ADDENDUM TO "FROBENIUS AND CARTIER ALGEBRAS OF STANLEY-REISNER RINGS" [J. ALGEBRA 358 (2012) 162-177]

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Abstract. We give a purely combinatorial characterization of complete Stanley-Reisner rings having a principally generated (equivalently, finitely generated) Cartier algebra.

1. Introduction

Let \((R, \mathfrak{m})\) be a complete local ring of prime characteristic \(p > 0\). The notion of Cartier algebra, introduced by K. Schwede [6] and developed by M. Blickle [3], has received a lot of attention due to its role in the study of test ideals. More precisely, the ring of Cartier operators on \(R\) is the graded, associative, not necessarily commutative ring

\[ C(R) := \bigoplus_{e \geq 0} \text{Hom}_R(F^e R, R), \]

where \(F^e R\) denotes the ring \(R\) with the left \(R\)-module structure given by the \(e\)-th iterated Frobenius map \(F^e : R \rightarrow R\), i.e. the left \(R\)-module structure given by \(r \cdot m := r^p m\).

One should mention that, using Matlis duality, the Cartier algebra of \(R\) corresponds to the Frobenius algebra of the injective hull of the residue field \(E_R(R/\mathfrak{m})\) introduced by G. Lyubeznik and K. E. Smith in [5].

Let \(S = K[x_1, \ldots, x_n]\) be the formal power series ring over a field \(K\). In this note we will assume that \(\text{char}(K) = p > 0\). Given a simplicial complex \(\Delta\) with vertex set \([n] := \{1, 2, \ldots, n\}\) one may associate a squarefree monomial ideal \(I_{\Delta} := (\prod_{i \in F} x_i \mid F \subseteq [n], F \notin \Delta)\) in \(S\) via the Stanley-Reisner correspondence. Building upon an example of M. Katzman [4], the first author together with A. F. Boix and S. Zarzuela [1] studied Cartier algebras of complete Stanley-Reisner rings \(R := S/I_{\Delta}\) associated to \(\Delta\). One of the main results obtained in [1] is that these Cartier algebras can be either principally generated or infinitely generated as an \(R\)-algebra.

**Theorem 1** ([1, Theorem 3.5]). With the above notation, set \(R := S/I_{\Delta}\). Assume that each \(x_i\) divides some minimal monomial generator of \(I_{\Delta}\). Then, the following are equivalent:

1. The Cartier algebra \(C(R)\) is principally generated.
2. \(I_{\Delta}^{[2]} : I_{\Delta} = I_{\Delta}^{[2]} + (x^1)\).

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Otherwise the Cartier algebra $\mathcal{C}(R)$ is infinitely generated. Here $I_\Delta^{[2]}$ denotes the second Frobenius power of $I_\Delta$ and $x^1 := x_1 x_2 \cdots x_n$.

**Remark 2.** Set $V := \{ i \mid x_i \text{ divides some minimal monomial generator of } I_\Delta \}$.

(i) The condition $x_i$ divides some minimal monomial generator of $I_\Delta$ (equivalently, $V = [n]$) is only used to simplify the notations of Theorem 1. If it is not satisfied, i.e., $\Delta$ is a cone over the vertex $i$, then we have that $\mathcal{C}(R)$ is principally generated if and only if $I_\Delta^{[2]} : I_\Delta = I_\Delta^{[2]} + (\prod_{i \in V} x_i)$. We can always reduce to the case $V = [n]$ since there is always a simplicial complex $\Delta'$ on $V$ such that $\Delta = \Delta' * 2^{[n]\setminus V}$ so the result follows from Lemma 3 below.

(ii) The original result in [1] has a slightly different formulation in terms of the colon ideals $I_\Delta^{[p]} : I_\Delta$, $e \geq 1$, but it was already noticed in [1, Remark 3.1.2] that one may reduce to the case $p = 2$ and $e = 1$. We also point out that Theorem 1 also holds in the case $\text{ht}(I_\Delta) = 1$ that was treated separately in [1] for clearness.

**Lemma 3.** Let $\Delta$ be a simplicial complex with vertex set $[n]$. Assume that there exists a simplicial complex $\Delta'$ on $V \subseteq [n]$ such that $\Delta = \Delta' * 2^{[n]\setminus V}$. Then, $\mathcal{C}(S/I_\Delta)$ is principally generated if and only if so does $\mathcal{C}(S/I_{\Delta'})$.

**Proof.** For $S' := K[x_i \mid i \in V]$, we have $S/I_\Delta \cong (S'/I_{\Delta'})[x_i \mid i \notin V]$. Then the result follows from the description of the Cartier algebra in terms of the colon ideals $I_\Delta^{[2]} : I_\Delta$ (see [1] and the references therein).

2. **A characterization of principally generated Cartier algebras**

The Cartier algebra of an $F$-finite complete Gorenstein local ring $R$ is principally generated as a consequence of [5, Example 3.7]. The converse holds true for $F$-finite normal rings (see [3]). Complete Stanley-Reisner rings are $F$-finite but, almost always non-normal and when discussing examples at the boundary of the Gorenstein property one can even find examples of principally generated Cartier algebras that are not even Cohen-Macaulay. The authors of [1] could not find the homological conditions that tackle this property so the aim of this note is to address this issue. Our main result is a very simple combinatorial criterion in terms of the simplicial complex $\Delta$. To this purpose we recall that a facet of a simplicial complex $\Delta$ is a maximal face with respect to inclusion. We say a face $F \in \Delta$ is subfacet if $F \cup \{i\}$ is a facet for some $i \notin F$.

**Theorem 4.** Under the same assumptions as in Theorem 1, the following are equivalent.

(a) The Cartier algebra $\mathcal{C}(R)$ is principally generated.

(b) Any subfacet of $\Delta$ is contained in at least two facets.

**Proof.** For a monomial $m = \prod_{i=1}^n x_i^{a_i} \in S$, set $\text{supp}(m) := \{ i \mid a_i \neq 0 \}$ and $\text{supp}_2(m) := \{ i \mid a_i \geq 2 \}$. Note that $m \in I_\Delta^{[2]}$ if and only if $\text{supp}_2(m) \notin \Delta$. Furthermore, under the assumption that $\text{supp}(m) \neq [n]$, we have $m \in I_\Delta^{[2]} + (x^1)$ if and only if $\text{supp}_2(m) \notin \Delta$.

\(^1\)Using Matlis duality.
(a) ⇒ (b): By Theorem 1, it suffices to show that $I_{\Delta}^{[2]} : I_{\Delta} = I_{\Delta}^{[2]} + (x^1)$ implies (b), and the same is true for the proof of the converse implication.

Assume that $\Delta$ does not satisfy (b). Then we may assume that $\{1, 2, \ldots, l\}$ is a subfacet, and it is contained in a unique facet $\{1, 2, \ldots, l+1\}$. Set

$$m := \left( \prod_{i=1}^{l} x_i^2 \right) \cdot \left( \prod_{i=l+2}^{n} x_i \right).$$

Clearly, $m \notin I_{\Delta}^{[2]} + (x^1)$. Take any monomial $n \in I_{\Delta}$. Since $\{1, 2, \ldots, l+1\} \in \Delta$, $n$ can be divided by $x_j$ for some $l+2 \leq j \leq n$. Then supp$_2(mn) \supseteq \{1, 2, \ldots, l, j\}$, which is not a face of $\Delta$. It follows that $mn \in I_{\Delta}^{[2]}$. Summing up, we have $m \in (I_{\Delta}^{[2]} : I_{\Delta}) \setminus I_{\Delta}^{[2]} + (x^1)$. Hence the condition (a) does not hold, and we are done.

(b) ⇒ (a): Assume that the condition (b) is satisfied. Since $I_{\Delta}^{[2]} : I_{\Delta} \supseteq I_{\Delta}^{[2]} + (x^1)$ always holds, it suffices to prove that $I_{\Delta}^{[2]} : I_{\Delta} \subseteq I_{\Delta}^{[2]} + (x^1)$, equivalently, $m \notin I_{\Delta}^{[2]} + (x^1)$ implies $m \notin I_{\Delta}^{[2]} : I_{\Delta}$. So take a monomial $m \in S$ with $m \notin I_{\Delta}^{[2]} + (x^1)$. If $\# \text{supp}(m) \leq n-2$ and $i \notin \text{supp}(m)$, then $x_im \notin I_{\Delta}^{[2]} + (x^1)$, and $x_im \notin I_{\Delta}^{[2]} : I_{\Delta}$ implies $m \notin I_{\Delta}^{[2]} : I_{\Delta}$. Hence we can replace $m$ by $x_im$ in this case. Repeating this operation, we may assume that $\# \text{supp}(m) = n-1$. Let $x_l$ be the only variable which does not divide $m$.

Set $F := \text{supp}_2(m)$. Since $m \notin I_{\Delta}^{[2]}$, we have $F \in \Delta$. Moreover, there is a facet $G \in \Delta$ with $G \supseteq F$ and $l \notin G$. To see this, take any facet $H \in \Delta$ with $H \supseteq F$. If $l \notin H$, then we can take $H$ as $G$. If $l \in H$, then the subfacet $H \setminus \{l\}$ is contained in a facet $H'$ other than $H$ by the condition (b). Clearly, we can take $H'$ as $G$. Replacing $m$ by $(\prod_{i \in G \setminus F} x_i) \cdot m$, we may assume that $F = \text{supp}_2(m)$ is a facet which does not contain $l$. Then $n := x_l \cdot \prod_{i \in F} x_i$ is contained in $I_{\Delta}$, since supp($n$) = $F \cup \{l\}$ is not a face of $\Delta$. However, it is easy to see that $\text{supp}_2(mn) = F \in \Delta$ and $mn \notin I_{\Delta}^{[2]}$. It follows that $m \notin I_{\Delta}^{[2]} : I_{\Delta}$. This is what we wanted to prove. □

**Remark 5.** Under the assumption that each variable $x_i$ divides some minimal monomial generator of $I_{\Delta}$, equivalently $\Delta$ is not a cone over any vertex, one may check out that the condition on the Cartier algebra of a complete Stanley-Reisner ring $R = S/I_{\Delta}$ being principally generated is a topological property of the geometric realization $X$ of $\Delta$. In fact, by Theorem 4, $C(R)$ is not principally generated if and only if there is an open subset $U \subset X$ which is homeomorphic to $\{ (x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_m \geq 0 \}$ for some $m \in \mathbb{N}$. However, the condition that $\Delta$ is not a cone over any vertex is not topological. In this sense, being principally generated is not a topological condition. This is quite parallel to the relation between Gorenstein and Gorenstein* properties of simplicial complexes where we have that $\Delta$ is Gorenstein if and only if $\Delta = \Delta' * 2^{[n]} \setminus V$ for some Gorenstein* complex $\Delta'$ on some $V \subseteq [n]$ and Gorenstein* is a topological property (See² [7, §II.5]).

Despite the fact that using Theorem 4 one may construct many simplicial complexes satisfying that the Cartier algebra $C(R)$ is principally generated, e.g. triangulations of $\Delta = \text{core} \Delta$ in [7, §II.5] corresponds to $V = [n]$ in our notation.

²The notation $\Delta = \text{core} \Delta$ in [7, §II.5] corresponds to $V = [n]$ in our notation.
manifolds without boundary, it seems that there is no tight relation to any homological conditions on \( R \). The best we can say in this direction is the following. For the definitions of Buchsbaum* complexes and undefined terminologies we refer to [7] and [2].

**Corollary 6.** If \( \Delta \) is Buchsbaum* (in particular, doubly Cohen-Macaulay, or Gorenstein*) over some field \( K \), then \( \mathcal{C}(R) \) is principally generated.

**Proof.** Suppose that \( \Delta \) is Buchsbaum* but \( \mathcal{C}(R) \) is not principally generated. Since \( \Delta \) is Buchsbaum*, \( \Delta \) is not a cone over any vertex. Hence there is a subfacet \( \sigma \) contained in a unique maximal face \( \tau \) by Theorem 4. Clearly, \( \text{cost}_\Delta(\sigma) = \Delta \setminus \{\sigma, \tau\} \) and \( \text{cost}_\Delta(\tau) = \Delta \setminus \{\tau\} \). Hence we have \( H_d(\Delta, \text{cost}_\Delta(\sigma); K) = 0 \) and \( H_d(\Delta, \text{cost}_\Delta(\tau); K) = K \), and the map \( H_d(\Delta, \text{cost}_\Delta(\sigma); K) \to H_d(\Delta, \text{cost}_\Delta(\tau); K) \) can not be surjective. It means that \( \Delta \) is not Buchsbaum* so we get a contradiction. \( \square \)

This result together with Lemma 3 allows us to give a direct proof of the fact that a complete Gorenstein Stanley-Reisner ring has a principally generated Cartier algebra since \( \Delta \) is Gorenstein if and only if \( \Delta = \Delta' * 2^{[n]\setminus V} \) for some Gorenstein* complex \( \Delta' \) on some \( V \subseteq [n] \).

**Example 7.** (i) Consider the 1-dimensional simplicial complex \( \Delta \) in Figure 1 below. \( \Delta \) is Cohen-Macaulay and \( \mathcal{C}(R) \) is principally generated, but \( \Delta \) is not doubly Cohen-Macaulay so it is not Buchsbaum* as well.

   (ii) Let \( \Delta \) be the simplicial complex with facets \( \{1, 2, 3\} \), \( \{1, 2, 4\} \), \( \{1, 3, 4\} \), \( \{2, 3, 4\} \), \( \{1, 5\} \) and \( \{2, 5\} \) (see Figure 2 below). Then, \( \mathcal{C}(R) \) is principally generated but \( \Delta \) is not pure.

![Figure 1](image1.png)

![Figure 2](image2.png)

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**References**