1(Z_2 + Z_L)}\right)^2 Z_L \end{aligned} \tag{11}$$

where M_{12} is the mutual inductance between coils.

Defining the efficiency as the ratio between the power dissipated in the load (P_L) and the total power ($P_1 + P_2 + P_L$):

$$\eta_{RLC} = \frac{(\omega M_{12})^2 Z_L}{Z_1(Z_2 + Z_L)^2 + (\omega M_{12})^2 (Z_2 + Z_L)} \tag{12}$$

The Lumped Circuit system can also be analyzed expressing each subsystem's power transfer functions as gains. Figure 3 shows the interrelations between them.

Using this model, the current that flows at first and second coils can be written as:

$$I_1 = (V_{ad} + I_2 G_m) G_1; \quad I_2 = (I_1 G_m G_2) \tag{13}$$

where G_m is the transfer function from intensity in coil 1 to voltage in coil 2 ($G_m = \omega M_{12}$), G_1 is the transfer function from Voltage to current in first coil and G_2 is the equivalent at second coil:

$$G_1 = \frac{s}{s^2 L_1 + s R_1 + 1/C_1}; \quad G_2 = \frac{s}{s^2 L_2 + s(R_2 + R_L) + 1/C_2} \tag{14}$$

where s is the complex frequency $s = j\omega$.

The efficiency of the system can be found by dividing the power transferred to the load by the total power available (power on the source coil plus power dissipated on R_2 and R_L).

$$\eta_{RLC,2} = \frac{I_2^2 R_L}{I_1^2 R_1 + I_2^2 (R_2 + R_L)} = \frac{R_L G_m^2}{R_1 (R_2 + R_L)^2 + G_m^2 (R_2 + R_L)} \tag{15}$$

Finally, knowing that the gain functions G_1, G_2 are the inverse of the impedances in coils 1 and 2: $G_1 = 1/Z_1$, $G_2 = 1/Z_2$, the efficiency can be found equivalent to the one obtained in Equation (12):

$$\eta_{RLC} = \frac{(\omega M_{12})^2 Z_L}{Z_1(Z_2 + Z_L)^2 + (\omega M_{12})^2 (Z_2 + Z_L)} = \frac{G_m^2 Z_L}{\frac{1}{G_1 G_2^2} + G_m^2 \frac{1}{G2}} = \eta_{RLC,2} \tag{16}$$

5. COMPARISON

The Power Transfer Efficiency of Coupled Mode Theory (6) can be found equivalent to the efficiency of Reflected Load Theory (9) using the relationships:

$$K_{12} = \frac{\omega k_{12}}{2}; \quad Q_{s,x} = \frac{\omega L_x}{R_x} = \frac{\omega}{2\Gamma_{s,x}}; \quad Q_{p,x} = \frac{R_x}{\omega L_x} = \frac{\omega}{2\Gamma_{p,x}} \tag{17}$$

$$\eta_{CMT} = \frac{\Gamma_L K_{12}^2}{\Gamma_1(\Gamma_2+\Gamma_L)^2 + (\Gamma_2+\Gamma_L)K_{12}^2} = \frac{k_{12}^2}{\frac{Q_L}{Q_1}\left(\frac{Q_2+Q_L}{Q_2Q_L}\right)^2 + k_{12}^2\frac{Q_2+Q_L}{Q_2}} = \eta_{RLT} \tag{18}$$

Similarly, it can be demonstrated that the efficiencies obtained using Lumped Circuit Theory are also equivalent to the ones obtained by CMT and RLT:

$$\eta_{RLC} = \frac{(\omega M_{12})^2 Z_L}{Z_1(Z_2+Z_L)^2 + (\omega M_{12})^2(Z_2+Z_L)} = \frac{k_{12}^2 Z_L}{\frac{\omega L_2}{Q_1}\left(\frac{Q_2+Q_L}{Q_2Q_L}\right)^2 + \omega k_{12}^2 L_2 \frac{Q_2+Q_L}{Q_2Q_L}} = \eta_{RLT} \tag{19}$$

6. OPTIMAL LOAD

Once the efficiencies of the different theoretical models have been found, it is interesting to analyze the optimal values of load that maximize them.

To find the optimal load in Coupled Mode Theory, we can differentiate the efficiency expression (6) with respect to Γ_L, obtaining:

$$\Gamma_{\mathrm{L},\eta_{\max}} = \sqrt{\Gamma_2^2 + \frac{\Gamma_2}{\Gamma_1}K_{12}^2} \tag{20}$$

Similarly, we can set the derivative of (9) with respect to Q_L to 0:

$$Q_{\mathrm{L},\eta_{\max}} = \frac{Q_2}{\sqrt{1+k_{12}^2 Q_1 Q_2}} \tag{21}$$

Using the relationships defined in (7), it can be demonstrated that resistor needed to match the optimal load Q factor to maximize the efficiency is the same as the one obtained using CMT theory [3].

$$\Gamma_{L,RLT} = \Gamma_2\sqrt{1 + \frac{4K_{12}^2}{\omega^2}\frac{\omega}{2\Gamma_1}\frac{\omega}{2\Gamma_2}} = \sqrt{\Gamma_2^2 + \frac{K_{12}^2\Gamma_2}{\Gamma_1}} = \Gamma_{L,CMT} \tag{22}$$

We could also differentiate the efficiency obtained using Lumped Circuit Theory (16)

$$R_{L,RLC} = \sqrt{R_2^2 + Gm^2\frac{R_2}{R_1}} \tag{23}$$

and using the Equation (17) it can be demonstrated that the equivalent optimal load found in CMT and RLT is equal to the load found in (23):

$$R_{L,CMT} = R_{L,RLT} = 2L_2\sqrt{\left(\frac{R_2}{2L_2}\right)^2 + \frac{\omega^2 k_{12}^2}{4}\frac{R_2 L_1}{R_1 L_2}} = \sqrt{R_2^2 + \omega^2 k_{12}^2 L_1 L_2 \frac{R_2}{R_1}} = R_{L,RLC} \tag{24}$$

To compare the load values obtained from optimizing the efficiency function to the ones from applying impedance matching techniques, we must first find the output impedance seen by the load. From the circuit model:

$$Z_{out} = j\omega L_2 - \frac{j}{\omega C_2} + R_2 + \frac{(\omega M_{12})^2}{j\omega L_1 - \frac{j}{\omega C_1} + R_1} \tag{25}$$

To maximize the efficiency using impedance matching techniques, it can be seen that the system must be totally matched, this is, the load should be conjugately matched to the output resistance: $Z_{out} = Z_L^*$.

$$Z_{L,RLC} = \left(\frac{j}{\omega C_2} - j\omega L_2\right) + \sqrt{R_2^2 + (\omega M_{12})^2\frac{R_2}{R_1}} \tag{26}$$

In resonance, $Z_L = Z_{out}*$ becomes purely real:

$$R_L = \sqrt{R_2^2 + (\omega M_{12})^2 \frac{R_2}{R_1}} = R_{L,CMT} = R_{L,RLT} = R_{L,RLC} \tag{27}$$

Therefore, we have demonstrated that using impedance matching techniques (conjugately matching the load to the output resistance) leads to the same optimal values than the ones obtained by setting the derivative of the efficiencies obtained using Coupled Mode Theory, Reflected Load Theory and Lumped Circuit Theory with respect to the load equal to zero.

7. CONCLUSIONS

We have successfully demonstrated that Coupled Mode Theory, Reflected Load Theory and Lumped Circuit theory are equivalent when calculating the Power Transfer Efficiency of a Resonant Inductive Coupled link in Steady-State. Also, the optimal values of load have been found for each theory, showing complete accordance. Finally, it has been demonstrated that applying impedance matching techniques (this is, conjugately matching the load to the output resistance of the system) is equivalent to optimizing the PTE efficiency by setting its derivative with respect to the load equal to zero.

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