

A NEW FUNCTIONAL FOR IMPROVING CELL AREA DISTRIBUTION

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Abstract. In this work we present a new area functional that help us improve the cell area distribution in structured grids over plane irregular regions, which avoids very small and very large cell areas as much as possible. We present some results and an implementation of this functional in a preliminary version of the latest UNAMalla system.

1 Introduction

One of the main problems in numerical grid generation is to improve grid quality. In many practical cases, this is equivalent to modify the grid geometry in such a way that the cell area values are as less spread around the mean as possible.

Before producing quality grids, a central issue that needed to be solved was the variational generation of convex grids. An important contribution on this direction is due to S. Ivanenko, who proposed an *ad hoc* modification of Winslow’s functional for generating convex grids [10]; it was improved later by Barrera *et al* in [5] by adding a general initialization procedure. A review on smoothness and convex functionals which have a strong control over the cell areas is presented in [3].

A discussion about the properties of the area functionals which are useful for generating convex grids can be found in [1] and [4]; however, all this functionals focus on controlling only the minimum cell area. A functional which controls both the minimum and maximum area cell values to produce quality grids was proposed in [2]. In this paper, we introduce

a new area functional which also controls the minimum and maximum area cell values but focusing only on those grid cells whose area values lie outside a control interval and, in consequence, improving the overall performance of the optimization process.

2 Problem formulation

In this section we describe briefly the main concepts required to present the new functional.

2.1 Basic theory for continuous mappings

Given a simply connected polygonal region Ω , it is possible to define a bijective mapping from the boundary of the unit square $B = [0, 1] \times [0, 1]$ onto the boundary of Ω . If this mapping is bijectively extended from B onto Ω , then a grid on B defines naturally a grid on Ω .

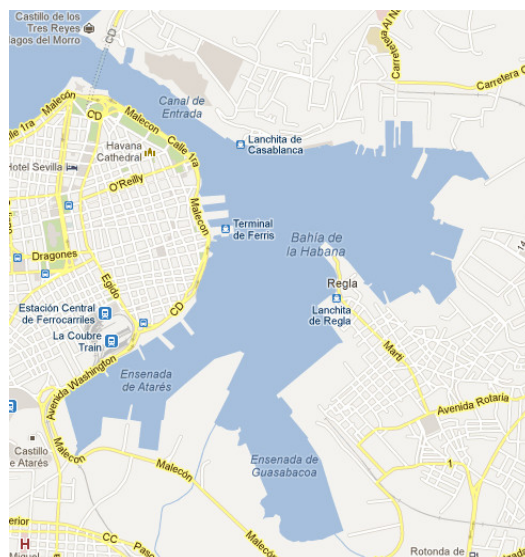


Figure 1: Image by Google maps on Habana bay.

For modelling purposes, we are mainly interested in simply connected domains Ω defined by irregular boundaries; an example is shown in figure 1. For such domains, a fundamental aspect is to avoid grid folding in the optimization process; in other words, we require homeomorphic mappings as described in the following theorem:

Theorem. If \mathbf{x} is a 2D mapping such that

- $\mathbf{x} : B \mapsto \Omega$
- $\mathbf{x}|_{\partial B} = \partial\Omega$
- $J(\xi, \eta) > 0, \quad \forall(\xi, \eta) \in B$

then $\mathbf{x}(\xi, \eta)$ is a homeomorphism from B onto Ω . For a proof of this theorem you can see Bobylev [7].

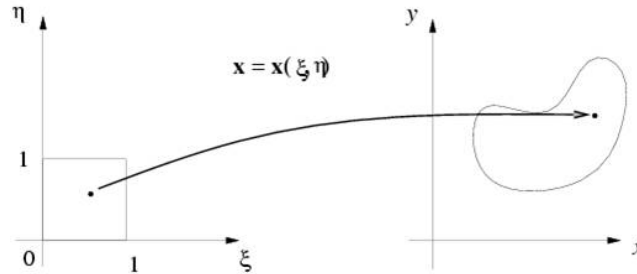


Figure 2: Mapping from a simple region onto a domain of interest.

Given a uniform mesh on B , the following theorem poses the conditions under which \mathbf{x} defines an unfolded grid for Ω as well as the guidelines for the selection of adequate mappings for meshing Ω .

Theorem. Let the unit square B be subdivided into nc simple regions B_i such that

1. $B = \cup_{i=1}^{nc} B_i, \quad \text{Int}(B_i) \cap \text{Int}(B_j) = \phi.$
2. $\mathbf{x} : B \mapsto \Omega$ is continuous.
3. \mathbf{x}_i is smooth on $B_i,$
4. $\mathbf{x}_i = \mathbf{x}|_{B_i},$
5. $\mathbf{x} : \partial B \mapsto \partial\Omega$ is a homeomorphism,
6. $J_i(\xi, \eta) > 0, \quad \forall(\xi, \eta) \in B_i, \quad \forall i = 1, \dots, nc,$

Then \mathbf{x} is a homeomorphism from B onto Ω .

2.2 Discrete formulation

Let B be the unit square and $U(m, n)$ the uniform mesh of size $m \times n$ on B given by

$$U(m, n) = \left\{ \left(\frac{i}{m}, \frac{j}{n} \right) \mid 0 \leq i \leq m, 0 \leq j \leq n \right\}$$

where

$$\partial U(m, n) = \partial B \cap U(m, n).$$

A discrete grid G of size $m \times n$ on Ω is a mapping

$$G : U(m, n) \mapsto \mathbb{R}^2$$

such that

$$G(\partial U) \subset \partial \Omega$$

and

$$\partial G = G(\partial U) \subset \partial \Omega.$$

By considering a positive orientation on the boundary of Ω , we get an induced orientation on the boundaries of the cells $c_{ij} = G(B_{ij})$, and also on the four triangles defined by the cell vertices.

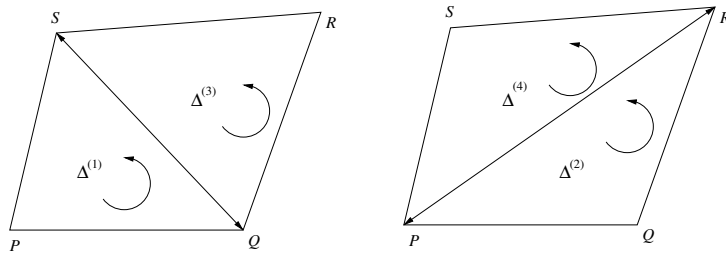


Figure 3: The four triangles defined by the vertices of a quadrilateral cell.

Besides, will also say that G is convex if each one of the triangles has positive area and non degenerate except, possibly in the corners cells, see [3]. In the other hand, we are interested on controlling the convexity, smoothness and orthogonality of the grid cells by minimizing a suitable functional defined on the set of all the discrete grids on Ω .

Minimization is the basis of the variational grid generation method, which is one of the few methods that can be successfully applied to produce a structured convex grid when the boundary of the domain Ω is an irregular curve; the standard functionals have the form

$$F(G) = \sum_{q=1}^N f(\Delta_q), \tag{1}$$

where $f(\Delta_q)$ depends on the vertices of the triangle Δ_q , and N is four times the total number of grid cells since the four triangles defined by the four vertices of every grid cell are considered by Barrera *et. al* [1, 3, 4, 5, 6], and Ivanenko *et. al.* [8, 9, 10].

When f is a function only of the areas of the triangles in the cells, F is referred to as an area functional. Its standard expression is given by

$$F(G) = \sum_{q=1}^N f(\alpha_q), \tag{2}$$

where G represents the grid where the functional is evaluated and α_q is the oriented area of the q -th grid triangle. As mentioned, there are four triangles in every grid cell defined by its vertices, as it can be seen in figure 3.

2.3 Variational setting

The variational grid generation problem can be posed as the minimization problem

$$G^* = \arg \min_{G \in M(\Omega)} \sum_{q=1}^N f(\Delta_q)$$

defined on the set of admissible grids $M(\Omega)$ for Ω

To solve this problem numerically, some quantities must be defined first. For the generic grid cell triangle $A, B, C \in \mathbb{R}^2$ we define the the length measure as

$$\lambda(\Delta(A, B, C)) = \|A - B\|^2 + \|C - B\|^2,$$

the area measure

$$\alpha(\Delta(A, B, C)) = (B - A)^t J_2 (B - C) = 2 \cdot \text{area}(\Delta(A, B, C)),$$

where

$$J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and the orthogonality is measured with

$$o(\Delta(A, B, C)) = (B - C)^t (B - C).$$

These quantities on the triangles define the classic area functional

$$F_A(G) = \sum_{q=1}^N \alpha(\Delta_q)^2,$$

the classic length functional

$$F_L(G) = \sum_{q=1}^N \lambda(\Delta_q),$$

and the classic orthogonality functional

$$F_O(G) = \sum_{q=1}^N o(\Delta_q)^2.$$

There are also some useful linear combinations. For instance, the area-orthogonality functional [11]:

$$F_{AO}(G) = \sum_{q=1}^N \left[\frac{\alpha(\Delta_q)^2}{2} + \frac{o(\Delta_q)^2}{2} \right],$$

and the area-length functional

$$F_{AL}(G) = \sum_{q=1}^N [\sigma\alpha(\Delta_q)^2 + (1 - \sigma)\lambda(\Delta_q)]$$

Some other functionals are continuous extensions of Winslow’s functional [10], like the smoothness functional

$$F_H(G) = \sum_{q=1}^N \frac{\lambda(\Delta_q)}{\alpha(\Delta_q)},$$

and the quasi-harmonic functional

$$F_{H\omega}(G) = \sum_{q=1}^N \frac{\lambda(\Delta_q) - 2\alpha(\Delta_q)}{\omega + \alpha(\Delta_q)},$$

which was designed with a flexible barrier to approximate the harmonic functional.

There is a existence theorem for optimal convex grids. In [1] , it is proven that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is a convex and decreasing positive function, then there exists $\omega > 0$ large enough, such that the minimizers of

$$S_w(G) = \sum_{q=1}^N f(w \cdot \alpha_q), \tag{3}$$

are convex grids.

For the implementation of an optimization process, we can even use a C^1 function $f_w(\alpha)$ such that $f \equiv 0$ for $\alpha \geq \alpha_l$ and $f'_l(\alpha) < 0$, for $\alpha < \alpha_l$.

A function f which turned out to be very useful is given by

$$f(\alpha) = \begin{cases} 1/\alpha, & \alpha \geq 1 \\ (\alpha - 1)(\alpha - 2) + 1, & \alpha < 1; \end{cases} \tag{4}$$

a thorough discussion on the functional S_w defined by (4) can be found in [4]; it features a “mobile” barrier which is the main tool to generate convex grids (See fig. 4).

2.4 Combination of functionals

In order to combine different geometrical properties in the optimal grids, it is convenient to minimize linear convex combinations of S_w with a classic functional F_c , where the latter is either the length, orthogonality or area-orthogonality functional:

$$F(G) = \sigma S_w(G) + (1 - \sigma)F_c(G).$$

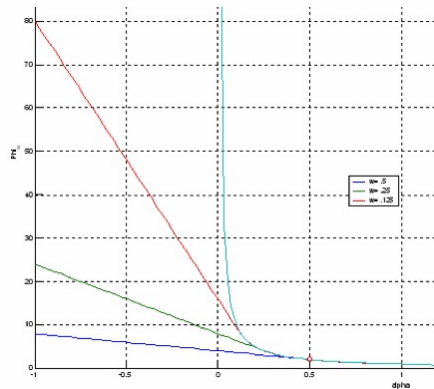


Figure 4: The functional S_ω for different values of ω .

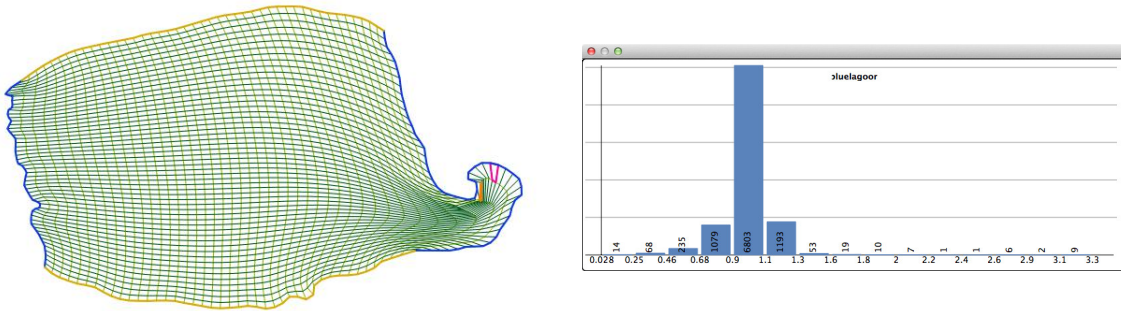


Figure 5: A grid of Blue Lagoon and its area distribution.

Even though it is possible to generate convex grids by minimizing S_ω , numerical experimentation has shown that in very irregular regions, the minimum value of α in the convex grids is very close to zero, an example of this phenomenon is sketched in figure 5.

Since these values of α are closely related to the accuracy of the numerical solution of partial differential equations using these convex grids, small values should be avoided if possible. This can be done using a convex function f which focuses only on those cells whose minimum value of α is less than a threshold α_l . In this way, the global values of smoothness and orthogonality are kept, whereas the area values are corrected (See fig. 6). This leads to propose the functional

$$f_l(\alpha) = A [(\alpha_l - \alpha)_+]^2 \quad (5)$$

where $A > 0$ is a relatively large coefficient.

3 Two-sided area functional

Next, the idea presented in the previous section can be extended to avoid relatively large α values in some cells, *i.e.*, we can impose an upper bound α_r for these values (See

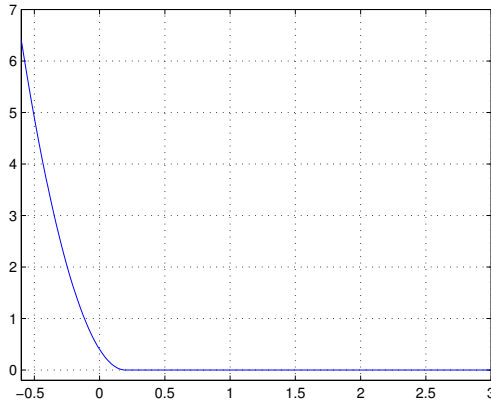


Figure 6: Quadratic truncated function.

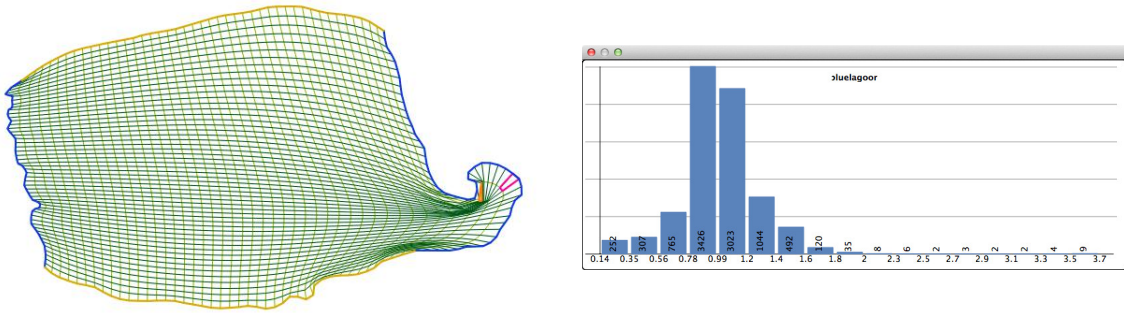


Figure 7: Optimal grid of Blue Lagoon and its area distribution.

fig. 8).

$$f_b(\alpha) = A_l[(\alpha_l - \alpha)_+]^2 + A_r[(\alpha - \alpha_r)_+]^2$$

The new functional F_b defined by f_b can be used in two ways:

- 1) As a convex grid generator first giving a larger weight to a classic functional to reflect its properties and controlling the α values at the end.
- 2) As a cell corrector for those cell whose α values lie outside of $[\alpha_l, \alpha_r]$.

In other words, F_b can be used either as a grid preprocessor or proprocessor.

4 Combination of F_b with classic functionals

F_b can also be combined with the classic functionals F_c of area, length and orthogonality

$$F(\omega G) = \sigma F_b(\omega G) + (1 - \sigma)F_c(G)$$

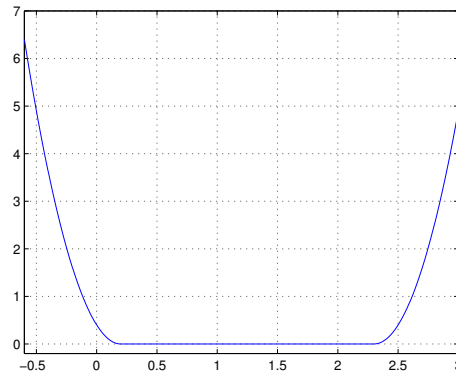


Figure 8: Sketch of f_b .

using an adequate normalization to reflect the properties of both functionals.

The optimization process produces a sequence of grids which converges to a convex grid. However, a large set of runs has provided enough empirical evidence to assert that the convergence rate is decreased in the last iterations due to the fact that $F_l(\omega G)$ becomes notably smaller than $F_c(G)$ since the former is positive only on small cells. Thus, to improve the convergence rate, an extra parameter σ_{eq} is added to reduce the value of the classic functional. This yields the functional

$$F(\omega G) = \sigma F_b(\omega G) + (1 - \sigma)\sigma_{eq}F_c(G);$$

initially, we set $\sigma_{eq} = 1000.0$ and in the optimization process σ_{eq} is updated accordingly to

$$\sigma_{eq} = \lambda \cdot \sigma_{eq}$$

where $0 < \lambda < 1$. Therefore, when σ_{eq} is close to one, the optimal grids of $F(\omega G)$ are close to the optimal grids of the classic functionals; as σ_{eq} decreases, F_b becomes the important component in the combination.

It must be noted that $F(\omega G)$ can also be used either as a pre or postprocessor, in both bases with very satisfactory results.

The functional $F(\omega G)$ is implemented in UNAMALLA [12], and it has proven to be a useful tool for generating and improving convex grids.

4.1 Examples

An important technical issue in UNAMALLA is the fact that the optimal grids generated satisfy the condition that their minimum value of α is larger than a preset critical

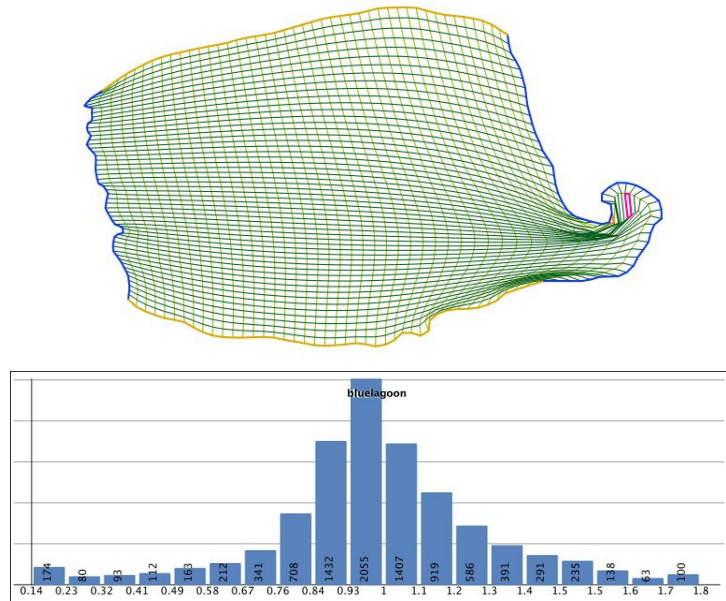


Figure 9: Optimal bilateral grid for Blue lagoon.

value ε ; in [4] this kind of optimal grids are referred to as ε -convex grid. The algorithm used for their generation is described in detail in [4].

Three examples of the grids generated by minimizing and the corresponding area distributions F_b are shown in figures 9, 10 and 11.

5 Conclusions

The proposed functional, used as a pre or post grid processor, is a powerful for the generation of convex grids having area control: if a nearly-convex grid with strong area, smoothness of area-orthogonality properties, F_b can modify the grid to force the α values to lie within the interval $[\alpha_l, \alpha_r]$ without losing the global properties we are looking for. The α histogram provides very useful information to decide how to select α_l and α_r .

An important aspect is the adequate selection of the four boundary segments which represent the geometrical sides of Ω , since this selection affects directly the properties of the grids and their cells.

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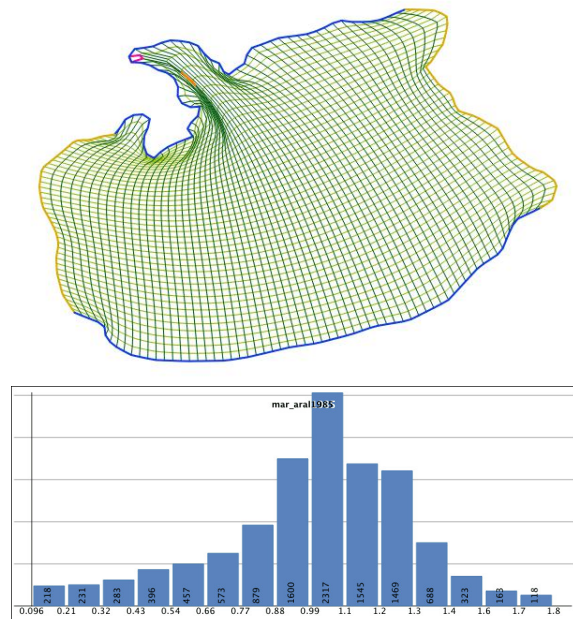


Figure 10: Optimal bilateral grid for Aral Seal on 1985.

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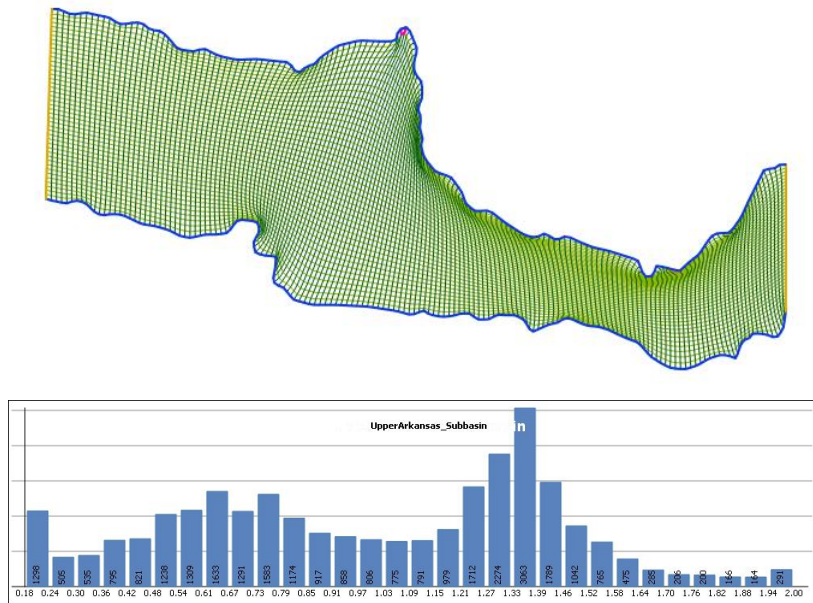


Figure 11: Optimal bilateral grid for the Upper Arkansas subbasins.

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