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KAM aspects of the quasi-periodic Hamiltonian Hopf bifurcation. Summary of results

Mercè Ollé[1]

merce.olle@upc.es

Juan R. Pacha[2]

joanr@vilma.upc.es

Jordi Villanueva[3]

jordi@vilma.upc.es

[1,2,3]Dept. de Matemàtica Aplicada I
Universitat Politècnica de Catalunya
08028 Barcelona (Spain)

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Abstract

In this work we consider a 1:-1 non semi-simple resonant periodic orbit of a three-degrees of freedom real analytic Hamiltonian system. From the formal analysis of the normal form, it is proved the branching off a two-parameter family of two-dimensional invariant tori of the normalised system, whose normal behaviour depends intrinsically on the coefficients of its low-order terms. Thus, only elliptic or elliptic together with parabolic and hyperbolic tori may detach form the resonant periodic orbit. Both patterns are mentioned in the literature as the direct and, respectively, inverse quasi-periodic Hopf bifurcation. In this report we focus on the direct case, which has many applications in several fields of science. Thus, we present here a summary of the results, obtained in the framework of KAM theory, concerning the persistence of most of the (normally) elliptic tori of the normal form, when the whole Hamiltonian is taken into account, and to give a very precise characterisation of the parameters labelling them, which can be selected with a very clear dynamical meaning. These results include sharp quantitative estimates on the "density" of surviving tori, when the distance to the resonant periodic orbit goes to zero, and state that the 4-dimensional Cantor manifold holding these tori admits a Whitney-C^∞ extension. In addition, an application to the Circular Spatial Three-Body Problem (CSRTBP) is reviewed.

**AMS Mathematics Subject Classification:** 37G05, 37G15

Contents

1 Introduction

This paper shows some results related with the persistence of the quasi-periodic Hopf bifurcation scenario in the Hamiltonian context. In its more simple formulation, we shall consider a real analytic three-degree of freedom Hamiltonian system, $\mathcal{H}$, with a 1:1 *non-degenerate* and *non semi-simple* resonant periodic orbit (often known in the literature as 1:−1 resonant), $\mathcal{M}_0$. Beyond the 1:1 order of the resonance which, in this context, implies that the non-trivial characteristic multipliers (i. e., those different from 1) of the orbit have multiplicity two, the non-degenerate and non semi-simple character mean respectively that, in the Jordan form of the monodromy matrix of $\mathcal{M}_0$: $M_0 = M(\mathcal{M}_0)$, diagonal blocks are not present, neither for the unit characteristic multipliers nor for the other ones (which come by two reciprocal pairs, since the system is Hamiltonian). Explicitly, let 1, λ_0, $1/\lambda_0$, with $|\lambda_0| = 1$, be the characteristic multipliers of $\mathcal{M}_0$ —and hence, the eigenvalues of M_0, all with multiplicity 2—, and let J_{M_0} be the Jordan form of M_0. Therefore J_{M_0} should be a block-diagonal matrix of type: $\mathrm{diag}(\mathfrak{J}_2(1), \mathfrak{J}_2(\lambda_0), \mathfrak{J}_2(\lambda_0^{-1}))$, being the $m \times m$ blocks: $\mathfrak{J}_m(\varpi) = \varpi I_m + N_m$, for $m \in \mathbb{N}$, $\varpi \in \mathbb{C}$, where I_m is the $m \times m$ identity matrix and $N^{(m)}$ is the $m \times m$ nilpotent matrix defined according to: $N_{i,j}^{(m)} = 1$, if $j = i - 1$ for $i = 2, 3, \ldots, m$ or $N_{i,j}^{(m)} = 0$ otherwise. In addition, *irrationality* of the resonance is also going to be assumed. More precisely: if $\lambda_0 = \exp(2\pi i \nu_0)$, then $\nu_0 \notin \mathbb{Q}$; in other words: the non-trivial (i. e., those different from 0) characteristic exponents, $\mu_0^{\pm} = \pm 2\pi \nu_0$, cannot be commensurable with 2π.

We know, however, that in Hamiltonian systems, periodic orbits are not usually isolated, but form one-parameter families. In fact, it can be seen that, under the generic conditions above, $\mathcal{M}_0$ is actually the *critical* periodic orbit of a one-parameter family, $\{\mathcal{M}_\sigma\}_{\sigma \in \mathbb{R}}$, which looses its stability, changing from stable (centre, purely elliptic) to a complex saddle. Hence, the characteristic multipliers of the family follow the evolution illustrated in figure 1. This fact is discussed —as a straightforward consequence of theorem 3.1— in section 3. The mechanism of instabilization just described is often referred in the literature as *stable-complex unstable* transition (see [19]), and has been also studied for families of four-dimensional symplectic maps, where an elliptic fixed point evolves to a complex saddle as the parameter of the family moves (see [12, 38]).

On the other hand, it turns out that, under the same general conditions, the branching off two-dimensional quasi-periodic solutions (respectively, invariant curves for mappings) from the resonant periodic orbit (respectively, the fixed point) has been described both numerically (in [11, 20, 32, 35, 36, 39]) and analytically (in [33, 37]). The analytic approach relies on the computation of the normal
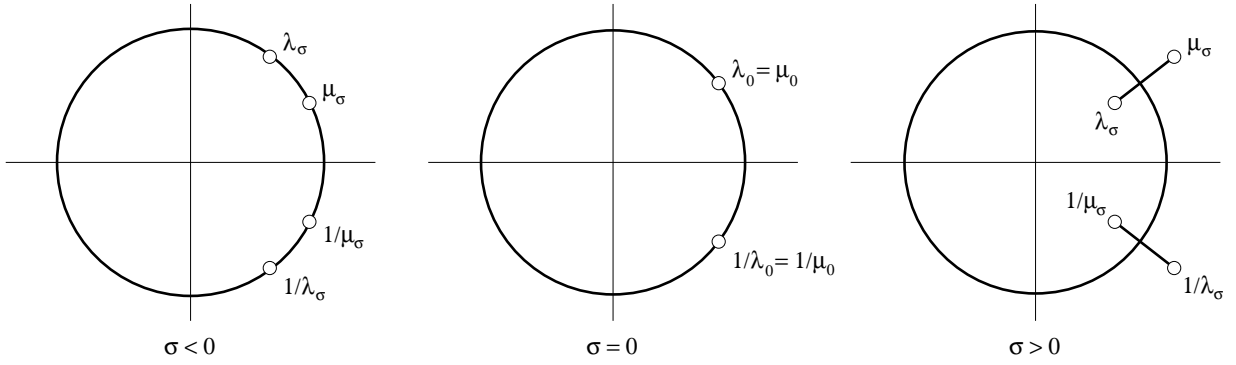
Figure 1: Evolution of the characteristic multipliers of the family. $\mathcal{M}_0$ is known as the *critical* or $1 : -1$ resonant periodic orbit.

form around the critical orbit. Thus, a biparametric family of two-dimensional invariant tori follows at once from the dynamics of the (integrable) normal form. This bifurcation can be *direct* or *inverse*. In our context, the direct case means that only normally elliptic tori unfold, while in the inverse case normally parabolic and hyperbolic tori are present as well. The type of bifurcation is determined by the coefficients of a low-order normal form.

Nevertheless, this bifurcation pattern cannot be directly stated for the complete Hamiltonian, since the normal form computed at all orders is, generically, divergent. If we stop the normalising process up to some finite order, the initial Hamiltonian is then casted (by means of a canonical transformation) into the sum of an integrable part plus a non-integrable remainder. Hence, the question is whether some quasi-periodic solutions of the integrable part survive in the whole system, and we know there are chances for this to happen if the remainder is sufficiently small to be thought of as a perturbation (see [26] for a nonperturbative approach to KAM theory).

This work discuss the persistence of the elliptic bifurcated tori in the direct case. In the inverse case, elliptic and hyperbolic tori can be dealt in a complete analogous way (see remark 3.6) whilst parabolic tori require a slightly different approach (we refer to [5, 17, 18] for works concerning the persistence of parabolic invariant tori).

For the direct case, in theorem 4.1 we state that there exists a two-parameter Cantor family of two-dimensional elliptic tori branching off the resonant periodic orbit. Moreover, we also give quantitative estimates on the (Lebesgue) measure, in the parameters space, of the holes between invariant tori and claim the Whitney-C^∞ smoothness of the 4D (Cantor) manifold holding them. Amid the features of this theorem, here we stress two. One is the precise description of the parameter set of "basic frequencies" for which we have an invariant bifurcated torus, i. e., the "geometry of the bifurcation". The other one is the sharp asymptotic measure estimates for the size of these holes, when the distance to the periodic orbit goes to zero.

This result has some straight applications, for instance in Celestial Mechanics, particularly in the Restricted Three Body Problem (an account of some results related with this field is given in section 5). For other applications, see [33] and references therein.

Even though proofs are not included in this report (the reader will be referenced to previous work and, particularly, those involved with KAM methods will be published somewhere else), some of the specificities of the problem at hand are comment below.

When computing the normal form of a Hamiltonian around maximal dimensional tori, elliptic fixed points or normally elliptic periodic orbits or tori, there are (standard) results providing exponentially small estimates for the size of the remainder as function of the distance, R, to the object (if the order of the normal form is chosen appropriately as function of R). These estimates can be translated into bounds for the relative measure of the complementary of the Cantor set of parameters for which we have invariant tori (see [4, 14, 21, 22, 23] for papers dealing with exponentially small measure

estimates in KAM theory). In the present context, the generic situation at the resonant periodic orbit is a non semi-simple structure for the Jordan blocks of the monodromy matrix associated to the colliding characteristic multipliers (see description above). This yields to homological equations in the normal form computations that cannot be reduced to diagonal form. When the homological equations are diagonal, it means that only one "small divisor" appears as a denominator of any coefficient of the solution. In the non semi-simple case, there are (at any order) some coefficients having as a denominator a small divisor power to the order of the corresponding monomial. This fact gives rise to very big "amplification factors" in the normal form computations, that do not allow to obtain exponentially small estimates for the remainder. In [34] it is proved that it decays with respect to R faster than any power of R, but with less sharp bounds than in the semi-simple case. This fact translates into poor measure estimates for the bifurcated tori.

To prove the persistence of these tori, we are faced with KAM methods for elliptic low-dimensional tori (see [6, 7, 15, 22, 23, 40, 43]). More precisely, the proof resembles those on the existence of invariant tori when adding to a periodic orbit the excitations of its elliptic normal modes (compare [15, 22, 41]), but with the additional intricacies due to the present bifurcation scenario. The main difficulty in tackling this persistence problem has to do with the choice of suitable parameters to characterise the tori of the family along the iterative KAM process. In this case one has three frequencies to control, the two intrinsic (those of the quasi-periodic motion) and the normal one, but only two parameters (those of the family) to keep track of them. So, we are bound to deal with the so-called "lack of parameters" problem for low-dimensional tori (see [6, 30, 42] for a general treatment of the problem, and (sub)section 4.1 to see how it is dealt in this specific case). However, some usual tricks for tackling elliptic tori cannot be applied directly to the problem at hand, for the reasons shown below.

When applying KAM techniques for invariant tori of Hamiltonian systems, it is usual to set a diffeomorphism between the intrinsic frequencies and the "parameter space" of the family of tori (typically the actions). In this way, in the case of elliptic low-dimensional tori, the normal frequencies can be expressed as a function of the intrinsic ones. Under these assumptions, the standard non-degeneracy conditions on the normal frequencies are to require that the denominators of the KAM process, which depend on the normal and intrinsic frequencies, "move" as function of the latter ones. These conditions —together with the Diophantine ones— can be fulfilled at each step of the KAM iterative process. Unfortunately, in the current context these transversality conditions are not defined at the critical orbit, due to the strong degeneracy of the problem. In few words, the elliptic invariant tori we study are too close to parabolic. This catch is worked out taking as vector of *basic frequencies* (those labelling the tori) not the intrinsic ones, say $\Omega = (\Omega_1, \Omega_2)$, but the vector $\Lambda = (\mu, \Omega_2)$, where μ is the normal frequency of the torus. Then, we put the other (intrinsic) frequency as a function of Λ, i. e., $\Omega_1 = \Omega_1(\Lambda)$. With this parametrisation, the denominators of the KAM process move with Λ even if we are close to the resonant periodic orbit. See (sub)section 4.1 for further details.

Another difficulty we have to face refers to the computation of the sequence of canonical transformations of the KAM scheme. At any step of this iterative process we compute the corresponding canonical transformation by means of the Lie method. Typically in the KAM context, the (homological) equations verified by the generating function of this transformation are coupled through a triangular structure, so we can solve them recursively. However, due to the forementioned proximity to parabolic, in the present case some equations —corresponding to the average of the system with respect to the angles of the tori— become simultaneously coupled, and have to be solved all together. Then, the resolution of the homological equations becomes a little more tricky, specially for what refers to the verification of the nondegeneracy conditions needed to solve them.

This work is organised as follows. We begin fixing the notation and introducing several definitions in section 2. In section 3, we state the normal form theorem (theorem 3.1) and show, in (sub)section 3.1, how this normal form can be used to describe the bifurcation of a two parameter family of 2D invariant tori and to distinguish between the two possible types of bifurcation (direct, inverse), which are now straightforward characterised from the coefficients of the normal form; next —in (sub)section 3.2—,

the (linear) normal behaviour of the bifurcated tori is analysed. As all the preceding analysis is done on the truncated normal form, it is necessary a result to state the preservation of these (formal) quasi-periodic solutions when the remainder of the normal form is added, and the whole Hamiltonian is considered. In fact, it constitutes the main result of this report: the theorem 4.1 —of KAM type, and which concerns the direct bifurcation—, is formulated in section 4. Morover, in (sub)section 4.1 we give some details on how the lack of parameters (commented above) is overcome in the present situation. Finally, section 5 is devoted to an application of theorem 4.1 to the Circular Spatial Restricted Three Body Problem (CSRTBP). There, we just give some results of our (numerical) approach. For more details on the methods involved, we refer the reader to the original work in [32].

2   Basic notation and definitions

Given a complex vector $u \in \mathbb{C}^n$, we denote by $|u|$ its supremum norm, $|u| = \sup_{1 \leq i \leq n} \{|u_i|\}$. We extend this notation to any matrix $A \in \mathbb{M}_{r,s}(\mathbb{C})$, so that $|A|$ means the induced matrix norm. Similarly, we write $|u|_1 = \sum_{i=1}^n |u_i|$ for the absolute norm of a vector and $|u|_2$ for its Euclidean norm. We denote by u^* and A^* the transpose vector and matrix, respectively. As usual, for any $u, v \in \mathbb{C}^n$, their bracket $\langle u, v \rangle = \sum_{i=1}^n u_i v_i$ is the inner product of $\mathbb{C}^n$. Moreover, $\lfloor \cdot \rfloor$ stands for the integer part of a real number.

We deal with analytic functions $f = f(\theta, x, I, y)$ defined in the domain

$$\mathcal{D}_{r,s}(\rho, R) = \{(\theta, x, I, y) \in \mathbb{C}^r \times \mathbb{C}^s \times \mathbb{C}^r \times \mathbb{C}^s : |\mathrm{Im}\,\theta| \leq \rho,\ |(x,y)| \leq R,\ |I| \leq R^2\}, \tag{1}$$

for some integers r, s and some $\rho > 0$, $R > 0$. These functions are 2π-periodic in θ and take values on $\mathbb{C}$, $\mathbb{C}^n$ or $\mathbb{M}_{n_1,n_2}(\mathbb{C})$.

The coordinates $(\theta, x, I, y) \in \mathcal{D}_{r,s}(\rho, R)$ are canonical through the symplectic form $d\theta \wedge dI + dx \wedge dy$. Hence, given scalar functions $f = f(\theta, x, I, y)$ and $g = g(\theta, x, I, y)$, we define their *Poisson bracket* by

$$\{f, g\} = (\nabla f)^* \mathcal{J}_{r+s} \nabla g,$$

where ∇ is the gradient with respect to (θ, x, I, y) and $\mathcal{J}_n$ the standard symplectic $2n \times 2n$ matrix. If $\Psi = \Psi(\theta, x, I, y)$ is a canonical transformation, close to the identity, then we consider the following expression of Ψ (according to its natural vector-components),

$$\Psi = \mathrm{Id} + (\Theta, \mathcal{X}, \mathcal{I}, \mathcal{Y}), \qquad \mathcal{Z} = (\mathcal{X}, \mathcal{Y}). \tag{2}$$

To generate such canonical transformations we mainly use the *Lie series method*. Thus, given a Hamiltonian $H = H(\theta, x, I, y)$ we denote by Ψ_t^H the flow time t of the corresponding vector field, $\mathcal{J}_{r+s} \nabla H$. We observe that if $\mathcal{J}_{r+s} \nabla H$ is 2π-periodic in θ, then also is $\Psi_t^H - \mathrm{Id}$.

3   The quantitative normal form

To study the dynamics around the resonant periodic orbit $\mathcal{M}_0$, we use normal forms. It is assumed that we have a system of symplectic coordinates specially suited for this orbit, so that the phase space is described by $(\theta, x, I, y) \in \mathbb{T}^1 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$, being $x = (x_1, x_2)$ and $y = (y_1, y_2)$, endowed with the 2-form $\mathrm{d}\theta \wedge \mathrm{d}I + \mathrm{d}x \wedge \mathrm{d}y$. In this reference system we want the periodic orbit to be given by the circle $I = 0$, $x = y = 0$. Such (local) coordinates can always be found for a given periodic orbit (see [9, 10] and [24] for an explicit example). In addition, a (symplectic) Floquet transformation is performed to reduce to constant coefficients the quadratic part of the Hamiltonian with respect to the *normal directions* (x, y) (see [37]). If the resonant eigenvalues of the monodromy matrix are non semi-simple, the Hamiltonian can be expressed in the new variables as

$$\mathcal{H}(\theta, x, I, y) = \omega_1 I + \omega_2 (x_2 y_1 - x_1 y_2) + \frac{1}{2}(y_1^2 + y_2^2) + \widehat{\mathcal{H}}(\theta, x, I, y), \tag{3}$$

(nevertheless, to achieve this form, yet an involution in time may be necessary, see [33]). Here, ω_1 is the angular frequency of the resonant periodic orbit and ω_2 its (only) normal frequency, so that its nontrivial characteristic multipliers are $\{\lambda_0, \lambda_0, 1/\lambda_0, 1/\lambda_0\}$, with $\lambda_0 = \exp(2\pi i \omega_2/\omega_1)$. The function $\widehat{\mathcal{H}}$ is 2π-periodic in θ, holds the higher order terms in (x, I, y), and can be analytically extended to a complex neighbourhood of the periodic orbit. From now on, we set $\mathcal{H}$ in (3) to be our initial Hamiltonian.

From here, our first step is to compute the normal form of (3) up to a suitable order. This order is chosen to minimise (as much as possible) the size of the non-integrable remainder of such normal form. Hence, for any $R > 0$ (small enough), a neighbourhood of "size" R of type (1) around $\mathcal{M}_0$ is considered and a normalising order, $r_{\text{opt}}(R)$, is selected so that the remainder of the normal form of $\mathcal{H}$ up to degree $r_{\text{opt}}(R)$ becomes as small as possible in this neighbourhood. As it has been pointed before, for an elliptic nonresonant periodic orbit it is possible to select this order so that the remainder becomes exponentially small in R. In the present resonant setting, the non semi-simple character of the homological equations leads to poor estimates of the remainder. The following result, which can be derived from [34], states the normal form up to "optimal" order and the bounds for the corresponding remainder.

Theorem 3.1 (The quantitative normal form). *We assume that the real analytic Hamiltonian* (3) *is defined in the complex domain* $\mathcal{D}_{1,2}(\rho_0, R_0)$, *for some* $\rho_0 > 0$, $R_0 > 0$, *and that the weighted norm* $|\mathcal{H}|_{\rho_0, R_0}$ *is finite. Moreover, we also assume that the vector* $\omega = (\omega_1, \omega_2)$ *satisfies the Diophantine condition*

$$|\langle k, \omega \rangle| \geq \gamma |k|_1^{-\tau}, \qquad \forall k \in \mathbb{Z}^1 \setminus \{(0,0)\} \tag{4}$$

(see remark 3.1) for some $\gamma > 0$ *and* $\tau > 1$. *Therefore, given any* $\varepsilon > 0$ *and* $\sigma > 1$, *both fixed, there exists* $0 < R^* < 1$ *such that, for any* $0 < R \leq R^*$, *there is a real analytic canonical diffeomorphism* $\widehat{\Psi}^{(R)}$ *verifying:*

(i) $\widehat{\Psi}^{(R)} : \mathcal{D}_{1,2}(\sigma^{-2}\rho_0/2, R) \to \mathcal{D}_{1,2}(\rho_0/2, \sigma R)$.

(ii) *If* $\widehat{\Psi}^{(R)} - \text{Id} = (\widehat{\Theta}^{(R)}, \widehat{\mathcal{X}}^{(R)}, \widehat{\mathcal{I}}^{(R)}, \widehat{\mathcal{Y}}^{(R)})$, *then all the components are* 2π-*periodic in* θ *and satisfy*

$$|\widehat{\Theta}^{(R)}|_{\sigma^{-2}\rho_0/2, R} \leq (1 - \sigma^{-2})\rho_0/2, \qquad |\widehat{\mathcal{I}}^{(R)}|_{\sigma^{-2}\rho_0/2, R} \leq (\sigma^2 - 1)R^2,$$
$$|\widehat{\mathcal{X}}_j^{(R)}|_{\sigma^{-2}\rho_0/2, R} \leq (\sigma - 1)R, \qquad |\widehat{\mathcal{Y}}_j^{(R)}|_{\sigma^{-2}\rho_0/2, R} \leq (\sigma - 1)R, \quad j = 1, 2.$$

(iii) *The transformed Hamiltonian by the action of* $\widehat{\Psi}^{(R)}$ *takes the form:*

$$\mathcal{H} \circ \widehat{\Psi}^{(R)}(\theta, x, I, y) = \mathcal{Z}^{(R)}(x, I, y) + \mathcal{R}^{(R)}(\theta, x, I, y), \tag{5}$$

where $\mathcal{Z}^{(R)}$ *(the normal form) is an integrable Hamiltonian system which looks like*

$$\mathcal{Z}^{(R)}(x, I, y) = \mathcal{Z}_2(x, I, y) + \widetilde{Z}^{(R)}(x, I, y), \tag{6}$$

with $\mathcal{Z}_2$ *given by*

$$\mathcal{Z}_2(x, I, y) = \omega_1 I + \omega_2 L + \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(aq^2 + bI^2 + cL^2) + dqI + eqL + fIL \tag{7}$$

and

$$\widetilde{Z}^{(R)}(x, I, y) = Z^{(R)}(q, I, L/2), \quad \text{with:} \quad q = (x_1^2 + x_2^2)/2 \quad \text{and} \quad L = y_1 x_2 - x_1 y_2. \tag{8}$$

The function $Z^{(R)}(u_1, u_2, u_3)$ *is analytic around the origin, with Taylor expansion starting at degree three. More precisely,* $Z^{(R)}(u_1, u_2, u_3)$ *is a polynomial of degree less than or equal to*

$\lfloor r_{\mathrm{opt}}(R)/2 \rfloor$, except by the affine part on u_1 and u_3, which allows generic dependence on u_2; therefore it can be written as:

$$Z^{(R)}(u_1, u_2, u_3) = u_1 \chi_1(u_2) + u_3 \chi_2(u_2) + F^{(R)}(u_1, u_2, u_3), \tag{9}$$

being χ_i real analytic functions with $\chi_i(0) = \chi_i'(0) = 0$ for $i = 1, 2$ whereas $F^{(R)}$ is a polynomial in u_1, u_2, u_3 with

$$3 \leq \mathrm{degree}(F^{(R)}) \leq \lfloor r_{\mathrm{opt}}(R)/2 \rfloor .$$

The remainder $\mathcal{R}^{(R)}$ contains terms in (x, I, y) of higher order than "the polynomial part" of $\mathcal{Z}^{(R)}$, being all of them of $\mathcal{O}_3(x, y)$.

(iv) The expression $r_{opt}(R)$ is given by

$$r_{\mathrm{opt}}(R) := 2 + \left\lfloor \exp\left(W \left(\log \left(\frac{1}{R^{1/(\tau+1+\varepsilon)}} \right) \right) \right) \right\rfloor, \tag{10}$$

with $W : (0, +\infty) \to (0, +\infty)$ defined from the equation $W(z) \exp(W(z)) = z$.

(v) $\mathcal{R}^{(R)}$ satisfies the bound

$$|\mathcal{R}^{(R)}|_{\sigma^{-2}\rho_0/2, R} \leq M^{(0)}(R) := R^{r_{\mathrm{opt}}(R)/2}. \tag{11}$$

In particular, $M^{(0)}(R)$ goes to zero with R faster than any algebraic order, that is

$$\lim_{R \to 0^+} \frac{M^{(0)}(R)}{R^n} = 0, \qquad \forall n \geq 1.$$

(vi) There exists a constant $\tilde{c}$ independent of R (but depending on ε and σ) such that

$$|\mathcal{Z}^{(R)}|_{0, R} \leq |\mathcal{H}|_{\rho_0, R_0}, \qquad |\widetilde{Z}^{(R)}|_{0, R} \leq \tilde{c} R^6. \tag{12}$$

Remark 3.1. *Note that the Lebesgue measure of the set of values $\omega \in \mathbb{R}^2$ for which condition (4) is not fulfilled is zero (see [27], appendix 4).*

Remark 3.2. *The function W corresponds to the principal branch of a special function $W : \mathbb{C} \to \mathbb{C}$ known as the Lambert W function (as a complex function $W(z)$ is multivaluated, with infinite number of branches, usually denoted by $W_k(z)$, $k \in \mathbb{Z}$). A detailed description of its properties can be found in many places, see for example [13].*

Remark 3.3. *The normal form $\mathcal{Z}^{(R)}(x, I, y)$ in (6) is integrable as a Hamiltonian system, since it can be shown that the three functions*
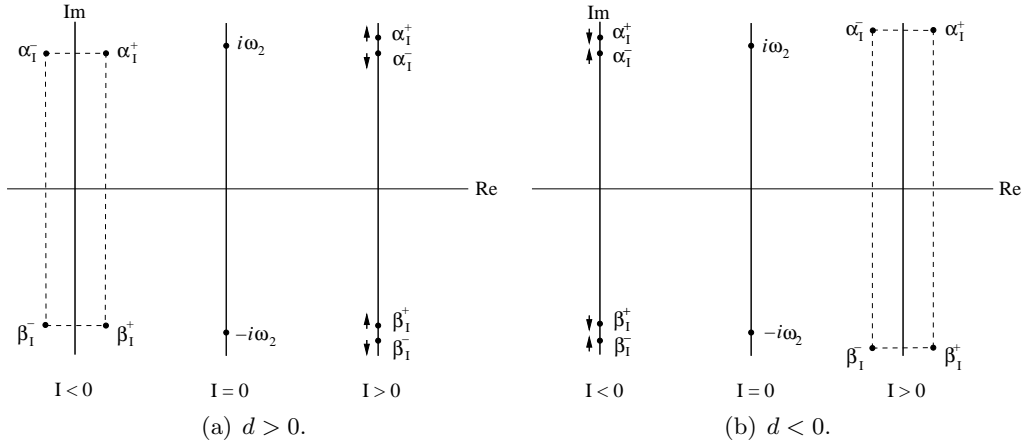
$$\mathcal{I}_1 = I, \quad \mathcal{I}_2 = L, \quad \mathcal{I}_3 = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(aq^2 + bI^2 + cL^2) + dqI + eqL + fIL + Z^{(R)}(q, I, L/2)$$

(with q and L given by (8)) are, outside the zero measure set defined by

$$\{y_1 = 0, y_2 = 0, aq + dI + eL + \partial_1 Z^{(R)}(q, I, L/2) = 0\},$$

three functional independent integrals in involution of the system:

$$\dot{\theta} = \partial_I \mathcal{Z}^{(R)}, \quad \dot{I} = -\partial_\theta \mathcal{Z}^{(R)}, \quad \dot{x}_i = \partial_{y_i} \mathcal{Z}^{(R)}, \quad \dot{y}_i = -\partial_{x_i} \mathcal{Z}^{(R)}, \quad i = 1, 2.$$

Figure 2: Evolution of the characteristic exponents. d is the coefficient of $\mathcal{Z}_2$ in (7).

Next, if we write the Hamiltonian equations from (5) taking (6), (7) and (9) into account:

$$
\begin{aligned}
\dot{\theta} &= \omega_1 + 2bI + (d + \chi_1'(I))q + \left(f + \frac{1}{2}\chi_2'(I) \right) L + \mathcal{O}_3(x,y), \\
\dot{I} &= \mathcal{O}_3(x,y), \\
\dot{x}_1 &= \left(\omega_2 + fI + \frac{1}{2}\chi_2(I) \right) x_2 + y_1 + \mathcal{O}_3(x,y), \\
\dot{x}_2 &= -\left(\omega_2 + fI + \frac{1}{2}\chi_2(I) \right) x_1 + y_2 + \mathcal{O}_3(x,y), \\
\dot{y}_1 &= -(2dI + 2\chi_1(I))x_1 + \left(\omega_2 + fI + \frac{1}{2}\chi_2(I) \right) y_2 + \mathcal{O}_3(x,y), \\
\dot{y}_2 &= -(2dI + 2\chi_1(I))x_2 - \left(\omega_2 + fI + \frac{1}{2}\chi_2(I) \right) y_1 + \mathcal{O}_3(x,y).
\end{aligned}
\tag{13}
$$

Therefore, it is clear that the system above has a family of periodic orbits $\{\mathcal{M}_I\}_{|I|\leq R}$, being

$$
\mathcal{M}_I = \mathbb{T}^1 \times \{(0,0)\} \times \{I\} \times \{(0,0)\},
\tag{14}
$$

in which the resonant periodic orbit $\mathcal{M}_0$ (corresponding to $I=0$), is embedded. After an straightforward computation it can be seen that the characteristic exponents of the orbit $\mathcal{M}_I$, $I \in [-R,R]$, of the family are given by

$$
\begin{aligned}
\alpha_I^{\pm} &= \left(\omega_2 + fI + \frac{1}{2}\chi_2(I) \right) \mathrm{i} \pm \sqrt{-2dI - 2\chi_1(I)}, \\
\beta_I^{\pm} &= -\left(\omega_2 + fI + \frac{1}{2}\chi_2(I) \right) \mathrm{i} \pm \sqrt{-2dI - 2\chi_1(I)}.
\end{aligned}
\tag{15}
$$

Here d and f are the coefficients of the terms in qI and IL, respectively, in $\mathcal{Z}_2$ —see (7)—. The evolution of these characteristic exponents is shown in figure 2. Thus, we see that when $d > 0$, the family $\{\mathcal{M}_I\}_{|I|\leq R}$ undergoes a transition from complex instability to stability as the action I increases from negative to positive values. Conversely, for $d < 0$, the periodic orbits in the family $\{\mathcal{M}_I\}_{|I|\leq R}$ change from stable to complex unstable as I increases from $I < 0$ to $I > 0$.

Actually, the full statement of theorem 3.1 is not explicitly contained in [34], but can be gleaned easily from the paper. Let us describe what are the new features we are talking about.

First, we have modified the action of the transformation $\widehat{\Psi}^{(R)}$ so that the family of periodic orbits (14) of $\mathcal{H}$, in which the resonant periodic orbit is embedded, and its normal (Floquet) behaviour, are fully described (locally) by the normal form $\mathcal{Z}^{(R)}$ of (9). Thus, the fact that the remainder, $\mathcal{R}^{(R)}$, is of $\mathcal{O}_3(x,y)$ implies that neither the family of periodic orbits nor its Floquet exponents (15) change in (5) —see sections 3.1 and 3.2—. To achieve this, we are forced to work not only with a polynomial expression for the normal for $\mathcal{Z}^{(R)}$ (as done in [34]), but to allow generic dependence on I for the coefficients of the affine part of the expression of $\mathcal{Z}^{(R)}$ in powers of q and L. For this purpose we have to extend the normal form criteria used in [34]. We do not plan to give here full details on these modifications, but we are going to summarise the main ideas below.

Let us consider the initial Hamiltonian $\mathcal{H}$ in (3). Then, we start applying a *partial normal form* process to it in order to reduce the remainder to $\mathcal{O}_3(x,y)$, and to arrange the affine part of the normal form in q and L. After this process, the family of periodic orbits of $\mathcal{H}$ and its Floquet behaviour (see (14) and (15)) remain the same if we compute them either in the complete transformed system or in the truncated one when removing the $\mathcal{O}_3(x,y)$ remainder. We point out that the divisors appearing in this (partial) normal form are $k\omega_1 + l\omega_2$, with $k \in \mathbb{Z}$ and $l \in \{0, \pm 1, \pm 2\}$ (excluding the case $k = l = 0$). As we are assuming irrational collision, these divisors are not "small divisors" at all, because all of them are uniformly bounded from below and go to infinity with k. Hence, we can ensure convergence of this normalising process in a neighbourhood of the periodic orbit.

After this convergent (partial) normal form is carried out on the Hamiltonian $\mathcal{H}$, we apply the result of [34] to the resulting system. In this way we establish the quantitative estimates, as function of R, for the normal form up to "optimal order". It is easy to realise that the normal form procedure of [34] does not "destroy" the $\mathcal{O}_3(x,y)$ structure of the remainder $\mathcal{R}^{(R)}$.

However, we want to emphasise that the particular structure for the normal form $\mathcal{Z}^{(R)}$ stated in theorem 3.1 is not necessary to apply KAM methods. We can prove the existence of the (Cantor) bifurcated family of 2D-tori only by using the polynomial normal form of [34]. The motivating reason for modifying the former normal form is only to characterise easily which bifurcated tori are real tori (see point *(iv)* in the statement of theorem 4.1).

The second remark on theorem 3.1 refers to the bound on $\widetilde{Z}^{(R)}$ given in the last point of the statement, which is neither explicitly contained in [34]. Again, it can can be easily derived from the paper (see chapter 2 in [37]).

3.1 Bifurcated family of 2D-tori of the normal form

It turns out that the normal form $\mathcal{Z}^{(R)}$ is integrable (see remark 3.3), but in this report we are only concerned with the two-parameter family of bifurcated 2D-invariant tori associated with this Hopf scenario. See [33] for a full description of the dynamics. To easily identify this family, we introduce new (canonical) coordinates $(\phi, q, J, p) \in \mathbb{T}^2 \times \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}$, with the 2-form $d\phi \wedge dJ + dq \wedge dp$, defined through the change:

$$
\begin{aligned}
\theta &= \phi_1, & x_1 &= \sqrt{2q}\cos\phi_2, & y_1 &= -\frac{J_2}{\sqrt{2q}}\sin\phi_2 + p\sqrt{2q}\cos\phi_2, \\
I &= J_1, & x_2 &= -\sqrt{2q}\sin\phi_2, & y_2 &= -\frac{J_2}{\sqrt{2q}}\cos\phi_2 - p\sqrt{2q}\sin\phi_2,
\end{aligned}
\tag{16}
$$

that casts the Hamiltonian (5) into (dropping, from now on, the superindex (R)):

$$
\breve{\mathcal{H}}(\phi, q, J, p) = \breve{\mathcal{Z}}(q, J, p) + \breve{\mathcal{R}}(\phi, q, J, p),
\tag{17}
$$

where,

$$
\breve{\mathcal{Z}}(q, J, p) = \langle \omega, J \rangle + qp^2 + \frac{J_2^2}{4q} + \frac{1}{2}(aq^2 + bJ_1^2 + cJ_2^2) + dqJ_1 + eqJ_2 + fJ_1J_2 + Z(q, J_1, J_2/2).
\tag{18}
$$

Let us consider the Hamilton equations of $\check{\mathcal{Z}}$:

$$\dot{\phi}_1 = \omega_1 + bJ_1 + dq + fJ_2 + \partial_2 Z(q, J_1, J_2/2), \qquad \dot{J}_1 = 0,$$

$$\dot{\phi}_2 = \omega_2 + \frac{J_2}{2q} + cJ_2 + eq + fJ_1 + \frac{1}{2}\partial_3 Z(q, J_1, J_2/2), \qquad \dot{J}_2 = 0,$$

$$\dot{p} = -p^2 + \frac{J_2^2}{4q^2} - aq - dJ_1 - eJ_2 - \partial_1 Z(q, J_1, J_2/2), \qquad \dot{q} = 2qp.$$

Next result sets precisely the bifurcated family of 2D-tori of $\check{\mathcal{Z}}$ (and hence of $\mathcal{Z}$).

Theorem 3.2. *With the same notations of theorem 3.1. If $d \neq 0$, there exists a real analytic function $\mathbb{I}(\xi, \eta)$ defined in a neighbourhood $\Gamma \subset \mathbb{C}^2$ of the origin, $(\xi, \eta) = (0, 0) \in \Gamma$, determined implicitly by the equation*

$$\eta^2 = a\xi + d\mathbb{I}(\xi, \eta) + 2e\xi\eta + \partial_1 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta), \tag{19}$$

with $\mathbb{I}(0, 0) = 0$ and such that, for any $\zeta = (\xi, \eta) \in \Gamma \cap \mathbb{R}^2$, the two-dimensional torus

$$\mathcal{T}_{\xi,\eta}^{(0)} = \{(\phi, q, J, p) \in \mathbb{T}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} : q = \xi, \ J_1 = \mathbb{I}(\xi, \eta), \ J_2 = 2\xi\eta, \ p = 0\}$$

is invariant under the flow of $\check{\mathcal{Z}}$ with parallel dynamics for ϕ determined by the vector $\Omega = (\Omega_1, \Omega_2)$ of intrinsic frequencies:

$$\Omega_1(\xi, \eta) = \omega_1 + b\mathbb{I}(\xi, \eta) + d\xi + 2f\xi\eta + \partial_2 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta) = \partial_{J_1}\check{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}}, \tag{20}$$

$$\Omega_2(\xi, \eta) = \omega_2 + \eta + 2c\xi\eta + e\xi + f\mathbb{I}(\xi, \eta) + \frac{1}{2}\partial_3 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta) = \partial_{J_2}\check{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}}. \tag{21}$$

Moreover, for $\xi > 0$, the corresponding tori of $\mathcal{Z}$ are real.

Remark 3.4. *If we set $\xi = 0$, then $\mathcal{T}_{0,\eta}^{(0)}$ corresponds to the family of periodic orbits of $\mathcal{Z}$ in which the critical one is embedded, but only those in the stable side of the bifurcation. These periodic orbits are parametrised by $q = p = J_2 = 0$ and $J_1 = \mathbb{I}(0, \eta) := \widetilde{\mathbb{I}}(\eta^2)$, and hence the periodic orbit given by η is the same given by $-\eta$. The angular frequency of the periodic orbit $\mathcal{T}_{0,\eta}^{(0)}$ is given by $\Omega_1(0, \eta) := \widetilde{\Omega}_1(\eta^2)$ and the two normal ones are $\Omega_2(0, \eta) := \eta + \widetilde{\Omega}_2(\eta^2)$ and $-\eta + \widetilde{\Omega}_2(\eta^2)$ (check it in the Hamiltonian equations of (6)). We observe that $\Omega_2(0, \eta)$ depends on the sign of η, but that when changing η by $-\eta$ only switches both normal frequencies. Moreover, the functions $\widetilde{\mathbb{I}}$, $\widetilde{\Omega}_1$ and $\widetilde{\Omega}_2$ are analytic around the origin and, as a consequence of the normal form criteria of theorem 3.1, they are independent of R and give the parametrisation of the family of periodic orbits of (5) and of their intrinsic and normal frequencies.*

The proof of theorem 3.2 follows directly by substitution in the Hamiltonian equations of $\check{\mathcal{Z}}$. Here we shall only stress that $d \neq 0$ is the only necessary hypothesis for the implicit function $\mathbb{I}$ to exist in a neighbourhood of $(0, 0)$. On its turn, the reality condition follows at once writing the invariant tori in the former coordinates (θ, x, I, y) (see (16)). Explicitly, the corresponding quasi-periodic solutions are

$$\theta = \Omega_1(\zeta)t + \phi_1^{(0)}, \qquad x_1 = \sqrt{2\xi}\cos(\Omega_2(\zeta)t + \phi_2^{(0)}), \qquad x_2 = -\sqrt{2\xi}\sin(\Omega_2(\zeta)t + \phi_2^{(0)}),$$

$$I = \mathbb{I}(\zeta), \qquad y_1 = -\eta\sqrt{2\xi}\sin(\Omega_2(\zeta)t + \phi_2^{(0)}), \qquad y_2 = -\eta\sqrt{2\xi}\cos(\Omega_2(\zeta)t + \phi_2^{(0)}).$$

Therefore, $\zeta = (\xi, \eta)$ are the parameters of the family of tori, so they "label" an specific invariant torus of $\mathcal{Z}$. Classically, when applying KAM methods, it is usual to require the *frequency map*, $\zeta \mapsto \Omega(\zeta)$, to be a diffeomorphism, so that we can label the tori in terms of its vector of intrinsic frequencies. This

is (locally) achieved by means of the standard Kolmogorov nondegeneracy condition, $\det(\partial_\zeta \Omega) \neq 0$. In the present case, simple computations show that:

$$\mathbb{I}(\xi, \eta) = -\frac{a}{d}\xi + \cdots, \quad \Omega_1(\xi, \eta) = \omega_1 + \left(d - \frac{ab}{d}\right)\xi + \cdots, \quad \Omega_2(\xi, \eta) = \omega_2 + \left(e - \frac{af}{d}\right)\xi + \eta + \cdots \quad (22)$$

(for higher order terms, see [33]). Then, Kolmogorov's condition computed at the resonant orbit reads as $d - ab/d \neq 0$. Although this is the classic approach, we shall be forced to choose a set of parameters on the family different from the intrinsic frequencies.

3.2 Normal behaviour of the bifurcated tori

Let us consider the variational equations of $\breve{\mathcal{Z}}$ around the family of (real) bifurcated tori $\mathcal{T}_{\xi,\eta}^{(0)}$ (with $\xi > 0$). The restriction of these equations to the normal directions (q, p) is given by a two dimensional constant coefficients linear system, with matrix

$$M_{\xi,\eta} = \begin{pmatrix} 0 & 2\xi \\ -\dfrac{2\eta^2}{\xi} - a - \partial_{1,1}^2 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta) & 0 \end{pmatrix}. \quad (23)$$

Then, the *characteristic exponents* (or *normal eigenvalues*) of this torus are

$$\lambda_\pm(\xi, \eta) = \pm\sqrt{-4\eta^2 - 2a\xi - 2\xi\partial_{1,1}^2 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta)}. \quad (24)$$
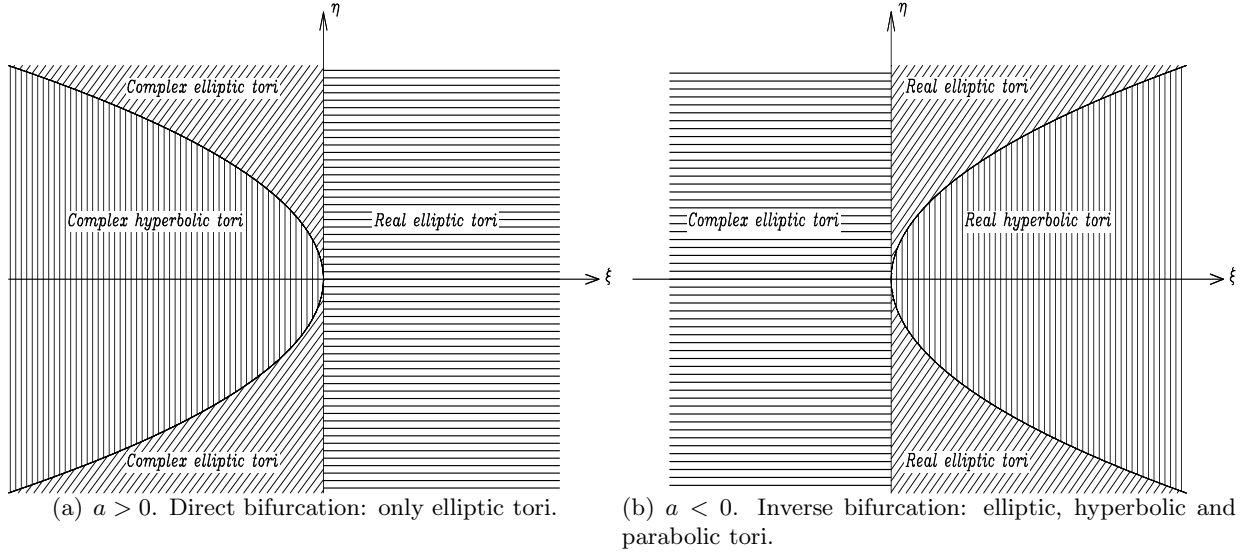
If $a > 0$, it is easy to realise that the eigenvalues $\lambda_\pm$ are purely imaginary if $\xi > 0$ and η are both small enough, and hence the family $\mathcal{T}_{\xi,\eta}^{(0)}$ holds only elliptic tori. If $a < 0$, then elliptic, hyperbolic and parabolic tori co-exist simultaneously in the family. In this report we are only interested in the case $a > 0$ (direct bifurcation). Figure 3 shows this two bifurcation patterns in the (ξ, η)-parameter plane. Particularly, here we shall only be concerned with elliptic tori, for which the main result (theorem 4.1) is formulated. Hence, we denote by $\mu = \mu(\xi, \eta) > 0$ the only *normal frequency* of the torus $\mathcal{T}_{\xi,\eta}^{(0)}$, so that $\lambda_\pm = \pm\mathrm{i}\mu$, with

$$\mu^2 := 2\xi\partial_{q,q}^2 \breve{\mathcal{Z}}|_{\mathcal{T}_\zeta^{(0)}} = 4\eta^2 + 2a\xi + 2\xi\partial_{1,1}^2 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta). \quad (25)$$

If we pick up the (stable) periodic orbit $(\pm\eta, 0)$, then $\mu = 2|\eta|$. Hence, it is clear that $\mu \to 0$ as we approach to the resonant orbit $\xi = \eta = 0$. Thus, the elliptic bifurcated tori of the normal form are very close to parabolic. This is the main source of problems when proving their persistence in the complete system.

Remark 3.5. *Besides those having $0 < \xi << 1$ and $|\eta| << 1$ we observe that, from formula (24), those tori having $\xi < 0$ but $4\eta^2 + 2a\xi + 2\xi\partial_{1,1}^2 Z(\xi, \mathbb{I}(\xi, \eta), \xi\eta) > 0$ are elliptic too, albeit they are complex tori when written in the original variables (recall that $\xi = 0$ corresponds to the stable periodic orbits of the family, see remark 3.4). However, when performing the KAM scheme, one can work with them all together (real or complex tori), because they turn to be real when written in the "action–angle" variables introduced in (16).*

Remark 3.6. *There is almost no difference in studying the persistence of elliptic tori in the inverse case. For hyperbolic tori, the same KAM methodology also works, only taking into account that now $\lambda_\pm = \pm\mu$, $\mu > 0$. Thus, in the hyperbolic case we can also use (suited) iterative KAM schemes, with the only difference that some of the divisors, appearing when solving the homological equations, are not "small divisors" at all, because their real part has an uniform lower bound in terms of μ. This fact simplifies a lot the measure estimates of the surviving tori. As pointed before in the introduction, the parabolic case, $\lambda_\pm = 0$, requires a different approach and it is not covered here.*

(a) $a > 0$. Direct bifurcation: only elliptic tori. (b) $a < 0$. Inverse bifurcation: elliptic, hyperbolic and parabolic tori.

Figure 3: Normal behaviour of the (unperturbed) bifurcated invariant tori. The separation curves correspond to *parabolic* tori.

4 Formulation of the main result

Once a direct quasi-periodic Hopf bifurcation is set, we can establish for the dynamics of $\mathcal{Z}_2$ and, in fact, for the dynamics of the truncated normal for up to an arbitrary order, the existence of a two-parameter family of two-dimensional elliptic tori branching off the resonant periodic orbit. Of course, due to the small divisors of the problem, it is not possible to expect full persistence of this family in the complete Hamiltonian system (3), but only a Cantor family of two-dimensional tori.

The precise result we have obtained about the persistence of this family is stated as follows, and constitutes the main result of this report.

Theorem 4.1. *Under the same conditions of theorem 3.1, i. e.: we assume that the real analytic Hamiltonian $\mathcal{H}$ in (3) is defined in the complex domain $\mathcal{D}_{1,2}(\rho_0, R_0)$, for some $\rho_0 > 0$, $R_0 > 0$, and that the weighted norm $|\mathcal{H}|_{\rho_0, R_0}$ is finite. Moreover, we also assume that the (real) coefficients a and d of its low-order normal form, $\mathcal{Z}_2$, in (7) verify $a > 0$, $d \neq 0$, and that the vector $\omega = (\omega_1, \omega_2)$ satisfies the Diophantine condition (4) for some $\gamma > 0$ and $\tau > 1$. Then, we have:*

(i) The $1{:}{-}1$ resonant periodic orbit $I = 0$, $x = y = 0$ of $\mathcal{H}$ is embedded into a one-parameter family of periodic orbits having a transition from stability to complex instability at this critical orbit.

(ii) There exists a Cantor set $\mathcal{E}^{(\infty)} \subset \mathbb{R}^+ \times \mathbb{R}$ such that, for any $\Lambda = (\mu, \Omega_2) \in \mathcal{E}^{(\infty)}$, the Hamiltonian system $\mathcal{H}$ has an analytic two-dimensional elliptic invariant torus —with vector of intrinsic frequencies $\Omega(\Lambda) = (\Omega_1^{(\infty)}(\Lambda), \Omega_2)$ and normal frequency μ— branching off the critical periodic orbit. However, for some values of Λ this torus is complex (i. e., a torus laying on the complex phase space but carrying out quasi-periodic motion for real time).

(iii) The "density" of the set $\mathcal{E}^{(\infty)}$ becomes almost one as we approach to the resonant periodic orbit. Indeed, there exist constants $c^* > 0$ and $\bar{c}^* > 0$ such that, if we define

$$\mathcal{V}(R) := \{\Lambda = (\mu, \Omega_2) \in \mathbb{R}^2 : 0 < \mu \leq c^* R, \ |\Omega_2 - \omega_2| \leq c^* R\}$$

and $\mathcal{E}^{(\infty)}(R) = \mathcal{E}^{(\infty)} \cap \mathcal{V}(R)$, then, for any given $0 < \alpha < 1/19$, there is $\breve{R}^* = \breve{R}^*(\alpha)$ such that

$$\mathrm{meas}\left(\mathcal{V}(R) \setminus \mathcal{E}^{(\infty)}(R)\right) \leq \bar{c}^* (M^{(0)}(R))^{\alpha/4}, \tag{26}$$

for any $0 < R \leq \breve{R}^*$. Here, meas *stands for the Lebesgue measure of $\mathbb{R}^2$ and the expression $M^{(0)}(R)$, which is defined precisely in the statement of theorem 3.1, goes to zero faster than any power of R (in spite of it is not exponentially small in R).*

(iv) *There exists a real analytic function $\widetilde{\Omega}_2$, with $\widetilde{\Omega}_2(0) = \omega_2$, such that the curves $\gamma_1(\eta) = (2\eta, \eta + \widetilde{\Omega}_2(\eta^2))$ and $\gamma_2(\eta) = (2\eta, -\eta + \widetilde{\Omega}_2(\eta^2))$, locally separate between the parameters $\Lambda \in \mathcal{E}^{(\infty)}$ giving rise to real or complex tori. Indeed, if $\Lambda = (\mu, \Omega_2) \in \mathcal{E}^{(\infty)}$ and $\mu = 2\eta > 0$, then real tori are those with $-\eta + \widetilde{\Omega}_2(\eta^2) < \Omega_2 < \eta + \widetilde{\Omega}_2(\eta^2)$. The meaning of the curves γ_1 and γ_2 are that their graphs represent, in the Λ-space, the periodic orbits of the family mentioned in (i), but only those in the stable side of the bifurcation. For a given $\eta > 0$, the periodic orbit labelled by $\gamma_1(\eta)$ is identified by the one labelled by $\gamma_2(\eta)$, being $\eta + \widetilde{\Omega}_2(\eta^2)$ and $-\eta + \widetilde{\Omega}_2(\eta^2)$ the two normal frequencies of the orbit ($\eta = 0$ corresponds to the critical one).*

(v) *The function $\Omega_1^{(\infty)} : \mathcal{E}^{(\infty)} \to \mathbb{R}$ is C^∞ in the sense of Whitney. Moreover, for each $\Lambda \in \mathcal{E}^{(\infty)}$, the following Diophantine conditions are fulfilled by the intrinsic frequencies and the normal frequency of the corresponding torus:*

$$|\langle k, \Omega^{(\infty)}(\Lambda) \rangle + \ell\mu| \geq (M^{(0)}(R))^{\alpha/2}|k|_1^{-\tau}, \qquad k \in \mathbb{Z}^2, \ell \in \{0, 1, 2\}, |k|_1 + \ell \neq 0.$$

(vi) *Let $\breve{\mathcal{E}}^{(\infty)}$ be the subset of $\mathcal{E}^{(\infty)}$ corresponding to real tori. There is a function $\Phi^{(\infty)}(\theta, \Lambda)$, defined as $\Phi^{(\infty)} : \mathbb{T}^2 \times \breve{\mathcal{E}}^{(\infty)} \to \mathbb{T} \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$, analytic in θ and Whitney-C^∞ with respect to Λ, giving a parametrisation of the Cantorian four-dimensional manifold defined by the real two-dimensional invariant tori of $\mathcal{H}$, branching off the critical periodic orbit. Precisely, for any $\Lambda \in \breve{\mathcal{E}}^{(\infty)}$, the function $\Phi^{(\infty)}(\cdot, \Lambda)$ gives a parametrisation of the corresponding two-dimensional invariant torus of $\mathcal{H}$, in such a way the pull-back of the dynamics on the torus to the variable θ is a linear quasi-periodic flow. Thus, for any $\theta^{(0)} \in \mathbb{T}^2$, then $t \in \mathbb{R} \mapsto \Phi^{(\infty)}(\Omega(\Lambda) \cdot t + \theta^{(0)}, \Lambda)$ is a solution of the Hamilton equations of $\mathcal{H}$. Moreover, $\Phi^{(\infty)}$ can be extended to a smooth function of $\mathbb{T}^2 \times \mathbb{R}^2$ —analytic in θ and C^∞ with respect to Λ—.*

As it was already said, the proof of this theorem is not included here. However, we shall explain below one of the keypoints of our approach: the choice of the normal and the second intrinsic frequencies as the basic frequencies to parametrise the family of bifurcated invariant tori along all the KAM iterative process.

4.1   Lack of parameters

One of the problems intrinsically linked to the perturbation of elliptic invariant tori is the so-called "lack of parameters". In fact, this is a common difficulty in the theory of quasi-periodic motions in dynamical systems (see [6, 30, 42]). Basically, it implies that one cannot construct a perturbed torus with a fixed set of (Diophantine) intrinsic and normal frequencies, for the system does not contain enough internal parameters to control them all simultaneously. All that one can expect is to build perturbed tori with only a given subset of *basic frequencies* previously fixed (equal to the numbers of parameters one has). The remaining frequencies have to be dealt (when possible) as function of the prefixed ones.

Let us suppose for the moment that, in our case, the two intrinsic frequencies could be the basic ones and that the normal frequency is function of the intrinsic ones (this is the standard approach). These three frequencies are present on the (small) denominators of the KAM iterative scheme (see (27)). It means that to carry out the first step of this process, one has to restrict the parameter set to the intrinsic frequencies so that them, together with the corresponding normal one of the unperturbed torus, satisfy the required Diophantine conditions. After this first step, we only can keep fixed the values of the intrinsic frequencies (assuming Kolmogorov nondegeneracy), but the function giving the

normal frequency of the new approximation to the invariant torus has changed. Thus, we cannot guarantee *a priori* that the new normal frequency is nonresonant with the former intrinsic ones.

To succeed in the iterative application of the KAM process, it is usual to ask for the denominators corresponding to the unperturbed tori to move when the basic frequencies do. In our context, with only one frequency to control, this is guaranteed if we can add suitable nondegeneracy conditions on the function giving the normal frequency. These *transversality conditions* avoid the possibility that one of the denominators falls permanently inside a resonance, and allows to obtain estimates for the Lebesgue measure of the set of "good" basic frequencies at any step of the iterative process. For 2D-elliptic low-dimensional tori with only one normal frequency, the denominators to be taken into account are (the so-called *Mel'nikov's second non-resonance condition*, see [28, 29])

$$\mathrm{i}\langle k, \Omega\rangle + \mathrm{i}\ell\mu, \qquad \forall k \in \mathbb{Z}^2 \setminus \{0\}, \quad \forall \ell \in \{0, \pm 1, \pm 2\}, \tag{27}$$

where $\Omega \in \mathbb{R}^2$ are the intrinsic frequencies and $\mu = \mu(\Omega) > 0$ the normal one.

Remark 4.1. *Indeed, other approaches are possible. For example, Bourgain showed in [1, 2] that conditions with $\ell = \pm 2$ can be omitted, but the proof becomes extremely involved.*

Now, we compute the gradient with respect to Ω of such divisors, and require them not to vanish. These transversality conditions are equivalent to $2\nabla_\Omega\mu(\Omega) \notin \mathbb{Z}^2 \setminus \{0\}$. For equivalent conditions in the "general" case see [22]. For nondegeneracy conditions of higher order see [6, 42].

This, however, does not work in the current situation. To realise, a glance at (22) shows that the first order expansion, at $\Omega = \omega$, of the inverse of the frequency map is

$$\xi = \frac{d}{d^2 - ab}(\Omega_1 - \omega_1) + \cdots, \qquad \eta = \frac{af - ed}{d^2 - ab}(\Omega_1 - \omega_1) + \Omega_2 - \omega_2 + \cdots.$$

Now, substitution in the expression (24) gives for the normal frequency

$$\mu(\Omega) = \sqrt{\frac{2ad}{d^2 - ab}(\Omega_1 - \omega_1) + \cdots},$$

so $\nabla_\Omega\mu(\Omega)$ is not well defined at the critical periodic orbit. Therefore we use different parameters on the family. From (22) and (24), it can be seen that ξ and η may be expressed as a function of μ and Ω_2,

$$\xi = \frac{\mu^2}{2a} - \frac{2}{a}(\Omega_2 - \omega_2)^2 + \cdots, \qquad \eta = \Omega_2 - \omega_2 + \left(\frac{f}{2d} - \frac{e}{2a}\right)\mu^2 + \left(\frac{2e}{a} - \frac{3f}{d}\right)(\Omega_2 - \omega_2)^2 + \cdots.$$

Now, let us denote $\Lambda = (\mu, \Omega_2)$ the new set of basic frequencies and write Ω_1 as function of them. Substitution in the expression for Ω_1 in (22) yields:

$$\Omega_1^{(0)}(\mu, \Omega_2) := \omega_1 + \left(\frac{d}{2a} - \frac{b}{2d}\right)\mu^2 + \left(\frac{3b}{d} - \frac{2d}{a}\right)(\Omega_2 - \omega_2)^2 + \cdots. \tag{28}$$

The derivatives with respect to Λ of the KAM denominators (27) are, at the critical (resonant) periodic orbit,

$$\nabla_\Lambda\left(k_1\Omega_1^{(0)}(\Lambda) + k_2\Omega_2 + \ell\mu\right)\Big|_{\Lambda=(0,\omega_2)} = (\ell, k_2), \qquad k_1, k_2, \ell \in \mathbb{Z}, \text{ with } |\ell| \le 2. \tag{29}$$

So, the divisors will change with Λ whenever the integer vector $(\ell, k_2) \ne (0, 0)$. But if $\ell = k_2 = 0$ then $k_1 \ne 0$, and the modulus of the divisor $k_1\Omega_1^{(0)}(\Lambda)$ will be bounded from below.

5 Application. The vertical family of L_4 in the CSRTBP

Theorem 4.1 has some straight applications. Particularly in Celestial Mechanics. We shall discuss here one of these applications, which appears in the Circular Spatial Restricted Three-Body Problem (CSRTBP). This model describes the motion, in a 3D (position) space, of a massless particle under the gravitational attraction of two massive bodies (called primaries) which are not attracted by the particle and move on a Keplerian circular periodic orbit about their common centre of mass (usually taken as the origin of coordinates). It is convenient to describe this problem in a rotating (synodical) system of coordinates such that, in suitable units of distance, mass and time, the primaries have masses $\mu \in (0, 1/2]$ and $1 - \mu$, and are fixed at the points $(\mu - 1, 0, 0)$ and $(\mu, 0, 0)$. Let (x, y, z) denote the position of the massless particle in the synodical system. The corresponding momenta are defined by $p_x = \dot{x} - y$, $p_y = \dot{y} + x$ and $p_z = \dot{z}$. Then, the equations of motion for the particle can be written as an autonomous Hamiltonian system of three degrees of freedom, whose Hamiltonian is given by (see [45])

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + yp_x - xp_y - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2} \tag{30}$$

being r_1 and r_2 the distances from the particle to the big and the small primaries respectively, i. e., $r_1^2 = (x - \mu)^2 + y^2 + z^2$ and $r_2^2 = (x - \mu + 1)^2 + y^2 + z^2$.

The CSRTBP has five equilibrium points: three *collinear* points $L_{1,2,3}$ —also known as the Eulerian points—, which lie on the x axis, and the *triangular* or Lagrangian points, which form an equilateral triangle with the primaries in the (x, y) plane so, in the position space, $L_{4,5} = (\mu - 1/2, \pm\sqrt{3}/2, 0)$. However, it turns out that the Hamiltonian vector field, X_H, corresponding to (30) is reversible with respect the involution $R = \mathrm{diag}(1, -1, 1, -1, 1, 1)$, i. e.: $X_H(x, y, z, p_x, p_y, p_z) = -RX_H(x, -y, z, -p_x, p_y, p_z)$; then, it suffices to study the motion around L_4.

The Jacobian matrix at the triangular points has the following characteristic exponents:

$$\lambda_1 = \mathrm{i}, \quad \lambda_{1,2} = \sqrt{-\frac{1}{2} \pm \sqrt{1 - 27\mu(1 - \mu))}}$$

and $\lambda_{j+3} = -\lambda_j$, $j = 1, 2, 3$. The pair $\pm\mathrm{i}$ gives rise to vertical oscillations with angular frequency equal to 1; and we also recall that the characteristics exponents are purely imaginary and different for $0 < \mu < \mu_{\mathrm{R}} = (1 - \sqrt{23/27})/2 \approx 0.03852$ (the Routh's mass parameter), so L_4 is linearly stable for this range of the mass parameter. For $\mu = \mu_{\mathrm{R}}$ the planar frequencies collide on the imaginary axis; this yields a change in the linear stability and for $\mu_{\mathrm{R}} < \mu \leq 1/2$, L_4 becomes complex-unstable.

On the other hand, due to the Lyapunov's centre theorem (see [43]), for any value of the mass parameter $0 < \mu \leq 1/2$, the linear vertical oscillations associated with the pair $\pm\mathrm{i}$ give rise to a family of periodic orbits of the CSRTBP, the so called *vertical family* of L_4. This family can be parametrised by the vertical amplitude of its orbits or by the value of $\dot{z}$ when they cross the hyperplane of $z = 0$ in positive sense.

For a given value of μ, the arc step method (see for instance [16, 32, 44]) is used for the numerical computation and continuation of the vertical family of periodic orbits. Actually, such family is regarded as a family of fixed points of 4D suitable isoenergetic Poincaré maps defined by $P_h : \Sigma_h \to \Sigma_h$, where

$$\Sigma_h = \{(x, y, z, p_x, p_y, p_z) \in \mathbb{R}^6 : z = 0\} \cap \{(x, y, z, p_x, p_y, p_z) \in \mathbb{R}^6 : H(x, y, z, p_x, p_y, p_z) = h\} \subset \mathbb{R}^6$$

and defined according to: $P_h(p) = q$ with $p, q \in \Sigma_h$ such that the solution of the CSRTBP, starting at p, has its second intersection with Σ_h at q. We remark that if $\mu \neq \mu_{\mathrm{R}}$, and at least for small vertical amplitudes, the linear stability of the periodic orbits of the family is the same as L_4. In particular, for a fixed $0 < \mu < \mu_{\mathrm{R}}$, the periodic orbits (for small amplitudes) of the vertical family are linearly stable, that is, for each periodic orbit, the corresponding four nontrivial —i. e., different from one— eigenvalues: $\{\lambda, 1/\lambda, \sigma, 1/\sigma\}$ of the monodromy matrix lie on the unit circle; however, for $\mu > \mu_{\mathrm{R}}$, the

periodic orbits of the vertical family are (for small amplitudes) complex-unstable, that is, the four eigenvalues leave the unit circle on a complex quadruple $\{\lambda, \sigma = \bar{\lambda}, 1/\lambda, 1/\bar{\lambda} = 1/\sigma\}$.

Nevertheless, the linear character may change for large enough amplitudes of the orbits, and in fact, it does. In order to determine the stability of the periodic orbits of the vertical family one can compute their *stability indices* α and β (see [8]), defined by the coefficients of the characteristic polynomial $p(z)$ of the corresponding monodromy matrix:

$$p(z) = (z - 1)^2 (z^4 + \alpha z^3 + \beta z^2 + \alpha z + 1),$$

and they satisfy

$$\alpha = -(\lambda + 1/\lambda + \sigma + 1/\sigma), \quad \beta = 2 + (\lambda + 1/\lambda)(\sigma + 1/\sigma).$$

To track the stability of a family of periodic orbits, it is usual to represent the stability indices in the so-called Broucke's Diagram, where the parbola $\beta = \alpha^2/4 + 2$ and the two staright lines $\beta = 2\alpha - 2$ and $\beta = -2\alpha - 2$ separate the seven possibly stability regions in the (α, β) plane (see [8]). This representation is carried out in figure 4. More precisely, we take $\mu = 0.04 > \mu_R$ and we continue numerically the vertical family starting at L_4 by increasing the value of $\dot{z}$ when $z = 0$, or equivalently, increasing the energy, h. For each orbit in the family, we have computed also the associated stability indices α and β and represent them in the Broucke's diagram of figure 4. There, it can be seen that the transition from stable to complex-unstable orbits appears. The marked points correspond to six selected orbits —whose projection on the position space (x, y, z) are plotted in figure 5—: 1 and 2 are complex-unstable, 3 is the resonant periodic orbit, 4 is stable and 5 and 6 are even semi-unstable (one positive reciprocal pair of eigenvalues on the real axis and the other complex conjugate reciprocal pair on the unit cercle). We observe how the vertical amplitue grows along the family. In particular, from figure 5 it can be appreciated
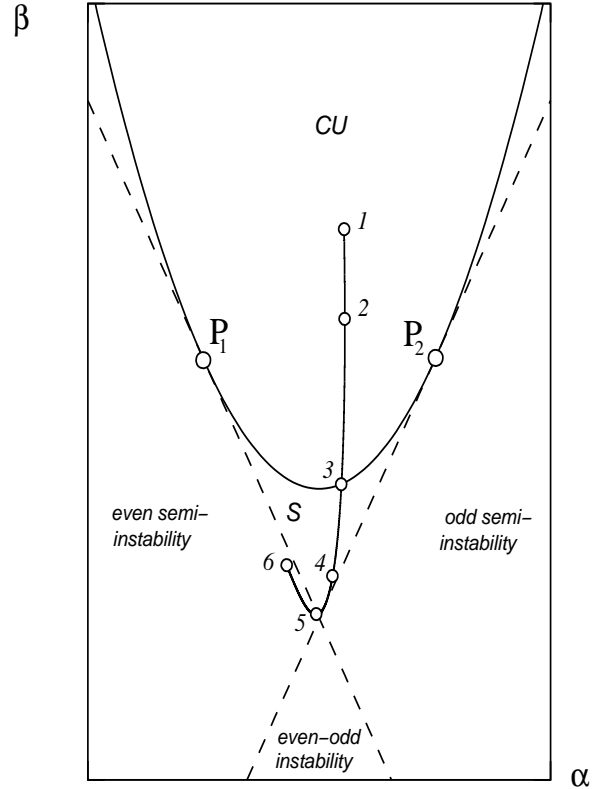

Figure 4: Broucke's diagram of the computed Lyapunov family. See the text for details.

that, for small amplitudes, the periodic orbits are complex-unstable and the transition to stability is reached only when the vertical amplitude has grown sufficiently.

In [32] the dynamics around this transition is studied numerically. Here we shall summarize the results obtained therein. Let h_{crit} denote the energy of the resonant periodic orbit (the orbit number three among the marked ones in figure 4). Therefore: for $h > h_{\text{crit}}$, the periodic orbit of the vertical family corresponding to this value of h is (linearly) stable, so, assuming for this orbit generic hypotheses of non resonance (involving the intrinsic frequency and the normal ones) and non degeneracy, there are Cantorian families of 3D invariant tori around this orbit; and also, there are the two Cantorian families of elliptic 2D tori that are born at the periodic orbit —the so called Lyapunov families of 2D tori—, which result from the quasi-periodic excitations of the vertical family in each of the two elliptic (normal) directions given by the normal frequencies (see [20, 22, 24]).

Hence, for $h > h_{\text{crit}}$ one has two of such families of 2D invarian tori. For $h = h_{\text{crit}}$, both families become one family and it detaches from the resonant periodic orbit, when $h < h_{\text{crit}}$, as a single family of 2D tori. Here, we remark that when h decreases and crosses the critical value h_{crit} (the value of the energy of the resonant periodic orbit), the elliptic tori unfold on the unstable side (at a finite distance
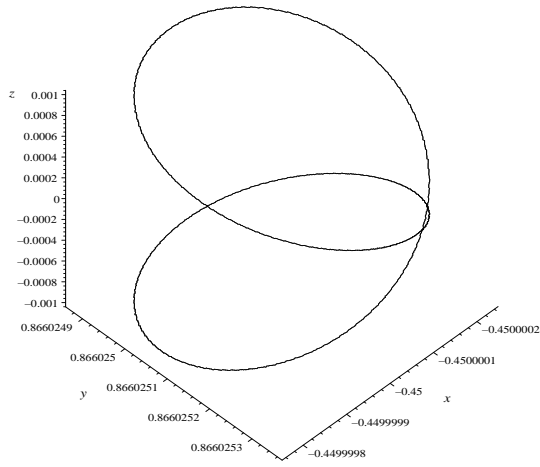
(a) Orbit # 1: *Complex-unstable*.
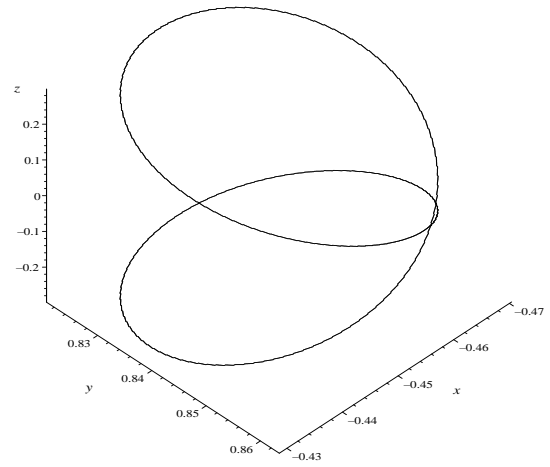
(b) Orbit # 2: *Complex-unstable*.
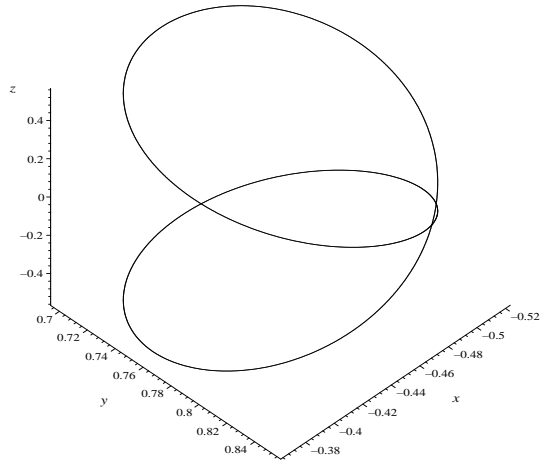
(c) Orbit # 3: *Resonant* 1:−1 *periodic orbit*.
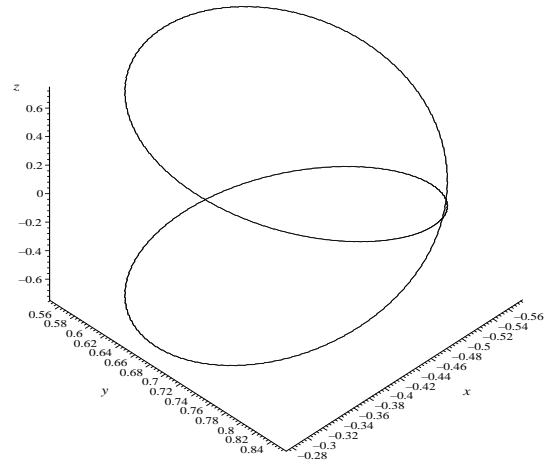
(d) Orbit # 4: *Stable*.
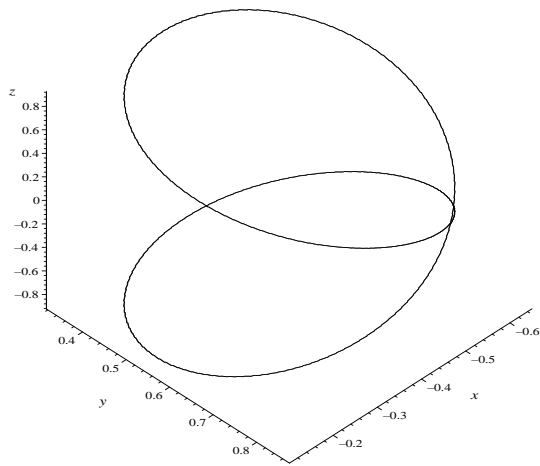
(e) Orbit # 5: *Even semi-unstable*.
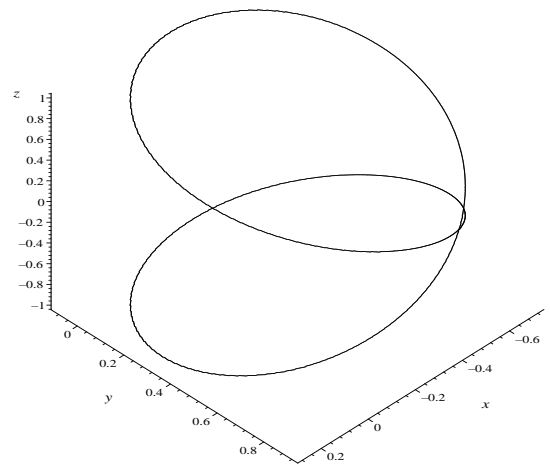
(f) Orbit # 6: *Even semi-unstable*.

Figure 5: Gallery of periodic orbits of the Lyapunov family for $\mu = 0.04$. These are the ones marked in figure 4.
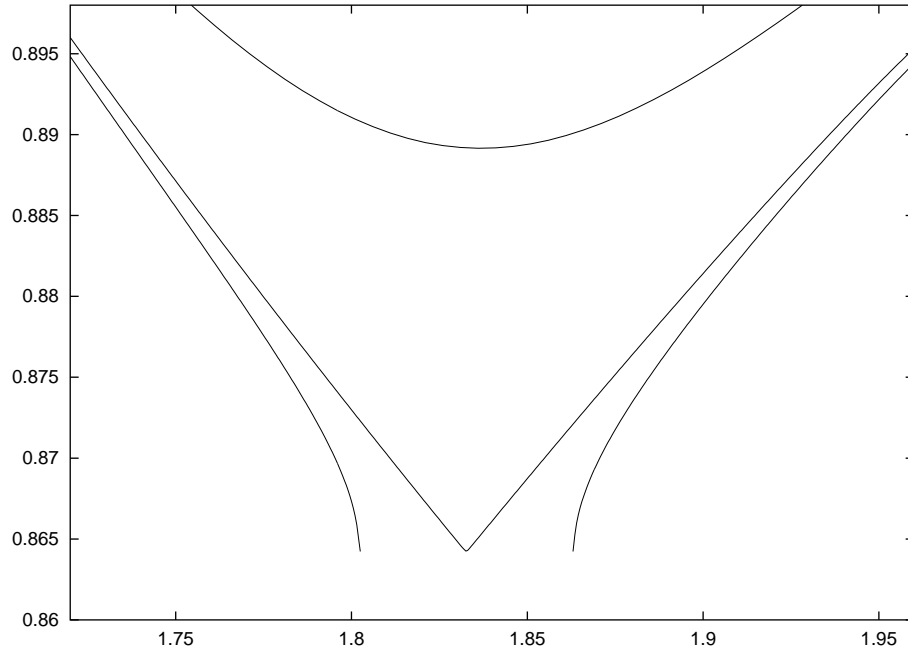
Figure 6: For $\mu = 0.04$, we plot in the $(\varpi, \widetilde{y})$ plane, the *two* Lyapunov families of invariant curves (the two lowermost —left and right— curves in the plot) for $h = -1.457018 > h_{\mathrm{crit}}$; when h decreases up to $h = h_{\mathrm{crit}}$, the two families meet at $(\widetilde{y}_{\mathrm{crit}}, \varpi_{\mathrm{crit}})$. We also plot the *detached* family (uppermost curve) for $h = -1.458308 < h_{crit}$. See the text for details.

from the periodic orbit) and, on the other hand, the 3D unstable and stable invariant manifold of each periodic orbit of the family become almost coincident (when h is very close to and less than h_{crit}) and as a consequence, the motion is confined for a very long time in a small neighbourhood of the unstable periodic orbit. This just described pattern of motion is known as the direct Hamiltonian Hopf bifurcation in dimension three (see [3, 31, 37]), and it resembles the standard (direct) Hopf bifurcation in dimension two, where an elliptic periodic orbit detaches from the equilibrium point when this equilibrium point becomes unstable (see, for example [25]).

Actually, what is computed are not the invariant tori directly, but their corresponding invariant curves —toghether with their (lineal) normal behavior—, for the isoenergetic Poincaré map $P_h : \Sigma_h \to \Sigma_h$ defined above (for the details on the numerical methods invloved, we refer again to [32] itself and references therein). In this way one gets, for a fixed value of the energy, h, a Cantorian 1-parameter family of invariant curves "living" in the 4D manifold Σ_h. For each invariant curve, we consider its parametrisation $X(\theta)$, its rotation number, ϖ, and the initial point $X(0) = (\widetilde{x}, \widetilde{y}, \widetilde{p}_x, \widetilde{p}_y)$; moreover —and just for technical reasons—, the value of $\widetilde{x}$ is fixed to $\widetilde{x} = -0.462$ for all the invariant curves (note that this is close to the value $x = \mu - 1/2$ of the point L_4 when $\mu = 0.04$). In order to represent each invariant curve, we consider the two quantities $(\varpi, \widetilde{y})$; therefore a family of invariant curves may be regarded as a curve in the $(\varpi, \widetilde{y})$ plane.

Thus, to show this direct Hopf bifurcation pattern, several families of invariant curves are found. We remark that, although these are Cantorian families of invariant curves, the holes are too small to be detected with the standard double precision arithmetic of the computer; so, from the numerical point of view, we deal with these families as if they were continuous. Figure 6 shows the two Lyapunov families of invariant curves, in the $(\varpi, \widetilde{y})$ plane, for a fixed $h = -1.457018 > h_{\mathrm{crit}}$ (the lowermost, left and right, curves in the figure): for this value of h, the corresponding periodic orbit has $\widetilde{y} = 0.86385432$ and its normal frequencies are $\omega_1 = 1.8028458$ and $\omega_2 = 1.8625327$; so, each family of invariant curves is born at the same periodic orbit (the same value of $\widetilde{y}$) but with the associated $\varpi_i = \omega_i$, $i = 1, 2$.

As mentioned above, when h decreases and tend to $h_{\mathrm{crit}} = 1.45714146$, the two normal frequencies
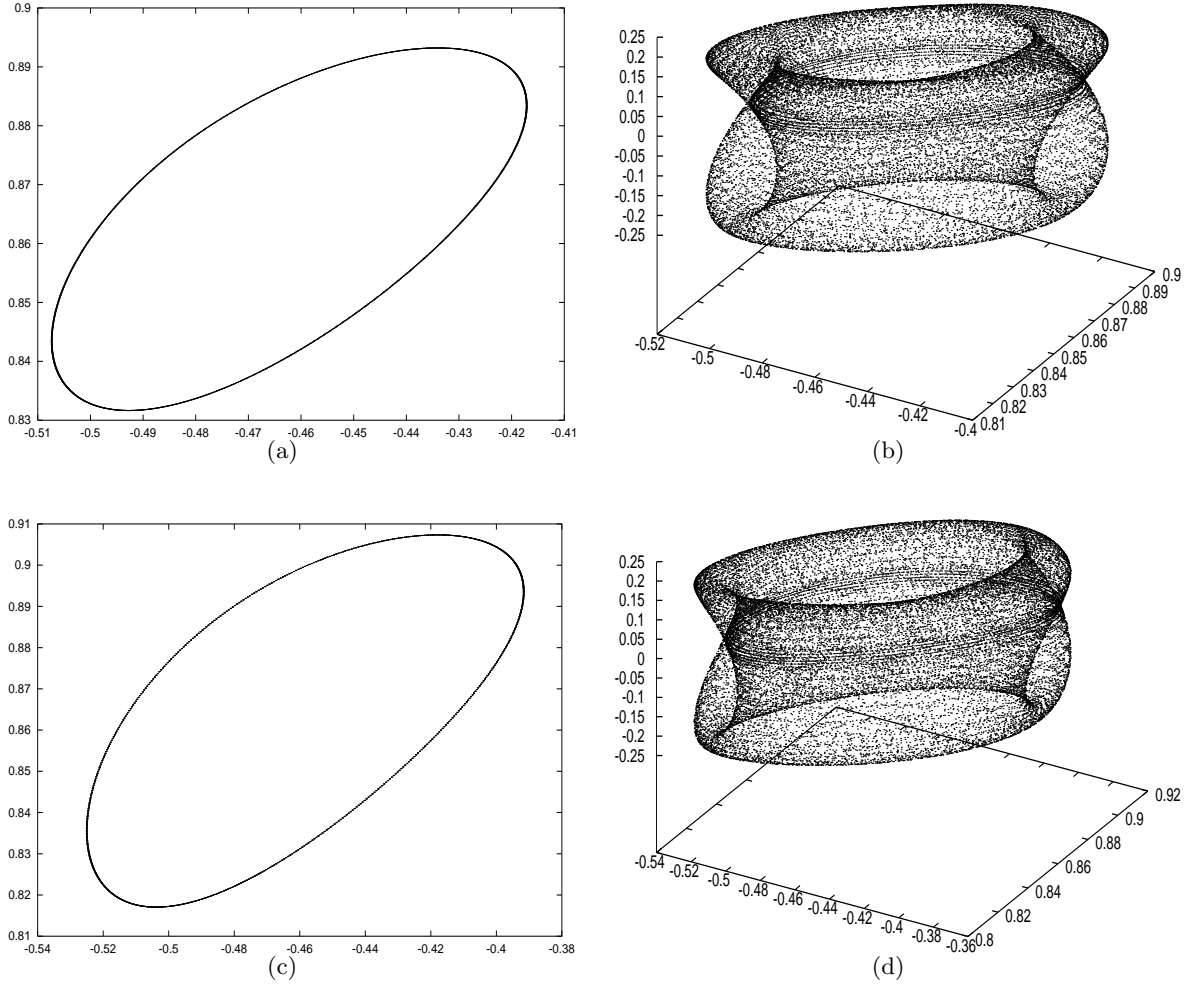
Figure 7: $\mu = 0.04$. 2D torus of the Lyapunov family of invariant tori on the stable region (top, with $\varpi = 1.90052495$) and on the complex unstable zone (bottom with $\varpi = 1.79602495$). Left: (x, y) projection of the invariant curve in the Poincaré section $z = 0$. Right: (x, y, z) projection of the corresponding torus under the flow.

tend to collide, and at the critical orbit we have $\widetilde{y} = \widetilde{y}_{\text{crit}} = 0.86386304$ and $\varpi = \omega_{\text{crit}} = 1.8326287$; therefore, the two families get closer and become a single family for $h = h_{\text{crit}}$. For $h < h_{\text{crit}}$, this family detaches from the periodic orbit, in the sense that the family of invariant curves exists at a finite (and different from zero) distance from the periodic orbit which now is complex-unstable. We see the evolution of this single family in figure 6 for $h = -1.458308 < h_{\text{crit}}$; as h decreases the distance of the invariant curves to the periodic orbit increases.

We also show, in figure 7 (top, left) the (x, y) projection of an invariant curve of the Lyapunov family with rotation number $\varpi = 1.90052495$, near the stable vertical periodic orbit corresponding to $h = -1.457018$ and (bottom, left) another curve of the family on the unstable region, with $h = -1.458308$ and $\varpi = 1.79602495$. If we follow the flow, the corresponding 2D tori are plotted —in the position space (x, y, z)—, in figure 7, at the right.

Acknowledgements

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