

THOMASON COHOMOLOGY OF CATEGORIES

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ABSTRACT. We investigate cohomology and homology theories of categories with general coefficients given by functors on simplex categories first studied by Thomason. These generalize Baues–Wirsching cohomology and homology of a small category, and coincide with Gabriel–Zisman cohomology and homology of the simplicial nerve of the category. Thus Baues–Wirsching cohomology of categories is seen to be a special case of simplicial cohomology. We analyze naturality and functoriality properties of these theories and construct associated spectral sequences for functors between small categories.

INTRODUCTION

In this article we introduce and investigate a new cohomology theory for small categories with very general coefficient systems. For any small category \mathcal{C} we define a cohomology theory with coefficients being functors from the simplex category Δ/\mathcal{C} to abelian categories. Functors of this kind were studied by Thomason in his notebooks in order to define very general notions of homotopy limits and to analyze their functoriality properties [32]. It turns out that these general coefficient systems, which we will call Thomason natural systems, provide also a systematic way to study the cohomology of small categories, especially with respect to general naturality and functoriality properties. This cohomology theory, called Thomason cohomology here, generalizes Baues–Wirsching cohomology, whose coefficients instead are functors from the factorization category $F\mathcal{C}$ to abelian categories. Baues–Wirsching cohomology was introduced in [2] in order to study systematically linear extensions of categories appearing in algebra and homotopy theory. Dually, we also introduce Thomason homology for small categories using contravariant Thomason natural systems. This generalizes Baues–Wirsching homology as introduced in [12].

An important property of Thomason homology of a category, which we prove here, is that it can be interpreted as simplicial homology, in the sense of Gabriel–Zisman [11] and Dress [8], of the nerve of the category. Dually, we show that Thomason cohomology can also be interpreted as a Gabriel–Zisman type simplicial cohomology. In this way we obtain the interesting

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result that Baues–Wirsching (co)homology of categories is a special case of Gabriel–Zisman simplicial (co)homology.

The structure of the article is as follows. In the first section, we introduce Thomason natural systems as functors from the simplex category of a given small category into any abelian category, define cohomology and homology with respect to these general coefficient systems, and analyze their basic properties. We then recall the simplicial homology theory considered by Gabriel–Zisman and Dress, and also introduce the corresponding cohomology theory. We show that Thomason cohomology and homology can be interpreted naturally in this simplicial setting. In the second section, we discuss how Thomason (co)homology generalizes Baues–Wirsching (co)homology and therefore also other classical theories like Hochschild–Mitchell (co)homology, and (co)homology of categories with local or constant coefficient systems. Finally, we analyze the functoriality of Thomason (co)homology with respect to any given functor between small categories, and construct Leray type spectral sequences in particular situations, including the cases when the functor is part of an adjoint pair and when it is a Grothendieck fibration.

1. THOMASON (CO)HOMOLOGY OF CATEGORIES

1.1. Categories of simplices. Recall that the *standard simplex category* Δ is the category whose objects are the nonempty finite ordered sets $[m] = \{0 < 1 < \dots < m\}$ and morphisms are order preserving functions. We may regard each $[m]$ as a category and Δ as the corresponding full subcategory of the category \mathfrak{Cat} of small categories.

Definition 1.1. The *simplex category* of a small category \mathcal{C} is the comma category Δ/\mathcal{C} . The object set is the set of pairs $([n], f)$ where $[n]$ is an object of Δ and $f : [n] \rightarrow \mathcal{C}$ is a functor. The morphisms

$$([n], f) \longrightarrow ([m], g)$$

are morphisms $\sigma : [n] \rightarrow [m]$ of Δ satisfying $f = \sigma^*g = g \circ \sigma$.

Objects $([n], f)$ of Δ/\mathcal{C} can therefore be interpreted as elements of the simplicial nerve $\mathcal{N}(\mathcal{C})$ of \mathcal{C} . We will often omit the $[n]$ from the notation and regard objects as diagrams

$$f = (C_0 \xleftarrow{f_1} C_1 \xleftarrow{f_2} \dots \xleftarrow{f_n} C_n).$$

The morphisms of Δ/\mathcal{C} are generated by omitting or repeating objects C_i in such diagrams.

The opposite category of the simplex category Δ/\mathcal{C} is a very special case of the *Grothendieck construction*, applied to the contravariant diagram of discrete categories given by the simplicial nerve,

$$\mathcal{N}(\mathcal{C}) : \Delta^{op} \rightarrow \mathfrak{Sets} \rightarrow \mathfrak{Cat}.$$

Recall, from [13, IX.3.4] for example, that the Grothendieck construction $\int_I F$ of a functor $F : I \rightarrow \mathcal{C}at$ is a category whose objects are pairs (i, f) where i is an object of I and f is an object of $F(i)$. In the special case that each $F(i)$ is a discrete category (that is, a set of objects with no non-identity arrows), then the morphisms $(j, g) \rightarrow (i, f)$ of the category $\int_I F$ are the morphisms $s : j \rightarrow i$ of I satisfying $f = (F(s))(g)$. The Grothendieck construction of the nerve $\mathcal{N}(\mathcal{C})$ may therefore be identified with the opposite of the simplex category,

$$(\Delta/\mathcal{C})^{op} \cong \int_{\Delta^{op}} \mathcal{N}(\mathcal{C}).$$

The *category of simplices* of an arbitrary simplicial set $X : \Delta^{op} \rightarrow \mathfrak{S}ets$ is defined as the comma category Δ/X , whose objects are pairs $([n], x)$ for $x \in X_n = X([n])$, and whose morphisms $([n], x) \rightarrow ([m], y)$ are morphisms $\sigma : [n] \rightarrow [m]$ of Δ with $x = \sigma^*y$ (see [13, I.2] or [11, II]). This comma category can again be interpreted in an elegant way as the opposite category to the Grothendieck construction of the functor $X : \Delta^{op} \rightarrow \mathfrak{S}ets \rightarrow \mathcal{C}at$, that is,

$$(\Delta/X)^{op} = \int_{\Delta^{op}} X.$$

It follows therefore, that the simplex category Δ/\mathcal{C} of a small category \mathcal{C} can be also identified with the category of simplices $\Delta/\mathcal{N}(\mathcal{C})$ of the simplicial nerve $\mathcal{N}(\mathcal{C})$ (see also [18]).

It was shown by Latch [19] that taking the category of simplices defines a functor from simplicial sets to small categories that is homotopy inverse to the nerve functor. Therefore constructions using the simplex category Δ/\mathcal{C} correspond to constructions with the category of simplices $\Delta/\mathcal{N}(\mathcal{C})$ of the nerve of \mathcal{C} .

1.2. (Co)homology of categories with Thomason natural systems as coefficients. We define cohomology and homology theories for small categories with general coefficient systems. These coefficient systems were introduced by Thomason to define abstract notions of homotopy limits and colimits [32].

Definition 1.2. Let \mathcal{M} be a category. A functor $T : \Delta/\mathcal{C} \rightarrow \mathcal{M}$ is called a (*covariant*) *Thomason natural system with values in \mathcal{M}* .

If $([n], f) \rightarrow ([m], g)$ is a morphism in the simplex category Δ/\mathcal{C} for some $\sigma : [n] \rightarrow [m]$ in Δ , where $f = \sigma^*g = g \circ \sigma$, then we will use the notation

$$\sigma_{\#} : T(f) \rightarrow T(g)$$

for the induced morphism in \mathcal{M} .

Here we will mainly consider Thomason natural systems with values in an abelian category \mathcal{A} . Particular cases include the category $\mathfrak{A}b$ of abelian groups and the abelian category $R\text{-Mod}$ of (left) R -modules for a ring R .

We define now the Thomason cohomology of a small category as follows:

Definition 1.3. Let \mathcal{C} be a small category and let $T : \Delta/\mathcal{C} \rightarrow \mathcal{A}$ be a Thomason natural system with values in a complete abelian category \mathcal{A} with exact products. We define the *Thomason cochain complex* $C_{Th}^*(\mathcal{C}, T)$ by

$$C_{Th}^n(\mathcal{C}, T) := \prod_{([n], f) \in \Delta/\mathcal{C}} T(f),$$

for each integer $n \geq 0$, with the differentials

$$d^n = \sum_{i=0}^{n+1} (-1)^i d_i^n : \prod_{[n] \xrightarrow{f} \mathcal{C}} T(f) \longrightarrow \prod_{[n+1] \xrightarrow{g} \mathcal{C}} T(g).$$

Here the morphisms d_i^n are induced by the coface maps $\delta^i : [n] \rightarrow [n+1]$,

$$d_i^n \left((a_f)_{[n] \xrightarrow{f} \mathcal{C}} \right) = \left((\delta^i)_\# (a_{g \circ \delta^i}) \right)_{[n+1] \xrightarrow{g} \mathcal{C}}$$

on all sequences of elements $a_f \in T(f)$.

The *n-th Thomason cohomology* is defined as the cohomology of this complex,

$$H_{Th}^n(\mathcal{C}, T) := H^n(C_{Th}^*(\mathcal{C}, T), d).$$

The Thomason cochain complex can equivalently be expressed as the cochain complex corresponding to a certain cosimplicial object in \mathcal{A} , defined by the cosimplicial replacement,

$$\prod^* T : \Delta \rightarrow \mathcal{A},$$

of the functor $T : \Delta/\mathcal{C} \rightarrow \mathcal{A}$. Recall from [32] (compare also [5, XI]) that the *cosimplicial replacement* of any functor $T : \Delta/\mathcal{C} \rightarrow \mathcal{M}$ is defined as the cosimplicial object $\prod^* T$, where

$$\prod^n T = \prod_{[n] \xrightarrow{f} \mathcal{C}} T(f).$$

For a morphism $\sigma : [n] \rightarrow [m]$ in Δ , the cosimplicial structure map $\prod^n T \rightarrow \prod^m T$ is defined by requiring that the component corresponding to $g : [m] \rightarrow \mathcal{C}$ is the projection onto the component corresponding to $\sigma^* g : [n] \rightarrow \mathcal{C}$ followed by $\sigma_\# : T(\sigma^* g) \rightarrow T(g)$.

For any abelian category \mathcal{A} , let $\mathfrak{NatS}^{Th}(\mathcal{A})$ be the category whose objects are the Thomason natural systems $T : \Delta/\mathcal{C} \rightarrow \mathcal{A}$ with values in \mathcal{A} . A morphism $(\varphi, \tau) : T_1 \rightarrow T_2$ between Thomason natural systems $\Delta/\mathcal{C}_i \xrightarrow{T_i} \mathcal{A}$ consists of a functor $\varphi : \mathcal{C}_2 \rightarrow \mathcal{C}_1$ together with a natural transformation $\tau : T_1 \circ \Delta/\varphi \rightarrow T_2$. The composition of morphisms is

given by following diagram,

$$\begin{array}{ccccc}
 \Delta/\mathcal{C}_1 & \xleftarrow{\Delta/\varphi} & \Delta/\mathcal{C}_2 & \xleftarrow{\Delta/\psi} & \Delta/\mathcal{C}_3 \\
 & \searrow & \downarrow & \swarrow & \\
 & T_1 & \xrightarrow{\tau} & T_2 & \xrightarrow{v} & T_3 \\
 & & & \downarrow & & \\
 & & & \mathcal{A} & &
 \end{array}$$

The Thomason cochain complex defines in fact a functor

$$C_{Th}^* : \mathfrak{NatS}^{Th}(\mathcal{A}) \rightarrow \mathfrak{CoChn}(\mathcal{A}), \quad C_{Th}^*(T) := C_{Th}^*(\mathcal{C}, T),$$

from the category of Thomason natural systems with values in the abelian category \mathcal{A} , to the category of cochain complexes in \mathcal{A} . The functor C_{Th}^* is defined on objects as above, and on morphisms by

$$\begin{aligned}
 C_{Th}^*(\varphi, \tau) : C_{Th}^*(\mathcal{C}_1, T_1) &\longrightarrow C_{Th}^*(\mathcal{C}_2, T_2), \\
 (af)_{[n] \xrightarrow{f} \mathcal{C}_1} &\longmapsto (\tau_g(a_{\varphi \circ g}))_{[n] \xrightarrow{g} \mathcal{C}_2}.
 \end{aligned}$$

Thomason cohomology therefore becomes a functor from $\mathfrak{NatS}^{Th}(\mathcal{A})$ to the category of graded objects in \mathcal{A} .

Dually, we can define the Thomason homology of a small category with coefficients in contravariant Thomason natural systems:

Definition 1.4. Let \mathcal{M} be a category. A functor $T : (\Delta/\mathcal{C})^{op} \rightarrow \mathcal{M}$ is called a (contravariant) Thomason natural system with values in \mathcal{M} .

If $([n], f) \rightarrow ([m], g)$ is a morphism in the simplex category Δ/\mathcal{C} for some $\sigma : [n] \rightarrow [m]$ in Δ , where $f = \sigma^* g = g \circ \sigma$, then we will use the notation

$$\sigma^\# : T(g) \rightarrow T(f)$$

for the induced morphism in \mathcal{M} under the contravariant functor T .

Definition 1.5. Let \mathcal{C} be a small category and let $T : (\Delta/\mathcal{C})^{op} \rightarrow \mathcal{A}$ be a contravariant Thomason natural system with values in a cocomplete abelian category \mathcal{A} with exact coproducts. We define the Thomason chain complex $C_*^{Th}(\mathcal{C}, T)$ by

$$C_n^{Th}(\mathcal{C}, T) := \bigoplus_{([n], f) \in \Delta/\mathcal{C}} T(f),$$

for each integer $n \geq 0$, with the differentials d_n given as:

$$\begin{aligned}
 d_n : \bigoplus_{[n+1] \xrightarrow{f} \mathcal{C}} T(f) &\longrightarrow \bigoplus_{[n] \xrightarrow{g} \mathcal{C}} T(g) \\
 af &\longmapsto \sum_{i=0}^{n+1} (-1)^i (\delta^i)^\#(af),
 \end{aligned}$$

where $(\delta^i)^\# : T(f) \rightarrow T(f \circ \delta^i)$ is induced by the coface map $\delta^i : [n] \rightarrow [n+1]$. The n -th Thomason homology is defined as the homology of this complex,

$$H_n^{Th}(\mathcal{C}, T) := H_n(C_*^{Th}(\mathcal{C}, T), d).$$

The Thomason chain complex can also be expressed as the chain complex corresponding to a certain simplicial object in \mathcal{A} ,

$$\coprod_* T : \Delta^{op} \rightarrow \mathcal{A},$$

given by the simplicial replacement of the functor $T : (\Delta/\mathcal{C})^{op} \rightarrow \mathcal{A}$. Here

$$\coprod_n T = \bigoplus_{[n] \xrightarrow{f} \mathcal{C}} T(f),$$

and for a morphism $\sigma : [m] \rightarrow [n]$ in Δ the simplicial structure map $\coprod_n T \rightarrow \coprod_m T$ is defined on the component corresponding to $f : [n] \rightarrow \mathcal{C}$ by $\sigma^\# : T(f) \rightarrow T(f \circ \sigma)$.

Let $\mathfrak{NatS}_{Th}(\mathcal{A})$ be the category with objects the *contravariant* Thomason natural systems $T : (\Delta/\mathcal{C})^{op} \rightarrow \mathcal{A}$, and in which a morphism $(\varphi, \tau) : T_1 \rightarrow T_2$ is given by a functor $\varphi : \mathcal{C}_1 \rightarrow \mathcal{C}_2$ together with a natural transformation $\tau : T_1 \rightarrow T_2 \circ \Delta/\varphi$. The composition of morphisms is described by the following diagram,

$$\begin{array}{ccccc} (\Delta/\mathcal{C}_1)^{op} & \xrightarrow{\Delta/\varphi} & (\Delta/\mathcal{C}_2)^{op} & \xrightarrow{\Delta/\psi} & (\Delta/\mathcal{C}_3)^{op} \\ & \searrow T_1 & \begin{array}{c} \xrightarrow{\tau} \\ T_2 \\ \downarrow \\ \mathcal{A} \end{array} & \xleftarrow{T_3} & \\ & & & & \end{array}$$

The Thomason chain complex defines a functor

$$C_*^{Th} : \mathfrak{NatS}_{Th}(\mathcal{A}) \rightarrow \mathfrak{Chn}(\mathcal{A}), \quad C_*^{Th}(T) := C_*^{Th}(\mathcal{C}, T),$$

where for morphisms we define

$$C_*^{Th}(\varphi, \tau) : C_*^{Th}(\mathcal{C}_1, T_1) \longrightarrow C_*^{Th}(\mathcal{C}_2, T_2)$$

using the maps

$$\tau_f : T_1(f) \longrightarrow T_2(\varphi \circ f).$$

1.3. Basic properties of Thomason (co)homology. Let $T : \Delta/\mathcal{C} \rightarrow \mathcal{A}$ be a Thomason natural system with values in a complete abelian category \mathcal{A} with exact products. There is a contravariant functor from \mathcal{A} to the functor category of \mathfrak{Ab} -valued Thomason natural systems on \mathcal{C} ,

$$\mathrm{Hom}_{\mathcal{A}}(-, T(-)) : \mathcal{A} \longrightarrow \mathfrak{Fun}(\Delta/\mathcal{C}, \mathfrak{Ab})^{op},$$

defined on objects A of \mathcal{A} by the functors $\mathrm{Hom}_{\mathcal{A}}(A, T(-)) : \Delta/\mathcal{C} \rightarrow \mathfrak{Ab}$ and on morphisms $f : A \rightarrow B$ in \mathcal{A} by the induced natural transformations $\eta_f : \mathrm{Hom}_{\mathcal{A}}(B, T(-)) \rightarrow \mathrm{Hom}_{\mathcal{A}}(A, T(-))$.

The right adjoint of this functor is

$$\overline{\text{Hom}}_{\Delta/\mathcal{C}}(-, T) : \mathfrak{Fun}(\Delta/\mathcal{C}, \mathfrak{Ab})^{op} \longrightarrow \mathcal{A},$$

and varying T gives the bifunctor

$$\overline{\text{Hom}}_{\Delta/\mathcal{C}}(-, -) : \mathfrak{Fun}(\Delta/\mathcal{C}, \mathfrak{Ab})^{op} \times \mathfrak{Fun}(\Delta/\mathcal{C}, \mathcal{A}) \longrightarrow \mathcal{A},$$

termed the *symbolic hom* functor (see [10], [17, Remark 2.3]). Adjointness is expressed by natural isomorphisms

$$\text{Hom}_{\mathfrak{Fun}(\Delta/\mathcal{C}, \mathfrak{Ab})}(-, \text{Hom}_{\mathcal{A}}(A, T)) \cong \text{Hom}_{\mathcal{A}}(A, \overline{\text{Hom}}_{\Delta/\mathcal{C}}(-, T)).$$

Proposition 1.6. *The symbolic hom functor satisfies:*

- (1) *If $\mathbb{Z}\text{Hom}_{\Delta/\mathcal{C}}(-, -) : (\Delta/\mathcal{C})^{op} \times \Delta/\mathcal{C} \rightarrow \mathfrak{Ab}$ is the bifunctor sending (f, g) to the free abelian group on the set $\text{Hom}_{\Delta/\mathcal{C}}(f, g)$, then there is a natural isomorphism*

$$\Psi_{T,f} : \overline{\text{Hom}}_{\Delta/\mathcal{C}}(\mathbb{Z}\text{Hom}_{\Delta/\mathcal{C}}(f, -), T(-)) \cong T(f)$$

for each $T : \Delta/\mathcal{C} \rightarrow \mathcal{A}$ and each object f of Δ/\mathcal{C} .

- (2) *If $\underline{\mathbb{Z}} : \Delta/\mathcal{C} \rightarrow \mathfrak{Ab}$ is the constant Thomason natural system, i.e. the functor with constant value the abelian group \mathbb{Z} , then there is a natural isomorphism*

$$\overline{\text{Hom}}_{\Delta/\mathcal{C}}(\underline{\mathbb{Z}}, T) \cong \lim_{\Delta/\mathcal{C}} T.$$

Proof. These results are well known in the special case of abelian groups, with $\mathcal{A} = \mathfrak{Ab}$, and the general case follows by standard arguments using the adjunction that defines the symbolic hom (see, for example, [25]).

To obtain the isomorphism of part (1) we first consider, for each object A of \mathcal{A} , the abelian group $\Omega(A)$ of natural transformations

$$\mathbb{Z}\text{Hom}_{\Delta/\mathcal{C}}(f, -) \longrightarrow \text{Hom}_{\mathcal{A}}(A, T(-)) \text{ in } \mathfrak{Fun}(\Delta/\mathcal{C}, \mathfrak{Ab}).$$

By the Yoneda Lemma, this is naturally isomorphic to the abelian group $\text{Hom}_{\mathcal{A}}(A, T(f))$. On the other hand, since $\text{Hom}_{\mathcal{A}}(-, T(-))$ is the left adjoint of $\overline{\text{Hom}}_{\Delta/\mathcal{C}}(-, T)$, there is a natural isomorphism between $\Omega(A)$ and the abelian group of morphisms $A \longrightarrow \overline{\text{Hom}}_{\Delta/\mathcal{C}}(\mathbb{Z}\text{Hom}_{\Delta/\mathcal{C}}(f, -), T)$ in \mathcal{A} . Altogether, we therefore have a natural isomorphism

$$\text{Hom}_{\mathcal{A}}(A, \overline{\text{Hom}}_{\Delta/\mathcal{C}}(\mathbb{Z}\text{Hom}_{\Delta/\mathcal{C}}(f, -), T)) \cong \text{Hom}_{\mathcal{A}}(A, T(f)).$$

If we take A to be $\overline{\text{Hom}}_{\Delta/\mathcal{C}}(\mathbb{Z}\text{Hom}_{\Delta/\mathcal{C}}(f, -), T)$, then the identity morphism on the left corresponds on the right to the isomorphism $\Psi_{T,f}$ we require. If we take A to be $T(f)$, the identity morphism on the right corresponds to $(\Psi_{T,f})^{-1}$ on the left.

The proof of (2) may be found for example in [27, Lemma 4.3.1] or [25]. \square

From the natural isomorphism

$$\overline{\text{Hom}}_{\Delta/\mathcal{C}}(\underline{\mathbb{Z}}, T) \cong \lim_{\Delta/\mathcal{C}} T$$

it follows that the corresponding right derived functors are also naturally isomorphic (see [25, III.1, III.7], [17, Sect. 2]), that is,

$$\mathrm{Ext}_{\Delta/\mathcal{C}}^n(\mathbb{Z}, T) \cong \lim_{\Delta/\mathcal{C}}^n T.$$

In the case $\mathcal{A} = \mathfrak{Ab}$ this gives the usual Ext-functor for diagrams of abelian groups over Δ/\mathcal{C} .

Remark 1.7. Though above we have worked, for convenience, only with Thomason natural systems, we can define for any small category \mathcal{C} and a complete abelian category with exact products \mathcal{A} , the *symbolic hom functor*

$$\overline{\mathrm{Hom}}_{\mathcal{C}}(-, -) : \mathfrak{Fun}(\mathcal{C}, \mathfrak{Ab})^{op} \times \mathfrak{Fun}(\mathcal{C}, \mathcal{A}) \longrightarrow \mathcal{A}.$$

It will have the same properties as in Proposition 1.6. In particular, we get an isomorphism

$$\overline{\mathrm{Hom}}_{\mathcal{C}}(\mathbb{Z}, D) \cong \lim_{\mathcal{C}} D$$

for the constant functor $\mathbb{Z} : \mathcal{C} \rightarrow \mathfrak{Ab}$ and any functor $D : \mathcal{C} \rightarrow \mathcal{A}$, which extends to an isomorphism of the associated right derived functors, i.e.

$$\mathrm{Ext}_{\mathcal{C}}^n(\mathbb{Z}, D) \cong \lim_{\mathcal{C}}^n D.$$

Dually, if \mathcal{A} is a cocomplete abelian category with exact coproducts, we have the *symbolic tensor product functor*

$$-\overline{\otimes}_{\mathcal{C}}- : \mathfrak{Fun}(\mathcal{C}^{op}, \mathfrak{Ab}) \times \mathfrak{Fun}(\mathcal{C}, \mathcal{A}) \longrightarrow \mathcal{A}.$$

We have an isomorphism

$$\mathbb{Z} \overline{\otimes}_{\mathcal{C}} D \cong \mathrm{colim}_{\mathcal{C}} D,$$

which extends to an isomorphism of the associated left derived functors

$$\mathrm{Tor}_n^{\mathcal{C}}(\mathbb{Z}, D) \cong \mathrm{colim}_n^{\mathcal{C}} D.$$

We refer the reader to [25, III.1, III.7] for the general theory and proofs (see also [17, Sect. 2], [20]).

We can now characterize Thomason cohomology and homology of small categories as cohomology and homology of the simplex category Δ/\mathcal{C} .

Theorem 1.8. *Let \mathcal{A} be complete abelian category with exact products. There are isomorphisms, natural in \mathcal{C} and in $T : \Delta/\mathcal{C} \rightarrow \mathcal{A}$, such that*

$$H_{Th}^n(\mathcal{C}, T) \cong \mathrm{Ext}_{\Delta/\mathcal{C}}^n(\mathbb{Z}, T) \cong \lim_{\Delta/\mathcal{C}}^n T = H^n(\Delta/\mathcal{C}, T).$$

Proof. We construct a chain complex which provides a projective resolution of the constant functor \mathbb{Z} in the abelian category $\mathfrak{Fun}(\Delta/\mathcal{C}, \mathfrak{Ab})$, as follows. Let $B_* = \coprod_* \mathbb{Z}\mathrm{Hom}_{\Delta/\mathcal{C}}$ be the simplicial replacement of the functor

$$\mathbb{Z}\mathrm{Hom}_{\Delta/\mathcal{C}} : (\Delta/\mathcal{C})^{op} \rightarrow \mathfrak{Fun}(\Delta/\mathcal{C}, \mathfrak{Ab}), \quad f \mapsto \mathbb{Z}\mathrm{Hom}_{\Delta/\mathcal{C}}(f, -).$$

We then take the chain complex associated to this simplicial object.

By definition of simplicial replacement, we have

$$B_n = \prod_n \mathbb{Z}\mathrm{Hom}_{\Delta/\mathcal{C}} = \bigoplus_{[n] \xrightarrow{f} \mathcal{C}} \mathbb{Z}\mathrm{Hom}_{\Delta/\mathcal{C}}(f, -) : \Delta/\mathcal{C} \rightarrow \mathfrak{Ab}.$$

We have to show that:

- (1) If $T : \Delta/\mathcal{C} \rightarrow \mathcal{A}$ is a Thomason natural system and $\prod^* T : \Delta \rightarrow \mathcal{A}$ is its cosimplicial replacement, there is a natural isomorphism

$$\overline{\mathrm{Hom}}_{\Delta/\mathcal{C}}(B_*, T) \cong \prod^* T.$$

- (2) The chain complex associated to B_* is a projective resolution of the constant Thomason natural system $\underline{\mathbb{Z}} : \Delta/\mathcal{C} \rightarrow \mathfrak{Ab}$.

It will then follow that

$$\mathrm{Ext}_{\Delta/\mathcal{C}}^*(\underline{\mathbb{Z}}, T) \cong H^*(\overline{\mathrm{Hom}}_{\Delta/\mathcal{C}}(B_*, T)) \cong H^*(\prod^* T) = H_{Th}^*(\mathcal{C}, T)$$

as required.

Point (1) holds because

$$\overline{\mathrm{Hom}}_{\Delta/\mathcal{C}}(B_n, T) \cong \prod_{[n] \xrightarrow{f} \mathcal{C}} \overline{\mathrm{Hom}}_{\Delta/\mathcal{C}}(\mathbb{Z}\mathrm{Hom}_{\Delta/\mathcal{C}}(f, -), T(-)) \cong \prod_{[n] \xrightarrow{f} \mathcal{C}} T(f).$$

For the second assertion, we observe that for fixed $g : [m] \rightarrow \mathcal{C}$, each $B_n(g)$ is the free abelian group on the set

$$\bigcup_{[n] \xrightarrow{f} \mathcal{C}} \mathrm{Hom}_{\Delta/\mathcal{C}}(f, g) \cong \mathrm{Hom}_{\Delta}([n], [m]).$$

That is, each $B_*(g)$ is the free contractible simplicial abelian group $\mathbb{Z}\Delta[m]$. The chain complex associated to B_* is therefore a projective resolution of the constant Thomason natural system $\underline{\mathbb{Z}}$ in $\mathfrak{Fun}(\Delta/\mathcal{C}, \mathfrak{Ab})$. \square

In section 2.1 we will see that our definition of Thomason cohomology generalises that of Baues–Wirsching cohomology as defined in [2]. The theorem above was proved there in the special case of Baues–Wirsching cohomology, and only for $\mathcal{A} = \mathfrak{Ab}$.

Dually, we have the following identification of Thomason homology of small categories:

Theorem 1.9. *Let \mathcal{A} be cocomplete abelian category with exact coproducts. There are isomorphisms, natural in \mathcal{C} and in $T : (\Delta/\mathcal{C})^{op} \rightarrow \mathcal{A}$, such that*

$$H_n^{Th}(\mathcal{C}, T) \cong \mathrm{Tor}_n^{(\Delta/\mathcal{C})^{op}}(\underline{\mathbb{Z}}, T) \cong \mathrm{colim}_n^{(\Delta/\mathcal{C})^{op}} T = H_n((\Delta/\mathcal{C})^{op}, T).$$

Proof. This follows along the same lines as Theorem 1.8 using the resolution of the constant functor $\underline{\mathbb{Z}}$ involving the dual notions, namely the symbolic tensor product functor and its derived Tor-functor as defined in Remark 1.7 for functors $T : (\Delta/\mathcal{C})^{op} \rightarrow \mathcal{A}$. \square

The following proposition describes the basic functoriality properties of Thomason cohomology with respect to a pair of adjoint functors between small categories:

Proposition 1.10. *Let \mathcal{C} and \mathcal{D} be small categories, \mathcal{A} a complete abelian category with exact products and (φ, ψ) a pair of adjoint functors:*

$$\varphi : \mathcal{D} \rightleftarrows \mathcal{C} : \psi.$$

Then for any Thomason natural system T on \mathcal{C} there is a natural isomorphism

$$H_{Th}^*(\mathcal{D}, \varphi^*(T)) \cong H_{Th}^*(\mathcal{C}, T).$$

In particular, an equivalence of categories $\varphi : \mathcal{D} \rightarrow \mathcal{C}$ induces a natural isomorphism

$$H_{Th}^*(\mathcal{D}, \varphi^*(T)) \cong H_{Th}^*(\mathcal{C}, T),$$

for any Thomason natural system T on \mathcal{C} .

Proof. We observe that the adjoint pair (φ, ψ) induces an adjoint pair $(\Delta/\varphi, \Delta/\psi)$ of functors between the associated simplex categories Δ/\mathcal{C} and Δ/\mathcal{D} :

$$\Delta/\varphi : \Delta/\mathcal{C} \rightleftarrows \Delta/\mathcal{D} : \Delta/\psi.$$

The statement now follows from Theorem 1.5 above and [9, Lemma 1.5, p. 10], with $\mathbf{C} = \Delta/\mathcal{C}$, $\mathbf{D} = \Delta/\mathcal{D}$ and $F = \underline{\mathbb{Z}}$ the constant Thomason natural system.

In particular, by [21, Theorem IV.4.1], an equivalence of categories φ is part of an adjunction between functors, and so the second statement follows. \square

Dually, using similar arguments, we have the following results regarding the basic functoriality of Thomason homology:

Proposition 1.11. *Let \mathcal{C} and \mathcal{D} be small categories, \mathcal{A} a cocomplete abelian category with exact coproducts and (φ, ψ) a pair of adjoint functors:*

$$\varphi : \mathcal{D} \rightleftarrows \mathcal{C} : \psi.$$

Then for any contravariant Thomason natural system T on \mathcal{C} there is a natural isomorphism

$$H_*^{Th}(\mathcal{D}, \varphi^*(T)) \cong H_*^{Th}(\mathcal{C}, T).$$

In particular, an equivalence of categories $\varphi : \mathcal{D} \rightarrow \mathcal{C}$ induces a natural isomorphism

$$H_*^{Th}(\mathcal{D}, \varphi^*(T)) \cong H_*^{Th}(\mathcal{C}, T),$$

for any contravariant Thomason natural system T on \mathcal{C} .

1.4. Thomason (co)homology as simplicial (co)homology. We will now give a characterization of Thomason (co)homology as simplicial (co)homology.

We will define cohomology and homology of simplicial sets with the following general coefficient systems, first described by Gabriel and Zisman [11, App. II.4] in the dual situation for homology of simplicial sets, which we will also discuss below. These coefficient systems are also discussed by Dress in [8].

Definition 1.12. Let X be a simplicial set and let \mathcal{M} be a category. A functor $T : \Delta/X \rightarrow \mathcal{M}$ is called a (covariant) *Gabriel–Zisman natural system with values in \mathcal{M}* .

In particular, if X is given as the nerve of a small category \mathcal{C} the notion of a Gabriel–Zisman natural system $T : \Delta/\mathcal{N}\mathcal{C} \rightarrow \mathcal{M}$ coincides with that of a Thomason natural system as defined above in section 1.2.

In analogy with the approach in [8, 11], we now define the cohomology of a simplicial set X with Gabriel–Zisman natural systems as coefficient systems.

Definition 1.13. Let X be a simplicial set and let $T : \Delta/X \rightarrow \mathcal{A}$ be a Gabriel–Zisman natural system with values in a complete abelian category \mathcal{A} with exact products. The *Gabriel–Zisman cochain complex* $C_{GZ}^*(X, T)$ of X is defined as

$$C_{GZ}^n(X, T) := \prod_{\sigma_n \in X_n} T(\sigma_n),$$

for each integer $n \geq 0$, with differential

$$d = \sum_{i=0}^{n+1} (-1)^i d^i : \prod_{\sigma_n \in X_n} T(\sigma_n) \longrightarrow \prod_{\sigma_{n+1} \in X_{n+1}} T(\sigma_{n+1}).$$

The components of these d^i are the morphisms

$$\delta_{\#}^i : T(\sigma_{n+1} \circ \delta^i) \rightarrow T(\sigma_{n+1})$$

induced by the coface maps $\delta^i : [n] \rightarrow [n+1]$. The n -th *Gabriel–Zisman cohomology* of X is the cohomology of this cochain complex,

$$H_{GZ}^n(X, T) = H^n(C_{GZ}^*(X, T), d).$$

Equivalently, $C_{GZ}^*(X, T)$ is the cochain complex associated to the cosimplicial object

$$\prod_{\sigma_0 \in X_0} T(\sigma_0) \begin{array}{c} \xleftarrow{s^0} \\ \xrightarrow{d^0} \\ \xrightarrow{d^1} \end{array} \prod_{\sigma_1 \in X_1} T(\sigma_1) \begin{array}{c} \xleftarrow{s^0} \\ \xleftarrow{s^1} \\ \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{d^2} \end{array} \prod_{\sigma_2 \in X_2} T(\sigma_2) \begin{array}{c} \xleftarrow{s^0} \\ \xleftarrow{s^1} \\ \xleftarrow{s^2} \\ \xrightarrow{d^0} \\ \xrightarrow{d^1} \\ \xrightarrow{d^2} \end{array} \prod_{\sigma_3 \in X_3} T(\sigma_3) \dots$$

defined by the cosimplicial replacement $\prod^* T$ of the functor $T : \Delta/X \rightarrow \mathcal{A}$.

From [11, Appendix II.4], it follows that this construction is the right Kan extension of T along the forgetful functor $p : \Delta/X \rightarrow \Delta$,

$$\begin{array}{ccc} \Delta/X & \xrightarrow{p} & \Delta \\ \downarrow T & \dashrightarrow p^*T = \Pi^*T & \\ \mathcal{A} & & \end{array}$$

Thomason cohomology of a small category \mathcal{C} can now be interpreted as Gabriel–Zisman cohomology of the associated simplicial nerve of \mathcal{C} as follows:

Theorem 1.14. *Let \mathcal{C} be a small category, \mathcal{A} a complete abelian category with exact products, and $T : \Delta/\mathcal{C} \rightarrow \mathcal{A}$ a Thomason natural system on \mathcal{C} . Then the Thomason cohomology of \mathcal{C} with coefficients in T is naturally isomorphic to the Gabriel–Zisman cohomology of the simplicial nerve $\mathcal{N}(\mathcal{C})$ of \mathcal{C} with coefficients in the associated Gabriel–Zisman natural system $T : \Delta/\mathcal{N}(\mathcal{C}) \rightarrow \mathcal{A}$,*

$$H_{Th}^*(\mathcal{C}, T) \cong H_{GZ}^*(\mathcal{N}(\mathcal{C}), T).$$

Proof. This is a direct consequence of the identification of the simplex category Δ/\mathcal{C} of the small category \mathcal{C} with the category of simplices $\Delta/\mathcal{N}(\mathcal{C})$ of the simplicial set $\mathcal{N}(\mathcal{C})$ as indicated in section 1.1. Under this identification, Thomason natural systems $T : \Delta/\mathcal{C} \rightarrow \mathcal{A}$ on \mathcal{C} correspond to Gabriel–Zisman natural systems $T : \Delta/\mathcal{N}(\mathcal{C}) \rightarrow \mathcal{A}$ and vice versa (and we denote them by the same letter T). It follows, therefore, that the associated cosimplicial replacements in the abelian category \mathcal{A} are the same, and we have

$$C_{Th}^*(\mathcal{C}, T) \cong C_{GZ}^*(\mathcal{N}(\mathcal{C}), T).$$

This induces an isomorphism in cohomology,

$$H_{Th}^*(\mathcal{C}, T) \cong H_{GZ}^*(\mathcal{N}(\mathcal{C}), T),$$

which is also natural with respect to functors $\varphi : \mathcal{D} \rightarrow \mathcal{C}$ and natural transformations of Thomason natural systems T . \square

Dually, we can also identify Thomason homology with simplicial homology. For this, we define homology of simplicial sets with coefficients in *contravariant* Gabriel–Zisman natural systems. These coefficients are in fact the original ones used by Gabriel and Zisman in [11, App. III.4] and by Dress in [8].

Definition 1.15. Let X be a simplicial set and let \mathcal{M} be a category. A functor $T : (\Delta/X)^{op} \rightarrow \mathcal{M}$ is called a (contravariant) Gabriel–Zisman natural system with values in \mathcal{M} .

Using these general coefficient systems, we can now define the Gabriel–Zisman homology of a given simplicial set.

Definition 1.16. Let X be a simplicial set and let $T : (\Delta/X)^{op} \rightarrow \mathcal{A}$ be a contravariant Gabriel–Zisman natural system with values in a cocomplete abelian category \mathcal{A} with exact coproducts. The *Gabriel–Zisman chain complex* $C_*^{GZ}(X, T)$ of X is defined as

$$C_n^{GZ}(X, T) := \bigoplus_{\sigma_n \in X_n} T(\sigma_n),$$

for each integer $n \geq 0$, with differentials

$$\begin{aligned} d_n : \bigoplus_{\sigma_{n+1} \in X_{n+1}} T(\sigma_{n+1}) &\longrightarrow \bigoplus_{\sigma_n \in X_n} T(\sigma_n) \\ a_f &\mapsto \sum_{i=0}^{n+1} (-1)^i (\delta^i)^\#(a_f), \end{aligned}$$

where $(\delta^i)^\# : T(\sigma_{n+1}) \rightarrow T(\sigma_{n+1} \circ \delta^i)$ is induced by the coface map $\delta^i : [n] \rightarrow [n+1]$. The n -th *Gabriel–Zisman homology* of X is defined as the homology of this chain complex,

$$H_n^{GZ}(X, T) := H_n(C_*^{GZ}(X, T), d).$$

Again, the Gabriel–Zisman chain complex is just the chain complex corresponding to a certain simplicial object in \mathcal{A} , given by the simplicial replacement of T . This allows us to interpret Thomason homology of a given small category directly as Gabriel–Zisman homology of its simplicial nerve:

Theorem 1.17. Let \mathcal{C} be a small category, \mathcal{A} a cocomplete abelian category with exact coproducts, and $T : (\Delta/\mathcal{C})^{op} \rightarrow \mathcal{A}$ a contravariant Thomason natural system on \mathcal{C} . Then the Thomason homology of \mathcal{C} with coefficients in T is isomorphic to the Gabriel–Zisman homology of the simplicial nerve $\mathcal{N}(\mathcal{C})$ of \mathcal{C} with coefficients in the associated Gabriel–Zisman natural system $T : (\Delta/\mathcal{N}(\mathcal{C}))^{op} \rightarrow \mathcal{A}$,

$$H_*^{Th}(\mathcal{C}, T) \cong H_*^{GZ}(\mathcal{N}(\mathcal{C}), T).$$

This isomorphism is natural in \mathcal{C} and T .

Proof. This follows directly by identifying the simplicial replacements of the Thomason and Gabriel–Zisman natural systems, dually to the proof of Theorem 1.14. This isomorphism of simplicial objects in \mathcal{A} induces isomorphic chain complexes via the Dold–Kan correspondence,

$$C_*^{Th}(\mathcal{C}, T) \cong C_*^{GZ}(\mathcal{N}(\mathcal{C}), T).$$

Therefore we have an isomorphism in homology,

$$H_*^{Th}(\mathcal{C}, T) \cong H_*^{GZ}(\mathcal{N}(\mathcal{C}), T),$$

which is natural in \mathcal{C} and T . \square

Let us discuss the example of group cohomology.

Example 1.18. Let G be a group, regarded as a category with one object $*$ and hom set G , and $T : \Delta/G \rightarrow \mathcal{A}$ a Thomason natural system.

The nerve of G is the simplicial set $B_\bullet G = \mathcal{N}(G)$ given as

$$\begin{array}{c} \begin{array}{ccccc} & & s_1 & & \\ & \curvearrowright & \curvearrowright & \curvearrowright & \\ & & s_0 & & \\ * & \xrightarrow{d^1} & G & \xrightarrow{d_0} & G^2 & \xrightarrow{\quad} & G^3 \dots \\ & \xleftarrow{d^0} & & \xleftarrow{d_1} & & \xleftarrow{\quad} & \\ & & d_2 & & & & \end{array} \end{array}$$

The Thomason natural system T may be regarded as a Gabriel–Zisman natural system $T : \Delta/\mathcal{N}(G) \rightarrow \mathcal{A}$, whose cosimplicial replacement is

$$\begin{array}{c} \begin{array}{ccccc} & & s^0 & & \\ & \curvearrowright & \curvearrowright & \curvearrowright & \\ & & s^1 & & \\ T(*) & \xrightarrow{d^0} & \prod_{\sigma \in G} T(\sigma) & \xrightarrow{d^1} & \prod_{\sigma_2 \in G^2} T(\sigma_2) & \xrightarrow{\quad} & \prod_{\sigma_3 \in G^3} T(\sigma_3) \dots \\ & \xleftarrow{d^1} & & \xleftarrow{d^2} & & \xleftarrow{\quad} & \end{array} \end{array}$$

The cohomology of this cosimplicial object gives the Gabriel–Zisman cohomology, and therefore by Theorem 1.14 computes simplicially the Thomason cohomology of the group G ,

$$H_{GZ}^*(B_\bullet G, T) \cong H_{Th}^*(G, T).$$

The right hand side can be viewed as a generalization of the classical cohomology of groups (see Theorem 2.1 and Remark 2.2 below). Dually, we can make similar considerations for Thomason homology of a group G .

The interpretations of Thomason (co)homology as simplicial (co)homology allow for a rich interplay between the homological algebra of small categories and the homotopy theory of simplicial sets, which will be the main topic of a sequel to this article.

2. THOMASON (CO)HOMOLOGY AND SPECTRAL SEQUENCES

In the first section we have defined Thomason (co)homology of categories, and related it to simplicial (co)homology which, in the case of homology, was considered by Gabriel–Zisman [11] and Dress [8]. The aim of this second section is twofold: We will show that Thomason (co)homology generalises Baues–Wirsching (co)homology and therefore other (co)homology theories of small categories considered in the literature. We then study general functoriality properties of Thomason (co)homology, by constructing spectral sequences associated to functors between small categories.

2.1. Comparison of Thomason (co)homology with Baues–Wirsching (co)homology. Thomason cohomology and homology generalise other cohomology and homology theories for small categories. The coefficient systems of these theories can all be interpreted as special cases of Thomason natural systems.

The most important example is the cohomology theory of small categories of Baues and Wirsching [2], and the corresponding homology theory as introduced in [12].

Coefficient systems for Baues–Wirsching (co)homology, called here *Baues–Wirsching natural systems*, are given by functors from a factorization category to an abelian category. The *factorization category* $F\mathcal{C}$ of a small category \mathcal{C} is the category whose object set is the set of morphisms of \mathcal{C} and whose Hom-sets $\text{Hom}_{F\mathcal{C}}(f, f')$ are the sets of pairs (α, β) such that $f' = \beta \circ f \circ \alpha$, that is, the following diagram commutes

$$\begin{array}{ccc} b & \xrightarrow{\beta} & b' \\ f \uparrow & & \uparrow f' \\ a & \xleftarrow{\alpha} & a' \end{array}$$

In order to compare Thomason and Baues–Wirsching theories, we introduce a comparison functor from the simplex category to the factorization category,

$$\nu : \Delta/\mathcal{C} \rightarrow F\mathcal{C},$$

defined on objects by

$$\nu \left(C_0 \xleftarrow{f_1} C_1 \xleftarrow{f_2} \dots \xleftarrow{f_m} C_m \right) = (C_0 \xleftarrow{f_1 \circ \dots \circ f_m} C_m)$$

and on morphisms by

$$\nu \left(([m], f) \xrightarrow{\sigma} ([n], g) \right) = (C_m \xleftarrow{g_{\sigma(m)+1} \circ \dots \circ g_n} D_n, D_0 \xleftarrow{g_1 \circ \dots \circ g_{\sigma(0)}} C_0).$$

For the special cases $\sigma(m) = n$ or $\sigma(0) = 0$, the terms on the right hand side of the last equation are the identity maps $C_m \leftarrow D_n$ and $D_0 \leftarrow C_0$, respectively.

For a fixed small category \mathcal{C} , we now have the following diagram of categories and functors between them, extending diagram (1.16) of [2]:

$$(1) \quad \Delta/\mathcal{C} \xrightarrow{\nu} F\mathcal{C} \xrightarrow{\pi} \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{p} \mathcal{C} \xrightarrow{q} \pi_1 \mathcal{C} \xrightarrow{t} \mathbb{1}.$$

Here $\mathbb{1}$ is the terminal category, consisting of one object and one morphism, π is the forgetful functor, p is the projection to the second factor and q is the localization functor into the fundamental groupoid $\pi_1 \mathcal{C}$. The functor ν was also considered by Thomason, and diagram (1) appears in [32, p. 542].

For any functor $\varphi : \mathcal{D} \rightarrow \mathcal{C}$ we have the following commutative diagram:

$$(2) \quad \begin{array}{ccccccccc} \Delta/\mathcal{D} & \xrightarrow{\nu} & F\mathcal{D} & \xrightarrow{\pi} & \mathcal{D}^{op} \times \mathcal{D} & \xrightarrow{p} & \mathcal{D} & \xrightarrow{q} & \pi_1 \mathcal{D} & \xrightarrow{t} & \mathbb{1} \\ \Delta/\varphi \downarrow & & \downarrow F\varphi & & \downarrow \varphi^{op} \times \varphi & & \downarrow \varphi & & \downarrow \pi_1 \varphi & & \parallel \\ \Delta/\mathcal{C} & \xrightarrow{\nu} & F\mathcal{C} & \xrightarrow{\pi} & \mathcal{C}^{op} \times \mathcal{C} & \xrightarrow{p} & \mathcal{C} & \xrightarrow{q} & \pi_1 \mathcal{C} & \xrightarrow{t} & \mathbb{1}. \end{array}$$

The first square in this diagram, for example, expresses the fact that we have functors $\Delta/-$ and $F : \mathcal{Cat} \rightarrow \mathcal{Cat}$ and that ν is a natural transformation

between them. Similarly, the other squares express the naturality of π , p , q and t .

Now let \mathcal{A} be a (co)complete abelian category with exact (co)products. The functor ν induces a functor between functor categories

$$\nu^* : \mathfrak{Fun}(F\mathcal{C}, \mathcal{A}) \rightarrow \mathfrak{Fun}(\Delta/\mathcal{C}, \mathcal{A}).$$

The image of a Baues–Wirsching natural system under ν^* is a Thomason natural system.

It is known that Baues–Wirsching cohomology generalizes the classical cohomology theories studied, for example, by Roos [29], Watts [31], Mitchell [7, 23], Quillen [28] and others. Recall from [2, Definition 1.18] that, if \mathcal{K} denotes one of the last four categories in the sequence (1), then elements of $\mathfrak{Fun}(\mathcal{K}, \mathcal{A})$ are \mathcal{C} -bimodules, \mathcal{C} -modules, local systems and trivial systems on \mathcal{C} respectively. The corresponding functors $\Delta/\mathcal{C} \rightarrow F\mathcal{C} \rightarrow \mathcal{K}$ induce functors

$$\mathfrak{Fun}(\mathcal{K}, \mathcal{A}) \rightarrow \mathfrak{Fun}(F\mathcal{C}, \mathcal{A}) \rightarrow \mathfrak{Fun}(\Delta/\mathcal{C}, \mathcal{A}).$$

That is, all these classical notions of coefficient systems are special cases of Baues–Wirsching and of Thomason natural systems.

We now show that Thomason (co)homology generalizes Baues–Wirsching (co)homology. It then follows from [2, Section 8] that it also generalises Hochschild–Mitchell (co)homology and the other classical (co)homology theories of categories where the coefficients are \mathcal{C} -modules, local systems or trivial systems on \mathcal{C} .

Theorem 2.1. *Let \mathcal{C} be a small category and \mathcal{A} a complete abelian category with exact products for cohomology, or a cocomplete abelian category with exact coproducts for homology. Then we have isomorphisms, natural in \mathcal{C} and D ,*

$$\begin{aligned} H_{BW}^n(\mathcal{C}, D) &\cong H_{Th}^*(\mathcal{C}, \nu^* D), \\ H_n^{BW}(\mathcal{C}, D) &\cong H_*^{Th}(\mathcal{C}, \nu^* D), \end{aligned}$$

between the Baues–Wirsching (co)homology of \mathcal{C} with coefficients in a (contravariant) Baues–Wirsching natural system D , and the Thomason (co)homology with coefficients in $\nu^ D$.*

Proof. This follows directly from the descriptions of the associated (co)simplicial replacements for the Baues–Wirsching natural system D and for the induced Thomason natural system $T = \nu^* D$, which give rise to the respective Thomason and Baues–Wirsching (co)chain complexes [2, 12]. It was observed in [32, Proposition 2.8] that both types of natural systems give the same (co)simplicial object, and therefore they induce isomorphic cohomologies.

Naturality of the isomorphisms follows now from the first commutative square of diagram (2) above, that is, $\nu^*(F\varphi)^* D = (\Delta/\varphi)^* \nu^* D$, for any functor $\varphi : \mathcal{D} \rightarrow \mathcal{C}$ and any Baues–Wirsching natural system D of $\mathfrak{Fun}(F\mathcal{C}, \mathcal{A})$. \square

Remark 2.2. Baues and Minian have shown that, in the special case in which the category \mathcal{C} is a group G , the functor $p \circ \pi : FG \rightarrow G$ induces an equivalence between the category $\mathfrak{Fun}(G, \mathfrak{Ab})$ of G -modules and the category $\mathfrak{Fun}(FG, \mathfrak{Ab})$ of Baues–Wirsching natural systems on G (see [3, Proposition 3.2]). Therefore Baues–Wirsching cohomology of a group can always be identified with classical group cohomology. In contrast, Thomason cohomology of a group is strictly more general. The functor $\nu : \Delta/G \rightarrow FG$ does not in general induce an equivalence between the category $\mathfrak{Fun}(FG, \mathfrak{Ab})$ of Baues–Wirsching natural systems on G and the category $\mathfrak{Fun}(\Delta/G, \mathfrak{Ab})$ of Thomason natural systems on G . For example, in the case $G = 1$, the former is equivalent to the category of abelian groups, while the latter is equivalent to the category of cosimplicial abelian groups.

As an immediate consequence, we obtain the following functoriality theorem, which in the case of cohomology is due to Muro [24] and was also proven independently by Pirashvili and Redondo [26]:

Corollary 2.3. *Let \mathcal{C} and \mathcal{D} be small categories and \mathcal{A} a complete abelian category with exact products for cohomology, or a cocomplete abelian category with exact coproducts for homology. Let (φ, ψ) be a pair of adjoint functors,*

$$\varphi : \mathcal{D} \rightleftarrows \mathcal{C} : \psi.$$

Then there are natural isomorphisms,

$$\begin{aligned} H_{BW}^*(\mathcal{D}, \varphi^*(C)) &\cong H_{BW}^*(\mathcal{C}, C), & H_*^{BW}(\mathcal{D}, \varphi^*(C)) &\cong H_*^{BW}(\mathcal{C}, C), \\ H_{BW}^*(\mathcal{C}, \psi^*(D)) &\cong H_{BW}^*(\mathcal{D}, D), & H_*^{BW}(\mathcal{C}, \psi^*(D)) &\cong H_*^{BW}(\mathcal{D}, D), \end{aligned}$$

for (contravariant) Baues–Wirsching natural systems C on \mathcal{C} and D on \mathcal{D} .

Proof. This follows from the previous theorem and the general functoriality properties of Thomason (co)homology in Propositions 1.10, 1.11. \square

It follows that Hochschild–Mitchell (co)homology of categories, and the classical (co)homology theories having as coefficients \mathcal{C} -modules, local systems or trivial systems, also enjoy these basic functoriality properties. Furthermore, our characterization of Thomason (co)homology as simplicial (co)homology (Theorems 1.14 and 1.17) gives similar characterizations for these more classical theories. The interpretation of Baues–Wirsching cohomology as simplicial cohomology is of particular interest because of its application to homotopy theory, and will be studied in a sequel to this article.

Remark 2.4. Thomason cohomology can also be viewed as *topos cohomology* in the sense of [16] (see also [22]), as follows:

Let \mathcal{C} be a small category, with associated simplex category Δ/\mathcal{C} , and consider the presheaf topos $\mathfrak{E} = \mathfrak{PSh}((\Delta/\mathcal{C})^{op}) = \mathfrak{Fun}(\Delta/\mathcal{C}, \mathfrak{Sets})$. This is a Grothendieck topos, that is, it is a category of sheaves on a site with an appropriate Grothendieck topology τ . For this, we simply endow the category $(\Delta/\mathcal{C})^{op}$ with the trivial Grothendieck topology τ_0 , where the only

covering families are the one-element families of isomorphisms. In this topology every presheaf $F : \Delta/\mathcal{C} \rightarrow \mathfrak{S}ets$ is automatically a sheaf, and therefore \mathfrak{E} is simply given as the category of sheaves $\mathfrak{S}h\mathfrak{v}((\Delta/\mathcal{C})^{op}, \tau_0)$ on the site $((\Delta/\mathcal{C})^{op}, \tau_0)$.

An abelian group object A of the presheaf topos is then simply a functor $A : \Delta/\mathcal{C} \rightarrow \mathfrak{Ab}$, that is, a Thomason natural system. It can be shown that the topos cohomology of \mathfrak{E} with coefficients in an abelian group object A of \mathfrak{E} is given as Thomason cohomology

$$H^n(\mathfrak{E}, A) \cong H_{Th}^n(\mathcal{C}, A).$$

This follows by using the characterization of Thomason cohomology as an Ext group as derived in Theorem 1.8, that is,

$$H_{Th}^n(\mathcal{C}, A) \cong \text{Ext}_{\Delta/\mathcal{C}}^n(\mathbb{Z}, A),$$

and observing that \mathbb{Z} is the free abelian group object over the terminal object of \mathfrak{E} (see also [2, Remark 8.3] for the special case of Baues–Wirsching cohomology).

2.2. Functoriality of Thomason (co)homology and spectral sequences. Given a functor between small categories, we will now derive several spectral sequences for Thomason cohomology and homology.

Let us consider a functor $u : \mathcal{E} \rightarrow \mathcal{B}$ between small categories and the following commutative diagram of categories and functors:

$$(3) \quad \begin{array}{ccccccccc} \Delta/\mathcal{E} & \xrightarrow{\nu} & F\mathcal{E} & \xrightarrow{\pi} & \mathcal{E}^{op} \times \mathcal{E} & \xrightarrow{p} & \mathcal{E} & \xrightarrow{q} & \pi_1 \mathcal{E} & \xrightarrow{t} & \mathbb{1} \\ \Delta/u \downarrow & & \downarrow Fu & & \downarrow u^{op} \times u & & \downarrow u & & \downarrow \pi_1 u & & \parallel \\ \Delta/\mathcal{B} & \xrightarrow{\nu} & F\mathcal{B} & \xrightarrow{\pi} & \mathcal{B}^{op} \times \mathcal{B} & \xrightarrow{p} & \mathcal{B} & \xrightarrow{q} & \pi_1 \mathcal{B} & \xrightarrow{t} & \mathbb{1} \end{array}$$

We deal with all the different types of coefficient systems at once, writing \mathcal{C} for any of the categories in the lower row of the above diagram and denoting by $u' : \Delta/\mathcal{E} \rightarrow \mathcal{C}$ the associated composition of functors. Furthermore, let \mathcal{A} be a complete abelian category with exact products. In each case we get a diagram of the form:

$$\begin{array}{ccc} \mathfrak{F}un(\Delta/\mathcal{E}, \mathcal{A}) & \begin{array}{c} \xleftarrow{u'^*} \\ \xrightarrow{u'_*} \end{array} & \mathfrak{F}un(\mathcal{C}, \mathcal{A}) \\ & \begin{array}{c} \searrow \text{lim}_{\Delta/\mathcal{E}} \\ \swarrow c \end{array} & \begin{array}{c} \nearrow c \\ \searrow \text{lim}_{\mathcal{C}} \end{array} \\ & & \mathcal{A} \end{array}$$

where c denotes the constant diagram functor, u'^* is pre-composition with u' , and the other functors in the diagram are the right adjoints of these, given by the limits $\text{lim}_{\Delta/\mathcal{E}}$, $\text{lim}_{\mathcal{C}}$ and by $u'_* = \text{Ran}_{u'}$, where $\text{Ran}_{u'}$ is the right Kan extension along the functor u' .

We obtain a spectral sequence for the derived functors of the composite functor

$$\lim_{\Delta/\mathcal{E}}(-) = \lim_{\mathcal{C}} u'_*(-)$$

which is an André spectral sequence as constructed in generality in [12, Section 1.1] (see also [1]). Here, it converges to the Thomason cohomology of \mathcal{E} with coefficients T being a Thomason natural system of $\mathfrak{Fun}(\Delta/\mathcal{E}, \mathcal{A})$. Therefore [12, Theorem 1.2] gives a first quadrant cohomology spectral sequence of the form:

$$E_2^{p,q} \cong H^p(\mathcal{C}, \text{Ran}_{u'}^q(T)) \Rightarrow H_{Th}^{p+q}(\mathcal{E}, T)$$

where $\text{Ran}_{u'}^q$ is the q -th right satellite of $\text{Ran}_{u'}$.

For each object c of \mathcal{C} , let $Q^{(c)} : c/u' \rightarrow \Delta/\mathcal{E}$ be the forgetful functor and denote by $\mathcal{H}^q(c/u', T \circ Q^{(c)})$ the derived limit

$$\lim^q \left(c/u' \xrightarrow{Q^{(c)}} \Delta/\mathcal{E} \xrightarrow{T} \mathcal{A} \right).$$

Then [12, Corollary 1.3] allows us to identify the terms in the E_2 -page of the spectral sequence as

$$E_2^{p,q} \cong H^p(\mathcal{C}, \mathcal{H}^q(-/u', T \circ Q^{(-)})) \Rightarrow H_{Th}^{p+q}(\mathcal{E}, T).$$

In particular cases, the E_2 -page can be simplified. For example, in the case $\mathcal{C} = \Delta/\mathcal{B}$ with $u' = \Delta/u$ considered above, we get the following Leray type spectral sequence:

Theorem 2.5. *Let \mathcal{E} and \mathcal{B} be small categories and $u : \mathcal{E} \rightarrow \mathcal{B}$ a functor. Let \mathcal{A} be a complete abelian category with exact products. Given a Thomason natural system $T : \Delta/\mathcal{E} \rightarrow \mathcal{A}$ on \mathcal{E} , there is a first quadrant cohomology spectral sequence*

$$E_2^{p,q} \cong H_{Th}^p(\mathcal{B}, (R^q \Delta/u_*)(T)) \Rightarrow H_{Th}^{p+q}(\mathcal{E}, T)$$

which is functorial with respect to natural transformations and where $R^q \Delta/u_* = \text{Ran}_{\Delta/u}^q$ is the q -th right satellite of $\text{Ran}_{\Delta/u}$, the right Kan extension along the induced functor $\Delta/u : \Delta/\mathcal{E} \rightarrow \Delta/\mathcal{B}$ between the simplex categories.

With the identification of the terms in the E_2 -page above we get therefore:

Corollary 2.6. *Let $u : \mathcal{E} \rightarrow \mathcal{B}$ be a functor between small categories and \mathcal{A} a complete abelian category with exact products. Let $T : \Delta/\mathcal{E} \rightarrow \mathcal{A}$ be a Thomason natural system on \mathcal{E} . Then there exists a first quadrant cohomology spectral sequence of the form*

$$E_2^{p,q} \cong H_{Th}^p(\mathcal{B}, \mathcal{H}^q(-/\Delta/u, T \circ Q^{(-)})) \Rightarrow H_{Th}^{p+q}(\mathcal{E}, T)$$

which is functorial with respect to natural transformations and where

$$\mathcal{H}^q(-/\Delta/u, T \circ Q^{(-)}) = \lim_{-/\Delta/u}^q (T \circ Q^{(-)}) : \Delta/\mathcal{B} \rightarrow \mathcal{A}.$$

In the special situation that the functor u is part of an adjoint pair of functors or an equivalence of categories this spectral sequence collapses in the E_2 -page and we simply recover Proposition 1.10.

For the particular case $\mathcal{C} = F\mathcal{B}$ and

$$u' = \nu \circ \Delta/u = Fu \circ \nu : \Delta/\mathcal{E} \longrightarrow \mathcal{C}$$

with a given Thomason natural system $T : \Delta/\mathcal{E} \rightarrow \mathcal{A}$ on \mathcal{E} we get the following spectral sequence as a special case:

$$E_2^{p,q} \cong H^p(F\mathcal{B}, R^q(Fu \circ \nu)_*(T)) \Rightarrow H_{Th}^{p+q}(\mathcal{E}, T).$$

After identifying the various terms involved, this gives a spectral sequence for Thomason cohomology in terms of Baues–Wirsching cohomology:

Theorem 2.7. *Let \mathcal{E} and \mathcal{B} be small categories and $u : \mathcal{E} \rightarrow \mathcal{B}$ a functor. Let \mathcal{A} be a complete abelian category with exact products. Given a Thomason natural system $T : \Delta/\mathcal{E} \rightarrow \mathcal{A}$, there is a first quadrant cohomology spectral sequence*

$$E_2^{p,q} \cong H_{BW}^p(\mathcal{B}, R^q(Fu \circ \nu)_*(T)) \Rightarrow H_{Th}^{p+q}(\mathcal{E}, T)$$

which is functorial with respect to natural transformations and where $R^q(Fu \circ \nu)_* = \text{Ran}_{Fu \circ \nu}^q$ is given as the q -th right satellite of $\text{Ran}_{Fu \circ \nu}$, the right Kan extension along $Fu \circ \nu$.

Again, we can identify the terms in the E_2 -page of the spectral sequence with concrete fiber data and get:

Corollary 2.8. *Let $u : \mathcal{E} \rightarrow \mathcal{B}$ be a functor between small categories. Let \mathcal{A} be a complete abelian category with exact products. Given a Thomason natural system $T : \Delta/\mathcal{E} \rightarrow \mathcal{A}$, there is a first quadrant cohomology spectral sequence*

$$E_2^{p,q} \cong H_{BW}^p(\mathcal{B}, \mathcal{H}^q(-/Fu \circ \nu, T \circ Q^{(-)})) \Rightarrow H_{Th}^{p+q}(\mathcal{E}, D)$$

which is functorial with respect to natural transformations and where

$$\mathcal{H}^q(-/Fu \circ \nu, T \circ Q^{(-)}) = \lim_{-/Fu \circ \nu}^q (T \circ Q^{(-)}) : F\mathcal{B} \rightarrow \mathcal{A}.$$

In the special case that $T = \nu^*D$ for a Baues–Wirsching natural system $D : F\mathcal{E} \rightarrow \mathcal{A}$ on \mathcal{E} we can further identify the E_2 -term of the spectral sequence in Theorem 2.7 and get a spectral sequence for Baues–Wirsching cohomology:

$$E_2^{p,q} \cong H_{BW}^p(\mathcal{B}, R^q(Fu_*)(D)) \Rightarrow H_{BW}^{p+q}(\mathcal{E}, D)$$

Similarly, we can identify the E_2 -page in terms of local fiber data as in Corollary 2.8. This spectral sequence describes directly the functorial behaviour of Baues–Wirsching cohomology for functors between small categories (see also [12, Theorem 1.14]).

Dually, with similar arguments as above, we can derive homological versions of the spectral sequences for Thomason homology of categories.

Theorem 2.9. *Let \mathcal{E} and \mathcal{B} be small categories and $u : \mathcal{E} \rightarrow \mathcal{B}$ a functor. Let \mathcal{A} be a cocomplete abelian category with exact coproducts. Given a contravariant Thomason natural system $T : \Delta/\mathcal{E} \rightarrow \mathcal{A}$ on \mathcal{E} , there is a third quadrant homology spectral sequence*

$$E_{p,q}^2 \cong H_p^{Th}(\mathcal{B}, (L_q \Delta/u^*)(T)) \Rightarrow H_{p+q}^{Th}(\mathcal{E}, T)$$

which is functorial with respect to natural transformations, where $L_q \Delta/u^* = Lan_q^{\Delta/u}$ is the q -th left satellite of $Lan^{\Delta/u}$, the left Kan extension along Δ/u .

Proof. This is simply dual to the statement of Theorem 2.5 and follows from the dual André homology spectral sequence involving the higher derived functors of $\text{colim}^{\Delta/\mathcal{E}} D$ in the description of Thomason homology $H_n^{Th}(\mathcal{E}, T)$ (see also [1] and [11, Appendix II.3]). \square

Again, we can identify the E^2 -page of this spectral sequence and get:

Corollary 2.10. *Let \mathcal{E} and \mathcal{B} be small categories and $u : \mathcal{E} \rightarrow \mathcal{B}$ a functor. Let \mathcal{A} be a cocomplete abelian category with exact coproducts. Given a contravariant Thomason natural system $T : \Delta/\mathcal{E} \rightarrow \mathcal{A}$ on \mathcal{E} , there is a third quadrant cohomology spectral sequence*

$$E_{p,q}^2 \cong H_p^{Th}(\mathcal{B}, \mathcal{H}_q(\Delta/u/-, T \circ Q_{(-)})) \Rightarrow H_{p+q}^{Th}(\mathcal{E}, T)$$

which is functorial with respect to natural transformations and where

$$\mathcal{H}_q(\Delta/u/-, T \circ Q_{(-)}) = \text{colim}_q^{\Delta/u/-} (T \circ Q_{(-)}) : \Delta/\mathcal{B} \rightarrow \mathcal{A}.$$

Proof. We can identify the E^2 -term in the above homology spectral sequence as follows (see [1], [6], [17])

$$Lan_q^{\Delta/u}(T) \cong \text{colim}_q^{\Delta/u/\beta} T \circ Q_{(\beta)}$$

which gives the desired spectral sequence for Thomason homology. \square

In the special situation that u is part of an adjoint pair of functors or an equivalence of categories we will recover Proposition 1.11.

Finally, in the particular case $\mathcal{E} = F\mathcal{B}$ with $u' = \nu \circ \Delta/u$ for a contravariant Thomason natural system $T : \Delta/\mathcal{E} \rightarrow \mathcal{A}$ on \mathcal{E} we get the homology spectral sequence

$$E_{p,q}^2 \cong H_p(F\mathcal{B}, L_q(\nu \circ \Delta/u)_*(T)) \Rightarrow H_{p+q}^{Th}(\mathcal{E}, T)$$

which after identifying the various terms gives the following spectral sequence for Baues–Wirsching homology, which is dual to the one of Theorem 2.7:

Theorem 2.11. *Let \mathcal{E} and \mathcal{B} be small categories and $u : \mathcal{E} \rightarrow \mathcal{B}$ a functor. Let \mathcal{A} be a cocomplete abelian category with exact coproducts. Given a contravariant Thomason natural system $T : \Delta/\mathcal{E} \rightarrow \mathcal{A}$, there is a third quadrant homology spectral sequence*

$$E_{p,q}^2 \cong H_p^{BW}(\mathcal{B}, L_q(Fu \circ \nu_*)(T)) \Rightarrow H_{p+q}^{Th}(\mathcal{E}, T)$$

which is functorial with respect to natural transformations and where $L_q(Fu \circ \nu)_* = \text{Lan}_q^{Fu \circ \nu}$ is given as the q -th left satellite of $\text{Lan}^{Fu \circ \nu}$, the left Kan extension along $Fu \circ \nu$.

We can again identify the terms in the E^2 -page of this spectral sequence and obtain:

Corollary 2.12. *Let $u : \mathcal{E} \rightarrow \mathcal{B}$ be a functor between small categories and \mathcal{A} a cocomplete abelian category with exact coproducts. Given a contravariant Thomason natural system $T : \Delta/\mathcal{E} \rightarrow \mathcal{A}$, there is a third quadrant homology spectral sequence*

$$E_{p,q}^2 \cong H_p^{BW}(\mathcal{B}, \mathcal{H}_q(Fu \circ \nu / -, T \circ Q_{(-)})) \Rightarrow H_{p+q}^{Th}(\mathcal{E}, T)$$

which is functorial with respect to natural transformations and where

$$\mathcal{H}_q(Fu \circ \nu / -, T \circ Q_{(-)}) = \text{colim}_q^{Fu \circ \nu / -} (T \circ Q_{(-)}) : \mathcal{B}^{op} \times \mathcal{B} \rightarrow \mathcal{A}.$$

As in the particular case $T = \nu^* D$, with a contravariant Baues–Wirsching natural system $D : F\mathcal{E} \rightarrow \mathcal{A}$ on \mathcal{E} , we identify the E^2 -term of this spectral sequence similarly as in Theorem 2.7 and obtain a spectral sequence for Baues–Wirsching homology in the form (see also [12, Theorem 1.18]):

$$E_{p,q}^2 \cong H_p^{BW}(\mathcal{B}, L_q(Fu_*)(D)) \Rightarrow H_{p+q}^{BW}(\mathcal{E}, D)$$

We can also derive cohomology and homology spectral sequences for the other types of coefficient systems described before, that is, \mathcal{E} -modules, local systems or trivial systems. The proofs follow the same line of arguments as in the case of Thomason natural systems discussed above.

The characterization of Thomason (co)homology as (co)homology of simplicial sets allows us to relate the above homological constructions of spectral sequences for functors between small categories directly with similar spectral sequences for general maps between simplicial sets. An important example is the Leray–Serre type spectral sequence for a fibration of simplicial sets, whose homology version was discussed by Gabriel and Zisman [11, App. III.4] and Dress [8]. Also the Lyndon–Hochschild–Serre spectral sequence for a group extension can be seen as a special case. The relation between all these spectral sequences will be discussed in detail in a sequel to this article.

2.3. Thomason (co)homology for Grothendieck fibrations. We will now study the constructed spectral sequences in the particular situation, where the functor between small categories is a Grothendieck fibration. Applying the spectral sequence constructions of the preceding paragraph to such a situation allows us to identify the E_2 -pages with simpler cohomology and homology data keeping track of the fiber of the functor. These constructions also generalize similar spectral sequences for Baues–Wirsching cohomology as discussed in [12].

For a given functor $u : \mathcal{E} \rightarrow \mathcal{B}$ between small categories \mathcal{E} and \mathcal{B} and a given object b of \mathcal{B} , recall that the *fiber category* $\mathcal{E}_b = u^{-1}(b)$ is the subcategory of \mathcal{E} which fits into the following pullback diagram

$$\begin{array}{ccc} \mathcal{E}_b & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow u \\ * & \xrightarrow{b} & \mathcal{B}. \end{array}$$

The objects of \mathcal{E}_b are those objects of \mathcal{E} which map onto b via the functor u and the morphisms are given by those which map to the identity 1_b .

We have the following notion of a fibration between small categories due to Grothendieck [15, Exposé VI]:

Definition 2.13. Let \mathcal{E} and \mathcal{B} be small categories. A *Grothendieck fibration* is a functor $u : \mathcal{E} \rightarrow \mathcal{B}$ such that the fibers $\mathcal{E}_b = u^{-1}(b)$ depend contravariantly and pseudofunctorially on the objects b of the category \mathcal{B} . The category \mathcal{E} is also called a *category fibered over \mathcal{B}* .

Recall that an assignment $b \mapsto \Phi(b) \in \mathfrak{Cat}$, for $b \in \mathcal{B}$, is *contravariantly pseudofunctorial* if we have functors $f^* : \Phi(b') \rightarrow \Phi(b)$ for each $f : b \rightarrow b'$ in \mathcal{B} such that $(1_b)^*$ is naturally isomorphic to the identity functor, and f^*g^* is naturally isomorphic to $(gf)^*$. The natural isomorphisms are required to be coherent; for full details we refer to [30, Section 3].

Grothendieck fibrations are characterized as pseudofunctors [14]. In fact, there is an equivalence of 2-categories

$$\mathfrak{Fib}(\mathcal{B}) \xrightarrow{\sim} \mathfrak{PsdFun}(\mathcal{B}^{op}, \mathfrak{Cat})$$

between the 2-category of Grothendieck fibrations $\mathfrak{Fib}(\mathcal{B})$ over a small category \mathcal{B} and the 2-category of contravariant pseudofunctors $\mathfrak{PsdFun}(\mathcal{B}^{op}, \mathfrak{Cat})$ from \mathcal{B} to the category \mathfrak{Cat} of small categories.

An alternative equivalent description of a Grothendieck fibration can be found in [15], [4, Vol. 2, 8.3.1]: a functor $u : \mathcal{E} \rightarrow \mathcal{B}$ is a Grothendieck fibration if for each object b of \mathcal{B} the inclusion functor from the fiber into the comma category

$$j_b : \mathcal{E}_b \rightarrow b/u, \quad e \mapsto (e, b \xrightarrow{=} ue)$$

is *coreflexive*, i.e. has a right adjoint left inverse.

From now on let us assume that the functor $u : \mathcal{E} \rightarrow \mathcal{B}$ is a Grothendieck fibration and $T : \Delta/\mathcal{E} \rightarrow \mathcal{A}$ a Thomason natural system on the category \mathcal{E} , where \mathcal{A} is again a complete abelian category with exact products.

We get a local system $\mathcal{H}_{Th}^q(G(-), T|_{\Delta/\mathcal{E}_{(-)}}) : \mathcal{B} \rightarrow \mathcal{A}$ from the associated pseudofunctor $G : \mathcal{B}^{op} \rightarrow \mathfrak{Cat}$ by assigning to every object b of the category \mathcal{B} the q -th Thomason cohomology of the category $G(b)$,

$$\mathcal{H}_{Th}^q(G(-), T|_{\Delta/\mathcal{E}_{(-)}}) : \mathcal{B} \rightarrow \mathcal{A}, \quad b \mapsto H_{Th}^q(G(b), T|_{\Delta/\mathcal{E}_b}).$$

Here the coefficients are given by the Thomason natural system $T|_{\Delta/\mathcal{E}_b} : \Delta/\mathcal{E}_b \rightarrow \mathcal{A}$.

For each object b of the base category \mathcal{B} we get a cartesian diagram

$$\begin{array}{ccc} \mathcal{E}_b & \xrightarrow{j_b} & b/u \\ \downarrow & & \downarrow Q^b \\ \mathcal{E}_b & \xrightarrow{i_b} & \mathcal{E} \\ \downarrow & & \downarrow u \\ * & \xrightarrow{b} & \mathcal{B}, \end{array}$$

where the horizontal arrows are the obvious inclusion functors.

Let R_b denote the right adjoint functor of the functor j_b and let T_b denote the Thomason natural system on b/u defined as the composition

$$T_b : \Delta/(b/u) \xrightarrow{\Delta/R_b} \Delta/\mathcal{E}_b \xrightarrow{\Delta/i_b} \Delta/\mathcal{E} \xrightarrow{T} \mathcal{A}.$$

This leads to the following first quadrant cohomology spectral sequence using the equivalence between Grothendieck fibrations and pseudofunctors:

$$E_2^{p,q} \cong H^p(\mathcal{B}, \mathcal{H}_{T_h}^q(-/u, T_{(-)})) \Rightarrow H_{T_h}^{p+q}(\mathcal{E}, T)$$

This spectral sequence is functorial with respect to 1-morphisms, i.e. natural transformations between Grothendieck fibrations. It generalizes a similar spectral sequence for Baues–Wirsching cohomology constructed by Pirashvili and Redondo [26].

To summarize, we have constructed the following particular cohomology spectral sequence:

Theorem 2.14. *Let \mathcal{E} and \mathcal{B} be small categories and let $u : \mathcal{E} \rightarrow \mathcal{B}$ be a Grothendieck fibration. Given a Thomason natural system $T : \Delta/\mathcal{E} \rightarrow \mathcal{A}$ on \mathcal{E} , where \mathcal{A} is a complete abelian category with exact products, there is a first quadrant spectral sequence*

$$E_2^{p,q} \cong H^p(\mathcal{B}, \mathcal{H}_{T_h}^q(-/u, T_{(-)})) \Rightarrow H_{T_h}^{p+q}(\mathcal{E}, T)$$

which is functorial with respect to 1-morphisms of Grothendieck fibrations.

To identify the E_2 -term of this spectral sequence with local data of the fiber category we introduce the following notion, corresponding to the property of h -locality for Baues–Wirsching natural systems introduced in [26].

Definition 2.15. Let \mathcal{A} be a complete abelian category with exact products. A Thomason natural system $T : \Delta/\mathcal{E} \rightarrow \mathcal{A}$ is called *local* if the adjoint functor R_b of the inclusion functor $j_b : \mathcal{E}_b \rightarrow b/u$ induces an isomorphism in Thomason cohomology

$$H_{T_h}^q(b/u, T_b) \cong H_{T_h}^q(\mathcal{E}_b, T \circ \Delta/i_b)$$

for every q and every object b of the base category \mathcal{B} , i.e. we have a natural isomorphism of local coefficient systems

$$\mathcal{H}_{Th}^q(-/u, T_{(-)}) \cong \mathcal{H}_{Th}^q(\mathcal{E}_{(-)}, T \circ \Delta/i_{(-)}).$$

Identifying the E_2 -page of the above spectral sequence, we get the following spectral sequence:

Theorem 2.16. *Let \mathcal{A} be a complete abelian category with exact products. Let \mathcal{E} and \mathcal{B} be small categories and $u : \mathcal{E} \rightarrow \mathcal{B}$ be a Grothendieck fibration. Given a local Thomason natural system $T : \Delta/\mathcal{E} \rightarrow \mathcal{A}$ on \mathcal{E} , where \mathcal{A} is a complete abelian category with exact products, there is a first quadrant spectral sequence*

$$E_2^{p,q} \cong H^p(\mathcal{B}, \mathcal{H}_{Th}^q(\mathcal{E}_{(-)}, T \circ \Delta/i_{(-)})) \Rightarrow H_{Th}^{p+q}(\mathcal{E}, T)$$

with local coefficient system

$$\mathcal{H}_{Th}^q(\mathcal{E}_{(-)}, T \circ \Delta/i_{(-)}) : \mathcal{B} \rightarrow \mathcal{A}, \quad b \mapsto H_{Th}^q(\mathcal{E}_b, T \circ \Delta/i_b).$$

Furthermore, the spectral sequence is functorial with respect to 1-morphisms of Grothendieck fibrations.

Proof. The construction of the cohomology spectral sequence in Theorem 2.14 and the definition of a local Thomason natural system $T : \Delta/\mathcal{E} \rightarrow \mathcal{A}$ on the total category \mathcal{E} give the desired identification of the E_2 -page of the spectral sequence. \square

Using the obvious dual notions of all the above constructions, we can also derive a homology version of the spectral sequence for a Grothendieck fibration:

Theorem 2.17. *Let \mathcal{A} be a cocomplete abelian category with exact coproducts. Let \mathcal{E} and \mathcal{B} be small categories and $u : \mathcal{E} \rightarrow \mathcal{B}$ be a Grothendieck fibration. Given a colocal contravariant Thomason natural system $T : \Delta/\mathcal{E} \rightarrow \mathcal{A}$ on \mathcal{E} , where \mathcal{A} is a cocomplete abelian category with exact coproducts, there is a third quadrant spectral sequence*

$$E_{p,q}^2 \cong H_p(\mathcal{B}, \mathcal{H}_q^{Th}(\mathcal{E}_{(-)}, T \circ \Delta/i_{(-)})) \Rightarrow H_{p+q}^{Th}(\mathcal{E}, T)$$

with the local coefficient system

$$\mathcal{H}_q^{Th}(\mathcal{E}_{(-)}, T \circ \Delta/i_{(-)}) : \mathcal{B} \rightarrow \mathcal{A}, \quad b \mapsto \mathcal{H}_q^{Th}(\mathcal{E}_b, T \circ \Delta/i_b).$$

Furthermore, the spectral sequence is functorial with respect to 1-morphisms of Grothendieck fibrations.

Using the commutative diagram (3) of categories and functors, we can also derive analogous (co)homology spectral sequences for Baues–Wirsching natural systems, bimodules, modules, local systems or trivial systems. These spectral sequences then relate the associated (co)homology in a given Grothendieck fibration (see also [12]). In turn, they all can be seen

as spectral sequences for simplicial (co)homology of maps or fibrations of simplicial sets, as will be discussed elsewhere.

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